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Research article

Stochastic dynamics of an elk–wolf system with inter-regional movement

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Abstract: This paper formulates and analyzes a stochastic predator-prey model focusing on the elk–wolf dynamics. The mathematical framework is motivated by the Banff–Bow Valley ecosystem, where the elk population is categorized into urban and valley subgroups, both subject to wolf predation. We extend the deterministic interaction model into a system of stochastic differential equations to incorporate environmental fluctuations. The global existence and uniqueness of positive solutions are established, ensuring the biological realism of the model. By applying logarithmic Itô calculus and Lyapunov function techniques, we derive explicit extinction criteria and prove moment as well as ultimate boundedness properties of the populations. Moreover, we introduce a practical parameter-based stochastic threshold that characterizes predator persistence and stochastic permanence, highlighting qualitative differences between deterministic coexistence and noise-induced extinction. Finally, numerical simulations based on the Euler–Maruyama scheme are provided to illustrate the theoretical results and confirm the predicted long-term stochastic behaviors.

Keywords: stochastic differential equations; predator–prey model; urban refuge; stochastic permanence; mathematical model; stationary distribution
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1. Introduction

Managing wildlife populations in human-dominated landscapes is a central challenge in ecological conservation, particularly in regions such as the Banff–Bow Valley in Canada. In this ecosystem, elk populations have exhibited a remarkable spatial division into two interacting subgroups: Valley elk occupying wilderness habitats and urban elk that increasingly exploit the Banff townsite as a refuge from wolf predation [1, 2]. Empirical evidence suggests that such behavioral shifts generate complex source–sink dynamics that strongly influence long-term predator–prey coexistence [3].

Early mathematical investigations by Goldberg et al. [3] and later by Maji et al. [4] provided a deterministic foundation for understanding this urban–valley structure. More broadly, the deterministic literature contains numerous studies of multi-species predator–prey systems incorporating prey migration and refuge mechanisms. For instance, Dai et al. [5] examined seasonal migration of prey between breeding and non-breeding regions, while Alebraheem et al. [6] analyzed two-prey–one-predator dynamics under prey immigration. Other works have highlighted the roles of fear, refuge behavior, and delayed predator responses in shaping persistence outcomes [7–9]. These contributions establish an important deterministic baseline for the Banff-type urban refuge structure.

However, deterministic frameworks inherently assume constant environmental conditions, thereby overlooking the randomness and variability that characterize natural ecosystems. In reality, biological parameters are continuously affected by fluctuations in climate, food availability, and anthropogenic disturbances. As emphasized in [10], ignoring such variability may lead to inaccurate predictions of persistence and extinction. Moreover, Clark et al. [11] demonstrated that stochasticity is not merely a technical perturbation, but can generate predator pits and local extinction events that deterministic approaches substantially underestimate.

Motivated by these limitations, stochastic population models have been widely developed to account for environmental variability. A substantial body of literature has shown that stochastic noise may induce extinction, shift permanence thresholds, or generate stationary distributions in ecological and epidemic systems [12, 13]. Recent works have further explored stochastic permanence and invariant measures in structured biological models, highlighting the crucial role of randomness in long-term ecosystem dynamics [14–16]. Beyond spatial dynamics, predators may also regulate ecosystems through selective removal of weak or diseased prey, sometimes referred to as a cleansing effect [17]. Since such interactions unfold in highly variable environments, a stochastic framework becomes essential for capturing the probabilistic nature of these ecological trade-offs. While many existing stochastic differential equation (SDE) models investigate harvesting effects [10] or delayed responses [15], a rigorous stochastic extension of the urban–valley elk–wolf system, together with explicit thresholds for extinction, permanence, and long-term coexistence, remains largely absent.

The present work addresses this gap by formulating and analyzing an SDE version of the elk–wolf model with inter-regional movement. In particular, we incorporate environmental perturbations acting on both prey subpopulations and predator dynamics, and we establish global existence, uniqueness, and positivity of solutions, ensuring biological well-posedness. Furthermore, we derive parameter-based extinction criteria for wolves under strong environmental fluctuations, revealing noise-induced predator die-out that is absent in deterministic coexistence theory. We also develop sufficient conditions for stochastic permanence and prove the existence of an ergodic stationary distribution, implying probabilistic stability around a coexistence regime.

These analytical results provide quantitative tools for wildlife managers to assess persistence probabilities under realistic environmental uncertainty, offering predictive insight beyond traditional deterministic approaches [4]. The remainder of the paper is organized as follows. Section 2 introduces the stochastic elk–wolf model. Section 3 establishes global existence and positivity of solutions, while Section 4 derives extinction criteria. Sections 5 and 6 address moment estimates and ultimate boundedness. Section 7 develops sufficient conditions for stochastic permanence through a parameter-based threshold. Section 8 proves the existence of an ergodic stationary distribution. Section 9 presents numerical simulations illustrating the theoretical findings. Finally, Section 10 concludes the paper.

2. Description of the stochastic model

Following [3, 4], we assume that both elk populations grow logistically. The urban elk $E(t)$ and valley elk $N(t)$ grow with intrinsic rates α and β, respectively, and are limited by carrying capacities K_1 (urban environment) and K_2 (wildlife valley environment). The parameter μ denotes the migration rate of elk from the townsite to the valley.

The urban elk population is additionally subject to harvesting, where q represents the catchability coefficient and ψ the harvesting effort. Wolves $P(t)$ prey on both elk groups with rates γ and ξ, while θ denotes the biomass conversion efficiency from prey consumption into predator reproduction. To incorporate environmental variability, we extend the deterministic model into the following SDE system:

$$\begin{cases} dE(t) = \Big[\alpha E\Big(1 - \dfrac{E}{K_1}\Big) - \gamma EP - \mu E - q\psi E\Big]\,dt + \sigma_1 E\,dW_1(t), \\ dN(t) = \Big[\beta N\Big(1 - \dfrac{N}{K_2}\Big) + \mu E - \xi NP\Big]\,dt + \sigma_2 N\,dW_2(t), \\ dP(t) = \Big[\theta(\gamma E + \xi N)P - \eta P\Big]\,dt + \sigma_3 P\,dW_3(t), \end{cases} \tag{2.1}$$

where $W_1(t)$, $W_2(t)$, and $W_3(t)$ are independent standard Wiener processes, and σ_i^2 denote the corresponding noise intensities. The initial values satisfy $E(0) = E_0 > 0$, $N(0) = N_0 > 0$, and $P(0) = P_0 > 0$.

