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Theory article

Classical and Bayesian estimation of stress–strength reliability under the discrete Alpha-power Weibull distribution with incomplete and record data: Application to high-voltage capacitors

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Abstract: In this paper, we explored the estimation of the stress–strength reliability parameter R = P(Y < X) when the stress and strength variables were independent and followed the Discrete Alpha-Power Weibull (DAPW) distribution. Statistical inference was developed under three practically important incomplete data structures: Type-II censored samples, upper record values, and mixed upper–lower record data. Classical and Bayesian estimation frameworks were employed. Maximum likelihood estimators (MLE) were obtained via numerical optimization, while Bayesian estimators were derived using non-informative prior distributions under the squared error loss function. To evaluate the finite-sample performance of the proposed estimators, a large-scale Monte Carlo simulation study was conducted, comparing bias, MSE, and interval precision across sample sizes and data structures. Bootstrap confidence intervals were constructed to quantify estimation uncertainty. The simulation results indicated that, within the considered settings, inference based on upper record values generally exhibited improved performance compared with Type-II censored samples and, in many configurations, also compared with the mixed record scheme, in terms of bias, MSE, and interval length. Moreover, Bayesian estimators tended to demonstrate superior performance relative to their classical counterparts, particularly for small and moderate sample sizes. The practical applicability of the proposed methodology was illustrated through the analysis of a real data set on breakdown voltages of high-voltage capacitors, where the Bayesian approach based on upper record values provided competitive and precise estimates of the stress–strength reliability parameter.

Keywords: discrete Alpha-power Weibull distribution; stress strength model; censored Type-II; upper records; upper-lower records; maximum likelihood estimation; Bayesian estimation
Mathematics Subject Classification: 62F10, 62F15, 60E05

1. Introduction

In the evaluation and design of engineering systems, medical devices, and industrial products, reliability analysis forms part of the core where it is necessary to know how the system will perform in cases of uncertainty. The stress-strength model is one of the most recognizable models of the reliability theory that assesses the likelihood that the intrinsic strength of a system is greater than the stress exerted on it. This likelihood is often simply abbreviated as $R = P(Y < X)$ and is a major measure of system safety and operational sufficiency and has been widely investigated since the original work of Birnbaum [1] Birnbaum and McCarty [2] later developed a distribution-free upper confidence bound for the probability $P(Y < X)$ based on independent samples of stress and strength.

Recently, several researchers have extended stress–strength reliability models under different statistical frameworks. Marei et al. [3] studied the estimation of stress–strength reliability for the exponential flexible Weibull extension distribution based on K-record values. Mohamed [4,5] investigated the estimation of the reliability function for different stress–strength models using record values and applied the methodology to real data. In addition, Alsadat et al. [6] discussed the stress–strength reliability estimate of the unit Gompertz distribution using ranked set sampling.

Over the past decades, substantial research effort has been directed toward the development of flexible probability distributions designed to capture complex lifetime behaviors that traditional models often fail to represent adequately. In recent years, several new generalized lifetime models have been proposed to further enhance modeling flexibility in applied sciences. For example, Zhang et al. [7] introduced exponent power sine Lomax distribution, highlighting the continued advancement of adaptable distributional families for real-world applications. For more details, see Abouelmagd et al. [8] studied the four-parameter Weibull distribution and demonstrated its usefulness in modeling lifetime data. Ahmad et al. [9] proposed the odd generalized N-H generated family of distributions and applied it to the exponential model. Chesneau [10] introduced a new bivariate trigonometric Gaussian distribution. Khalaf et al. [11] studied Bayesian and non-Bayesian approaches for estimating the extended exponential distribution. Ragab and Elgarhy [12] discussed Type II half logistic Ailamujia distribution. In addition, Benkhelifa [13] studied the Modi linear failure rate distribution. The alpha-power family of distributions has been one of these advances, becoming one of the most effective generalization methods, providing greater flexibility in skewness modeling, tail modeling, and hazard rate modeling. The alpha-power Weibull (APW) distribution and its discrete version, the DAPW distribution, have been shown to perform better than the traditional models based on Weibull in a wide range of applications. Nassar et al. [14] introduced the alpha-power Weibull distribution and investigated its statistical properties along with several practical applications. Mohamed et al. [15] later proposed the discrete alpha-power Weibull distribution and studied its characteristics with applications to real datasets. Specifically, discrete lifetime models are particularly useful in reliability modeling where count data or inspection cycles or measurements on a discrete scale are observed.

Simultaneously with the progress in distributional models, contemporary reliability research is facing more incomplete data structures. Practically, it is not always possible to have complete lifetime observations because it is expensive, time-consuming, or because it is destructive testing. One of the most common censoring schemes in reliability engineering is type-II censoring in which experiments are terminated once a pre-set number of failures has occurred. Also, record values, especially upper record values, occur naturally in the context of stress-testing experiments, environmental analysis, quality control processes, etc., where only the extreme observations are kept. Record-based statistical inference has received a lot of attention because it is efficient and relevant in extreme-value analysis. Chandler [16] studied the distribution and frequency of record values and established important theoretical results for record statistics. Ahsanullah [17] provided a comprehensive introduction to record statistics and discussed several inferential procedures based on record values. Arnold et al. [18] presented a comprehensive treatment of life data analysis and discussed several statistical methods for reliability analysis and lifetime modeling.

Despite the growing literature on stress–strength reliability models, several important gaps remain. First, while stress–strength reliability has been extensively studied under continuous distributions and standard censoring schemes, relatively limited attention has been given to discrete lifetime distributions, particularly under Type-II censoring and upper record sampling frameworks. Second, studies rarely provide a unified comparative investigation of classical and Bayesian estimation methods for stress–strength reliability when stress and strength follow flexible discrete distributions. Third, the efficiency of upper record data relative to Type-II censored samples in estimating stress–strength reliability has not been thoroughly explored within the context of discrete alpha-power models.

Motivated by these gaps, we investigate the estimation of the stress–strength reliability parameter when the stress and strength variables are independent and follow the DAPW distribution. Classical and Bayesian inferential frameworks are employed to estimate the reliability measure under three practically important incomplete data scenarios: Type-II censored samples, upper record values, and mixed upper–lower record data. MLE are derived from the corresponding likelihood functions, while Bayesian estimators are obtained under the squared error loss function using non-informative prior distributions. Owing to the analytical complexity of the resulting estimating equations under these sampling schemes, numerical optimization techniques are employed to obtain parameter estimates. To the best of our knowledge, we are the first to examine stress–strength reliability under the DAPW distribution across Type-II censored samples, upper record values, and mixed upper–lower record data within unified classical and Bayesian frameworks.

To evaluate the finite-sample performance of the proposed estimators, an extensive Monte Carlo simulation study is conducted to assess bias, MSE, and interval precision across sample sizes and data structures. In addition, bootstrap confidence intervals are constructed to quantify estimation uncertainty and to compare interval performance among the competing methods. The practical applicability of the proposed methodology is illustrated through the analysis of a real data set comprising breakdown voltages of high-voltage capacitors.

The major contributions of this study can be summarized as follows:

(i) It extends stress–strength reliability analysis under the DAPW distribution to multiple incomplete data structures, including mixed upper–lower record schemes.

(ii) It provides a comprehensive comparative investigation of MLE and Bayesian estimation methods under Type-II censored, upper record, and mixed record data.

(iii) It shows that, within the scope of the present study, inference based on upper record values generally exhibits improved performance in terms of bias, MSE, and confidence interval length relative to the alternative data structures considered.

The remainder of this paper is organized as follows: In Section 2, we introduce the stress–strength reliability model under the DAPW distribution and presents its series representation. In Section 3, we develop maximum likelihood estimation under Type-II censored samples, upper record values, and mixed upper–lower record data. In Sections 4–6, we present the corresponding Bayesian estimation procedures for Type-II censored samples, upper record data, and mixed record schemes, respectively. In Section 7, we describe the bootstrap confidence interval methodology. In Section 8, we report the results of the Monte Carlo simulation study. In Section 9, we present the real data application, and we conclude in Section 10 with key findings and directions for future research.

2. The stress–strength reliability model under the DAPW distribution

Let X and Y denote the strength and stress random variables, respectively. Assume that X and Y are independent and follow the DAPW distribution, as introduced by Nassar et al. [14] and discretized by Mohamed et al. [15].

The probability mass function (PMF) of $X$with parameter vector $(\alpha, \theta_1, \beta_1)$ is given by

$$f_X(x) = \frac{\alpha}{\alpha - 1}\left[(1 - \alpha^{-\theta_1^{x^{\beta_1}}}) - (1 - \alpha^{-\theta_1^{(x+1)^{\beta_1}}})\right], x \geq 0 \tag{1}$$

with $0 < \theta_1 < 1$, $\alpha > 0$, $\beta_1 > 0$, and $\alpha \neq 1$. The corresponding cumulative distribution function (CDF) is

$$F_X(x) = 1 - \frac{\alpha}{\alpha - 1}\left[1 - \alpha^{-\theta_1^{(x+1)^{\beta_1}}}\right], x \geq 0 \tag{2}$$

with $0 < \theta_1 < 1$, $\alpha > 0$, $\beta_1 > 0$, and $\alpha \neq 1$, and These parameter constraints ensure the validity of the probability mass function and are consistent with the original formulation of the DAPW distribution.

Similarly, the stress variable Y follows a DAPW distribution with parameter vector $(\alpha, \theta_2, \beta_2)$.

The stress–strength reliability parameter is defined as $R = P(Y < X)$, which measures the likelihood that the natural strength of the system is greater than the stress on the system, and that the system will work without failure. This has been one of the foundations of reliability theory since the seminal works of Birnbaum [1] and Birnbaum and McCarty [2] and is extensively used in the continuous and discrete settings [19].

