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*Research article*

# Stability and convergence of common fixed point algorithms for a countable infinite family of enriched nonexpansive mappings

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**Abstract:** This work intoduced a modified Halpern iterate for a countably infinite family of enriched nonexpansive mappings within convex metric spaces. Under the Aoyama-Kimura-Takahashi-Toyoda (AKTT) condition, we established that the generated sequence serves as an approximating common fixed point sequence of enriched nonexpansive mappings. Furthermore, strong convergence theorems were presented, ensuring that the iterative sequence converges to a common fixed point of the countably infinite family of enriched nonexpansive mappings in convex metric spaces, provided that the AKTT and the Song-Zheng (SZ) conditions are satisfied. Additionally, the concept of a W-mapping was extended from Banach spaces to the convex metric spaces, thereby broadening and refining existing results in the literature. We also gave a numerical illustration in the framework of convex metric spaces to show the efficiency of our proposed algorithm.

**Keywords:** enriched nonexpansive mapping; convex metric space; common fixed point; AKTT and SZ conditions; $\Delta-$convergence; strong convergence
**Mathematics Subject Classification:** 47H09, 47H10, 47J25, 54E40, 54H25

## 1. Introduction

Here, we first present some basic definitions in normed linear vector spaces, and then these definitions will be extended and adapted to metric spaces as part of the subsequent work.

Consider a real normed linear space $\mathcal{W}$ and let $\wp$ represent a nonempty closed and convex subset of $\mathcal{W}$. A mapping $\Theta : \wp \to \wp$ is called nonexpansive if

$$\|\Theta\zeta - \Theta\xi\| \leq \|\zeta - \xi\| \text{ for all } \zeta, \xi \in \wp. \tag{1.1}$$

The set of fixed points $\Theta$ is shown as $F_\Theta = \{\zeta \in \wp : \zeta = \Theta\zeta\}$.

**Definition 1.1.** [8] Suppose that $(\mathcal{W}, \|.\|)$ is a normed space. An enriched nonexpansive mapping $\Theta : \mathcal{W} \to \mathcal{W}$ is characterized by the existence of $\mathsf{b} \in [0, +\infty)$ for which the following condition holds true:

$$\|\mathsf{b}(\zeta - \xi) + \Theta\zeta - \Theta\xi\| \leq (1 + \mathsf{b})\|\zeta - \xi\| \text{ for all } \zeta, \xi \in \mathcal{W}. \tag{1.2}$$

It is worth mentioning that the class of enriched nonexpansive mappings is more general than the class of nonexpansive mappings. It has been shown in [8] that if $\mathsf{b} = 0$ in (1.2), one obtains (1.1). Moreover, if $\mathsf{b} = \frac{1}{\upsilon} - 1$ and $\upsilon \in (0, 1]$, then the above inequality can be expressed as follows:

$$\|(1 - \upsilon)(\zeta - \xi) + \upsilon\Theta\zeta - \upsilon\Theta\xi\| \leq \|\zeta - \xi\| \text{ for all } \zeta, \xi \in \mathcal{W}. \tag{1.3}$$

Denote $\Theta_\upsilon\zeta = (1 - \upsilon)\zeta + \upsilon\Theta\zeta$, and therefore, the inequality (1.3) can be subsequently transformed as follows:

$$\|\Theta_\upsilon\zeta - \Theta_\upsilon\xi\| \leq \|\zeta - \xi\| \text{ for all } \zeta, \xi \in \mathcal{W}, \tag{1.4}$$

that is, the averaged operator $\Theta_\upsilon$ is nonexpansive. The term *averaged mapping* was introduced by Baillon et al. [7].