Table 1. Biological descriptions of the model parameters.

Parameter	Description
α	Intrinsic growth rate of urban elk (Banff townsite).
β	Intrinsic growth rate of valley elk (Bow Valley).
K_1	Carrying capacity of the urban environment.
K_2	Carrying capacity of the wildlife valley environment.
μ	Migration rate from townsite to valley.
q, ψ	Harvesting catchability and effort (urban elk).
γ	Predation rate on urban elk.
ξ	Predation rate on valley elk.
θ	Biomass conversion efficiency from elk to wolves.
η	Natural mortality rate of wolves.
σ_i	Environmental noise intensity for population i.

It is important to emphasize that system (2.1) is not merely a technical extension of the deterministic elk–wolf framework, but leads to qualitatively different ecological implications. In the deterministic setting, the long-term behavior is typically governed by equilibrium states, where populations persist and converge to constant steady densities whenever critical thresholds are exceeded.

In contrast, the stochastic formulation incorporates random environmental fluctuations through multiplicative noise terms of the form $\sigma_i X_i\,dW_i(t)$. This gives rise to several phenomena absent in deterministic models. In particular, sufficiently strong environmental perturbations may induce predator extinction even when deterministic theory predicts stable coexistence. Moreover, persistence thresholds are effectively reduced by stochastic correction terms such as $-\sigma_i^2/2$, so that classical deterministic conditions are no longer sufficient. Unlike deterministic trajectories converging to fixed equilibria, the stochastic system exhibits long-term random oscillations and, under suitable conditions established later, admits an ergodic stationary distribution. Therefore, the proposed stochastic elk–wolf model provides a more realistic framework for assessing extinction risk, ecosystem resilience, and persistence under environmental uncertainty, which cannot be captured solely within the deterministic equilibrium approach.

3. Global existence and positivity

Initially, we show that the elk–wolf system (2.1) is well-posed, possessing a unique global positive solution as stated in the theorem below.

Theorem 3.1. *For any initial value* $(E_0, N_0, P_0) \in \mathbb{R}^3_+$, *system* (2.1) *admits a unique global solution a.s. and the solution remains in* $\mathbb{R}^3_+$ *for all* $t \geq 0$.

Proof. The coefficients of system (2.1) are locally Lipschitz continuous on $\mathbb{R}^3_+$. Thus, for any initial value $(E_0, N_0, P_0) \in \mathbb{R}^3_+$, there exists a unique local solution on $[0, \tau_e)$, where τ_e denotes the explosion time. To show that this solution is global (i.e., $\tau_e = \infty$ a.s.), we define the following C^2-Lyapunov function $V : \mathbb{R}^3_+ \to \mathbb{R}_+$:

$$V(E, N, P) = E - 1 - \ln E + N - 1 - \ln N + \frac{1}{\theta}(P - 1 - \ln P).$$

Applying Itô's formula to V, we obtain

$$dV = LV\,dt + \sigma_1(E - 1)\,dW_1(t) + \sigma_2(N - 1)\,dW_2(t) + \frac{\sigma_3}{\theta}(P - 1)\,dW_3(t),$$

where the operator L acting on V is given by

$$\begin{aligned} LV = &\left(1 - \frac{1}{E}\right)\left[\alpha E\Big(1 - \frac{E}{K_1}\Big) - \gamma EP - (\mu + q\psi)E\right] + \frac{\sigma_1^2}{2} \\ &+ \left(1 - \frac{1}{N}\right)\left[\beta N\Big(1 - \frac{N}{K_2}\Big) + \mu E - \xi NP\right] + \frac{\sigma_2^2}{2} \\ &+ \frac{1}{\theta}\left(1 - \frac{1}{P}\right)[\theta(\gamma E + \xi N)P - \eta P] + \frac{\sigma_3^2}{2\theta}. \end{aligned}$$

Expanding the terms and simplifying, we have

$$\begin{aligned} LV = &\left[\alpha E - \frac{\alpha E^2}{K_1} - (\mu + q\psi)E - \alpha + \frac{\alpha E}{K_1} + \gamma P + \mu + q\psi + \frac{\sigma_1^2}{2}\right] \\ &+ \left[\beta N - \frac{\beta N^2}{K_2} + \mu E - \beta + \frac{\beta N}{K_2} - \frac{\mu E}{N} + \xi P + \frac{\sigma_2^2}{2}\right] \\ &+ \left[-\gamma E - \xi N - \frac{\eta P}{\theta} + \frac{\eta}{\theta} + \frac{\sigma_3^2}{2\theta}\right]. \end{aligned}$$

By regrouping the terms and noting that $-\frac{\mu E}{N} \leq 0$, we obtain

$$LV \leq -\frac{\alpha}{K_1}E^2 + C_1 E - \frac{\beta}{K_2}N^2 + C_2 N + \left(\gamma + \xi - \frac{\eta}{\theta}\right)P + K,$$

where $C_1 = \alpha + \frac{\alpha}{K_1} + \mu$, $C_2 = \beta + \frac{\beta}{K_2}$, and $K = \mu + q\psi + \frac{\eta}{\theta} + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2\theta}$.
To obtain a uniform upper bound for LV, we use the elementary inequality

$$-ax^2 + bx \;\leq\; -a\left(x - \frac{b}{2a}\right)^2 + \frac{b^2}{4a}, \qquad a > 0,$$

which follows from completing the square. Applying this inequality to the terms $-\frac{\alpha}{K_1}E^2 + C_1 E$ and $-\frac{\beta}{K_2}N^2 + C_2 N$ yields

$$-\frac{\alpha}{K_1}E^2 + C_1 E \leq -\frac{\alpha}{K_1}\left(E - \frac{K_1 C_1}{2\alpha}\right)^2 + \frac{K_1 C_1^2}{4\alpha},$$

and similarly,

$$-\frac{\beta}{K_2}N^2 + C_2 N \leq -\frac{\beta}{K_2}\left(N - \frac{K_2 C_2}{2\beta}\right)^2 + \frac{K_2 C_2^2}{4\beta}.$$

Therefore, LV admits the global bound stated below.

