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Research article

Bayesian and non-Bayesian estimation for the multiple-stress model under progressively type-II hybrid censoring with optimal design

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Abstract: In this paper, estimation of the accelerated life testing (ALT) for the stress model in multiple case is considered. Maximum likelihood estimation is employed to obtain the estimated parameters of the multiple-stress model by a maximum likelihood approach. A Bayesian estimation procedure is proposed to estimate the parameters of the multiple-stress model. Estimation methods are considered under a progressively type-II hybrid censoring scheme with generalized inverted exponential distribution. Different criteria are discussed to determine the optimal design of the progressively type-II hybrid censoring scheme for the multiple-stress model. A simulation study is conducted to obtain results for the maximum likelihood and Bayesian estimates of the parameters of the multiple-stress model. Real data application involving the breakdown voltage of insulating oil is introduced to analyze the performance of the multiple-stress model under progressively type-II hybrid censoring scheme with generalized inverted exponential distribution.

Keywords: accelerated life testing; multiple-stress; generalized inverted exponential distribution; progressively type-II hybrid censoring; maximum likelihood estimation; Bayesian estimation; Markov chain Monte Carlo method; Metropolis–Hastings algorithm
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1. Introduction

For products that are highly reliable, acquiring failure information under actual use conditions can be quite challenging, if not impossible. Consequently, to minimize the time needed to gather failure information, it is advisable to test products under a typical stress levels; this testing approach is referred to as accelerated life testing (ALT). The failure time data obtained from such ALT is analyzed and interpreted to estimate the life characteristics under actual use conditions. There are three primary types of ALTs: constant-stress ALT (CSALT), step-stress ALT (SSALT), and progressive-stress ALT (PSALT). The key distinction among these three types of ALTs lies in the relationship between stress loading and testing duration. In CSALT, the stress remains constant over time, whereas in SSALT, the stress is incrementally increased at predetermined intervals. In contrast, PSALT involves a continuous increase in stress over time. ALT can also be classified based on the number of stress levels into two categories: simple and multiple ALT. Simple ALT consists of only two stress levels, while multiple ALT encompasses more than two stress levels.

In an effort to identify effective ALT strategies in reliability demonstration experiments, a variety of ALT design methodologies have been introduced over the past few decades. For example, Doksum and Hbyland [1] formulated models for variable-stress ALT design based on a fatigue failure framework. Chaloner and Larntz [2] examined ALT design by utilizing two distribution models, specifically lognormal and Weibull, for the failure times of test units. Bagdonavicius et al. [3] addressed ALT design strategies and analyzed ALT data (i.e., failure times of test samples) by employing the changing shape and scale model, which serves as a natural advancement of the well-known accelerated failure time model (AFT). Dorp and Mazzuchi [4] devised a comprehensive Bayes Weibull inference model for ALT, assuming that failure times adhere to a Weibull distribution under a consistent stress level. Zhu and Elsayed [5] applied the design of equivalent accelerated life testing plans for different applications of stress. Furthermore, Han and Bai [6] proposed a step-stress ALT design method that takes into account non-uniform step durations. In their research, they posit that the elevated testing stress level has a maximum threshold and address type-I censoring. Abd El-Raheem et al. [7] presented accelerated life testing under Pareto IV lifetime distribution. Rahman et al. [8] discussed a statistical inference on step-stress partially accelerated life testing based on multiple censoring approaches. Nelson [9] discussed the accelerated life testing with step and varying stress.

The two primary methods of censoring are recognized as type-I and type-II censoring methods. In the traditional type-I censoring method, the trial proceeds until a predetermined time T. Conversely, in the traditional type-II censoring method, testing carries on until a predetermined count of failures takes place. The combination of type-I and type-II censoring methods is referred to as a hybrid censoring method. In recent years, the progressive type-II censoring method has gained considerable attention. A significant disadvantage of this progressive censoring approach is that the duration of the experiment may be extensive when the items exhibit high reliability. Reliability engineering is utilized to evaluate data derived from experiments that involve multiple items. It integrates type-I and type-II censoring schemes to improve experimental efficiency while upholding statistical rigor. Type-II progressive censoring scheme allows experiments to conclude at a predetermined time, rendering it appropriate for data that is highly reliable.

The topic of censoring has been discussed in many papers in literature. Pradhan and Kundu [10] introduced the progressively censored generalized exponential distribution. Salah [11] presented a parameter estimation of the Marshall-Olkin exponential distribution under type-II hybrid censoring schemes. Sen et al. [12] presented a statistical inference for lognormal distribution with type I progressive hybrid censored data. Arabi and Noori [13] discussed estimation based on progressively type-I hybrid censored data from the Burr XII distribution. El-Sherpieny et al. [14] discussed the progressive type-II hybrid censored schemes based on maximum product spacing applied to power Lomax distribution. Zhu [15] introduced a statistical inference of Weibull distribution based on generalized progressively hybrid censored data. Berzborn and Cramer [16] presented an inference for type-I and type-II hybrid censored minimal repair and record data.

The investigation into ALT goes beyond fundamental models that rely on censored samples. Riad et al. [17] studied the step-stress accelerated life testing for the Burr-XII distribution using a cumulative exposure model under the progressive type-II censoring. Abushal and Abdel-Hamid [18] introduced an inference on a new distribution under progressive-stress accelerated life tests and progressive type-II censoring based on a series-parallel system. Yousef et al. [19] presented an inference on partially accelerated life testing for the inversed Kumaraswamy distribution based on type-II progressive censoring data. Ismail [20] introduced a progressive stress accelerated life test for the inverse Weibull failure model. Alotaibi et al. [21] introduced classical and Bayesian inference of a progressive-stress model for the Nadarajah-Haghighi distribution with type-II progressive censoring and different loss functions. Moala and Chagas [22] introduced a Bayesian analysis for multiple step-stress accelerated life testing under gamma lifetime distribution subject to type-II censoring. Mahto et al. [23] presented a Bayesian estimation and prediction under progressive-stress accelerated life test for a log-logistic model.

The optimal design of ALTs has attracted significant research interest. Barton [24] introduced a change in the optimum ALT plans described by Nelson and other authors, and discussed how to minimize the accelerated test stress. Bai and Chung [25] introduced optimal designs based on partially accelerated life tests (PALTs), in which the test can be altered when there are changes in acceleration and use conditions. Bai et al. [26] proposed an optimal simple ramp-test for Weibull distribution under type-I censoring considering different linearly increasing stresses. Haghighi [27] proposed an optimal design of accelerated life tests for an extension of exponential distribution. Guan et al. [28] proposed an optimal multiple constant-stress accelerated life tests for generalized exponential distribution. Abd El-Raheem [29] suggested an optimal design of multiple accelerated life testing for generalized half-normal distribution under type-I censoring. Abd El-Raheem [30] presented an optimal design of multiple constant-stress accelerated life testing for the extension of the exponential distribution under type-II censoring.

The generalized inverted exponential distribution (GIED) represents a modification of the inverse exponential distribution (IED). This adaptation allows for a more accurate fitting of lifetime data. The GIED was first introduced by Abouammoh and Alshingiti [31]. This distribution features a non-constant hazard rate function that is unimodal and positively skewed. These characteristics enable the distribution to effectively model various shapes of failure rates associated with aging criteria. Singh et al. [32] suggested utilizing the maximum product of spacing method for point estimation of the GIED parameter. Al-Omari [33] formulated acceptance sampling plans based on truncated lifetimes, assuming that the lifespan of an item adheres to a generalized inverted exponential distribution. Dube et al. [34] conducted a Monte Carlo simulation for the GIED to evaluate the performance of the estimations. Singh et al. [35] examined the point estimation of the GIED parameters under progressively type-II censored test units. Dey et al. [36] generated samples from the GIED and calculated the Bayes estimates. Njeri and Njenga [37] explored the maximum likelihood estimates (MLEs) of the GIED in the context of progressively type-II censored test units. Chen and Gui [38] proposed a statistical inference of the generalized inverted exponential distribution under joint progressively type-II censoring.