Since, X and Y are independent and discrete, the reliability, R, can be represented as the following:

$$R = \sum_{x=0}^{\infty} f_X(x) P(Y < x) = \sum_{x=0}^{\infty} f_X(x) \sum_{y=0}^{x-1} f_Y(y). \tag{3}$$

Noting that $P(Y < x) = F_Y(x - 1)$ for $x \geq 1$ and defining $F_Y(-1) = 0$, we obtain the computationally equivalent formulation:

$$R = \sum_{x=0}^{\infty} f_X(x) F_Y(x - 1), \tag{4}$$

where $F_Y(x - 1)$ is evaluated using the CDF of the DAPW distribution for Y.

In our formulation, the stress variable Y and the strength variable X are modeled by DAPW distributions that share a common shape parameter α, while the remaining parameters are allowed to differ between stress and strength. The shared parameter α provides a parsimonious and interpretable structure and may be justified in applications where stress and strength exhibit a similar baseline shape behavior, whereas differences in the remaining parameters capture the distinct mechanisms governing X and Y.

Substituting the explicit forms of $f_X(x)$ and $F_Y(x-1)$ yields the final series representation for R in terms of the model parameters $(\alpha, \theta_1, \beta_1, \theta_2, \beta_2)$:

$$R = \sum_{x=0}^{\infty} \left\{ \frac{\alpha}{\alpha - 1} \left[(1 - \alpha^{-\theta_1^{x^{\beta_1}}}) - (1 - \alpha^{-\theta_1^{(x+1)^{\beta_1}}}) \right] \right\} . G(x), \tag{5}$$

where

$$G(x) = \begin{cases} 0, & x = 0, \\ 1 - \dfrac{\alpha}{\alpha - 1} \left(1 - \alpha^{-\theta_2^{(x)^{\beta_2}}} \right), & x \geq 1. \end{cases}$$

The parameters are constrained within $0 < \theta_1, \theta_2 < 1$, $\beta_1, \beta_2 > 0$, and $\alpha > 0$, $\alpha \neq 1$.

Due to the absence of a closed-form expression for the stress–strength reliability parameter R, numerical evaluation is required. In this study, the infinite series representation in Eq (1) is truncated at a sufficiently large upper bound K, such that the absolute difference between successive partial sums satisfies $| R_K - R_{K-1} | < 10^{-6}$.

The truncation level K is selected adaptively to ensure numerical stability and accuracy across all parameter configurations. All computations are performed using double-precision arithmetic, and convergence of the numerical approximation is verified empirically.

3. MLE of stress–strength reliability under incomplete data

In this section, we present MLE of the stress–strength reliability parameter $R = P(Y < X)$ under three incomplete data scenarios: Type-II censored samples, upper record values, and mixed upper–lower record data.

3.1. MLE of stress–strength reliability based on under Type-II censoring schemes

Complete lifetime data is frequently not available in reliability and survival analysis because of time or cost reasons. They are thus used as censoring schemes to make inferences using partially observed data. In this research, Type-II censoring will be considered, and the experiment is stopped when a preset number of failures occurs. The scheme is especially beneficial in testing high value or critical components because it ensures a constant number of failures, and it is possible to control the time course of the experiment.

Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ represent the first r ordered failure times from a random sample of size n drawn from the strength distribution $\mathrm{DAPW}(\alpha, \theta_1, \beta_1)$. Similarly, let $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(s)}$ denote the first s ordered failure times from an independent sample of size m drawn from the stress distribution $\mathrm{DAPW}(\alpha, \theta_2, \beta_2)$. Here, $r \leq n$ and $s \leq m$ are the predetermined numbers of observed failures for strength and stress, respectively.

Given the independence of the two samples, the combined likelihood function for the observed Type-II censored data is:

$$L(\Theta \mid data) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f_X(x_{(i)}, \alpha, \theta_1, \beta_1) \left[1 - F_X(x_{(r)}, \alpha, \theta_1, \beta_1) \right]^{n-r} \tag{6}$$

$$\times \frac{m!}{(m-s)!}\prod_{j=1}^{s} f_Y(y_{(j)}, \alpha, \theta_2, \beta_2)\left[1 - F_Y(y_{(s)}, \alpha, \theta_2, \beta_2)\right]^{m-s},$$

where $\Theta = (\alpha, \theta_1, \beta_1, \theta_2, \beta_2)$ is the vector of unknown parameters, f_X and f_Y are the PMF, and F_X and F_Y are the corresponding CDF.

The log-likelihood function, after omitting additive constants, is:

$$\ell(\Theta) = \sum_{i=1}^{r} \ln\left[f_X(x_{(i)}, \alpha, \theta_1, \beta_1)\right] + (n-r)\ \ln\left[\left[1 - F_X(x_{(r)}, \alpha, \theta_1, \beta_1)\right]\right]$$
$$+\sum_{j=1}^{s} \ln\left[f_Y(y_{(j)}, \alpha, \theta_2, \beta_2)\right] + (m-s)\ \ln\left[\left[1 - F_Y(y_{(s)}, \alpha, \theta_2, \beta_2)\right]\right]. \quad (7)$$

MLE of $\hat{\Theta} = (\hat{\alpha}, \hat{\theta}_1, \hat{\beta}_1, \hat{\theta}_2, \hat{\beta}_2)$ are obtained by maximizing $\ell(\Theta)$ or, equivalently, by solving the score equations:

$$\frac{\partial \ell}{\partial \alpha} = 0, \quad \frac{\partial \ell}{\partial \theta_1} = 0, \quad \frac{\partial \ell}{\partial \theta_2} = 0, \quad \frac{\partial \ell}{\partial \beta_1} = 0, \quad \frac{\partial \ell}{\partial \beta_2} = 0.$$

Due to the complex, nonlinear form of these equations, closed-form solutions are not attainable. Consequently, numerical optimization techniques must be employed. Careful choice of initial values is required to ensure convergence to the global maximum. Standard errors of the estimates can be approximated by inverting the observed Fisher information matrix evaluated at the MLE.

Once the MLE are obtained, MLE of the stress-strength reliability $R = P(Y < X)$ is computed by plugging $\hat{\Theta}$ into the series representation derived in Section 2:

$$\hat{R}_c = \sum_{x=0}^{\infty} \hat{f}_X(x)\hat{F}_Y(x-1), \quad (8)$$

where $\hat{f}_X(x) = f_X(x; \hat{\alpha}_c, \hat{\theta}_{1_c}, \hat{\beta}_{1_c})$, $\hat{F}_Y(x-1) = F_Y(x-1; \hat{\alpha}_c, \hat{\theta}_{2_c}, \hat{\beta}_{2_c})$, and $F_Y(-1) = 0$.

3.2. MLE of stress–strength reliability based on upper record values

In many practical reliability engineering scenarios, particularly in stress testing and extreme value analysis, observations are recorded only when they exceed all previous measurements. Such data are known as upper record values and arise naturally in applications where only extreme events are of interest or can be observed. The mathematical theory of record values was pioneered by Chandler [9], with comprehensive treatments provided by Ahsanullah [17] and Arnold et al. [18]. Record-based inference is especially relevant when dealing with destructive testing or when monitoring systems for unprecedented performance levels.

Let X_1, $X_2, \ldots$ be an independent and identically distributed (i.i.d.) sequence from the strength distribution DAPW$(\alpha, \theta_1, \beta_1)$. Let $U_X = (U_{X,1}, U_{X,2}, \ldots, U_{X,n})$ denote the first $n$upper record values from this sequence, satisfying $U_{X,1} < U_{X,2} < \cdots < U_{X,n}$. Similarly, let $Y_1, Y_2, \ldots$be an i.i.d. sequence from the stress distribution DAPW$(\alpha, \theta_2, \beta_2)$, and let $U_Y = (U_{Y,1}, U_{Y,2}, \ldots, U_{Y,m})$ denote the first m upper record values from the stress sequence, satisfying $U_{Y,1} < U_{Y,2} < \cdots < U_{Y,m}$.

Assuming independence between the strength and stress sequences, the joint likelihood function for the observed upper record data $D = (U_X, U_Y)$ is given by the product of the corresponding upper record likelihood functions for the strength and stress components.

$$L(\Theta|\underline{U},\underline{s}) = \left(f_X(U_n)\prod_{i=1}^{n-1}\frac{f_X(U_i)}{1-F_X(U_i)}\right)\left(f_Y(s_m)\prod_{j=1}^{m-1}\frac{f_Y(s_j)}{1-F_Y(s_j)}\right), \tag{9}$$

where $\Theta = (\alpha, \theta_1, \beta_1, \theta_2, \beta_2)$ is the parameter vector, and f_X, F_X (respectively f_Y, F_Y) denote PMF and CDF of the DAPW distribution.