Given the assumption that $\gamma + \xi < \frac{\eta}{\theta}$, the term containing P becomes non-positive, leading to:

$$\begin{aligned} LV \leq &-\frac{\alpha}{K_1}\left[E - \frac{K_1 C_1}{2\alpha}\right]^2 + \frac{K_1 C_1^2}{4\alpha} - \frac{\beta}{K_2}\left[N - \frac{K_2 C_2}{2\beta}\right]^2 + \frac{K_2 C_2^2}{4\beta} + K \\ \leq &\frac{K_1 C_1^2}{4\alpha} + \frac{K_2 C_2^2}{4\beta} + K := M. \end{aligned}$$

Since M is a positive constant, we have $LV \leq M$ for all $(E, N, P) \in \mathbb{R}_+^3$. According to the standard comparison theorem for SDEs [18–20], this uniform bound implies that $\tau_e = \infty$ a.s. Consequently, the solution is global and remains in $\mathbb{R}_+^3$ for all $t \geq 0$ with probability one. □

4. Long-term properties: Extinction criteria

In this section, we establish the conditions under which urban elk, valley elk, and wolf populations will become extinct.

Theorem 4.1. *(Extinction of populations) Consider the solution to the elk–wolf system* (2.1).

(i) *If*

$$\alpha - \mu - q\psi - \frac{\sigma_1^2}{2} < 0, \tag{4.1}$$

then $\lim_{t\to\infty} E(t) = 0$ *a.s.*

(ii) *If*

$$\beta - \frac{\sigma_2^2}{2} + \mu\frac{K_1}{K_2} < 0, \tag{4.2}$$

then $\lim_{t\to\infty} N(t) = 0$ *a.s.*

(iii) *If*

$$\theta(\gamma K_1 + \xi K_2) < \eta + \frac{\sigma_3^2}{2}, \tag{4.3}$$

then $\lim_{t\to\infty} P(t) = 0$ *a.s.*

Proof. The proof follows the standard approach in Mao [18] by applying Itô's formula to logarithmic functions and then using the strong law of large numbers for martingales. Applying Itô's formula to $\ln E(t)$ yields

$$d\ln E(t) = \Big[\alpha\Big(1 - \frac{E}{K_1}\Big) - \gamma P - \mu - q\psi - \frac{\sigma_1^2}{2}\Big]dt + \sigma_1\, dW_1(t).$$

Integration and time-averaging over $[0, t]$ gives

$$\frac{\ln E(t)}{t} = \frac{\ln E_0}{t} + \alpha - \mu - q\psi - \frac{\sigma_1^2}{2} - \frac{\alpha}{K_1 t}\int_0^t E(s)\, ds - \frac{\gamma}{t}\int_0^t P(s)\, ds + \sigma_1 \frac{W_1(t)}{t}.$$

Since $\lim_{t\to\infty} \frac{W_1(t)}{t} = 0$ a.s. and the integral terms are nonnegative, we obtain

$$\limsup_{t\to\infty} \frac{\ln E(t)}{t} \le \alpha - \mu - q\psi - \frac{\sigma_1^2}{2}.$$

Condition (4.1) yields

$$\limsup_{t\to\infty} \frac{1}{t} \ln E(t) < 0 \quad \text{a.s.}$$

Therefore, $E(t)$ decays exponentially in the sample-path sense and, in particular,

$$\lim_{t\to\infty} E(t) = 0 \quad \text{a.s.}$$

by the standard extinction criterion for 1D SDEs (negative sample Lyapunov exponent); see Mao [18].

By applying Itô's formula to $\ln N(t)$, we obtain

$$d\ln N(t) = \left(\beta\Big(1 - \frac{N(t)}{K_2}\Big) + \mu\frac{E(t)}{N(t)} - \xi P(t) - \frac{\sigma_2^2}{2}\right)dt + \sigma_2\, dW_2(t).$$

Integrating from 0 to t and dividing by t yields

$$\frac{\ln N(t)}{t} = \frac{\ln N_0}{t} + \frac{1}{t}\int_0^t \left(\beta - \frac{\sigma_2^2}{2} + \mu\frac{E(s)}{N(s)} - \xi P(s) - \frac{\beta}{K_2}N(s)\right)ds + \frac{\sigma_2 W_2(t)}{t}.$$

Since $W_2(t)/t \to 0$ almost surely as $t \to \infty$, and $N(s) \ge 0$, we obtain the estimate

$$\limsup_{t\to\infty} \frac{\ln N(t)}{t} \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(\beta - \frac{\sigma_2^2}{2} + \mu\frac{E(s)}{N(s)} - \xi P(s)\right)ds.$$

To derive a condition depending only on parameters, we use the boundedness of the prey populations. In particular, ultimate boundedness implies that

$$E(t) \le K_1, \qquad N(t) \le K_2, \qquad \text{a.s. for all } t \ge 0.$$

Hence,

$$\frac{E(t)}{N(t)} \leq \frac{K_1}{K_2}, \qquad -\xi P(t) \leq 0.$$

Therefore,

$$\beta - \frac{\sigma_2^2}{2} + \mu\frac{E(t)}{N(t)} - \xi P(t) \leq \beta - \frac{\sigma_2^2}{2} + \mu\frac{K_1}{K_2}.$$

If the parameter-based condition

$$\beta - \frac{\sigma_2^2}{2} + \mu\frac{K_1}{K_2} < 0$$

holds, then we conclude that

$$\limsup_{t\to\infty} \frac{\ln N(t)}{t} < 0 \quad \text{a.s.},$$

which implies $\ln N(t) \to -\infty$ and therefore

$$\lim_{t\to\infty} N(t) = 0 \quad \text{a.s.}$$

This completes the proof of part (ii). □

It follows from Itô's formula for $\ln P(t)$ that

$$d\ln P(t) = \Big[\theta(\gamma E(t) + \xi N(t)) - \eta - \frac{\sigma_3^2}{2}\Big]dt + \sigma_3\, dW_3(t).$$

Integration and time-averaging over $[0, t]$ gives

$$\frac{\ln P(t)}{t} = \frac{\ln P(0)}{t} + \frac{1}{t}\int_0^t \Big[\theta(\gamma E(s) + \xi N(s)) - \eta - \frac{\sigma_3^2}{2}\Big]ds + \sigma_3\frac{W_3(t)}{t}.$$

Since $\lim_{t\to\infty} \frac{W_3(t)}{t} = 0$ a.s. and the elk populations are ultimately bounded by their carrying capacities, $E(t) \leq K_1$ and $N(t) \leq K_2$ almost surely for large t, it follows that

$$\limsup_{t\to\infty} \frac{\ln P(t)}{t} \leq \theta(\gamma K_1 + \xi K_2) - \Big(\eta + \frac{\sigma_3^2}{2}\Big).$$

Condition (4.3) implies that the right-hand side is strictly negative, and hence

$$\lim_{t\to\infty} P(t) = 0 \quad \text{a.s.}$$

□

5. Moment boundedness

In this section, we show that the moments of the solution to the elk–wolf system (2.1) are globally bounded. This property is essential for ensuring that the population sizes do not grow to infinity in an average sense.