In this article, accelerated life testing (ALT) in a stress model across various scenarios is discussed. Maximum likelihood estimation method is employed to derive the estimated parameters of the multiple-stress model. A Bayesian estimation approach is applied for determining the parameters related to the multiple-stress model. Estimation techniques are examined in the context of a progressively type-II hybrid censoring scheme utilizing a generalized inverted exponential distribution. Various criteria are presented to find the optimal censoring scheme. A simulation study is conducted to generate results pertaining to the maximum likelihood and Bayesian estimates for the parameters of the multiple-stress model. An application using real data examining the breakdown voltage of insulating oil is included to demonstrate the use of the multiple-stress model within the framework of a progressively type-II hybrid censoring scheme with generalized inverted exponential distribution.

2. Generalized inverted exponential distribution

Abouammoh and Alshingiti [31] introduced the generalized inverted exponential distribution (GIED). This distribution exhibits a positively skewed, unimodal, non-constant hazard rate function. These characteristics allow the distribution to model various forms of aging criteria and failure rates.

GIED serves as a generalization of the one-parameter exponential distribution, developed by researchers to enhance the model's adaptability for a range of applications. GIE distribution is distinguished by its capacity to accommodate various shapes of failure rates, rendering it appropriate for modeling real-world situations where the data displays skewness, including left-skewed and right-skewed distributions, as well as symmetric or reversed-J shapes. This distribution has found applications in numerous fields such as reliability engineering, economics, forecasting, astronomy, demography, and insurance.

GIED provides a more reliable model for data analysis and decision-making, making it an invaluable tool for researchers and practitioners across a range of disciplines. The probability density function for a random variable ξ that follows GIED is given by:

$$f(\xi) = \frac{\alpha\vartheta}{\xi^2} e^{-\vartheta/\xi} \left(1 - e^{-\vartheta/\xi}\right)^{\alpha-1}, \xi > 0, \alpha, \vartheta > 0. \tag{1}$$

The cumulative distribution function of ξ is given by:

$$F(\xi) = 1 - \left(1 - e^{-\vartheta/\xi}\right)^{\alpha}, \xi > 0, \alpha, \vartheta > 0. \tag{2}$$

3. Progressively type-II hybrid censoring

Recently, the type-II progressive censoring scheme has gained significant popularity for the analysis of highly reliable data. However, a notable disadvantage of the type-II progressive censoring scheme is that the duration of the experiment can be extensive. Kundu and Joarder [39] proposed a type-II progressively hybrid censored scheme where the experiment terminates at a pre-specified time.

Assume N items are assigned to a life test consisting of k stress levels, say $s_0, s_1, \dots, s_k$, where $s_0 < s_1 < \cdots < s_k$ and be the normal stress is s_0. For a stress level $s_i, i = 1, \dots, k$, there are n_i items assigned for testing such that $\sum_{i=1}^{k} n_i = \mathrm{N}$.

According to progressively type-II hybrid censoring, suppose that:

1) m_i identical items are put on a test at stress level s_i, and the lifetime distributions of the m_i items are denoted by $\xi_{ij}, \dots, \xi_{im_i}$.
2) The integers $m_i < n_i$ are fixed at the beginning of the experiment.
3) The censored schemes at each level are R_{ij}^* such that $\sum_{j=1}^{m_i} R_{ij}^* + m_i = n_i$.
4) The time points $T_i, i = 1, \dots, k$ are fixed at the beginning of the experiment.
5) At the first level, at the time of first failure $\xi_{1j:m_1:n_1}$, R_{i1}^* of the remaining units are removed from the experiment.
6) Similarly, at the time of the second failure $\xi_{2j:m_2:n_2}$, R_{i2}^* of the remaining units are removed and so on.
7) If the m_i failure occurs before the time point T_i, then the experiment stops at the time point $\xi_{im_i:m_i:n_i}$.
8) If the m_i failure does not occur before time point T_i and only J_i failures occur before the time point T_i, where $0 \le J_i < m_i$, then at the time point T_i, all remaining $R_{iJ_i}^*$ units are removed, and the experiment terminates at the time point T_i where $R_{iJ_i}^* = m_i - \sum_{\varepsilon=1}^{J_i} R_{i\varepsilon}^* - J_i$.
9) Each unit in the model has lifetime that follows GIED with parameters α and ϑ_i for each stress level.
10) Consider the following exponential model that connects the parameter ϑ_i and the stress s_i: $\vartheta_i = \vartheta_0 \psi^{\eta_i}, i = 1, \dots, k,$ where ϑ_0 and ψ are unknown parameters to be estimated, and η_i is a function of s_i.

4. Maximum likelihood estimation

The maximum likelihood estimation method can be implemented to obtain the MLE estimates for the parameters of the multiple-stress model under progressively type-II hybrid censoring as follows:

$$\mathcal{L} = \prod_{i=1}^{k} \prod_{j=1}^{d_i} f\left(\xi_{ij:m_i:n_i}\right) \left[\bar{F}\left(\xi_{ij:m_i:n_i}\right)\right]^{R_{ij}^*} \bar{F}^{\delta_i R_{d_i}^*}(t_i) \tag{3}$$

where

$$d_i = \begin{cases} m_i, & \xi_{im_i:m_i:n_i} < t_i, \\ J_i, & \xi_{iJ_i:m_i:n_i} < t_i < \xi_{iJ_{i+1}:m_i:n_i}. \end{cases}$$

and

$$\delta_i = \begin{cases} 0, & \xi_{im_i:m_i:n_i} < t_i, \\ 1, & \xi_{iJ_i:m_i:n_i} < t_i < \xi_{iJ_{i+1}:m_i:n_i}, \end{cases} \quad R_{iJ_i}^* = n_i - J_i - \sum_{\varepsilon=1}^{J_i} R_{i\varepsilon}^*.$$

Applying GIED, the likelihood function will take the form:

$$\mathcal{L} = \prod_{i=1}^{k} \prod_{j=1}^{d_i} \frac{\alpha \vartheta_0 \psi^{\eta_i}}{\xi_{ij}^2} e^{-\frac{\vartheta_0 \psi^{\eta_i}}{\xi_{ij}}} \left(1 - e^{-\frac{\vartheta_0 \psi^{\eta_i}}{\xi_{ij}}}\right)^{\alpha\left(R_{ij}^*+1\right)-1} \left(1 - e^{-\frac{\vartheta_0 \psi^{\eta_i}}{t_i}}\right)^{\delta_i \alpha R_{d_i}^*} \tag{4}$$

Taking the logarithmic of the likelihood function, the following equation will be obtained:

$$\log\mathcal{L} = \sum_{i=1}^{k}\sum_{j=1}^{d_i}\log\alpha + \sum_{i=1}^{k}\sum_{j=1}^{d_i}\log\vartheta_0 + \sum_{i=1}^{k}\sum_{j=1}^{d_i}\eta_i\log\psi - \sum_{i=1}^{k}\sum_{j=1}^{d_i}\log\xi_{ij}^2 - \vartheta_0\sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\psi^{\eta_i}}{\xi_{ij}}$$
$$+\sum_{i=1}^{k}\sum_{j=1}^{d_i}\left(\alpha\left(R_{ij}^*+1\right)-1\right)\log\left(1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}\right)$$
$$+\sum_{i=1}^{k}\delta_i\alpha R_{d_i}^*\log\left(1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}\right) \tag{5}$$