Substituting the explicit forms of the DAPW density and cumulative functions yields:

$$L(\Theta|\underline{U},\underline{s}) = \left(\frac{\alpha}{\alpha-1}\left[\alpha^{-\theta_1^{(U_n+1)^{\beta_1}}} - \alpha^{-\theta_1^{(U_n)^{\beta_1}}}\right]\prod_{i=1}^{n-1}\frac{\alpha^{-\theta_1^{U_i^{\beta_1}}} - \alpha^{-\theta_1^{(U_i+1)^{\beta_1}}}}{\alpha^{-\theta_1^{(U_i+1)^{\beta_1}}}}\right)$$

$$\times\left(\frac{\alpha}{\alpha-1}\left[\alpha^{-\theta_2^{(s_m+1)^{\beta_2}}} - \alpha^{-\theta_2^{(s_m)^{\beta_2}}}\right]\prod_{j=1}^{m-1}\frac{\alpha^{-\theta_2^{s_j^{\beta_2}}} - \alpha^{-\theta_1^{(s_j+1)^{\beta_2}}}}{\alpha^{-\theta_2^{(s_j+1)^{\beta_2}}}}\right). \tag{10}$$

The corresponding log-likelihood function, after omitting additive constants, simplifies to

$$\begin{aligned}\ell(\Theta) = {}& \ln\left[\alpha^{-\theta_1^{(U_n+1)^{\beta_1}}} - \alpha^{-\theta_1^{(U_n)^{\beta_1}}}\right] + \ln\left[\alpha^{-\theta_2^{(s_m+1)^{\beta_2}}} - \alpha^{-\theta_2^{(s_m)^{\beta_2}}}\right] \\ &+ \sum_{i=1}^{n-1}\ln\left[1-\alpha^{-\theta_1^{U_i^{\beta_1}}+\theta_1^{(U_i+1)^{\beta_1}}}\right] + \sum_{j=1}^{m-1}\ln\left[1-\alpha^{-\theta_2^{s_j^{\beta_2}}+\theta_2^{(s_j+1)^{\beta_2}}}\right] \\ &+ (n+m)\ln\left(\frac{\alpha}{\alpha-1}\right).\end{aligned} \tag{11}$$

MLE of $\hat{\Theta}_U = (\hat{\alpha}_U, \hat{\theta}_{1_U}, \hat{\beta}_{1_U}, \hat{\theta}_{2_U}, \hat{\beta}_{2_U})$ are obtained by maximizing $\ell(\Theta)$ with respect to the parameters. This requires solving the system of score equations:

$$\frac{\partial\ell}{\partial\alpha} = 0, \quad \frac{\partial\ell}{\partial\theta_1} = 0, \quad \frac{\partial\ell}{\partial\theta_2} = 0, \quad \frac{\partial\ell}{\partial\beta_1} = 0, \quad \frac{\partial\ell}{\partial\beta_2} = 0.$$

Due to the nonlinear and implicit nature of these equations, closed-form solutions are not available. Therefore, numerical optimization techniques are employed. The observed Fisher information matrix, evaluated at the MLE, provides approximate standard errors for the parameter estimates. Once the MLE of the parameters are obtained, MLE of the stress–strength reliability $R = P(Y < X)$ based on upper record data is computed by plugging $\hat{\Theta}_U$ into the series representation given in Eq (4):

$$\hat{R}_U = \sum_{x=0}^{\infty}\hat{f}_X(x)\hat{F}_Y(x-1), \tag{12}$$

where $\hat{f}_X(x) = f_X(x; \hat{\alpha}_U, \hat{\theta}_{1_U}, \hat{\beta}_{1_U})$, $\hat{F}_Y(x-1) = F_Y(x-1; \hat{\alpha}_U, \hat{\theta}_{2_U}, \hat{\beta}_{2_U})$, and $F_Y(-1) = 0$.

MLE under both Type-II censored samples and upper record values are obtained by maximizing the corresponding log-likelihood functions using numerical optimization techniques. Due to the highly nonlinear nature of the score equations, closed-form solutions are not available. Therefore, the Newton–Raphson algorithm combined with a line search strategy is employed to solve the resulting system of equations.

To enhance numerical stability and avoid convergence to local maxima, multiple sets of initial values are considered. Initial values for the model parameters are selected using moment-based estimators derived from uncensored approximations of the DAPW distribution, as well as simplified Weibull-type estimates obtained by setting the alpha-power parameter equal to unity. Convergence is assessed by monitoring the relative change in the log-likelihood function and parameter estimates, with termination declared when successive updates differ by less than 10^{-6}.

All numerical procedures are implemented using standard optimization routines, and convergence to a global maximum is verified through sensitivity analysis with respect to different starting values.

3.3. MLE of stress–strength reliability under mixed upper–lower record data

In reliability engineering, the mechanisms governing stress and strength are fundamentally asymmetric. System failure is typically initiated by extreme stress events, which correspond to the upper tail of the stress distribution. Conversely, system integrity is governed by its weakest points, corresponding to the lower tail of the strength distribution. This practical reality motivates a statistical framework based on mixed record data: Utilizing upper record values to model stress and lower record values to model strength. Such an approach is parsimonious and relevant for scenarios like accelerated life testing or structural health monitoring, where only extreme values are recorded or of primary interest.

Let $\{X_i\}_{i\geq 1}$ and $\{Y_i\}_{i\geq 1}$ denote two independent sequences of independent and identically distributed random variables representing the strength and stress observations, respectively. It is assumed that both variables follow the DAPW distribution. To enable a parsimonious yet flexible modeling structure, the strength and stress distributions are assumed to share a common shape parameter α, while the remaining parameters are allowed to differ.

In the Monte Carlo simulation study, the common shape parameter α is treated as known and fixed standard practice that enhances numerical stability and facilitates a clearer assessment of the finite-sample performance of the remaining parameters. Accordingly, the parameter vectors for strength and stress are defined as $\eta_X = (\beta_1, \theta_1), \eta_Y = (\beta_2, \theta_2)$, while α is considered a known constant. The full parameter vector for the stress–strength model is thus $\Theta = (\alpha, \beta_1, \theta_1, \beta_2, \theta_2)$.

Accordingly, the PMF and CDF of the strength and stress variables are denoted by $f_X(\cdot;\eta_X)$, $F_X(\cdot;\eta_X)$ and $f_Y(\cdot;\eta_Y)$, $F_Y(\cdot;\eta_Y)$, respectively. The available data consist of the first m lower record values from the strength sequence, $L_X = (L_{X,1}, \dots, L_{X,m})$, and the first u upper record values from the stress sequence, $U_Y = (U_{Y,1}, \dots, U_{Y,u})$. These are defined recursively.

Assuming independence between the strength and stress processes, the joint likelihood function for the observed mixed record data $\mathrm{D} = (L_X, U_Y)$ can be expressed as

$$\mathcal{L}(\Theta \mid \mathrm{D}) = \mathcal{L}_L(\eta_X)\ \mathcal{L}_U(\eta_Y), \tag{13}$$

where the likelihood for the lower records is

$$\mathcal{L}_L(\eta_X) = f_X(L_{X,1};\eta_X,\alpha)\ \prod_{i=2}^{m} \frac{f_X(L_{X,i};\eta_X,\ \alpha)}{F_X(L_{X,i-1};\eta_X,\alpha)}, \tag{14}$$

and the likelihood for the upper records is

$$\mathcal{L}_U(\eta_Y) = f_Y(U_{Y,1};\eta_Y,\alpha)\ \prod_{j=2}^{u} \frac{f_Y(U_{Y,j};\eta_Y,\alpha)}{1 - F_Y(U_{Y,j-1};\eta_Y,\alpha)}. \tag{15}$$

The above likelihood formulation follows the standard theory of record values (see Arnold et al. [18];

Ahsanullah [17]). Following the numerical determination of the MLE $\hat{\Theta} = (\hat{\eta}_X, \hat{\eta}_Y)$, the corresponding maximum likelihood estimator for the stress–strength reliability $R = P(Y < X)$ is derived by substituting these estimates into the discrete series representation developed in Section 2. This yields the estimator

$$\hat{R}_M = \sum_{x=0}^{\infty} \hat{f}_X(x) \hat{F}_Y(x-1), \tag{16}$$

where $\hat{f}_X(x) = f_X(x; \alpha, \hat{\beta}_1, \hat{\theta}_1)$ is the estimated probability mass function of the strength component, and $\hat{F}_Y(x-1) = F_Y(x-1; \alpha, \hat{\beta}_2, \hat{\theta}_2)$ is the estimated cumulative distribution function of the stress component. For the evaluation to be well-defined at the lower bound, the convention $F_Y(-1) = 0$ is employed. In practice, the infinite sum in (16) is truncated at a sufficiently large upper bound to ensure numerical stability and accuracy.

4. Bayesian estimation for Type-II censored data

While maximum likelihood estimation provides a natural and widely used framework for parameter estimation, its performance may deteriorate in the presence of small or moderately sized samples and incomplete data structures, such as Type-II censoring. In such situations, Bayesian inference offers a flexible and powerful alternative by formally incorporating parameter uncertainty and yielding full posterior distributions rather than point estimates alone.

In this section, a Bayesian estimation framework is developed for the stress–strength reliability model under Type-II censored samples if both stress and strength variables follow the DAPW distribution. Non-informative prior distributions are adopted to reflect the absence of strong prior knowledge and to enable the observed data to dominate posterior inference. Bayesian estimation is carried out under the squared error loss function, for which posterior means provide optimal estimators of the model parameters.

Due to the non-conjugate structure of the likelihood function arising from Type-II censoring and the complexity of the DAPW distribution, closed-form expressions for the posterior distributions are not available. Consequently, posterior inference is performed using adaptive numerical integration techniques, which provide accurate and stable approximations to posterior expectations without the computational overhead associated with simulation-based methods such as MCMC. Bayesian credible intervals for both the model parameters and the stress–strength reliability parameter are also constructed to quantify estimation uncertainty.

4.1. Specification of prior distributions

In the Bayesian paradigm, parameters are treated as random variables with associated probability distributions that encode pre-experimental knowledge or beliefs. To maintain objectivity and enable the observed data to dominate the posterior inference, we employ independent non-informative prior distributions for all parameters of the DAPW model, respecting their respective domains.