In 1967, Halpern [18] presented an iteration in the setting of Hilbert spaces for approximating the fixed point of $\Theta$.

$$\zeta_{m+1} = \alpha_m\zeta_0 + (1 - \alpha_m)\Theta\zeta_m \text{ for all } m \in \mathbb{N}, \tag{1.5}$$

where $\{\alpha_m\}$ is a sequence in $[0, 1]$, $\zeta_0, \zeta_1 \in \wp$ are arbitrarily selected, and $\mathbb{N}$ refers to the set of positive integers. Wittmann [46] studied the algorithm (1.5) in Hilbert spaces and proved that the iteration strongly converges. Subsequently, Reich [24, 25, 27] and Shioji and Takahashi [33] generalized these findings to real Banach spaces, showing that the convergence behavior persists in this broader setting.

Several mathematicians have thoroughly examined the modified Halpern iteration. In this regard, Kim and Xu [21] examined the sequence $\{\zeta_m\}$ given as:

$$\begin{cases} \xi_m = \alpha_m\zeta_m + (1 - \alpha_m)\Theta\zeta_m, \\ \zeta_{m+1} = \beta_m\zeta_0 + (1 - \beta_m)\xi_m \text{ for all } m \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\zeta_0, \zeta_1 \in \wp$ are arbitrary chosen and $\{\alpha_m\}, \{\beta_m\}$ are two sequences in the closed interval $[0, 1]$. They established that the iterative algorithm (1.6) converges strongly in the uniformly convex Banach space. The Halpern iterative approach was proposed by Aoyama et al. [6] in 2007. The intended purpose of this approach was finding a common fixed point for a countably infinite family of nonexpansive mappings in Banach spaces. It is stated as follows:

$$\zeta_{m+1} = \alpha_m\zeta_0 + (1 - \alpha_m)\Theta_m\zeta_m \text{ for all } m \in \mathbb{N}, \tag{1.7}$$

where $\{\alpha_m\}$ is a sequence in the closed interval $[0, 1]$, $\{\Theta_m\}$ is the sequence of nonexpansive mappings under particular conditions, and $\zeta_0, \zeta_1 \in \wp$ are arbitrarily selected. It was determined that the sequence $\{\zeta_m\}$ generated through algorithm (1.7) converges strongly to the common fixed point of the sequence $\{\Theta_m\}$. In 2021, Uba et al. [45] introduced a hybrid iterative algorithm for approximating common fixed points of a countable family of generalized nonexpansive-type mappings combined with solutions to equilibrium and variational inequality problems. They proved strong convergence results under suitable conditions and demonstrated the practical relevance of their method. In 2022, Shukla and Panicker [34] studied generalized enriched nonexpansive mappings in Banach spaces and established fixed point theorems by extending classical results (see [9, 17, 30, 43, 44] for details). Their findings provided a broader framework for analyzing iterative processes. In the context of a $CAT(0)$ space, a specific space of convex metric space, Saejung [29] in 2010 expanded upon the results of Halpern [18], Wittmann [46], Aoyama et al. [6], Reich and Shafrir [26], Reich [27] and Shioji and Takahashi [33]. The results of Kim and Xu [21] were extended to the $CAT(0)$ space in 2011 by Cuntavepanit and Panyanak [14]. In 2022, Huang and Qian [19] focused on nonlinear contractive mappings and proved the existence of common fixed points using new contractive conditions, enhancing the theory of convergence in convex settings (see [3, 10, 35, 36, 37, 39] and the references therein). In 2023, Abbas et al. [1] examined enriched asymptotically nonexpansive mappings in $CAT(0)$ spaces, presenting strong convergence theorems and iterative schemes applicable in geodesic settings. Recently, Anjum and Abbas [4] analyzed b-enriched nonexpansive mappings and established fixed point results that generalized several known theorems, offering new perspectives on convergence in metric spaces with relaxed structures. Together, these works significantly enrich the understanding of countably infinite families of enriched nonexpansive mappings in convex metric spaces.

Takahashi [40] proposed the idea of convex metric spaces by employing the concept of convex structure in the following manner.