Differentiating Eq (5), with respect to the parameters $\alpha, \vartheta_0,$ and ψ, yields:

$$\frac{\partial\log\mathcal{L}}{\partial\alpha} = \sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{1}{\alpha} + \sum_{i=1}^{k}\sum_{j=1}^{d_i}\left(R_{ij}^*+1\right)\log\left(1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}\right) + \sum_{i=1}^{k}\delta_i R_{d_i}^*\log\left(1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}\right) \tag{6}$$

$$\frac{\partial\log\mathcal{L}}{\partial\vartheta_0} = \sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{1}{\vartheta_0} - \sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\psi^{\eta_i}}{\xi_{ij}} + \sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\left(\alpha\left(R_{ij}^*+1\right)-1\right)\left(\frac{\psi^{\eta_i}}{\xi_{ij}}\right)e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}}{1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}}$$
$$+\sum_{i=1}^{k}\frac{\delta_i\alpha R_{d_i}^*\left(\frac{\psi^{\eta_i}}{t_i}\right)e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}}{1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}} \tag{7}$$

$$\frac{\partial\log\mathcal{L}}{\partial\psi} = \sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\eta_i}{\psi} - \vartheta_0\sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\eta_i\psi^{\eta_i-1}}{\xi_{ij}} + \sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\left(\alpha\left(R_{ij}^*+1\right)-1\right)\left(\frac{\vartheta_0\eta_i\psi^{\eta_i-1}}{\xi_{ij}}\right)e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}}{1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}}$$
$$+\sum_{i=1}^{k}\frac{\delta_i\alpha R_{d_i}^*\left(\frac{\vartheta_0\eta_i\psi^{\eta_i-1}}{t_i}\right)e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}}{1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}} \tag{8}$$

It can be observed that Eqs (6)–(8) are nonlinear with respect to $\alpha, \vartheta_0,$ and ψ and can not to be solved analytically. So, the numerical method of Newton–Raphson can be employed to find the maximum likelihood estimates for the parameters $\alpha, \vartheta_0,$ and ψ.

5. Fisher information matrix

Fisher information matrix measures the amount of data that an experiment offers concerning the parameters that are being estimated. The observed Fisher information matrix for the parameters $\alpha, \vartheta_0,$ and ψ can be defined as:

$$I = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

where

$$I_{11}=-\frac{\partial^2 log\mathcal{L}}{\partial\alpha^2}, I_{12}=I_{21}=-\frac{\partial^2 log\mathcal{L}}{\partial\alpha\partial\vartheta_0}, I_{13}=I_{31}=-\frac{\partial^2 log\mathcal{L}}{\partial\alpha\partial\psi},$$

$$I_{22}=-\frac{\partial^2 log\mathcal{L}}{\partial\vartheta_0^2}, I_{23}=I_{32}=-\frac{\partial^2 log\mathcal{L}}{\partial\vartheta_0\partial\psi}, I_{33}=-\frac{\partial^2 log\mathcal{L}}{\partial\psi^2}.$$

The second derivatives of the parameters $\alpha, \vartheta_0,$ and ψ can be obtained by taking the derivative of the Eqs (6)–(8), so the following equations will be obtained:

$$\frac{\partial^2 log\mathcal{L}}{\partial\alpha^2}=-\sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{1}{\alpha^2} \tag{9}$$

$$\frac{\partial^2 log\mathcal{L}}{\partial\alpha\partial\vartheta_0}=\sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\left(R_{ij}^*+1\right)\left(\frac{\psi^{\eta_i}}{\xi_{ij}}\right)e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}}{1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}}+\sum_{i=1}^{k}\frac{\delta_i R_{d_i}^*\left(\frac{\psi^{\eta_i}}{t_i}\right)e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}}{1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}} \tag{10}$$

$$\frac{\partial^2 log\mathcal{L}}{\partial\alpha\partial\psi}=\sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\left(R_{ij}^*+1\right)\left(\frac{\vartheta_0\eta_i\psi^{\eta_i-1}}{\xi_{ij}}\right)e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}}{1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}}+\sum_{i=1}^{k}\frac{\delta_i R_{d_i}^*\left(\frac{\vartheta_0\eta_i\psi^{\eta_i-1}}{t_i}\right)e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}}{1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}} \tag{11}$$

$$\frac{\partial^2 log\mathcal{L}}{\partial\vartheta_0^2}=-\sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{1}{\vartheta_0^2}-\sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\left(\alpha\left(R_{ij}^*+1\right)-1\right)\left(\frac{\psi^{\eta_i}}{\xi_{ij}}\right)^2 e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}}{\left(1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}\right)^2}-\sum_{i=1}^{k}\frac{\delta_i\alpha R_{d_i}^*\left(\frac{\psi^{\eta_i}}{t_i}\right)^2 e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}}{\left(1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}\right)^2} \tag{12}$$

$$\frac{\partial^2 log\mathcal{L}}{\partial\psi\partial\vartheta_0}=-\sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\eta_i\psi^{\eta_i-1}}{\xi_{ij}}+\sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\left(\alpha\left(R_{ij}^*+1\right)-1\right)\left(\frac{\eta_i\psi^{\eta_i-1}}{\xi_{ij}}\right)e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}\left(1-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}-e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}\right)}{\left(1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}\right)^2}+$$

$$+\sum_{i=1}^{k}\frac{\delta_i\alpha R_{d_i}^*\left(\frac{\eta_i\psi^{\eta_i-1}}{t_i}\right)e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}\left(1-\frac{\vartheta_0\psi^{\eta_i}}{t_i}-e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}\right)}{\left(1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}\right)^2} \tag{13}$$

$$\frac{\partial^2 log\mathcal{L}}{\partial\psi^2}=-\sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\eta_i}{\psi^2}-\vartheta_0\sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\eta_i(\eta_i-1)\psi^{\eta_i-2}}{\xi_{ij}} \tag{14}$$

$$+\sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\left(\alpha\left(R_{ij}^*+1\right)-1\right)\vartheta_0\eta_i\left(\frac{\psi^{\eta_i-1}}{\xi_{ij}}\right)e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}\left[\left(\frac{\eta_i-1}{\psi}\right)\left(1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}\right)-\vartheta_0\eta_i\left(\frac{\psi^{\eta_i-1}}{\xi_{ij}}\right)\right]}{\left(1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}\right)^2}$$

$$+\sum_{i=1}^{k}\frac{\delta_i\alpha R_{d_i}^{*}\vartheta_0\eta_i\left(\frac{\psi^{\eta_i-1}}{t_i}\right)e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}\left[\left(\frac{\eta_i-1}{\psi}\right)\left(1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}\right)-\vartheta_0\eta_i\left(\frac{\psi^{\eta_i-1}}{t_i}\right)\right]}{\left(1-e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}\right)^2}.$$

6. Optimal design methods

Prior to choosing a specific sampling strategy, it is essential to ascertain which progressive censoring technique provides the most insightful information regarding the unknown parameters we are trying to estimate. So, different methods of optimality will be discussed to obtain the optimal design of the progressively type-II hybrid censoring scheme to estimate the unknown parameters of the multiple stress model.

Method 1: The observed Fisher information matrix measures the extent of information that a study delivers regarding the estimated parameters. In this regard, an increased value indicates that the data is more informative. In this method, the objective is to maximize the trace of the observed Fisher information matrix $\{Max\ trace(I)\}$, where the trace is defined as the sum of the diagonal elements of the observed Fisher information matrix.

Method 2: This method is designed to reduce the ambiguity in the evaluations. This is achieved by decreasing the trace of the inverse of the observed Fisher information matrix $\{Min\ trace(I)^{-1}\}$. Smaller values signify a reduction in uncertainty.

Method 3: This approach aims to decrease the uncertainty present in the assessments. This goal is accomplished by lowering the determinant of the inverse of the observed Fisher information matrix $\{Min\ det(I)^{-1}\}$. Smaller values indicate a decrease in ambiguity.