For the common shape parameter $\alpha_c > 0$, we adopt the standard scale-invariant Jeffreys prior:

$$\pi(\alpha_c) \propto \frac{1}{\alpha_c}, \qquad \alpha_c > 0. \tag{17}$$

For the scale parameters θ_1 (strength) and θ_2 (stress), which are constrained to the interval $(0, 1)$, we specify a non-informative prior that explicitly depends on the parameter value. Following the principle of minimal prior information, we use:

$$\pi(\theta_{k_c}) \propto \frac{1}{\theta_{k_c}}, \qquad 0 < \theta_{k_c} < 1, \qquad k = 1,2. \tag{18}$$

For the shape parameters $\beta_{1_c} > 0$ and $\beta_{2_c} > 0$, we similarly employ Jeffreys priors:

$$\pi(\beta_{k_c}) \propto \frac{1}{\beta_{k_c}}, \qquad \beta_{k_c} > 0, \qquad k = 1,2. \tag{19}$$

Assuming prior independence across parameters, the joint prior distribution for the complete parameter vector $\Theta_c = (\alpha_c, \theta_{1_c}, \beta_{1_c}, \theta_{2_c}, \beta_{2_c})$ is:

$$\pi(\Theta_c) \propto \frac{1}{\alpha_c.\theta_{1_c}.\beta_{1_c}.\theta_{2_c}.\beta_{2_c}}. \tag{20}$$

It is important to note that these priors are improper (i.e., their integrals over the parameter space diverge). However, in Bayesian practice, improper non-informative priors are widely accepted when the resulting posterior distribution remains proper, a condition that typically holds given sufficiently informative data. Assuming prior independence across parameters, the resulting posterior distribution does not admit a closed-form expression due to the non-conjugate structure of the likelihood and is therefore evaluated numerically.

4.2. Posterior distribution

Let $\mathcal{D}_c$ denote the observed Type-II censored data, consisting of the first r ordered failure times from a strength sample of size n and the first s ordered failure times from an independent stress sample of size m. By Bayes' theorem, the joint posterior distribution is proportional to the product of the likelihood and the joint prior:

$$\pi(\Theta_c \mid \mathcal{D}_c) \propto L(\Theta \mid \mathcal{D}_c) \times \pi(\Theta_c), \tag{21}$$

where $L(\Theta \mid \mathcal{D}_c)$ is the likelihood function derived in Eq (6).

Owing to the non-conjugate structure of the likelihood function and the use of non-informative prior distributions, the resulting posterior distribution does not admit a closed-form expression. Given the moderate dimensionality of the parameter space, posterior inference in this study is carried out using adaptive numerical integration techniques rather than simulation-based approaches such as MCMC.

Posterior expectations required for Bayesian estimation under squared error loss are approximated through numerical integration over the parameter space. This approach provides accurate and stable estimates of posterior means while avoiding the additional computational burden and convergence diagnostics associated with MCMC sampling. Similar numerical integration strategies have been widely adopted in Bayesian reliability analysis when the posterior surface is well-behaved, and the parameter dimension is moderate.

4.3. Bayesian parameter estimation

Under the squared error loss function, the Bayesian estimator for any parameter is its posterior mean. For the parameter vector $\Theta_c = (\alpha_c, \theta_{1_c}, \beta_{1_c}, \theta_{2_c}, \beta_{2_c})$, these estimates are obtained as:

$$
\begin{aligned}
\hat{\alpha}_{c_{\text{Bayes}}} &= E[\alpha_c \mid \mathcal{D}_c] = \int \alpha\, \pi(\Theta_c \mid \mathcal{D}_c)\, d\Theta, \\
\hat{\theta}_{1_{c,\text{Bayes}}} &= E[\theta_{1_c} \mid \mathcal{D}_c] = \int \theta_1\, \pi(\Theta_c \mid \mathcal{D}_c)\, d\Theta, \\
\hat{\beta}_{1_{c,\text{Bayes}}} &= E[\beta_{1_c} \mid \mathcal{D}_c] = \int \beta_1\, \pi(\Theta_c \mid \mathcal{D}_c)\, d\Theta, \\
\hat{\theta}_{2_{c,\text{Bayes}}} &= E[\theta_{2_c} \mid \mathcal{D}_c] = \int \theta_2\, \pi(\Theta_c \mid \mathcal{D}_c)\, d\Theta, \\
\hat{\beta}_{2_{c,\text{Bayes}}} &= E[\beta_{2_c} \mid \mathcal{D}_c] = \int \beta_2\, \pi(\Theta_c \mid \mathcal{D}_c)\, d\Theta.
\end{aligned} \tag{22}
$$

4.4. *Bayesian reliability estimate*

The Bayesian estimator of stress-strength reliability $R = P(Y < X)$ is obtained by substituting the Bayesian parameter estimates into the series representation of R from Eq (4):

$$
\hat{R}_{\text{Bayes}} = \sum_{x=0}^{\infty} f_X\left(x; \hat{\alpha}_{c_{\text{Bayes}}}, \hat{\theta}_{1_{c,\text{Bayes}}}, \hat{\beta}_{1_{c,\text{Bayes}}}\right) F_Y\left(x-1; \hat{\alpha}_{c_{\text{Bayes}}}, \hat{\theta}_{2_{c,\text{Bayes}}}, \hat{\beta}_{2_{c,\text{Bayes}}}\right). \tag{23}
$$

4.5. *Bayesian credible intervals for the stress–strength reliability parameter*

Bayesian credible intervals for the model parameters and the stress–strength reliability parameter R are constructed based on the posterior distribution obtained via numerical integration. Specifically, equal-tailed credible intervals are computed by determining the lower and upper quantiles of the posterior distribution corresponding to the desired credibility level. These intervals provide a probabilistic measure of uncertainty that naturally accounts for parameter variability and data uncertainty. The same procedure is employed for both Type-II censored samples and upper record data.

5. Bayesian estimation for upper record data

The Bayesian framework for estimating the stress–strength reliability parameter R based on upper record data follows the same methodology outlined in Section 4 for Type-II censored samples, with the only modification arising from the likelihood function associated with record-based sampling. Bayesian credible intervals for the stress–strength reliability parameter are constructed using the same posterior quantile-based approach described in Section 4.5.

Given the upper record sequences $\underline{r} = (r_1, \dots, r_n)$ for strength and $\underline{s} = (s_1, \dots, s_m)$ for stress, the joint posterior distribution is:

$$
\pi(\Theta_U \mid \underline{r}, \underline{s}) \propto L(\Theta_U \mid \underline{r}, \underline{s}) \times \pi(\Theta_U), \tag{24}
$$

where:

- $L(\Theta_U \mid \underline{r}, \underline{s})$ is the likelihood function for upper record data derived in Eq (9).
- $\pi(\Theta_U)$ is the same joint non-informative prior specified in Section 4.1, namely:

$$
\pi(\Theta_U) \propto \frac{1}{\alpha_U \theta_{1_U} \theta_{2_U} \beta_{1_U} \beta_{2_U}}. \tag{25}
$$

Under squared error loss, the Bayesian parameter estimates are the posterior means, denoted

$\hat{\alpha}_{U,\text{Bayes}}, \hat{\theta}_{1_{U,\text{Bayes}}}, \hat{\beta}_{1_{U,\text{Bayes}}}, \hat{\theta}_{2_{U,\text{Bayes}}}, \hat{\beta}_{2_{U,\text{Bayes}}}$. The Bayesian estimator for stress-strength reliability is then obtained by substituting these estimates into the series representation from Eq (5):

$$\hat{R}_{U,\text{Bayes}} = \sum_{x=0}^{\infty} f_X\left(x; \hat{\alpha}_{U,\text{Bayes}}, \hat{\theta}_{1_{U,\text{Bayes}}}, \hat{\beta}_{1_{U,\text{Bayes}}}\right) F_Y\left(x-1; \hat{\alpha}_{U,\text{Bayes}}, \hat{\theta}_{2_{U,\text{Bayes}}}, \hat{\beta}_{2_{U,\text{Bayes}}}\right). \quad (26)$$

All computational considerations, including the implementation of numerical integration techniques to evaluate the posterior expectations, remain identical to those discussed for the Type-II censored case.

6. Bayesian estimation for mixed upper–lower record data

In many practical reliability applications, extreme stress events are responsible for system failure, while system survival is governed by the weakest components. This asymmetry naturally motivates a mixed record framework, where upper record values are used for the stress variable and lower record values are used for the strength variable. In this section, we develop a Bayesian inference procedure for the stress–strength reliability parameter $R = P(Y < X)$ under this mixed record sampling scheme.

6.1. *Specification of prior distributions*

The Bayesian framework for estimating the stress–strength reliability parameter R under the mixed upper–lower record sampling scheme follows the same general methodology outlined in Section 4 for Type-II censored samples, with appropriate modifications arising from the mixed record likelihood structure. Bayesian credible intervals for the stress–strength reliability parameter are constructed using the same posterior quantile-based approach described in Section 4.5, conditional on the known value of the common shape parameter α.

Let the vector of unknown parameters be $\Theta_M = (\theta_{1,M}, \beta_{1,M}, \theta_{2,M}, \beta_{2,M})$. The joint prior distribution is specified as

$$\pi(\Theta_M) \propto \frac{1}{\theta_{1,M}\, \beta_{1,M}\, \theta_{2,M}\, \beta_{2,M}}. \quad (27)$$

6.2. *Posterior distribution*

Let $\mathcal{D}_M$ denote the observed mixed record data, consisting of lower record values for the strength component and upper record values for the stress component. The likelihood function corresponding to $\mathcal{D}_M$, conditional on the known value of α, is given in subsection 3.3. By Bayes' theorem, the joint posterior distribution is given by $\pi(\Theta_M \mid \mathcal{D}_M) \propto L(\Theta_M \mid \mathcal{D}_M, \alpha) \times \pi(\Theta_M)$, where $L(\Theta_M \mid \mathcal{D}_M, \alpha)$ denotes the likelihood function for the mixed upper–lower record data.

Due to the non-conjugate structure of the likelihood and the use of non-informative priors, the posterior distribution does not admit a closed-form expression and is therefore evaluated numerically using adaptive integration techniques.