Let $(\mathcal{W}, \mathtt{d})$ be a metric space. A convex structure on $\mathcal{W}$ is defined as a mapping $\mathtt{W} : \mathcal{W} \times \mathcal{W} \times [0,1] \to \mathcal{W}$ if for all $\zeta, \xi \in \mathcal{W}$ and $\upsilon \in [0,1]$, we have

$$\mathtt{d}(q, \mathtt{W}(\zeta, \xi, \upsilon)) \leq \upsilon \mathtt{d}(q, \zeta) + (1-\upsilon)\mathtt{d}(q, \xi) \text{ for all } q \in \mathcal{W}. \tag{1.8}$$

Let $(\mathcal{W}, \mathtt{d})$ be a metric space equipped with a convex structure $\mathtt{W}$. The triplet $(\mathcal{W}, \mathtt{d}, \mathtt{W})$ is referred to as a convex metric space. A subset $\wp \subseteq \mathcal{W}$ is said to be convex if for any two elements $\zeta, \xi \in \wp$ and any $\upsilon \in [0,1]$, the convex combination $\mathtt{W}(\zeta, \xi, \upsilon)$ also belongs to $\wp$. It is clear that every normed space and all of its convex subsets are always convex metric spaces. However, the reverse implication does not necessarily hold.

Now, we propose a novel iterative approach to establish the common fixed point for a countably infinite family of enriched nonexpansive self-mappings $\{\Theta_m\}$ of $\wp$ in a convex metric space as follows:

$$\begin{cases} \xi_m = \mathtt{W}(\zeta_0, \Theta_m \zeta_m, \alpha_m), \\ \zeta_{m+1} = \mathtt{W}(\xi_m, \Theta_m \xi_m, \beta_m), \text{ for all } m \in \mathbb{N}, \end{cases} \tag{1.9}$$

where $\Theta_m$ is a countably infinite family of enriched nonexpansive mappings and $\zeta_0, \zeta_1 \in \wp$ are selected arbitrarily and $\{\alpha_m\}$ and $\{\beta_m\}$ are two sequences in the closed interval $[0,1]$. Under certain conditions, the aim of this article is to establish the strong convergence theorem of $\{\zeta_m\}$ defined by algorithm (1.9) to the common fixed point of a countably infinite family of enriched nonexpansive mappings in $CAT(0)$ spaces and convex metric spaces.

## 2. Preliminaries

We discuss certain fundamental concepts and key lemmas to support our main results.

**Lemma 2.1.** *(See [5, 40] for more information). Let* $(\mathcal{W}, \mathtt{d}, \mathtt{W})$ *be a convex metric space. For any* $\zeta, \xi \in \mathcal{W}$ *and* $\upsilon, \upsilon_1, \upsilon_2 \in [0,1]$, *we have the following properties:*

*(i)* $\mathtt{W}(\zeta,\zeta,\upsilon) = \zeta, \quad \mathtt{W}(\zeta,\xi,0) = \xi,$ *and* $\mathtt{W}(\zeta,\xi,1) = \zeta.$
*(ii)* $\mathtt{d}(\zeta, \mathtt{W}(\zeta,\xi,\upsilon)) = (1-\upsilon)\mathtt{d}(\zeta,\xi)$ *and* $\mathtt{d}(\xi, \mathtt{W}(\zeta,\xi,\upsilon)) = \upsilon \mathtt{d}(\zeta,\xi).$
*(iii)* $\mathtt{d}(\zeta,\xi) = \mathtt{d}(\zeta, \mathtt{W}(\zeta,\xi,\upsilon)) + \mathtt{d}(\mathtt{W}(\zeta,\xi,\upsilon),\xi).$
*(iv)* $|\upsilon_1 - \upsilon_2|\mathtt{d}(\zeta,\xi) \le \mathtt{d}(\mathtt{W}(\zeta,\xi,\upsilon_1), \mathtt{W}(\zeta,\xi,\upsilon_2)).$

*We say that the convex metric space* $(\mathcal{W}, \mathtt{d}, \mathtt{W})$ *possesses the following properties:*