7. Bayesian estimation

Choosing a prior for the unknown model parameter is a crucial aspect of Bayesian analysis. The gamma prior density class is notably versatile as it allows for the modeling of diverse prior information; for further details, refer to Xu and Wang [40] and Zhu et al. [41]. Therefore, in order to conduct the Bayesian estimation, the prior distribution for the parameters $\alpha, \vartheta_0,$ and ψ is assumed to be gamma distribution according to the following:

$$\pi(\alpha)=\alpha^{\zeta_1-1}e^{-\nu_1\alpha},\quad \zeta_1,\nu_1>0$$

$$\pi(\vartheta_0)=\vartheta_0^{\zeta_2-1}e^{-\nu_2\vartheta_0},\quad \zeta_2,\nu_2>0$$

$$\pi(\psi)=\psi^{\zeta_3-1}e^{-\nu_3\psi},\quad \zeta_3,\nu_3>0$$

The conditional posterior distributions of the parameters $\alpha, \vartheta_0,$ and ψ are derived as follows:

$$\pi(\alpha|\vartheta_0,\psi)\propto\alpha^{\sum_{i=1}^{k}d_i+\zeta_1-1}e^{-\alpha\Upsilon_1}, \tag{15}$$

$$\pi(\vartheta_0|\psi)\propto\vartheta_0^{\sum_{i=1}^{k}d_i+\zeta_2-1}e^{-\vartheta_0\Upsilon_2}, \tag{16}$$

$$\pi(\psi|\vartheta_0) \propto \psi^{\sum_{i=1}^{k} d_i\eta_i+\zeta_3-1} e^{-\Upsilon_3}. \tag{17}$$

From Eqs (15) and (16), it can be concluded that:

$$\alpha \sim Gamma\left(\textstyle\sum_{i=1}^{k} d_i + \zeta_1, \Upsilon_1\right), \tag{18}$$

$$\vartheta_0 \sim Gamma\left(\textstyle\sum_{i=1}^{k} d_i + \zeta_2, \Upsilon_2\right), \tag{19}$$

where

$$\Upsilon_1 = \nu_1 - \sum_{i=1}^{k}\sum_{j=1}^{d_i}\left(R_{ij}^* + 1\right)\log\left(1 - e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}\right) - \sum_{i=1}^{k}\delta_i R_{d_i}^* \log\left(1 - e^{-\frac{\vartheta_0\psi^{\eta_i}}{t_i}}\right),$$

$$\Upsilon_2 = \nu_2 + \sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{\psi^{\eta_i}}{\xi_{ij}},$$

$$\Upsilon_3 = \psi\nu_3 + \sum_{i=1}^{k}\sum_{j=1}^{d_i}\log\left(1 - e^{-\frac{\vartheta_0\psi^{\eta_i}}{\xi_{ij}}}\right).$$

It is obvious that the conditional posterior distribution for α and ϑ_0 can be generated from gamma distribution. However, ψ cannot be directly simulated from its posterior distribution as it is not in known form and in this case the Metropolis-Hastings algorithm can be applied to simulate random samples from the posterior density for ψ.

8. Markov chain Monte Carlo method

Markov chain Monte Carlo (MCMC) techniques include a range of algorithms used for sampling from a probability distribution. By creating a Markov chain where the target distribution acts as its equilibrium distribution, it is possible to gather samples from the intended distribution by observing states of the chain. Increasing the number of steps taken leads to a sample distribution that more accurately reflects the actual target distribution. There are several algorithms available for creating these chains, one of which is the Metropolis–Hastings (MH) algorithm. The steps for applying the MCMC method and the Metropolis–Hastings algorithm are as follows:

Algorithm 1.

Step 1: Put initial values for the parameters $\alpha, \vartheta_0,$ and ψ, say $\alpha^{(0)}, \vartheta_0^{(0)},$ and $\psi^{(0)}$.
Step 2: Set $i = 1$.
Step 3: Generate $\alpha^{(i)}$ from $Gamma\left(\sum_{i=1}^{k} d_i + \zeta_1, \Upsilon_1\right)$.
Step 4: Generate $\vartheta_0^{(i)}$ from $Gamma\left(\sum_{i=1}^{k} d_i + \zeta_2, \Upsilon_2\right)$.
Step 5: Generate $\psi^{(i)}$ from $\pi(\psi|\vartheta_0)$ using the MH algorithm as follows:

(i) Generate proposals $\psi^{(i)}$ from $N\left(\psi^{(i-1)}, Var(\psi^{(i-1)})\right), i = 1, \dots, k$.

(ii) Evaluate the acceptance probabilities $z = min\left\{1, \frac{\pi\left(\psi^{(i)}|\vartheta_0\right)}{\pi\left(\psi^{(i-1)}|\vartheta_0\right)}\right\}$.

(iii) Generate u from *uniform*(0, 1).

(iv) If $u \leq z$, accept the proposal and set $\psi^{(i)} = \psi^{(i)}$, else set $\psi^{(i)} = \psi^{(i-1)}$.
Step 6: Set $i = i + 1$ and repeat steps 3 to 6, Ω times.
Step 7: Calculate the Bayesian estimates of α, ϑ_0, and ψ from the following formulas:

$$\tilde{\alpha} = \frac{1}{\Omega}\sum_{i=1}^{\Omega} \alpha^{(i)}, \tilde{\vartheta}_0 = \frac{1}{\Omega}\sum_{i=1}^{\Omega} \vartheta_0^{(i)}, \tilde{\psi} = \frac{1}{\Omega}\sum_{i=1}^{\Omega} \psi^{(i)}.$$

Step 8: To construct the credible interval for the parameters α, ϑ_0, and ψ, order $\alpha^{(i)}$, $\vartheta_0^{(i)}$, and $\psi^{(i)}$ in ascending order. Then, $100(1-\varepsilon)\%$ credible intervals for the parameters α, ϑ_0, and ψ are given by:

$$\left[\tilde{\alpha}^{[\Omega(\varepsilon/2)]}, \tilde{\alpha}^{[\Omega(1-\varepsilon/2)]}\right], \quad \left[\tilde{\vartheta}_0^{[\Omega(\varepsilon/2)]}, \tilde{\vartheta}_0^{[\Omega(1-\varepsilon/2)]}\right], \quad \left[\tilde{\psi}^{[\Omega(\varepsilon/2)]}, \tilde{\psi}^{[\Omega(1-\varepsilon/2)]}\right].$$

9. Simulation study

The simulation demonstrates how maximum likelihood and Bayesian estimation methods perform in different scenarios using progressively type-II hybrid censoring. To apply a simulation study, the following algorithm is proposed to show the steps for obtaining the maximum likelihood estimates (MLEs) of the multiple-stress model.

Algorithm 2.
Step 1: Set $k, m_i, n_i, \eta_i, R_{ij}^*, t_i, i = 1, 2, \ldots, k, j = 1,2, \ldots, d_i$.
Step 2: Put initial values for the parameters α, ϑ_0, and ψ.
Step 3: Suppose that $u_{ij} \sim uniform(d_i, 0, 1)$, $j = 1, \ldots, d_i$, $i = 1, 2, \ldots, k$.
Step 4: Set $Z_{ij} = -\log(1 - u_{ij}), j = 1, \ldots, d_i$, $i = 1, 2, \ldots, k$.
Step 5: Set

$$\Lambda_{ij} = \Lambda_{ij-1} + \frac{Z_{ij}}{n_i - \sum_{j=1}^{j-1} R_{ij}^* - j + 1}, \qquad i = 1, 2, \ldots, k$$

Where $\Lambda_{11} = \frac{Z_{11}}{m_1}$.
Step 6: Set $H_{ij} = 1 - e^{-\Lambda_{ij}}$.
Step 7: Generate the random variables ξ_{ij} using the following formula:

$$\xi_{ij} = -\frac{\vartheta_0 \psi^{\eta_i}}{\log\left[1 - (1 - H_{ij})^{\frac{1}{\alpha}}\right]} \quad for\ j = 1, \ldots, d_i, i = 1, 2, \ldots, k.$$

Step 8: Find the maximum likelihood estimates for the parameters α, ϑ_0, and ψ by using the Eqs (6)–(8) and apply the Newton–Raphson method in R program using, package optim().
Step 9: Repeat steps (3)–(8), 5000 times and then calculate the bias and mean squared error (MSE) using the following formulas:

$$Bias = \frac{1}{5000}\sum_{i=1}^{5000} (\Theta_i - \Theta), MSE = \frac{1}{5000}\sum_{i=1}^{5000} (\Theta_i - \Theta)^2,$$

where $\Theta = (\alpha, \vartheta_0, \psi)$.