Theorem 5.1. *For any $p > 2$, there exists a constant $C(p) > 0$ such that for any initial value $(E_0, N_0, P_0) \in \mathbb{R}^3_+$, the solution of* (2.1) *satisfies*

$$\limsup_{t\to\infty} \mathbb{E}[E^p(t) + N^p(t) + P^p(t)] \le C(p).$$

Proof. Let $X(t) = E(t) + N(t) + \frac{1}{\theta}P(t)$ and define the C^2-Lyapunov function $V(X) = (1+X)^p$. Applying Itô's formula, the operator LV is:

$$\begin{aligned} LV =& p(1+X)^{p-1}\left(\alpha E - \frac{\alpha E^2}{K_1} + \beta N - \frac{\beta N^2}{K_2} - q\psi E - \frac{\eta}{\theta}P\right) \\ &+ \frac{p(p-1)}{2}(1+X)^{p-2}\left(\sigma_1^2 E^2 + \sigma_2^2 N^2 + \frac{\sigma_3^2}{\theta^2}P^2\right). \end{aligned}$$

Observe that the drift part contains terms of order $-(1+X)^{p-1}X^2 \approx -X^{p+1}$ due to the logistic growth constraints. In contrast, the diffusion part consists of terms of order $(1+X)^{p-2}X^2 \approx X^p$. Since the negative term of order $p+1$ dominates all positive terms of lower orders as $|(E, N, P)| \to \infty$, there exist positive constants L_1 and L_2 such that:

$$LV \le L_1 - L_2(1+X)^p = L_1 - L_2 V.$$

By the comparison theorem for SDEs, we have

$$d(\mathbb{E}[V(t)]) = \mathbb{E}[LV]dt \le (L_1 - L_2\mathbb{E}[V(t)])dt,$$

which yields $\mathbb{E}[V(t)] \le \frac{L_1}{L_2} + [V(0) - \frac{L_1}{L_2}]e^{-L_2 t}$. Then we take the limit superior:

$$\limsup_{t\to\infty} \mathbb{E}[(E + N + \frac{1}{\theta}P)^p] \le \frac{L_1}{L_2}.$$

Applying the C_r-inequality, $E^p + N^p + P^p \le C \cdot (E + N + \frac{1}{\theta}P)^p$, we conclude that all moments of the population densities are uniformly bounded. □

6. Stochastic ultimate boundedness

Before establishing ultimate boundedness, we first show that the generator of system (2.1) is dissipative in the sense of Lyapunov, ensuring that the stochastic dynamics are driven toward a bounded region of the state space.

Lemma 6.1. *(Dissipativity of the generator) Let f and G denote the drift and diffusion coefficients of the stochastic system. There exist positive constants $c_0 > 0$ and $K_0 > 0$ such that*

$$2\langle X, f(X)\rangle + \operatorname{trace}(G(X)G(X)^\top) \le K_0 - c_0|X|^2, \qquad \forall X \in \mathbb{R}^3_+.$$

Proof. Consider the Lyapunov function $V(X) = |X|^2 = E^2 + N^2 + P^2$ for $X = (E, N, P)^\top \in \mathbb{R}^3_+$. The infinitesimal generator $\mathcal{L}$ associated with the stochastic system acts on V as follows:

$$\mathcal{L}V(X) = 2\langle X, f(X)\rangle + \operatorname{trace}(G(X)G(X)^\top).$$

The trace term, representing the quadratic variation of the noise, is given by:

$$\mathrm{trace}(G(X)G(X)^\top) = \sigma_1^2 E^2 + \sigma_2^2 N^2 + \sigma_3^2 P^2 \le \bar{\sigma}^2 |X|^2,$$

where $\bar{\sigma}^2 := \max\{\sigma_1^2, \sigma_2^2, \sigma_3^2\}$. Expanding the inner product $\langle X, f(X)\rangle$ based on the system dynamics, we obtain:

$$\begin{aligned}\langle X, f(X)\rangle = &(\alpha - (\mu + q\psi))\, E^2 - \frac{\alpha}{K_1} E^3 + \beta N^2 - \frac{\beta}{K_2} N^3 \\ &+ \mu EN - \gamma E^2 P - \xi N^2 P + (\theta(\gamma E + \xi N) - \eta)\, P^2.\end{aligned}$$

By applying Young's inequality, we have $\mu EN \le \frac{\mu}{2} E^2 + \frac{\mu}{2} N^2$. Thus, the quadratic terms can be bounded by:

$$(\alpha - (\mu + q\psi))\, E^2 + \beta N^2 + \mu EN \le \left(\alpha + \tfrac{\mu}{2}\right) E^2 + \left(\beta + \tfrac{\mu}{2}\right) N^2 \le C_3 (E^2 + N^2),$$

where $C_3 = \max\{\alpha + \frac{\mu}{2}, \beta + \frac{\mu}{2}\} > 0$. Observing the structure of $\langle X, f(X)\rangle$, the negative cubic terms (logistic damping) $-\frac{\alpha}{K_1} E^3$ and $-\frac{\beta}{K_2} N^3$ dominate the growth and noise terms, which are at most quadratic. Consequently, the following asymptotic property holds:

$$\lim_{|X|\to\infty} \frac{\mathcal{L}V(X)}{|X|^2} = -\infty.$$

This implies that there exist constants $R_0 > 0$ and $c_0 > 0$ such that for all $|X| \ge R_0$

$$\mathcal{L}V(X) \le -c_0 |X|^2.$$

To ensure the inequality holds over the entire domain $\mathbb{R}^3_+$, we define:

$$K_0 := \sup_{|X| \le R_0} \left\{ \mathcal{L}V(X) + c_0 |X|^2 \right\}.$$

Since $\mathcal{L}V(X)$ is continuous on the compact set $\{|X| \le R_0\}$, the supremum K_0 is finite. Therefore, for all $X \in \mathbb{R}^3_+$,

$$\mathcal{L}V(X) \le K_0 - c_0 |X|^2,$$

which completes the proof of ultimate boundedness. □

Using the dissipativity of the drift established in Lemma 6.1, we can now prove that the system is stochastically ultimately bounded. This property ensures that the population trajectories do not explode to infinity.