*(A)* $\mathtt{W}(\zeta,\xi,\upsilon) = \mathtt{W}(\xi,\zeta,1-\upsilon)$, *for all* $\zeta,\xi \in \mathcal{W}$ *and* $\upsilon \in (0,1)$,
*(B)* $\mathtt{d}(\mathtt{W}(\zeta,\xi,\upsilon_1), \mathtt{W}(\zeta,\xi,\upsilon_2)) \le |\upsilon_1 - \upsilon_2|\mathtt{d}(\zeta,\xi)$, *for all* $\zeta,\xi \in \mathcal{W}$ *and* $\upsilon_1,\upsilon_2 \in (0,1)$,
*(E)* $\mathtt{d}(\mathtt{W}(\zeta,\xi,\upsilon), \mathtt{W}(\zeta,q,\upsilon)) \le (1-\upsilon)\mathtt{d}(\xi,q)$, *for all* $\zeta,\xi,q \in \mathcal{W}$ *and* $\upsilon \in (0,1)$,
*(R)* $\mathtt{d}(\mathtt{W}(\zeta,\xi,\upsilon), \mathtt{W}(q,w,\upsilon)) \le \upsilon\mathtt{d}(\zeta,q) + (1-\upsilon)\mathtt{d}(\xi,w)$, *for all* $\zeta,\xi,q,w \in \mathcal{W}$ *and* $\upsilon \in (0,1)$.

It can be seen from the properties above, condition $(A)$ and condition $(E)$ imply continuity of a convex structure $\mathtt{W}$. Moreover, condition $(E)$ is implied by the condition $(R)$. In 2005, Aoyama et al. [5] established that a convex metric space satisfying requirements $(A)$ and (E) also fulfills condition $(R)$.

In convex metric spaces, the concept of uniform convexity was presented, and many characteristics of these specific spaces were examined by Shimizu and Takahashi [32] in 1996. A convex metric space $(\mathcal{W}, \mathtt{d}, \mathtt{W})$ is considered uniformly convex if, for any $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that for any $\gamma > 0$ and $\zeta,\xi,q \in \mathcal{W}$, $\mathtt{d}(q,\zeta) \le \gamma$, $\mathtt{d}(q,\xi) \le \gamma$, and $\mathtt{d}(\zeta,\xi) \ge \gamma\epsilon$, which implies that $\mathtt{d}(q, \mathtt{W}(\zeta,\xi,1/2)) \le (1-\delta)\gamma$. A metric space is assured to be uniformly convex if a Banach space is uniformly convex. It is stated in [20] that the condition (B) is valid in metric spaces that are uniformly convex.

**Lemma 2.2.** *[23] For uniformly convex metric spaces, property* $(A)$ *is true.*

According to Lemma 2.2, a uniformly convex metric space, denoted as $(\mathcal{W}, \mathtt{d}, \mathtt{W})$, satisfies both condition $(E)$ and condition $(R)$. The convex structure is continuous as well.

Now, we examine $CAT(0)$ spaces known as the special space of convex metric spaces. Let $(\mathcal{W}, \mathrm{d})$ be a metric space. For any $\mathrm{t}_1, \mathrm{t}_2 \in [0,l]$ with $\mathfrak{c}(0) = \zeta$ and $\mathfrak{c}(l) = \xi$, and satisfying $\mathtt{d}(\mathfrak{c}(\mathrm{t}_1), \mathfrak{c}(\mathrm{t}_2)) = |\mathrm{t}_1 - \mathrm{t}_2|$, a geodesic path from $\zeta \in \mathcal{W}$ to $\xi \in \mathcal{W}$, or simply a geodesic from $\zeta$ to $\xi$, is a function $\mathfrak{c}$ from a closed interval $[0,l] \subset \mathbb{R}$ to $\mathcal{W}$. Particularly, $\mathtt{d}(\zeta,\xi) = l$ and $\mathfrak{c}$ represents an isometry. A geodesic segment or metric segment joining $\zeta$ and $\xi$ is the image $\alpha$ of $\wp$. When unique, this geodesic is represented as $[\zeta,\xi]$. The space $(\mathcal{W}, \mathrm{d})$ is considered a geodesic metric space if each pair of points of $\mathcal{W}$ are joined by a geodesic and $\mathcal{W}$ is defined as uniquely geodesic if there exists precisely one geodesic joining $\zeta$ and $\xi$ for all $\zeta,\xi \in \mathcal{W}$. A subset $Y$ of $\mathcal{W}$ is termed *convex* if it contains every geodesic segment that joins any pair of its points.