The simulation study for the progressively type-II hybrid censoring will be implemented after assuming the following:

$$\alpha = 0.1, \vartheta_0 = 0.5,\ \psi = 0.1,\ \eta_i = 1.5, t_i = \xi_{im_i:m_i:n_i} + 1, i = 1,2,3, j = 1,2,3,4,5.$$

Considering the following different schemes:
Scheme (1): $R_{ij}^* = 0\ for\ j \neq m_i, R_{ij}^* = n_i - m_i\ for\ j = m_i$.
Scheme (2): $R_{ij}^* = 0\ for\ j\ is\ even, R_{ij}^* = 1\ for\ j\ is\ odd, R_{im_i}^* = n_i - m_i - \sum_{\varepsilon=1}^{m_i-1} R_{i\varepsilon}^*$.
Scheme (3): $R_{ij}^* = 1\ for\ j\ is\ even, R_{ij}^* = 0\ for\ j\ is\ odd, R_{im_i}^* = n_i - m_i - \sum_{\varepsilon=1}^{m_i-1} R_{i\varepsilon}^*$.

The results for the values of the maximum likelihood (MLEs) and Bayesian estimates for the parameters of the multiple-stress model under the progressively type II hybrid censoring with (bias and MSE) at different schemes are given in Table 1, assuming different values of n_i and m_i for $i = 1,2,3$ and $k = 3$.

Table 1. The results for MLEs and Bayesian estimates for the parameters with bias and MSE at different schemes.

(n_1, n_2, n_3)	(m_1, m_2, m_3)	Scheme	MLEs			Bayes E		
			$\hat{\alpha}$	$\hat{\vartheta}_0$	$\hat{\psi}$	$\tilde{\alpha}$	$\tilde{\vartheta}_0$	$\tilde{\psi}$
(10, 10, 10)	(5, 5, 5)	(1)	0.60007	0.49169	0.36085	0.86032	0.81621	0.50071
			(-0.12112)	(0.13429)	(0.45092)	(0.59514)	(0.13300)	(3.97229e-06)
			(0.07104)	(0.06655)	(0.43161)	(0.53468)	(0.09523)	(2.46161e-07)
		(2)	0.35004	0.48691	0.33224	0.97805	0.67327	0.50001
			(-0.13474)	(0.12544)	(0.40157)	(0.47805)	(0.17327)	(1.80579e-05)
			(0.04501)	(0.05323)	(0.40706)	(0.36645)	(0.11939)	(2.42596e-07)
		(3)	0.40607	0.57392	0.33720	0.94326	0.64886	0.49980
			(-0.09566)	(0.11979)	(0.41382)	(0.52446)	(0.14886)	(1.05054e-05)
			(0.04441)	(0.06571)	(0.41061)	(0.42414)	(0.10134)	(2.47352e-07)
(20, 20, 20)	(10, 10, 10)	(1)	0.40283	0.58509	0.32650	1.26244	0.70983	0.50043
			(0.05979)	(0.03174)	(0.36987)	(0.17381)	(0.02854)	(3.08472e-07)
			(0.04012)	(0.03355)	(0.21071)	(0.03639)	(0.02526)	(2.40950e-07)
		(2)	0.33591	0.57976	0.41323	1.00602	0.58888	0.50014
			(-0.07410)	(0.06336)	(0.30748)	(0.13983)	(0.05782)	(7.47326e-06)
			(0.02086)	(0.02104)	(0.12978)	(0.02398)	(0.03009)	(2.41792e-07)
		(3)	0.47313	0.52948	0.18142	0.92933	0.55125	0.50056
			(-0.05381)	(0.05631)	(0.31274)	(0.14973)	(0.05125)	(9.47940e-07)
			(0.02113)	(0.02102)	(0.14001)	(0.02729)	(0.03002)	(2.38108e-07)
(30, 30, 30)	(15, 15, 15)	(1)	0.68523	0.53169	0.46344	1.41768	0.49962	0.50002
			(-0.00204)	(0.02763)	(0.27131)	(0.09769)	(-0.00037)	(2.58968e-07)
			(0.03289)	(0.01478)	(0.09480)	(0.01078)	(0.01418)	(2.20655e-07)
		(2)	0.43185	0.51938	0.36714	1.09458	0.52850	0.49982
			(-0.05426)	(0.03711)	(0.27622)	(0.10139)	(0.02850)	(1.71573e-08)
			(0.01740)	(0.01006)	(0.09881)	(0.02195)	(0.01738)	(2.38869e-07)
		(3)	0.38628	0.44028	0.43691	1.27027	0.52088	0.50002
			(-0.03448)	(0.03244)	(0.27578)	(0.08797)	(0.02088)	(1.75741e-08)
			(0.01692)	(0.01028)	(0.09716)	(0.02001)	(0.01649)	(2.35538e-07)

The results for the 95% credible intervals (with length) for the parameters of the multiple-stress model under progressively type-II hybrid censoring at different schemes are given in Table 2, assuming different values of n_i and m_i for $i = 1, 2, 3$ and $k = 3$.

Table 2. The results for the 95% credible intervals (with length) at different schemes.

(n_1, n_2, n_3)	(m_1, m_2, m_3)	Scheme	C. I. $(\tilde{\alpha})$	C. I. $(\tilde{\vartheta}_0)$	C. I. $(\tilde{\psi})$
(10, 10, 10)	(5, 5, 5)	(1)	[0.47874, 2.14033] (1.66158)	[0.25921, 1.32604] (1.06683)	[0.49901, 0.50097] (0.00195)
		(2)	[0.43263, 1.86904] (1.43641)	[0.27254, 1.42919] (1.15665)	[0.49902, 0.50098] (0.00196)
		(3)	[0.46180, 1.93291] (1.47110)	[0.27197, 1.33382] (1.06185)	[0.49902, 0.50098] (0.00196)
(20, 20, 20)	(10, 10, 10)	(1)	[0.72163, 2.15247] (1.43083)	[0.29044, 0.89754] (0.60710)	[0.49901, 0.50095] (0.00193)
		(2)	[0.64306, 1.87114] (1.22808)	[0.30470, 0.94844] (0.64374)	[0.49906, 0.50098] (0.00192)
		(3)	[0.66793, 1.93635] (1.26842)	[0.30742, 0.95224] (0.64482)	[0.49906, 0.50097] (0.00191)
(30, 30, 30)	(15, 15, 15)	(1)	[0.87637, 2.18425] (1.30787)	[0.30593, 0.77645] (0.47051)	[0.49910, 0.50096] (0.00186)
		(2)	[0.76864, 1.85469] (1.08604)	[0.32454, 0.82275] (0.49820)	[0.49907, 0.50099] (0.00192)
		(3)	[0.78830, 1.97651] (1.18820)	[0.32175, 0.81688] (0.49513)	[0.49908, 0.50099] (0.00190)

The results for the optimal design of the progressively type-II hybrid censoring scheme in estimating the unknown parameters of the multiple-stress model at different schemes are illustrated in Table 3, assuming different values of n_i and m_i for $i = 1, 2, 3$ and $k = 3$.

Table 3. The results for the optimal design of the progressively type-II hybrid censoring scheme at different schemes.