6.3. Bayesian parameter estimation

Under the squared error loss function, the Bayesian estimator of any unknown parameter is given by its posterior mean. For the mixed record parameter vector $\Theta_M = (\theta_{1,M}, \beta_{1,M}, \theta_{2,M}, \beta_{2,M})$, Bayesian estimates are obtained as

$$\begin{aligned}\hat{\theta}_{1,M,\mathrm{Bayes}} &= \mathrm{E}\big(\,\theta_{1,M} \mid \mathcal{D}_M\,\big) = \int \theta_{1,M}\, \pi(\,\Theta_M \mid \mathcal{D}_M\,)\, d\Theta_M,\\ \hat{\beta}_{1,M,\mathrm{Bayes}} &= \mathrm{E}\big(\,\beta_{1,M} \mid \mathcal{D}_M\,\big) = \int \beta_{1,M}\, \pi(\,\Theta_M \mid \mathcal{D}_M\,)\, d\Theta_M,\\ \hat{\theta}_{2,M,\mathrm{Bayes}} &= \mathrm{E}\big(\,\theta_{2,M} \mid \mathcal{D}_M\,\big) = \int \theta_{2,M}\, \pi(\,\Theta_M \mid \mathcal{D}_M\,)\, d\Theta_M,\\ \hat{\beta}_{2,M,\mathrm{Bayes}} &= \mathrm{E}(\beta_{2,M} \mid \mathcal{D}_M) = \int \beta_{2,M}\, \pi(\Theta_M \mid \mathcal{D}_M)\, d\Theta_M.\end{aligned} \tag{28}$$

6.4. Bayesian reliability estimate

The Bayesian estimator of the stress–strength reliability parameter $R = P(Y < X)$ under the mixed upper–lower record sampling scheme, conditional on the known value of α, is obtained by substituting the Bayesian parameter estimates into the series representation of Rgiven in Eq (4):

$$\hat{R}_{\mathrm{M,\ Bayes}} = \sum_{x=0}^{\infty} f_X\left(x; \alpha_{M_{\mathrm{Bayes}}}, \hat{\theta}_{1_{M,\mathrm{Bayes}}}, \hat{\beta}_{1_{M,\mathrm{Bayes}}}\right) F_Y\left(x-1; \alpha_{M_{\mathrm{Bayes}}}, \hat{\theta}_{2_{M,\mathrm{Bayes}}}, \hat{\beta}_{2_{M,\mathrm{Bayes}}}\right). \tag{29}$$

7. Bootstrap confidence intervals

Bootstrap methods provide a flexible and effective approach for assessing the variability of estimators, especially when analytical variance expressions are difficult to obtain. The bootstrap technique, originally introduced by Efron [20], is based on repeatedly resampling from the observed data and recalculating the estimators of interest. In this study, bootstrap procedures are employed to construct confidence intervals for the model parameters under both Type-II censored samples and upper record values.

7.1. Bootstrap confidence intervals under Type-II censored data

The bootstrap procedure for Type-II censored samples is implemented as follows:

1. Compute MLE of the DAPW model parameters based on the original Type-II censored sample.
2. Generate B independent bootstrap samples from the fitted DAPW distribution under the same Type-II censoring scheme.
3. For each bootstrap sample $b = 1,2,\dots,B$, compute the corresponding bootstrap estimate, denoted by $\hat{\Theta}^{(b)}$.
4. The bootstrap estimates are then used to evaluate the bias and MSE of the estimator.

Here, $\hat{\Theta}$ denotes the estimate obtained from the original sample, $\hat{\Theta}^{(b)}$ denotes the estimate obtained from the b-th bootstrap resample, and B is the total number of bootstrap resamples.

The bias and MSE of the estimator are computed as follows:

$$Bias(\hat{\Theta}) = \frac{1}{B} \sum_{b=1}^{B} (\hat{\Theta}^{(b)} - \hat{\Theta}). \tag{30}$$

$$MSE(\hat{\Theta}) = \frac{1}{B} \sum_{b=1}^{B} (\hat{\Theta}^{(b)} - \hat{\Theta})^2. \tag{31}$$

The percentile bootstrap confidence intervals are given by

$$CI = \left(\hat{\Theta}^* \left(\frac{\alpha}{2}\right), \hat{\Theta}^* \left(1 - \frac{\alpha}{2}\right)\right). \tag{32}$$

where $\Theta^*(p)$ denotes the p-th empirical quantile of the bootstrap estimates.

7.2. Bootstrap confidence intervals under upper record values

The bootstrap procedure for upper record data follows the same general framework as that described for Type-II censored samples, with the likelihood function modified to reflect the record-based sampling scheme. The steps are summarized as follows:

1. Obtain MLE estimates of the DAPW parameters based on the observed upper record values.
2. Generate B bootstrap samples of upper record data from the fitted DAPW distribution.
3. For each bootstrap sample, compute the corresponding parameter estimates.
4. Use the bootstrap estimates to evaluate bias, MSE, and confidence intervals.

The bias, MSE, and percentile bootstrap confidence intervals for upper record data are computed using the same formulas as those given in Subsection 7.1.

7.3. Bootstrap confidence intervals under mixed upper–lower record data

The bootstrap procedure for mixed upper–lower record data follows the same general framework adopted for Type-II censored samples and upper record values, with appropriate modifications to account for the mixed record sampling scheme. In this framework, lower record values are used for the strength component, while upper record values are used for the stress component.

In the Monte Carlo simulation study, the common shape parameter α is assumed to be known. Accordingly, bootstrap inference is carried out conditional on the fixed value of α, and resampling is performed for the remaining model parameters only.

The bootstrap algorithm is implemented as follows:

1. Compute MLE of the unknown parameters $(\beta_1, \theta_1, \beta_2, \theta_2)$ based on the observed mixed record data.
2. Generate B independent bootstrap samples of mixed upper–lower record data from the fitted DAPW distributions, preserving the original mixed record structure.
3. For each bootstrap sample $b = 1,2, \dots, B$, compute the corresponding bootstrap estimates of the model parameters and the stress–strength reliability parameter.

The bootstrap bias, MSE, and percentile bootstrap confidence intervals for the mixed record case are computed using the same formulas as those given in subsection 7.1.

8. Monte Carlo simulation study

In this section, we present a Monte Carlo simulation study to investigate the finite-sample performance of the proposed estimators for the parameters of the DAPW distribution. The comparison is conducted between the MLE and Bayesian estimators under Type-II censored samples, upper record sampling schemes, and mixed upper–lower record data structures. Estimator performance is evaluated using bias, MSE, and bootstrap confidence intervals.

8.1. Simulation design

Random samples are generated from the DAPW distribution for each parameter configuration. Here, n denotes the sample size, and t represents the number of Monte Carlo replications. For every case, the simulation experiment is repeated $t = 1000$. The sample sizes considered are $n = 20, 50, 100,$ and 150. In this simulation setup, sample sizes $n = 20$ and $n = 50$ are regarded as small and moderate, respectively, while $n = 100$ and $n = 150$ represent larger sample sizes.

For each generated sample, the unknown parameters are estimated using the MLE and Bayesian methods under Type-II censored samples, upper record samples, and mixed upper–lower record data. In the mixed record case, the common shape parameter α is assumed to be known and fixed throughout the simulation study. The performance of an estimator $\hat{\Theta}$ is evaluated using the bias and MSE, defined as

$$\text{Bias}(\hat{\Theta}) = \frac{1}{t}\sum_{t=1}^{R}(\hat{\Theta}_t - \Theta), \tag{33}$$

$$\text{MSE}(\hat{\Theta}) = \frac{1}{t}\sum_{t=1}^{R}(\hat{\Theta}_t - \Theta)^2. \tag{34}$$

8.2. Bias and MSE results

The bias and MSE values of the estimators are reported in Tables 1–4, arranged as follows:

Simulation scenarios parameter table

Scenario 1:

- Type-II Censored: $\beta_1 = 1, \beta_2 = 2, \alpha = 1, \theta_1 = 0.9, \theta_2 = 0.4$.
- Upper Records: $\beta_1 = 1, \beta_2 = 2, \alpha = 1, \theta_1 = 0.9, \theta_2 = 0.4$.
- Mixed Records: $\beta_1 = 1, \beta_2 = 2, \alpha = 1 (known), \theta_1 = 0.9, \theta_2 = 0.4$.

Scenario 2:

- Type-II Censored: $\beta_1 = 2, \beta_2 = 0.3, \alpha = 2, \theta_1 = 0.9, \theta_2 = 0.8$.
- Upper Records: $\beta_1 = 2, \beta_2 = 0.3, \alpha = 2, \theta_1 = 0.9, \theta_2 = 0.8$.
- Mixed Records: $\beta_1 = 2, \beta_2 = 0.3, \alpha = 2\ \ (known), \theta_1 = 0.9, \theta_2 = 0.8$.

Scenario 3:

- Type-II Censored: $\beta_1 = 1.5, \beta_2 = 0.9, \alpha = 0.9, \theta_1 = 0.5, \theta_2 = 0.5$.
- Upper Records: $\beta_1 = 1.5, \beta_2 = 0.9, \alpha = 0.9, \theta_1 = 0.5, \theta_2 = 0.5$.
- Mixed Records: $\beta_1 = 1.5, \beta_2 = 0.9, \alpha = 0.9\ \ (known), \theta_1 = 0.5, \theta_2 = 0.5$.

Scenario 4:

- Type-II Censored: $\beta_1 = 0.2, \beta_2 = 2, \alpha = 0.3, \theta_1 = 0.4, \theta_2 = 0.4$.
- Upper Records: $\beta_1 = 0.2, \beta_2 = 2, \alpha = 0.3, \theta_1 = 0.4, \theta_2 = 0.4$.
- Mixed Records: $\beta_1 = 0.2, \beta_2 = 2, \alpha = 0.3\ \ (known), \theta_1 = 0.4, \theta_2 = 0.4$.

In the mixed-records analysis, the shape parameter α is treated as known and fixed at the true value for each scenario.