In a geodesic metric space $(\mathcal{W}, \mathrm{d})$, a geodesic triangle $\triangle(\zeta_1,\zeta_2,\zeta_3)$ consists of three points $\zeta_1,\zeta_2,\zeta_3 \in \mathcal{W}$ referred to as its vertices together with geodesic segments connecting each pair of these points, which represent its edges. The comparison triangle for the geodesic triangle $\triangle(\zeta_1,\zeta_2,\zeta_3)$ in $(\mathcal{W}, \mathrm{d})$ forms a triangle $\overline{\triangle}(\zeta_1,\zeta_2,\zeta_3)$. We assume that $\mathrm{d}_{\mathbb{E}^2}(\overline{\zeta_i},\overline{\zeta_j}) = \mathtt{d}(\zeta_i,\zeta_j)$ for $i,j \in \{1,2,3\}$ and $\triangle(\overline{\zeta_1},\overline{\zeta_2},\overline{\zeta_3})$ is in the Euclidean plane $\mathbb{E}^2$.

When every geodesic triangle satisfies the comparison axiom, the geodesic metric space is referred to as a $CAT(0)$ space. Let $\triangle$ be a geodesic triangle in $\mathcal{W}$ and let $\overline{\triangle}$ be a comparison triangle for $\triangle$, respectively. The $CAT(0)$ inequality is thus said to be satisfied by $\triangle$ if for each $\zeta, \xi \in \triangle$ and all the comparison points $\overline{\zeta}, \overline{\xi} \in \overline{\triangle}$, $\mathrm{d}(\zeta, \xi) \leq \mathrm{d}_{\mathbb{E}^2}(\overline{\zeta}, \overline{\xi})$.

If the segment $[\zeta, \xi]$ has $m$ as its midpoint and $q, \zeta$, and $\xi$ are points in a $CAT(0)$ space, then the $CAT(0)$ inequality is given as follows:

$$\mathrm{d}(q, m)^2 \leq \frac{1}{2}\mathrm{d}(q, \zeta)^2 + \frac{1}{2}\mathrm{d}(q, \xi)^2 - \frac{1}{4}\mathrm{d}(\zeta, \xi)^2. \tag{2.1}$$

According to Bruhat and Tits [13], the Convexity–Normal ($CN$) inequality is as follows:

$$\mathrm{d}(q, \upsilon\zeta \oplus (1-\upsilon)\xi)^2 \leq \upsilon \mathrm{d}(q, \zeta)^2 + (1-\upsilon)\mathrm{d}(q, \xi)^2 - \upsilon(1-\upsilon)\mathrm{d}(\zeta, \xi)^2, \tag{2.2}$$

for each $\upsilon \in [0, 1]$, where the unique point in $[\zeta, \xi]$ is indicated by $\upsilon\zeta \oplus (1-\upsilon)\xi$. Inequality (2.2) has been referenced in [15]. Applying inequality (2.1), it is clear that $CAT(0)$ spaces possess uniform convexity. According to [11], the geodesic metric space is the $CAT(0)$ space if it fulfills inequality (2.1).

Now, we proceed by recalling some fundamental concepts about $CAT(0)$ spaces.

Let $\mathcal{W}$ be a complete $CAT(0)$ space and let $\zeta_m$ be a bounded sequence in $\mathcal{W}$. For $\zeta \in \mathcal{W}$, we set

$$r(\zeta, \{\zeta_m\}) = \limsup_{m\to\infty} \mathrm{d}(\zeta, \zeta_m).$$

The asymptotic radius of the sequence $\{\zeta_m\}$ is defined as

$$r(\{\zeta_m\}) = \inf\{r(\zeta, \{\zeta_m\}) : \zeta \in \mathcal{W}\}.$$

The asymptotic center of the sequence $\zeta_m$ is given by

$$A(\{\zeta_m\}) = \{\zeta \in \mathcal{W} : r(\zeta, \{\zeta_m\}) = r(\{\zeta_m\})\}.$$

It is well established that in a complete $CAT(0)$ space, the asymptotic center contains precisely one point (see [16]).