(n_1, n_2, n_3)	(m_1, m_2, m_3)	Scheme	Method 1	Method 2	Method 3
(10, 10, 10)	(5, 5, 5)	(1)	56911.5500	0.18748	3.52984e-08
		(2)	1772.23600	0.26474	1.66452e-12
		(3)	7555.76900	4.403e-05	1.74818e-13
(20, 20, 20)	(10, 10, 10)	(1)	3753.34300	0.06010	1.30441e-09
		(2)	3374.10500	0.13688	4.225e-21
		(3)	2716.17800	6.980e-06	1.86569e-15
(30, 30, 30)	(15, 15, 15)	(1)	4392.34800	0.04546	3.26728e-10
		(2)	4506.31900	0.04934	2.25515e-10
		(3)	4537.6400	0.09129	6.04290e-10

The results for the values of the maximum likelihood (MLEs) and Bayesian estimates for the parameters of the multiple-stress model under the progressively type-II hybrid censoring with (bias and MSE) at different schemes are given in Table 4, assuming different values of n_i and m_i for $i = 1, 2, 3, 4$ and $k = 4$.

Table 4. The results for the MLEs and Bayesian estimates for the parameters with bias and MSE at different schemes.

(n_1, n_2, n_3, n_4)	(m_1, m_2, m_3, m_4)	Scheme	MLE			Bayes E		
			$\hat{\alpha}$	$\hat{\vartheta}_0$	$\hat{\psi}$	$\tilde{\alpha}$	$\tilde{\vartheta}_0$	$\tilde{\psi}$
(10, 10, 10, 10)	(5, 5, 5, 5)	(1)	0.32046	0.53745	0.37688	1.10308	0.62882	0.50000
			(-0.05153)	(0.09339)	(0.40516)	(0.60308)	(0.12882)	(1.16455e-06)
			(0.03708)	(0.04499)	(0.30469)	(0.54913)	(0.09314)	(2.47947e-07)
		(2)	0.39229	0.59654	0.17391	0.97081	0.66724	0.50001
			(-0.15707)	(0.11256)	(0.34816)	(0.47081)	(0.16724)	(1.64786e-05)
			(0.03916)	(0.03112)	(0.21045)	(0.35318)	(0.11377)	(2.52149e-07)
		(3)	0.33027	0.45662	0.32997	0.79553	0.93468	0.49965
			(-0.12354)	(0.10012)	(0.35889)	(0.52879)	(0.15178)	(-1.86966e-06)
			(0.03539)	(0.03289)	(0.20564)	(0.43660)	(0.10426)	(2.47234e-07)
(20, 20, 20, 20)	(10, 10, 10, 10)	(1)	0.49551	0.51291	0.40486	1.22254	0.74124	0.50001
			(0.03186)	(0.02437)	(0.34486)	(0.17160)	(0.02995)	(6.92176e-07)
			(0.02429)	(0.02060)	(0.16629)	(0.03388)	(0.01961)	(2.41447e-07)
		(2)	0.56299	0.54886	0.31370	1.03264	0.624316	0.49939
			(-0.09818)	(0.05786)	(0.27982)	(0.13778)	(0.06194)	(5.81686e-06)
			(0.01951)	(0.01252)	(0.09786)	(0.03854)	(0.02508)	(2.36789e-07)
		(3)	0.40114	0.47953	0.43294	1.17513	0.48802	0.50072
			(-0.07176)	(0.04921)	(0.29051)	(0.14568)	(0.05462)	(1.68075e-08)
			(0.01701)	(0.01323)	(0.10445)	(0.02488)	(0.02484)	(2.30763e-07)
(30, 30, 30, 30)	(15, 15, 15, 15)	(1)	0.77296	0.41067	0.47444	1.21900	0.66815	0.49984
			(-0.02484)	(0.00317)	(0.32774)	(0.07894)	(-0.00182)	(1.62166e-08)
			(0.02253)	(0.01376)	(0.13946)	(0.01543)	(0.01442)	(2.32943e-07)
		(2)	0.45884	0.53531	0.32950	1.32953	0.54050	0.49977
			(-0.06815)	(0.03725)	(0.26113)	(-0.11002)	(0.02847)	(-6.37606e-09)
			(0.01538)	(0.00702)	(0.07891)	(0.02329)	(0.01629)	(2.31755e-07)
		(3)	0.38046	0.56316	0.33596	1.25905	0.63874	0.50002
			(-0.04944)	(0.03269)	(0.26727)	(-0.09818)	(0.02270)	(-8.37014e-09)
			(0.01449)	(0.00726)	(0.08496)	(0.02124)	(0.01686)	(2.29022e-07)

The results for the 95% credible intervals (with length) for the parameters of the multiple-stress model under progressively type-II hybrid censoring at different schemes are given in Table 5, assuming different values of n_i and m_i for $i = 1, 2, 3, 4$ and $k = 4$.

Table 5. The results for the 95% credible intervals (with length) at different schemes.

(n_1, n_2, n_3, n_4)	(m_1, m_2, m_3, m_4)	Scheme	C. I. $(\tilde{\alpha})$	C. I. $(\tilde{\vartheta}_0)$	C. I. $(\tilde{\psi})$
(10, 10, 10, 10)	(5, 5, 5, 5)	(1)	[0.48566, 2.14717] (1.66150)	[0.26541, 1.29576] (1.03035)	[0.49904, 0.50099] (0.00194)
		(2)	[0.42915, 1.82277] (1.39361)	[0.27810, 1.36767] (1.08956)	[0.49902, 0.50100] (0.00198)
		(3)	[0.45353, 1.98703] (1.53349)	[0.27259, 1.36348] (1.09089)	[0.49901, 0.50099] (0.00198)
(20, 20, 20, 20)	(10, 10, 10, 10)	(1)	[0.78628, 2.05891] (1.27262)	[0.32074, 0.84252] (0.52178)	[0.49902, 0.50097] (0.00194)
		(2)	[0.70239, 1.72321] (1.02082)	[0.33492, 0.89435] (0.55942)	[0.49902, 0.50095] (0.00192)
		(3)	[0.71258, 1.80828] (1.09570)	[0.33311, 0.88733] (0.55421)	[0.49908, 0.50097] (0.00188)
(30, 30, 30, 30)	(15, 15, 15, 15)	(1)	[0.87343, 2.04108] (1.16764)	[0.31073, 0.78910] (0.47836)	[0.49908, 0.50100] (0.00192)
		(2)	[0.73236, 1.72141] (0.98905)	[0.32882, 0.81607] (0.48725)	[0.49908, 0.50097] (0.00188)
		(3)	[0.81939, 1.90934] (1.08994)	[0.32099, 0.81129] (0.49029)	[0.49909, 0.50097] (0.00187)

The results for the optimal design of the progressively type-II hybrid censoring scheme in estimating the unknown parameters of the multiple-stress model at different schemes are illustrated in Table 6, assuming different values of n_i and m_i for $i = 1, 2, 3, 4$ and $k = 4$.

Table 6. The results for the optimal design of the progressively type-II hybrid censoring scheme at different schemes.

(n_1, n_2, n_3, n_4)	(m_1, m_2, m_3, m_4)	Scheme	Method 1	Method 2	Method 3
(10, 10, 10, 10)	(5, 5, 5, 5)	(1)	4433.84400	0.20803	1.19172e-08
		(2)	8374.30000	0.18305	2.22749e-08
		(3)	3164.99900	0.17996	7.55078e-09
(20, 20, 20, 20)	(10, 10, 10, 10)	(1)	3052.17900	0.08671	4.28128e-09
		(2)	4315.25400	0.10769	5.85480e-14
		(3)	3571.69900	0.06319	7.77824e-10
(30, 30, 30, 30)	(15, 15, 15, 15)	(1)	3714.60400	0.24012	4.06307e-09
		(2)	6099.04300	0.07144	2.20841e-15
		(3)	9766.60900	0.09757	1.29204e-09

The results presented in Tables 1, 2, 4, and 5 emphasize a number of important aspects:

1) Typically, MSE in both MLEs and Bayesian estimates decreases as the size of the sample size increases. This is probably due to the fact that having more data offers a better understanding of underlying relationships.
2) In general, Bayesian estimates tend to be more precise than MLEs regarding accuracy. This implies that adding prior knowledge about the parameters can be beneficial.