Theorem 6.2. *(Ultimate boundedness in mean square) There exists a constant* $M > 0$ *such that for any initial condition* $X(0) \in \mathbb{R}^3_+$*, the solution* $X(t) = (E(t), N(t), P(t))$ *of* (2.1) *satisfies*

$$\limsup_{t\to\infty} \mathbb{E}[|X(t)|^2] \le M.$$

Proof. Let $V(X) = |X|^2$ for $X \in \mathbb{R}^3_+$. Then, by Itô's formula, it holds that

$$dV(X(t)) = \mathcal{L}V(X(t))\,dt + 2\langle X(t), G(X(t))\,dW(t)\rangle,$$

where

$$\mathcal{L}V(X) = 2\langle X, f(X)\rangle + \mathrm{trace}(G(X)G(X)^\top).$$

By Lemma 6.1, there exist constants $c_0, K_0 > 0$ such that

$$\mathcal{L}V(X) \le K_0 - c_0|X|^2 = K_0 - c_0V(X), \qquad \forall X \in \mathbb{R}^3_+.$$

Taking expectations and using that the stochastic integral has zero mean, we obtain

$$\frac{d}{dt}\mathbb{E}[V(X(t))] = \mathbb{E}[\mathcal{L}V(X(t))] \le K_0 - c_0\mathbb{E}[V(X(t))].$$

By Grönwall's inequality,

$$\mathbb{E}[V(X(t))] \le e^{-c_0t}\mathbb{E}[V(X(0))] + \frac{K_0}{c_0}(1 - e^{-c_0t}).$$

Consequently,

$$\limsup_{t\to\infty}\mathbb{E}[|X(t)|^2] = \limsup_{t\to\infty}\mathbb{E}[V(X(t))] \le \frac{K_0}{c_0} =: M,$$

which proves the ultimate boundedness in mean square. □

7. Stochastic permanence via a parameter-based threshold

In this section, we establish a practical sufficient criterion for stochastic permanence of the elk–wolf ecosystem. The goal is to provide explicit persistence thresholds that depend only on measurable biological parameters, avoiding conditions involving unknown time averages of the solution.

7.1. A stochastic predator persistence threshold

By Theorem 6.2, the prey populations remain ultimately bounded and satisfy

$$E(t) \le K_1, \qquad N(t) \le K_2, \qquad \text{a.s. for all } t \ge 0.$$

Consequently,

$$\gamma E(t) + \xi N(t) \le \gamma K_1 + \xi K_2.$$

Definition 7.1. *(Parameter-based stochastic reproduction number) We define the parameter-based stochastic predator reproduction number by*

$$\mathcal{R}_0^s := \frac{\theta(\gamma K_1 + \xi K_2)}{\eta + \frac{\sigma_3^2}{2}}.$$

This quantity represents a conservative persistence threshold for the wolf population, incorporating both prey availability through carrying capacities and the destabilizing effect of environmental noise through σ_3. The stochastic threshold $\mathcal{R}_0^s$ establishes a threshold behavior for the predator's long-term survival. Specifically, if $\mathcal{R}_0^s < 1$, the stochastic mortality term dominates predator growth, leading to the almost sure extinction of wolves as predicted by Theorem 4.1(iii). This scenario reflects situations where high environmental noise intensity or insufficient prey availability force the system toward collapse. Conversely, if $\mathcal{R}_0^s > 1$, the prey resources are sufficient on average to support predator survival, rendering stochastic coexistence possible. The following theorem establishes the conditions for the long-term persistence of the entire elk–wolf community.

Theorem 7.2. *Assume that the elk–wolf system* (2.1) *is ultimately bounded in probability, and that both prey populations persist in probability. If*

$$\mathcal{R}_0^s > 1,$$

then the wolf population persists in probability, and the full system is stochastically permanent.

Proof. We use a logarithmic Itô argument for the predator component. Applying Itô's formula to $\ln P(t)$ yields

$$\ln P(t) = \ln P(0) + \int_0^t \Big[\theta(\gamma E(s) + \xi N(s)) - \eta - \frac{\sigma_3^2}{2}\Big]ds + \sigma_3 W_3(t). \tag{7.1}$$

Dividing (7.1) by t and using the strong law property $\lim_{t\to\infty} \frac{W_3(t)}{t} = 0$ a.s., we obtain

$$\liminf_{t\to\infty} \frac{1}{t} \ln P(t) \geq \liminf_{t\to\infty} \frac{1}{t} \int_0^t \Big[\theta(\gamma E(s) + \xi N(s)) - \eta - \frac{\sigma_3^2}{2}\Big]ds \quad \text{a.s.} \tag{7.2}$$

Next, we connect the drift term to the prey persistence assumption. Since $E(t)$ and $N(t)$ persist in probability, for any $\varepsilon \in (0, 1)$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\liminf_{t\to\infty} \mathbb{P}\{E(t) \geq \delta,\ N(t) \geq \delta\} \geq 1 - \varepsilon. \tag{7.3}$$

Moreover, the logistic structure of the prey equations implies the natural carrying-capacity bounds $E(t) \leq K_1$ and $N(t) \leq K_2$ (in particular, this is consistent with ultimate boundedness in probability and can be made rigorous using a standard comparison argument for the prey dynamics). Hence, on the event $\{E(t) \geq \delta,\ N(t) \geq \delta\}$, we have

$$\theta(\gamma E(t) + \xi N(t)) - \eta - \frac{\sigma_3^2}{2} \geq \theta(\gamma\delta + \xi\delta) - \eta - \frac{\sigma_3^2}{2}.$$

The condition $\mathcal{R}_0^s > 1$ means that the predator has a positive net growth potential under bounded prey densities, i.e.,

$$\theta(\gamma K_1 + \xi K_2) - \eta - \frac{\sigma_3^2}{2} > 0.$$

Consequently, there exists $\varepsilon_0 > 0$ such that whenever prey densities remain in the biologically feasible range and away from the axes (as ensured in probability by (7.3)), the drift term in (7.1) admits a positive margin on events of arbitrarily large probability for large times. Combining this with (7.2) and the fact that the noise term grows sublinearly, it follows that $P(t)$ cannot converge to 0 in probability. In other words, for any $\varepsilon \in (0,1)$, there exists $\delta_P = \delta_P(\varepsilon) > 0$ such that

$$\liminf_{t\to\infty} \mathbb{P}\{P(t) \geq \delta_P\} \geq 1-\varepsilon,$$

which proves predator persistence in probability. Finally, stochastic permanence of the full system follows by combining predator persistence in probability with the assumed prey persistence in probability and ultimate boundedness in probability. □

8. Stationary distribution

In the following theorem, we establish the sufficient conditions for the existence of a unique ergodic stationary distribution of the stochastic elk–wolf system (2.1) by employing the Lyapunov functional method developed by Khasminskii. As suggested by Caraballo [21] and Kloeden and Ji et al. [22], in the presence of environmental fluctuations, investigating the stationary distribution provides a more robust framework for understanding long-term population stability than the classical analysis of asymptotically stable equilibria [23].