Table 1. Bias and MSE of the proposed estimators under Type-II censored samples, upper record values, and mixed upper–lower record data.

n	m	Parameter	Type II Bias (MLE)	MSE (MLE)	Bias (Bayes)	MSE (Bayes)	Upper record Bias (MLE)	MSE (MLE)	Bias (Bayes)	MSE (Bayes)	Mixed record Bias (MLE)	MSE (MLE)	Bias (Bayes)	MSE (Bayes)
20	20	β_1	0.059	0.130	0.018	0.222	0.056	0.123	0.015	0.221	0.053	0.121	0.011	0.218
		β_2	0.077	0.589	0.052	0.375	0.075	0.583	0.051	0.371	0.071	0.581	0.045	0.367
		α	0.051	0.187	0.049	0.174	0.043	0.182	0.041	0.171	——	——	——	——
		θ_1	0.049	0.316	0.031	0.174	0.043	0.313	0.025	0.171	0.041	0.311	0.022	0.170
		θ_2	0.023	0.107	0.021	0.107	0.021	0.101	0.018	0.101	0.016	0.099	0.014	0.098
50	50	β_1	0.043	0.128	0.006	0.185	0.041	0.121	0.001	0.181	0.037	0.118	0.008	0.178
		β_2	0.063	0.523	0.045	0.302	0.062	0.521	0.042	0.288	0.060	0.517	0.040	0.282
		α	0.047	0.172	0.036	0.147	0.046	0.162	0.032	0.143	——	——	——	——
		θ_1	0.037	0.305	0.027	0.136	0.034	0.301	0.021	0.132	0.031	0.299	0.018	0.127
		θ_2	0.017	0.106	0.012	0.086	0.013	0.101	0.009	0.084	0.012	0.098	0.008	0.081
100	100	β_1	0.039	0.108	0.002	0.055	0.036	0.106	0.001	0.052	0.032	0.103	0.001	0.045
		β_2	0.056	0.432	0.034	0.277	0.055	0.431	0.031	0.271	0.051	0.427	0.028	0.267
		α	0.038	0.166	0.028	0.136	0.034	0.161	0.022	0.131	——	——	——	——
		θ_1	0.033	0.176	0.018	0.106	0.031	0.171	0.016	0.101	0.028	0.168	0.012	0.098
		θ_2	0.005	0.073	0.007	0.056	0.003	0.071	0.002	0.052	0.001	0.066	0.0008	0.045

Table 2. Bias and MSE of the proposed estimators under Type-II censored samples, upper record values, and mixed upper–lower record data.

n	m	Parameter	Type II Bias (MLE)	MSE (MLE)	Bias (Bayes)	MSE (Bayes)	Upper record Bias (MLE)	MSE (MLE)	Bias (Bayes)	MSE (Bayes)	Mixed record Bias (MLE)	MSE (MLE)	Bias (Bayes)	MSE (Bayes)
20	20	β_1	0.056	0.273	0.026	0.222	0.046	0.271	0.021	0.221	0.043	0.270	0.016	0.219
		β_2	0.104	0.041	0.054	0.066	0.101	0.034	0.048	0.062	0.098	0.031	0.040	0.056
		α	0.038	0.025	0.022	0.045	0.032	0.021	0.021	0.042	——	——	——	——
		θ_1	0.014	0.031	0.009	0.018	0.011	0.024	0.008	0.013	0.009	0.021	0.002	0.010
		θ_2	0.011	0.009	0.011	0.061	0.007	0.002	0.009	0.054	0.002	0.001	0.003	0.051
50	50	β_1	0.041	0.222	0.021	0.202	0.034	0.118	0.013	0.218	0.031	0.215	0.002	0.204
		β_2	0.064	0.028	0.041	0.020	0.061	0.031	0.034	0.019	0.057	0.027	0.033	0.016
		α	0.039	0.014	0.014	0.034	0.032	0.009	0.009	0 .031	——	——	——	——
		θ_1	0.008	0.030	0.007	0.013	0.002	0.011	0.006	0.002	0.001	0.018	0.007	0.009
		θ_2	0.019	0.002	0.011	0.045	0.012	0.001	0.008	0.041	0.001	0.001	0.003	0.038
100	100	β_1	.030	0.173	0.009	0.201	0.026	0.105	0.003	0.205	0.023	0.012	0.001	0.014
		β_2	0.046	0.012	0.034	0.010	0.042	0.021	0.034	0.013	0.041	0.025	0.032	0.011
		α	0.026	0.011	0.008	0.034	0.022	0.009	0.008	0.031	——	——	——	——
		θ_1	0.003	0.022	0.004	0.011	0.001	0.005	0.004	0.002	0.001	0.011	0.002	0.005
		θ_2	0.007	0.001	0.010	0.030	0.003	0.001	0.010	0.041	0.001	0.001	0.001	0.023

Table 3. Bias and MSE of the proposed estimators under Type-II censored samples, upper record values, and mixed upper–lower record data.

n	m	Parameter	Type II				Upper record				Mixed record			
			Bias (MLE)	MSE (MLE)	Bias (Bayes)	MSE (Bayes)	Bias (MLE)	MSE (MLE)	Bias (Bayes)	MSE (Bayes)	Bias (MLE)	MSE (MLE)	Bias (Bayes)	MSE (Bayes)
20	20	β_1	0.029	0.028	0.008	0.016	0.025	0.023	0.002	0.013	0.024	0.017	0.0017	0.01
		β_2	0.025	0.065	0.015	0.048	0.022	0.061	0.011	0.043	0.021	0.055	0.008	0.04
		α	0.024	0.054	0.022	0.110	0.021	0.051	0.021	0.108	——	——	——	——
		θ_1	0.072	0.167	0.056	0.101	0.071	0.162	0.052	0.100	0.070	0.156	0.049	0.097
		θ_2	0.019	0.078	0.015	0.077	0.014	0.072	0.013	0.074	0.013	0.066	0.01	0.071
50	50	β_1	0.012	0.011	0.004	0.013	0.009	0.009	0.001	0.011	0.008	0.003	0.0007	0.008
		β_2	0.013	0.044	0.011	0.035	0.011	0.042	0.006	0.032	0.01	0.036	0.003	0.029
		α	0.019	0.044	0.014	0.106	0.012	0.041	0.011	0.102	——	——	——	——
		θ_1	0.049	0.158	0.045	0.101	0.044	0.152	0.043	0.082	0.043	0.146	0.040	0.056
		θ_2	0.007	0.068	0.006	0.068	0.005	0.062	0.004	0.062	0.004	0.056	0.001	0.059
100	100	β_1	0.008	0.004	0.006	0.001	0.005	0.001	0.002	0.0001	0.001	0.0003	0.0017	0.007
		β_2	0.004	0.021	0.010	0.015	0.002	0.019	0.006	0.011	0.007	0.026	0.002	0.029
		α	0.019	0.035	0.013	0.067	0.008	0.032	0.007	0.062	——	——	——	——
		θ_1	0.041	0.105	0.039	0.068	0.038	0.101	0.033	0.002	0.037	0.095	0.039	0.014
		θ_2	0.008	0.058	0.018	0.056	0.001	0.052	0.013	0.053	0.0009	0.046	0.001	0.050

Table 4. Bias and MSE of the proposed estimators under Type-II censored samples, upper record values, and mixed upper–lower record data.

n	m	Parameter	Type II				Upper record				Mixed record			
			Bias (MLE)	MSE (MLE)	Bias (Bayes)	MSE (Bayes)	Bias (MLE)	MSE (MLE)	Bias (Bayes)	MSE (Bayes)	Bias (MLE)	MSE (MLE)	Bias (Bayes)	MSE (Bayes)
20	20	β_1	0.045	0.242	0.035	0.203	0.035	0.232	0.025	0.103	0.034	0.226	0.0247	0.1023
		β_2	0.037	0.339	0.025	0.342	0.027	0.329	0.015	0.242	0.026	0.323	0.012	0.239
		α	0.042	0.035	0.041	0.055	0.032	0.025	0.031	0.045	——	——	——	——
		θ_1	0.031	0.111	0.029	0.022	0.021	0.101	0.021	0.012	0.02	0.095	0.018	0.009
		θ_2	0.011	0.236	0.008	0.245	0.009	0.226	0.003	0.235	0.008	0.22	0.0007	0.232
50	50	β_1	0.035	0.241	0.025	0.167	0.025	0.231	0.015	0.101	0.024	0.225	0.0147	0.1019
		β_2	0.022	0.322	0.024	0.303	0.012	0.312	0.014	0.101	0.011	0.306	0.011	0.208
		α	0.023	0.034	0.015	0.035	0.013	0.024	0.013	0.025	——	——	——	——
		θ_1	0.028	0.047	0.022	0.004	0.018	0.037	0.012	0.002	0.017	0.031	0.009	0.002
		θ_2	0.011	0.213	0.005	0.211	0.009	0.203	0.002	0.201	0.008	0.197	0.0002	0.198
100	100	β_1	0.018	0.158	0.008	0.125	0.013	0.148	0.002	0.1009	0.012	0.142	0.0017	0.1011
		β_2	0.026	0.253	0.022	0.267	0.016	0.243	0.012	0.057	0.010	0.237	0.009	0.204
		α	0.027	0.025	0.018	0.022	0.017	0.015	0.013	0.012	——	——	——	——
		θ_1	0.029	0.021	0.024	0.002	0.019	0.011	0.014	0.002	0.012	0.009	0.001	0.001
		θ_2	0.008	0.211	0.002	0.200	0.006	0.201	0.002	0.189	0.002	0.155	0.0001	0.156

8.3. Bootstrap confidence intervals

In this section, bootstrap confidence intervals are constructed for the model parameters under both Type-II censored samples and upper record values. Interval performance is evaluated using the average interval length (AIL), while coverage properties are discussed qualitatively based on the observed stability of the intervals across sample sizes. Bayesian credible intervals, obtained via posterior numerical integration, are discussed separately in the Bayesian inference and real data application sections for comparative purposes.

The bootstrap confidence intervals for the model parameters are presented in Tables 5–8. These tables correspond to the same parameter configurations and simulation settings described earlier for Tables 1–4, ensuring direct comparability of results across estimation methods.