3) The width of the credible intervals generally shrinks as the data size increases. This indicates a more precise range for the true parameter values.

10. Application of real data

The proposed real data application is inspired by a genuine case derived from the insulating oil examination introduced by Nelson [42]. The researcher raised the voltage gradually over time at a determined speed (V/s), to facilitate the quicker breakdown of samples, and the breakdown voltage measurements were recorded. These measurements consist of three categories, with each category comprising 60 items and the failures have been recorded. These datasets were also discussed in [43] and are given as follows:

Dataset I (breakdown voltage at 10 V/s): 34, 34, 34, 35, 35, 35, 36, 38, 38, 38, 38, 39, 39, 39, 40, 40, 40, 40, 41, 41, 41, 41, 41, 41, 42, 42, 42, 42, 42, 43, 43, 43, 43, 43, 44, 44, 44, 44, 44, 44, 44, 45, 45, 46, 46, 46, 46, 46, 47, 47, 47,47, 47, 48, 49, 49, 49, 50, 51, 52

Dataset II (breakdown voltage at 100 V/s): 34, 36, 37, 39, 42, 43, 43, 43, 44, 45, 45, 45, 45, 45, 46, 46, 46, 46, 46, 47, 47, 47, 47, 48, 48, 48, 49, 49, 49, 49, 49, 50, 50, 50, 50, 50, 50, 51, 51, 52, 52, 52, 52, 52, 53, 53, 53, 53, 53, 53, 53, 53, 53, 54, 54, 54, 55, 55, 55, 58

Dataset III (breakdown voltage at 1000 V/s): 41, 41, 51, 51, 51, 51, 53, 53, 53, 53, 54, 54, 54, 54, 55, 55, 55, 55, 56, 56, 56, 56, 57, 57, 57, 57, 57, 57, 57, 58, 58, 58, 59, 59, 59, 59, 60, 60, 60, 60, 60, 60, 61, 61, 62, 62, 62, 62, 63, 63, 63, 63, 63, 64, 64, 65, 65, 65, 65, 69

The estimates with standard errors, log L, Akaike information criterion (AIC), and Kolmogorov-Smirnov (K-S) test with p-value are calculated for the datasets I, II, and III and the results are shown in Table 7.

Table 7. Estimates, log L, AIC, and K-S for datasets I, II, and III.

Dataset	Estimates	Std. error	Log L	AIC	K-S	P-value
I	$\hat{\alpha}$ =6433.7040	2.9660	-175.6577	355.3154	0.09853	0.60490
	$\hat{\vartheta}$ =387.2800	5.9320				
II	$\hat{\alpha}$ =4580.8605	2.09720	-182.5539	369.1078	0.16042	0.09116
	$\hat{\vartheta}$ =423.8739	0.29290				
III	$\hat{\alpha}$ =9291.0544	5.93600	-187.2877	378.5755	0.17467	0.05140
	$\hat{\vartheta}$ =544.4542	0.22660				

From the results obtained in Table 7, it can be observed that the three datasets I, II and III fit the generalized inverted exponential distribution well. Datasets I, II, and III are graphically presented in the form of histogram, probability density function (PDF), cumulative distribution function (CDF), P-P plot, and Q-Q plot in Figures 1–3.

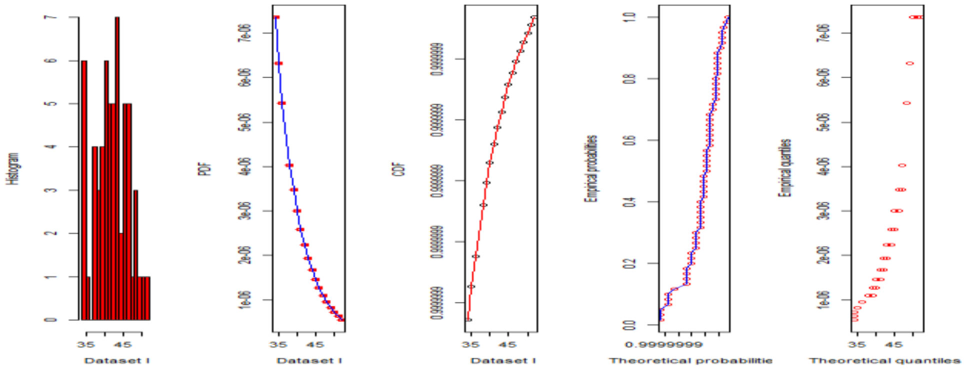

Figure 1. Histogram, PDF, CDF, P-P plot, and Q-Q plot for dataset I.

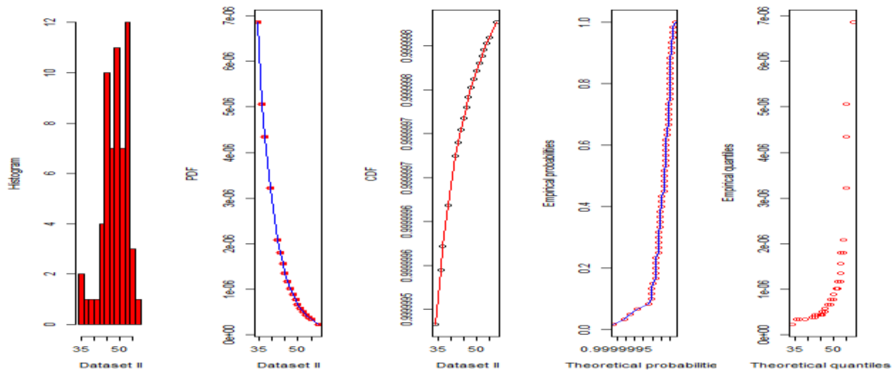

Figure 2. Histogram, PDF, CDF, P-P plot, and Q-Q plot for dataset II.

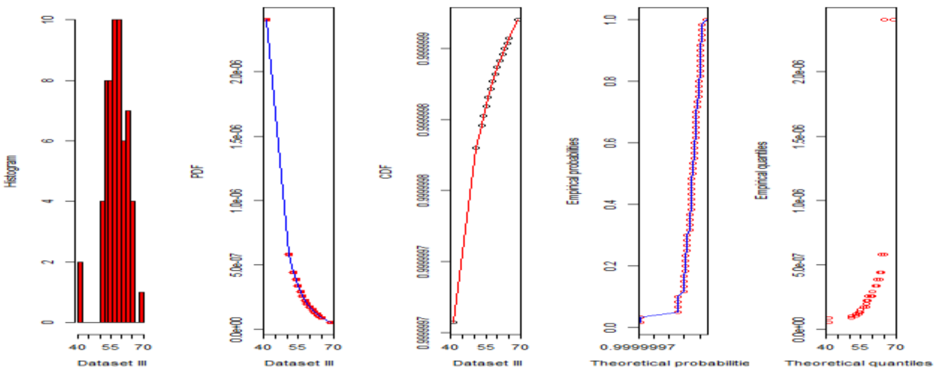

Figure 3. Histogram, PDF, CDF, P-P plot, and Q-Q plot for dataset III.

In order to obtain the results for the MLEs and Bayes estimates, different schemes will be considered in Tables 8 and 9. In Table 8, the considered times are $T_1 = 53, T_2 = 59$, and $T_3 = 70$. In Table 9, the considered times are $T_1 = 43.5$, $T_2 = 49.5$, and $T_3 = 58.5$.