Theorem 8.1. *Assume that $\mathcal{R}_0^s > 1$. Then, for any initial value $(E_0, N_0, P_0) \in \mathbb{R}_+^3$, system* (2.1) *admits a unique stationary distribution $\pi(\cdot)$ on $\mathbb{R}_+^3$, and this stationary distribution is ergodic.*

Proof. The diffusion matrix is diagonal:

$$A(E,N,P) = \mathrm{diag}(\sigma_1^2E^2, \sigma_2^2N^2, \sigma_3^2P^2),$$

which is uniformly positive definite on any compact set bounded away from the axes. Let $D_\varepsilon = [\varepsilon, 1/\varepsilon]^3$ for $\varepsilon \in (0,1)$; then A is non-degenerate on $\overline{D_\varepsilon}$. Define $V(E,N,P) = M_0V_1 + V_2$,

$$V_1 = -\ln E - \ln N - \ln P, \qquad V_2 = \frac{1}{k+1}(E+N+P)^{k+1},$$

where $k \in (0,1)$ and $M_0 > 0$ are sufficiently large. By Itô's formula, one can show that

$$\begin{aligned} LV_1 =& -\Big[\alpha\Big(1-\frac{E}{K_1}\Big) - \gamma P - \mu - q\psi - \frac{\sigma_1^2}{2}\Big] \\ &- \Big[\beta\Big(1-\frac{N}{K_2}\Big) + \frac{\mu E}{N} - \xi P - \frac{\sigma_2^2}{2}\Big] \\ &- \Big[\theta(\gamma E + \xi N) - \eta - \frac{\sigma_3^2}{2}\Big]. \end{aligned}$$

Under the condition $\mathcal{R}_0^s > 1$, there exist constants $\varepsilon > 0$ and $c_1 > 0$ such that

$$LV_1(E,N,P) \leq -c_1 \quad \text{whenever} \quad (E,N,P) \in \mathbb{R}_+^3 \setminus D_\varepsilon \text{ and } \min\{E,N,P\} < \varepsilon.$$

For the polynomial term, we obtain

$$LV_2 \leq (E+N+P)^k\Big[\alpha E - \frac{\alpha}{K_1}E^2 + \beta N - \frac{\beta}{K_2}N^2 - \eta P\Big]$$

$$+\frac{k}{2}(E+N+P)^{k-1}(\sigma_1^2E^2+\sigma_2^2N^2+\sigma_3^2P^2).$$

As $|(E,N,P)|\to\infty$, the negative quadratic terms dominate, and hence there exist constants $R>0$ and $c_2>0$ such that

$$LV_2(E,N,P)\le -c_2 \quad \text{for all} \quad |(E,N,P)|\ge R.$$

Let $D:=D_\varepsilon\cap\{(E,N,P):|(E,N,P)|<R\}$. By choosing M sufficiently large, we obtain

$$LV(E,N,P)\le -1, \quad \forall(E,N,P)\in\mathbb{R}_+^3\setminus D.$$

Therefore, the conditions of Has'minskii's theorem are satisfied (see Mao [18] and Khasminskii [19]), which yields existence of a unique ergodic stationary distribution. □

9. Numerical results

In this section, we illustrate the long-term behavior predicted by the theoretical analysis of system (2.1) and provide numerical verification of the main parameter-based threshold conditions derived in the previous sections.

The stochastic system is discretized using the Euler–Maruyama method with time step $\Delta t=0.01$ over the horizon $T=500$. Unless stated otherwise, parameters are chosen to be biologically plausible for the Banff–Bow Valley ecosystem: $\alpha=0.5,\beta=0.4,\gamma=0.01,K_1=1000,K_2=2000,\theta=0.006$, $\eta=0.15$, $\xi=0.09$, $\mu=0.05$, $q=0.02$, $\psi=0.5$, and noise intensities $\sigma_1=\sigma_2=\sigma_3=0.05$. All simulations are initialized at $E(0)=300$, $N(0)=150$, and $P(0)=20$. Figures 1 and 2 display representative single sample paths, while the empirical stationary distributions in Figure 3 are computed from $M=50$ Monte Carlo simulations after discarding a burn-in interval $T_{\text{burn}}=200$.

Figure 1 shows that urban elk $E(t)$ and valley elk $N(t)$ remain persistent under environmental fluctuations, while the wolf population $P(t)$ survives at a lower density. This behavior is consistent with the parameter-based stochastic permanence criterion established in Theorem 7.2, since the corresponding threshold satisfies $\mathcal{R}_0^s>1$ for the chosen parameter set. From an ecological perspective, the observed irregular fluctuations around positive mean levels are not an artifact of the model, but rather reflect realistic predator–prey dynamics in large protected ecosystems. Similar temporal and spatial variability in wolf–ungulate interactions has been documented in the Yellowstone ecosystem, where population densities do not converge to fixed equilibria but instead exhibit persistent fluctuations driven by environmental heterogeneity and stochastic influences [24]. This qualitative agreement supports the interpretation of stochastic persistence and stationary distributions as a biologically meaningful notion of ecosystem stability.

The ultimate boundedness result (Theorem 6.2) is supported by the attractor-like bounded region displayed in the 3D phase portrait in Figure 2. Long-run histograms in Figure 3 provide numerical evidence for convergence toward an invariant probabilistic regime, in agreement with the stationary distribution result proved in Theorem 8.1.

To validate the extinction condition in Theorem 4.1, we increase the predator noise intensity to $\sigma_3=0.25$ while keeping all other parameters unchanged. As shown in Figure 4, the wolf population approaches extinction, whereas both elk populations remain persistent. Ecologically, this highlights the vulnerability of top predators under strong environmental variability: Stochastic disturbances may induce predator collapse even when the deterministic model predicts stable coexistence.