Table 5. Bootstrap confidence intervals under Type-II censored samples, upper record values, and mixed upper–lower record sampling schemes.

n	m	Parameter	Type II Bootstrap CI	Upper record Bootstrap CI	Mixed record Bootstrap CI
20	20	β_1	(0.342,1.476)	(0.522,1.578)	(0.621,1.474)
		β_2	(1.238,3.632)	(1.258,3.511)	(1.355,3.498)
		α	(0.364,1.611)	(0.378,1.605)	———
		θ_1	(0.365,1.622)	(0.380,1.611)	(0.470,1.511)
		θ_2	(0.313,0.489)	(0.318,0.485)	(0.414,0.480)
50	50	β_1	(0.353,1.467)	(0.544,1.565)	(0.643,1.465)
		β_2	(1.245,3.623)	(1.256,3.501)	(1.552,3.499)
		α	(0.366,1.620)	(0.380,1.607)	———
		θ_1	(0.368,1.620)	(0.382,1.607)	(0.478,1.501)
		θ_2	(0.321,0.481)	(0.324,0.475)	(0.466,0.441)
100	100	β_1	(0.366,1.443)	(0.553,1.543)	(0.654,1.454)
		β_2	(1.248,3.601)	(1.259,3.499)	(1.387,3.376)
		α	(0.369,1.618)	(0.383,1.601)	———
		θ_1	(0.371,1.618)	(0.386,1.601)	(0.498,1.578)
		θ_2	(0.326,0.479)	(0.327,0.479)	(0.424,0.459)
150	150	β_1	(0.375,1.438)	(0.564,1.521)	(0.661,1.443)
		β_2	(1.253,3.599)	(1.264,3.488)	(1.361,3.398)
		α	(0.372,1.612)	(0.388,1.589)	———
		θ_1	(0.373,1.612)	(0.388,1.589)	(0.487,1.483)
		θ_2	(0.332,0.468)	(0.334,0.461)	(0.454,0.456)

Table 6. Bootstrap confidence intervals under Type-II censored samples, upper record values, and mixed upper–lower record sampling schemes.

n	m	Parameter	Type II Bootstrap CI	Upper record Bootstrap CI	Mixed record Bootstrap CI
20	20	β_1	(1.3418,2.475)	(1.521,2.577)	(1.532,2.571)
		β_2	(0.13,80.531)	(0.159,0.410)	(0.165,0.401)
		α	(1.364,2.610)	(1.379,2.604)	————
		θ_1	(0.865,1.221)	(0.881,1.210)	(0.889,1.201)
		θ_2	(0.366,1.632)	(0.378,1.639)	(0.382,1.631)
50	50	β_1	(1.3543,2.466)	(1.544,2.460)	(1.549,2.451)
		β_2	(0.145,0.522)	(0.157,0.400)	(0.165,0.409)
		α	(1.366,2.621)	(1.381,2.606)	————
		θ_1	(0.868,1.221)	(0.883,1.206)	(0.888,1.202)
		θ_2	(0.379,1.629)	(0.381,1.631)	(0.386,1.627)
100	100	β_1	(1.362,2.440)	(1.551,2.541)	(1.555,2.534)
		β_2	(0.148,0.500)	(0.160,0.398)	(0.167,0.392)
		α	(1.371,2.617)	(1.384,2.600)	————
		θ_1	(0.871,1.317)	(0.887,1.300)	(0.892,1.299)
		θ_2	(0.386,1.625)	(0.386,1.626)	(0.389,1.621)
150	150	β_1	(1.301,2.433)	(1.562,2.521)	(1.569,2.517)
		β_2	(0.153,0.598)	(0.165,0.387)	(0.169,0.381)
		α	(1.372,2.611)	(1.389,2.588)	————
		θ_1	(0.873,1.311)	(0.889,1.388)	(0.892,1.380)
		θ_2	(0.389,1.614)	(0.393,1.612)	(0.398,1.608)

Table 7. Bootstrap confidence intervals under Type-II censored samples, upper record values, and mixed upper–lower record sampling schemes.

n	m	Parameter	Type II Bootstrap CI	Upper record Bootstrap CI	Mixed record Bootstrap CI
20	20	β_1	(1.233,1.566)	(1.513,1.568)	(1.517,1.562)
		β_2	(0.711,0.995)	(0.715,0.996)	(0.716,0.992)
		α	(1.343,1.576)	(1.523,1.578)	————
		θ_1	(0.263,0.521)	(0.272,0.501)	(0.275,0.499)
		θ_2	(0.368,0.887)	(0.373,0.977)	(0.375,0.972)
50	50	β_1	(1.244,1.557)	(1.435,1.555)	(1.437,1.551)
		β_2	(0.719,0.983)	(0.722,0.985)	(0.727,0.981)
		α	(1.354,1.567)	(1.366,1.565)	————
		θ_1	(0.264,0.529)	(0.276,0.517)	(0.279,0.513)
		θ_2	(0.388, 0.684)	(0.353,0.764)	(0.357,0.761)

Continued on next page

n	m	Parameter	Type II Bootstrap CI	Upper record Bootstrap CI	Mixed record Bootstrap CI
100	100	β_1	(1.257,1.533)	(1.464,1.533)	(1.468,1.531)
		β_2	(0.722,0.979)	(0.725,0.972)	(0.729,0.970)
		α	(1.367,1.543)	(1.374,1.543)	———
		θ_1	(0.269,0.587)	(0.281,0.589)	(0.288,0.582)
		θ_2	(0.332, 0.554)	(0.361,0.558)	(0.367,0.553)
150	150	β_1	(1.266,1.528)	(1.455,1.511)	(1.458,1.510)
		β_2	(0.725,0.975)	(0.729,0.971)	(0.731,0.970)
		α	(1.376,1.538)	(1.379,1.521)	———
		θ_1	(0.271,0.516)	(0.288,0.589)	(0.294,0.585)
		θ_2	(0.349, 0.594)	(0.383,0.579)	(0.388,0.576)

Table 8. Bootstrap confidence intervals under Type-II censored samples, upper record values, and mixed upper–lower record sampling schemes.

n	m	Parameter	Type II Bootstrap CI	Upper record Bootstrap CI	Mixed record Bootstrap CI
20	20	β_1	(0.139,0.432)	(0.159,0.311)	(0.162,0.310)
		β_2	(1.433,2.566)	(1.613,2.668)	(1.616,2.662)
		α	(2.111,3.395)	(2.115,3.401)	———
		θ_1	(0.265,0.522)	(0.255,0.521)	(0.257,0.518)
		θ_2	(0.365,0.432)	(0.355,0.429)	(0.357,0.426)
50	50	β_1	(0.146,0.423)	(0.157,0.301)	(0.159,0.298)
		β_2	(1.444,2.557)	(1.635,2.655)	(1.638,2.652)
		α	(2.119,3.383)	(2.122,3.400)	———
		θ_1	(0.355,0.568)	(0.346,0.666)	(0.349,0.663)
		θ_2	(0.276,0.544)	(0.265,0.531)	(0.268,0.528)
100	100	β_1	(0.149,0.401)	(0.160,0.299)	(0.167,0.297)
		β_2	(1.457,2.533)	(1.644,2.633)	(1.646,2.638)
		α	(2.122,3.379)	(2.125,3.399)	———
		θ_1	(0.368,0.544)	(0.355,0.644)	(0.358,0.641)
		θ_2	(0.278,0.556)	(0.271,0.539)	(0.277,0.536)
150	150	β_1	(0.154,0.399)	(0.165,0.288)	(0.169,0.282)
		β_2	(1.476,2.528)	(1.665,2.611)	(1.667,2.609)
		α	(2.125,3.375)	(2.129,3.381)	———
		θ_1	(0.377,0.539)	(0.366,0.622)	(0.369,0.620)
		θ_2	(0.283,0.560)	(0.280,0.545)	(0.287,0.543)

A comparative inspection of Tables 5–8 indicates that both upper record and mixed upper–lower record sampling schemes generally outperform Type-II censoring in terms of interval precision. In particular, the mixed record scheme frequently produces the shortest confidence intervals across parameter configurations and sample sizes, reflecting improved estimation efficiency under extreme-value-based data structures.

Figure 1 illustrates the comparative performance of the MLE and Bayesian estimators of the stress–strength reliability parameter R in terms of bias and MSE under Type-II censoring, upper record, and mixed record sampling schemes across sample sizes.

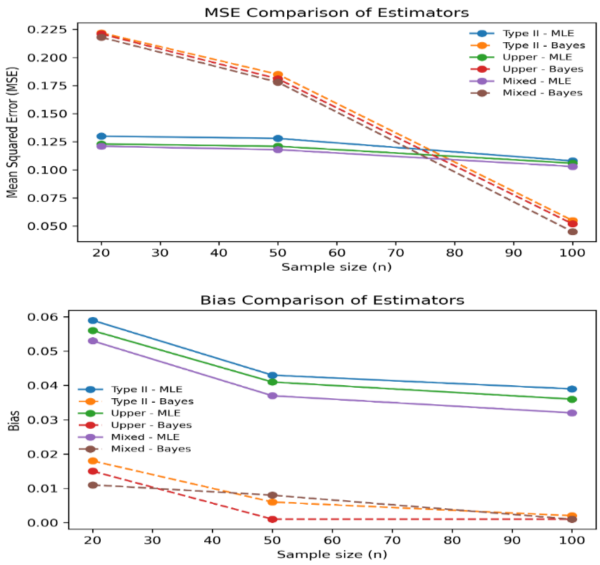

Figure 1. Bias and MSE comparison of MLE and Bayesian estimators for the stress–strength reliability parameter R under different sampling schemes.

Figure 1 shows a monotonic decline in both bias and MSE with increasing sample size for all estimators. The visual comparison indicates that one estimator achieves lower error measures in small and moderate samples, while the performance differences become negligible for larger sample sizes.