Table 8. Different schemes and the censored data at $T_1 = 53, T_2 = 59$, and $T_3 = 70$.

	Scheme	**Censored data**
(1)	**(30,0*^{29})**	Dataset I: (34, 43, 43, 43, 44, 44, 44, 44, 44, 44, 44, 45, 45, 46, 46, 46, 46, 46, 47, 47, 47, 47, 47, 48, 49, 49, 49, 50, 51, 52)
		Dataset II: (34, 50, 50, 50, 50, 50, 50, 51, 51, 52, 52, 52, 52, 52, 53, 53, 53, 53, 53, 53, 53, 53, 53, 54, 54, 54, 55, 55, 55, 58)
		Dataset III: (41, 58, 59, 59, 59, 59, 60, 60, 60, 60, 60, 60, 61, 61, 62, 62, 62, 62, 63, 63, 63, 63, 63, 64, 64, 65, 65, 65, 65, 69)
(2)	**(0*^{29},30)**	Dataset I: (34, 34, 34, 35, 35, 35, 36, 38, 38, 38, 38, 39, 39, 39, 40, 40, 40, 40, 41, 41, 41, 41, 41, 41, 42, 42, 42, 42, 42, 43)
		Dataset II: (34, 36, 37, 39, 42, 43, 43, 43, 44, 45, 45, 45, 45, 45, 46, 46, 46, 46, 46, 47, 47, 47, 47, 48, 48, 48, 49, 49, 49, 49)
		Dataset III: (41, 41, 51, 51, 51, 51, 53, 53, 53, 53, 54, 54, 54, 54, 55, 55, 55, 55, 56, 56, 56, 56, 57, 57, 57, 57, 57, 57, 57, 58)
(3)	**(15, 0*^{28},15)**	Dataset I: (34, 40, 40, 41, 41, 41, 41, 41, 41, 42, 42, 42, 42, 42, 43, 43, 43, 43, 43, 44, 44, 44, 44, 44, 44, 44, 45, 45, 46, 46)
		Dataset II: (34, 46, 46, 46, 47, 47, 47, 47, 48, 48, 48, 49, 49, 49, 49, 49, 50, 50, 50, 50, 50, 50, 51, 51, 52, 52, 52, 52, 52, 53)
		Dataset III: (41, 55, 55, 56, 56, 56, 56, 57, 57, 57, 57, 57, 57, 57, 58, 58, 58, 59, 59, 59, 59, 60, 60, 60, 60, 60, 60, 61, 61, 62)

Table 9. Different schemes and the censored data at $T_1 = 43.5$, $T_2 = 49.5$, and $T_3 = 58.5$.

	Scheme	Censored data
(1)	(0*^{34}, $R^*_{d_1}$ =26)	Dataset I: (34, 34, 34, 35, 35, 35, 36, 38, 38, 38, 38, 39, 39, 39, 40, 40, 40, 40, 41, 41, 41, 41, 41, 41, 42, 42, 42, 42, 42, 43, 43, 43, 43, 43)
	(0*^{31}, $R^*_{d_2}$ =29)	Dataset II: (34, 36, 37, 39, 42, 43, 43, 43, 44, 45, 45, 45, 45, 45, 46, 46, 46, 46, 46, 47, 47, 47, 47, 48, 48, 48, 49, 49, 49, 49, 49)
	(0*^{32}, $R^*_{d_3}$ =28)	Dataset III: (41, 41, 51, 51, 51, 51, 53, 53, 53, 53, 54, 54, 54, 54, 55, 55, 55, 55, 56, 56, 56, 56, 57, 57, 57, 57, 57, 57, 57, 58, 58, 58)
(2)	(5, 0*^{28}, $R^*_{d_1}$ =26)	Dataset I: (34, 36, 38, 38, 38, 38, 39, 39, 39, 40, 40, 40, 40, 41, 41, 41, 41, 41, 41, 42, 42, 42, 42, 42, 43, 43, 43, 43, 43)
	(5, 0*^{25}, $R^*_{d_2}$ =29)	Dataset II: (34, 43, 43, 44, 45, 45, 45, 45, 45, 46, 46, 46, 46, 46, 47, 47, 47, 47, 48, 48, 48, 49, 49, 49, 49, 49)
	(5, 0*^{26}, $R^*_{d_3}$ =28)	Dataset III: (41 , 53, 53, 53, 53, 54, 54, 54, 54, 55, 55, 55, 55, 56, 56, 56, 56, 57, 57, 57, 57, 57, 57, 57, 58, 58, 58)
(3)	(0*^{6},5,0*^{22}, $R^*_{d_1}$ =26)	Dataset I: (34, 34, 34, 35, 35, 35 , 39, 39, 39, 40, 40, 40, 40, 41, 41, 41, 41, 41, 41, 42, 42, 42, 42, 42, 43, 43, 43, 43, 43)
	(0*^{6},5,0*^{19},$R^*_{d_2}$ =29)	Dataset II: (34, 36, 37, 39, 42, 43, 43, 43, 44, 45, 45, 45, 45, 45, 46, 46, 46, 46, 46, 47, 47, 47, 47, 48, 48, 48, 49, 49, 49, 49, 49)
	(0*^{6},5,0*^{20},$R^*_{d_3}$ =28)	Dataset III: (41, 41, 51, 51, 51, 51, 53 , 54, 54, 55, 55, 55, 55, 56, 56, 56, 56, 57, 57, 57, 57, 57, 57, 57, 58, 58, 58)

The results for the MLEs and Bayes estimates with MSE according to different times and schemes discussed in Tables 8 and 9 are summarized in Table 10.

Table 10. The results for the MLEs and Bayes estimates (with MSE) at different times and schemes.

Time	Scheme	$\hat{\alpha}$	$\hat{\vartheta}_0$	$\hat{\psi}$	$\tilde{\alpha}$	$\tilde{\vartheta}_0$	$\tilde{\psi}$
$T_1 = 53$	(1)	0.91693	0.69447	0.02259	0.93243	1.48251	0.50209
$T_2 = 59$		(0.33303)	(0.10725)	(0.003633)	(0.00700)	(0.02445)	(2.37220e-05)
$T_3 = 70$	(2)	0.87506	0.60678	0.01085	0.89365	1.27992	0.50228
		(0.59430)	(0.18726)	(0.00336)	(0.00915)	(0.01781)	(2.15937e-05)
	(3)	0.89875	0.33048	0.03102	0.96459	1.38793	0.50229
		(0.39854)	(0.06417)	(0.00617)	(0.00778)	(0.02074)	(2.33489e-05)
$T_1 = 43.5$	(1)	0.80459	0.49954	0.03369	0.82729	1.19270	0.50204
$T_2 = 49.5$		(0.14957)	(0.03263)	(0.00272)	(0.00784)	(0.01546)	(2.44477e-05)
$T_3 = 58.5$	(2)	1.01019	0.45515	0.02427	1.28067	1.31382	0.50232
		(0.35493)	(0.05692)	(0.00361)	(0.01974)	(0.02075)	(1.95838e-05)
	(3)	0.896928	0.31523	0.03294	1.23067	1.21514	0.50240
		(0.36327)	(0.05249)	(0.00568)	(0.01909)	(0.01748)	(2.03209e-05)

11. Conclusions

In this paper, estimation of the accelerated life testing for the multiple-stress model was introduced under progressively type-II hybrid censoring scheme and generalized inverted exponential distribution. Estimation methods included the maximum likelihood and Bayesian estimation, which were implemented to obtain the estimates of the parameters of the multiple-stress model. Different optimality criteria were discussed in order to obtain the optimal designs for the progressively type-II hybrid censoring schemes. A simulation study was presented to obtain results for the parameters of the multiple-stress model assuming different schemes. Real data application was presented to show the performance of the multiple-stress model.

Use of Generative-AI tools declaration

The author declares she has not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that she has no conflict of interest in this paper.

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