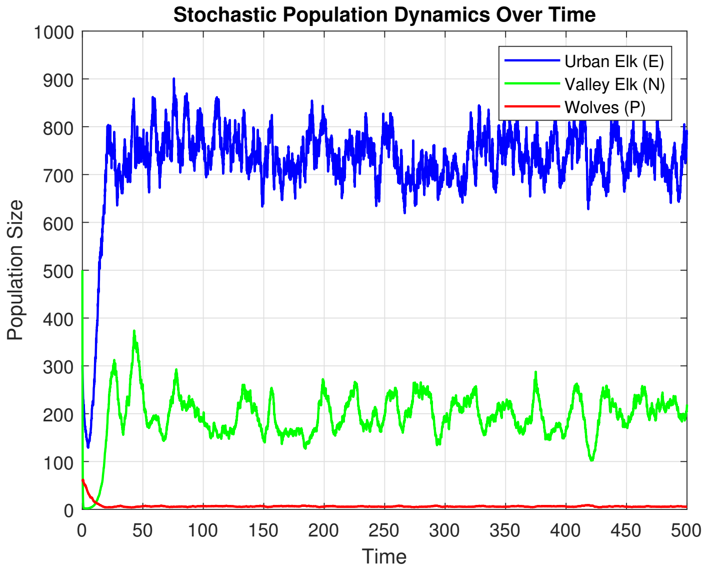

Figure 1. Time series of the stochastic elk–wolf system illustrating long-term persistence of urban elk $E(t)$, valley elk $N(t)$, and wolves $P(t)$ under environmental fluctuations. The figure reflects a realistic coexistence regime in the Banff–Bow Valley ecosystem, where prey populations remain abundant while the predator persists at a lower density.

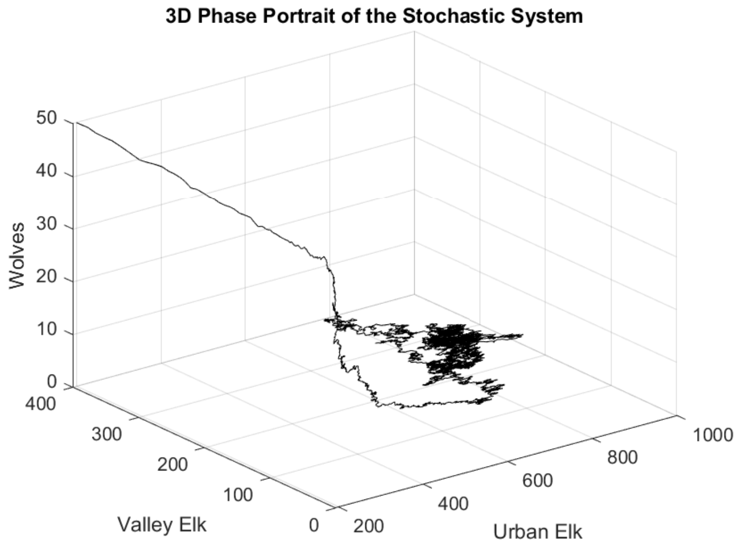

Figure 2. 3D phase portrait of the stochastic elk–wolf system. The attractor-like bounded region indicates that all population trajectories remain confined within ecologically feasible ranges, supporting long-term coexistence and resilience of the ecosystem under stochastic environmental variability.

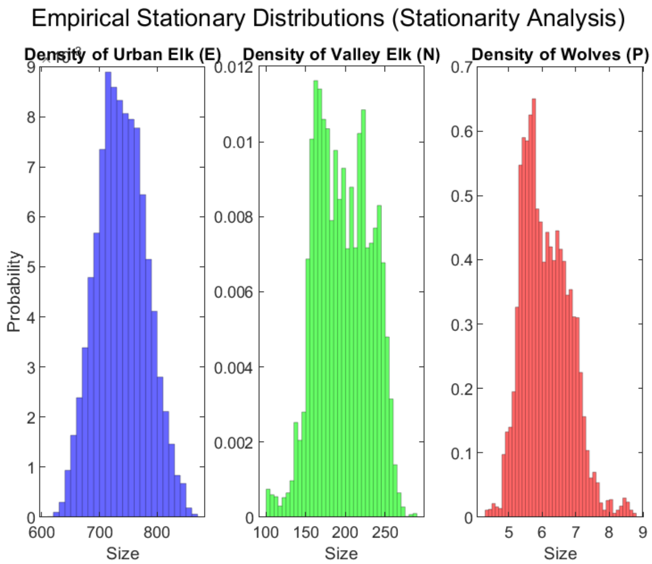

Figure 3. Empirical stationary marginal distributions of urban elk E, valley elk N, and wolves P obtained from long-run simulations. The unimodal distributions indicate probabilistic stability around typical population levels rather than convergence to fixed equilibria, consistent with field observations in fluctuating ecosystems.

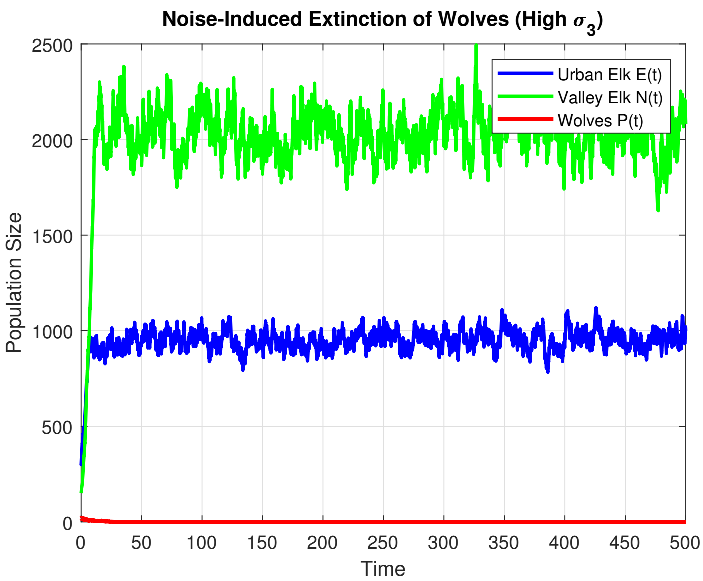

Figure 4. Noise-induced extinction of wolves under strong environmental fluctuations ($\sigma_3 = 0.25$), while both elk subpopulations remain persistent. This outcome highlights the vulnerability of top predators to stochastic disturbances and illustrates how environmental variability can trigger predator collapse even when deterministic models predict coexistence.