8.4. Discussion

Based on the results reported in Tables 1–4, the bias and the MSE of all estimators exhibit a clear monotonic decrease as the sample size increases, indicating the consistency and stability of the proposed estimation procedures. This pattern is visually confirmed in Figure 1, where a systematic downward trend in both bias and MSE is observed across all sampling schemes.

Figure 1 further demonstrates that, in small and moderate sample sizes, the Bayesian estimators consistently outperform their classical counterparts by achieving lower bias and smaller MSE values. This advantage is particularly evident under record-based sampling schemes, with upper record values generally providing the most accurate and stable estimates among the designs considered.

Moreover, estimators derived from upper record samples tend to yield smaller bias and MSE compared to those based on Type-II censored data, reflecting the greater informational efficiency inherent in record observations. As the sample size increases, the performance differences between the Bayesian and classical estimators gradually diminish, indicating their asymptotic equivalence.

In addition, the bootstrap confidence intervals presented in Tables 5–8 become progressively narrower as the sample size grows, further supporting the improved precision and convergence properties of the estimators.

9. Real data application

In real data applications, the true values of the model parameters and the stress–strength reliability parameter R are inherently unknown. Consequently, classical measures such as bias and MSE cannot be evaluated in a strict theoretical sense. In this study, the reported bias and MSE values for the real data analysis are computed with respect to high-precision reference estimates of the reliability parameter obtained under the Bayesian upper record framework. These reference values are treated as plug-in benchmarks and are used solely for comparative purposes to assess the relative performance of the competing estimation methods rather than to provide absolute measures of estimation accuracy.

Consider a real data set as an application of the estimation method described in this paper. In two separate experiments, the breakdown voltage of high-voltage capacitors is measured as the capacitors are subjected to steadily increasing electrical load until failure occurred. For each experiment, 50 capacitors of a specific design are tested, and the exact breakdown voltage (in kilovolts) is recorded. In the first experiment, we examine the breakdown voltages of the capacitor type optimized for high-frequency applications, while in the second, we examine the breakdown voltages of the capacitor type designed for long-term energy storage. The observed data are as follows:

The observed breakdown voltages for the capacitor type optimized for high-frequency applications are as follows (Table 9).

Table 9. The observed breakdown voltages for the capacitor type optimized for high-frequency applications.

0.923	0.93	0.965	0.93	0.96
0.931	0.953	0.928	0.942	0.963
0.948	0.936	0.923	0.925	0.946
0.938	0.927	0.946	0.956	0.943
0.958	0.929	0.949	0.945	0.96
0.956	0.939	0.959	0.949	0.945
0.965	0.952	0.944	0.954	0.955
0.949	0.923	0.955	0.948	0.93
0.934	0.935	0.943	0.929	0.944
0.959	0.948	0.955	0.941	0.953

The observed breakdown voltages for the capacitor type designed for long-term energy storage are as follows (Table 10).

Table 10. The observed breakdown voltages for the capacitor type designed for long-term energy storage.

1.06	1.054	1.057	1.056	1.059
1.06	1.059	1.053	1.053	1.061
1.054	1.054	1.057	1.055	1.053
1.057	1.054	1.051	1.053	1.059
1.051	1.06	1.054	1.06	1.056
1.06	1.053	1.059	1.058	1.06
1.053	1.056	1.053	1.059	1.054
1.059	1.057	1.056	1.06	1.056
1.053	1.051	1.061	1.053	1.057
1.06	1.056	1.056	1.055	1.054

In this application, X represents the breakdown voltages of the capacitor type optimized for high-frequency applications (strength), while Y represents the breakdown voltages of the capacitor type designed for long-term energy storage (stress). Thus, $R = P(Y < X)$ is the probability that a randomly selected unit from the strength design withstands a randomly selected stress level from the stress design.

Table 11 reports MLE and Bayesian estimates of $R = P(Y < X)$ for the real data set under both Type-II censoring and upper record sampling schemes. It is evident that, for classical and Bayesian approaches, inference based on upper record values yields smaller bias and substantially smaller MSE compared to the corresponding estimates obtained from Type-II censored samples. It should be emphasized that the reported bias and MSE values in Table 11 are relative measures based on reference estimates and are intended to facilitate methodological comparison among the competing estimation approaches.

In particular, the Bayesian estimator based on upper record data provides the most accurate estimate of R, achieving the smallest bias and MSE among all considered methods. This superior performance is further illustrated in Figure 2, which displays the point and interval estimates of R and shows that the Bayesian upper record approach produces the narrowest confidence interval, indicating the highest estimation precision.

Table 11. Point estimates of the stress–strength reliability parameter R obtained using MLE and Bayesian methods under Type-II censored samples and upper record values for the real data set.

Type II						Upper record					
MLE			Bayes			MLE			Bayes		
Bias	MSE	$\hat{R}_c$	Bias	MSE	$\hat{R}_c$	Bias	MSE	$\hat{R}_U$	Bias	MSE	$\hat{R}_U$
0.060	0.0086	0.7591	0.005	0.00071	0.8821	0.023	0.0023	0.8421	0.0023	0.00001	0.9991

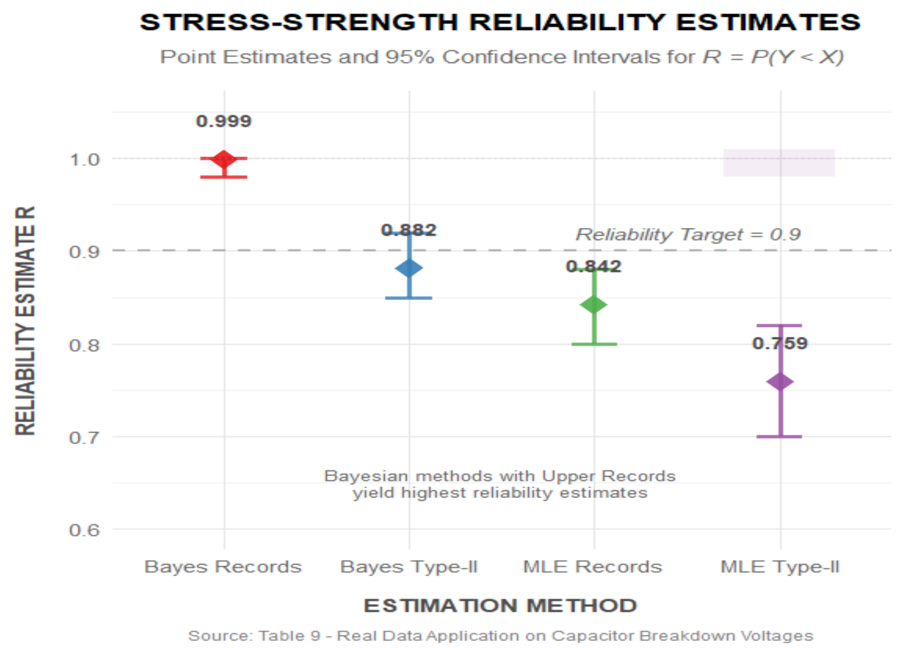

Figure 2. Point and interval estimates of the stress–strength reliability parameter $R = P(Y < X)$ for the real data under Type-II censoring and upper record sampling schemes.

10. Conclusions

The problem of estimating the stress–strength reliability parameter $R = P(Y < X)$, when both stress and strength follow the DAPW distribution, has been investigated in this study. Motivated by practical situations where complete data are rarely available, the analysis was conducted under three incomplete data structures: Type-II censored samples, upper record values, and mixed upper–lower record data. Classical MLE and Bayesian estimation frameworks were developed for these sampling schemes.

Owing to the nonlinear structure of the likelihood functions, numerical optimization techniques were employed to obtain the parameter estimates. Within the Bayesian framework, non-informative prior distributions were adopted, and estimation was performed under the squared error loss function. To evaluate the performance of the proposed estimators, an extensive Monte Carlo simulation study was conducted, and bootstrap confidence intervals were constructed to assess estimation uncertainty.

The results indicated that, within the considered settings, inference based on upper record data generally exhibits improved performance in terms of bias, MSE, and interval length compared with Type-II censored samples. The mixed upper–lower record scheme provides a flexible and practically motivated framework for capturing asymmetric extreme behavior in stress–strength systems, and, in several configurations, yields competitive or shorter interval estimates.

The practical applicability of the proposed methodologies was illustrated through the analysis of real data on breakdown voltages of high-voltage capacitors. The empirical findings were consistent with the simulation results and highlight the effectiveness of record-based inference for stress–strength reliability modeling under incomplete data structures.

Several directions for future research arise from this study. In many reliability and industrial applications, stress and strength variables may depend on high-dimensional covariates, making variable selection and screening an important methodological challenge. Extending the proposed stress–strength framework to high-dimensional regression settings, where relevant covariates can be systematically identified and incorporated, represents a promising avenue for further investigation.

In addition, standard stress–strength models typically assume that system strength remains static over time, whereas in real-world applications, it often deteriorates due to aging, fatigue, or environmental influences. Thus, integrating stochastic degradation processes into the stress–strength reliability framework would enable a more realistic characterization of failure mechanisms. Bayesian degradation modeling and online learning approaches provide flexible tools that could be incorporated within the proposed framework [21].

Furthermore, advancements in reliability modeling emphasize multi-state system structures with performance sharing mechanisms, offering additional extensions for complex system analysis [22]. Moreover, developments in high-dimensional inference and covariate screening methods also provide complementary directions for expanding the applicability of this model [23].

Author contributions

Ghareeb A. Marei: Conceptualization, methodology, writing–review and editing, supervision; Bassant Elkalzah: Methodology, formal analysis, investigation, data curation, writing–original draft preparation; M. O. Mohamed: Methodology, formal analysis; Khaled Elsharkawy: Investigation; Eman Osman: Investigation; A. Aldukeel: Writing–review and editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of this paper.

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