Let $\mathcal{W}$ be a $CAT(0)$ space. A sequence $\{\zeta_m\}$ in $\mathcal{W}$ is said to $\Delta$-converge to a point $\zeta \in \mathcal{W}$ denoted as

$$\Delta - \lim_{m\to\infty}\{\zeta_m\} = \zeta,$$

if $\zeta$ is the unique asymptotic center of every subsequence $\{u_m\}$ of $\{\zeta_m\}$.

Let $\{\zeta_m\} \subset \mathcal{W}$ be a sequence that $\Delta$-converges to $\zeta \in \mathcal{W}$. By the uniqueness of the asymptotic center, for any $\xi \in \mathcal{W}$ with $\xi \neq \zeta$, we have

$$\limsup_{m\to\infty} \mathrm{d}(\zeta_m, \zeta) < \limsup_{m\to\infty} \mathrm{d}(\zeta_m, \xi).$$

This confirms that every $CAT(0)$ space possesses the Opial property.

**Lemma 2.3.** *[2, Lemma 1.4]*

*(i) In any $CAT(0)$ space $\mathcal{W}$, a bounded sequence always has a subsequence that $\Delta$-converges.*

*(ii) If $\wp$ is a closed convex subset of $\mathcal{W}$ and $\zeta_m$ is a bounded sequence contained in $\wp$, then its asymptotic center is also in $\wp$.*

*(iii)* $\wp$ *is a closed convex subset of* $\mathcal{W}$ *and* $G : \wp \to \mathcal{W}$ *is a nonexpansive mapping. If* $\zeta_m$ $\Delta$*-converges to* $\zeta$ *and satisfies* $d(\zeta_m, G\zeta_m) \to 0$*, then it follows that* $\zeta \in \wp$ *and* $G(\zeta) = \zeta$*.*

In the $CAT(0)$ space, we now define the enriched nonexpansive mapping.

**Definition 2.1.** [2] Consider a complete $CAT(0)$ space $\mathcal{W}$ with $\wp$ as a nonempty and convex subset of $\mathcal{W}$ and let $\Theta$ be the self-mapping of $\wp$. The mapping $\Theta$ is known as enriched nonexpansive if for any $\zeta, \xi \in \wp$ and $\mathtt{b} \in [0, \infty)$,

$$\mathsf{d}(\frac{\mathtt{b}}{\mathtt{b}+1}\zeta \oplus \frac{1}{\mathtt{b}+1}\Theta\zeta, \frac{\mathtt{b}}{\mathtt{b}+1}\xi \oplus \frac{1}{\mathtt{b}+1}\Theta\xi) \leq \mathsf{d}(\zeta, \xi). \tag{2.3}$$

A point $\zeta \in \wp$ can be considered a *fixed point* of $\Theta$ if $\zeta = \Theta\zeta$.

Furthermore, for any $\upsilon \in [0, 1]$, if $\mathcal{W}$ is a $CAT(0)$ space and $\zeta, \xi \in \mathcal{W}$, then there is a unique point $\upsilon\zeta \oplus (1-\upsilon)\xi \in [\zeta, \xi]$ such that

$$\mathsf{d}(q, \upsilon\zeta \oplus (1-\upsilon)\xi) \leq \upsilon\mathsf{d}(q, \zeta) + (1-\upsilon)\mathsf{d}(q, \xi), \text{ for any } q \in \mathcal{W}. \tag{2.4}$$

Consequently, $CAT(0)$ spaces possess a convex structure defined by $\mathtt{W}(\zeta, \xi, \upsilon) = \upsilon\zeta \oplus (1-\upsilon)\xi$ (See [28] for more information). Clearly, for $CAT(0)$ spaces, $(A)$, $(B)$ and $(R)$ are fulfilled (see [11, 22]). This holds for Banach spaces as well.