To assess the ecological role of migration, Figure 5 compares the urban elk dynamics in the baseline case ($\mu > 0$) and the migration-free scenario ($\mu = 0$). When migration is present, urban elk are continuously redistributed toward the valley habitat, reducing accumulation in the townsite refuge. In contrast, removing migration leads to higher urban elk densities fluctuating closer to the carrying capacity, emphasizing the importance of movement in balancing prey distribution across habitats.

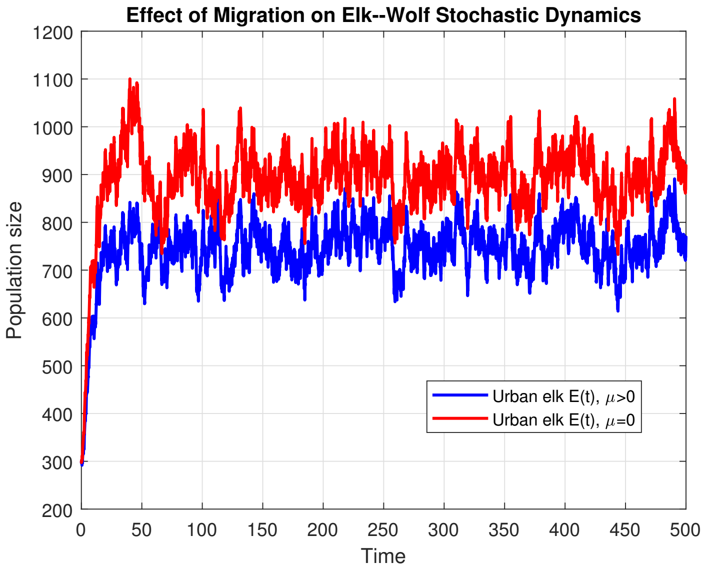

Figure 5. Effect of prey migration on the urban elk population $E(t)$. When migration from the townsite to the valley is present ($\mu > 0$), urban elk densities are reduced through spatial redistribution. In the absence of migration ($\mu = 0$), elk accumulate in the urban refuge and fluctuate near the carrying capacity, highlighting the regulatory role of movement between habitats.

Finally, Figure 6 presents an empirical finite-time Lyapunov-type growth-rate diagnostic for $E(t)$, stabilizing near a small negative value. This supports stochastic stability and rules out explosive behavior, consistent with the global well-posedness results in Section 3.

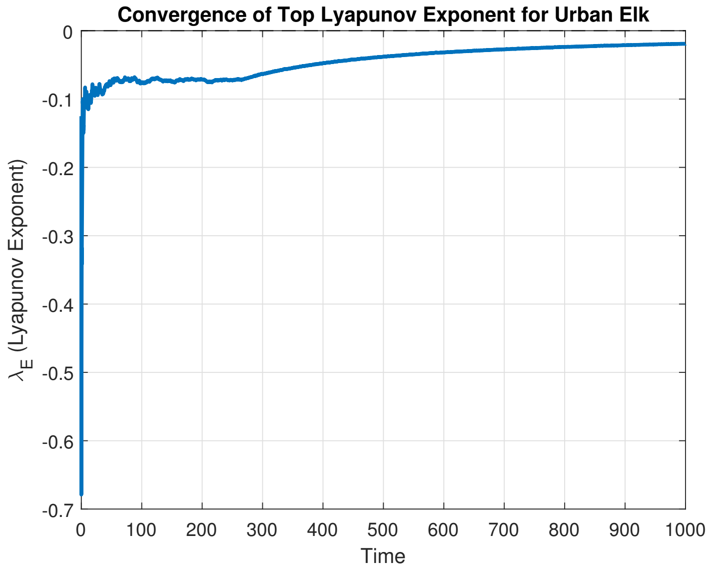

Figure 6. Empirical finite-time Lyapunov-type growth-rate diagnostic for the urban elk population $E(t)$. The convergence toward a negative value indicates stochastic stability and excludes unbounded population growth, supporting sustainable prey persistence under environmental noise.

10. Conclusions

In this work, we extended a deterministic urban–valley elk–wolf interaction model to a stochastic framework in order to account for environmental variability in the Banff–Bow Valley ecosystem. The analysis established global existence and positivity of solutions, explicit extinction criteria via logarithmic Itô calculus, and boundedness properties supporting long-term stability. In addition, we characterized coexistence through probabilistic concepts, showing that long-term dynamics are governed by stochastic persistence and stationary distributions rather than convergence to a single deterministic equilibrium. Numerical simulations corroborated the theoretical predictions, illustrating bounded trajectories, stochastic persistence, and the possibility of noise-induced predator extinction under strong environmental perturbations. These findings emphasize that random environmental disturbances can qualitatively alter ecosystem outcomes compared with classical deterministic coexistence theory.

From a biological perspective, the Banff–Bow Valley system represents a distinctive urban-refuge predator–prey environment, where elk redistribute between predator-exposed wilderness habitats and predator-free townsite zones. Our results quantify how such refuge-driven movement, combined with stochastic fluctuations, can increase predator vulnerability and reshape prey distribution across habitats. In particular, the derived extinction thresholds indicate that wolves, as top predators, may face an elevated risk of collapse under climatic or anthropogenic disturbances, even when average prey abundance remains high.

Furthermore, the permanence conditions provide explicit criteria under which wolf persistence can be maintained, thereby supporting ecosystem balance and mitigating the risk of uncontrolled elk overpopulation. The existence of an ergodic stationary distribution reinforces that wildlife stability should be interpreted probabilistically, with populations fluctuating around typical coexistence levels rather than settling at fixed equilibrium states.

Overall, this study contributes biologically meaningful indicators of persistence versus collapse that can be linked to measurable ecological parameters, offering wildlife managers quantitative thresholds for decision-making under environmental uncertainty. While the analytical techniques employed are classical, their integration within a Banff-motivated urban-refuge stochastic framework yields management-oriented insights and thresholds that are not captured by existing deterministic models.

An important direction for future research is to incorporate time delays into the proposed stochastic elk–wolf framework. In realistic ecosystems, such delays may arise from gestation and maturation periods, delayed predator responses, or time-lagged migration effects. Time delays are known to generate richer predator–prey dynamics, including periodic oscillations and stability switches, even in deterministic settings [25]. Extending the present analysis to stochastic predator–prey models with discrete or distributed delays therefore represents a natural and challenging avenue for further investigation.

Author contributions

Yousef Alnafisah: Methodology, software, investigation, writing–original draft, writing–review and editing; Moustafa El-Shahed: Conceptualization, methodology, investigation, writing–original draft, writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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