Suppose $(j_1, j_2, \cdots) \in \ell^\infty$ and $\mu$ is a continuous linear functional on $\ell^\infty$, the Banach space of all bounded real sequences. Instead of using $\mu(j_1, j_2, \cdots)$, we express $\mu_m(j_m)$. For each $(j_1, j_2, \cdots) \in \ell^\infty$, $\mu$ is a Banach limit if $\|\mu\| = \mu(1, 1, \cdots) = 1$ and $\mu_m(j_m) = \mu_m(j_{m+1})$. It is known that for a Banach limit $\mu$,

$$\liminf_{m\to\infty} j_m \leq \mu_m(j_m) \leq \limsup_{m\to\infty} j_m \quad \text{for all } (j_1, j_2, \cdots) \in \ell^\infty.$$

Accordingly, $\mu_m(j_m) = k$ if $(j_1, j_2, \cdots) \in \ell^\infty$ with $\lim_{m\to\infty} j_m = k$ (see also [12, 42] for more explanation).

**Lemma 2.4.** *([33, Proposition 2]) Let* $(j_1, j_2, \cdots) \in \ell^\infty$ *satisfy* $\mu_m(j_m) \leq 0$ *for all Banach limits* $\mu$*. If* $\limsup_{m\to\infty}(j_{m+1} - j_m) \leq 0$*, then* $\limsup_{m\to\infty} j_m \leq 0$*.*

**Lemma 2.5.** *([6, Lemma 2.3]) Let* $\{s_m\}$ *be a sequence of non-negative real numbers. Assume that* $\{\alpha_m\}$ *represents a sequence of real numbers within the interval* $[0, 1]$ *such that* $\sum_{m=1}^{\infty}\alpha_m = \infty$ *and* $\{\delta_m\}$ *is a sequence of nonnegative real numbers for which* $\sum_{m=1}^{\infty}\delta_m < \infty$*, while* $\{\gamma_m\}$ *indicates a sequence of real numbers satisfying* $\limsup_{m\to\infty}\gamma_m \leq 0$*. Assume that*

$$s_{m+1} \leq (1-\alpha_m)s_m + \alpha_m\gamma_m + \delta_m \textit{ for all } m \in \mathbb{N}. \tag{2.5}$$

*Consequently,* $\lim_{m\to\infty} s_m = 0$*.*

**Lemma 2.6.** *([31, Lemma 1]). Let the uniformly convex metric space with continuous structure* $\mathtt{W}$ *be symbolized as* $(\mathcal{W}, \mathsf{d}, \mathtt{W})$*. Then,* $\eta = \eta(\epsilon) > 0$ *exists for arbitrary positive values* $\epsilon$ *and* $\gamma$*:*

$$d(q, \mathtt{W}(\zeta, \xi, \upsilon)) \leq \gamma(1 - 2\min\{\upsilon, 1-\upsilon\}\eta), \tag{2.6}$$

*for each* $\zeta, \xi, q \in \mathcal{W}$ *with* $\mathsf{d}(q, \zeta) \leq \gamma$*,* $\mathsf{d}(q, \xi) \leq \gamma$*,* $\mathsf{d}(\zeta, \xi) \geq \gamma\epsilon$*, where* $\upsilon \in [0, 1]$*.*

**Remark 2.7.** *The aforementioned lemma is also valid for uniformly convex metric spaces that satisfy condition* $(E)$*.*

## 3. Main results

Let $\{\Theta_m\}$ denote a countably infinite family of mappings from $\wp$ to itself and consider $\wp \neq \emptyset$ to be a closed and convex subset of the complete convex metric space $(\mathcal{W}, \mathrm{d}, \mathtt{W})$. We say that $\{\Theta_m\}$ satisfies the AKTT condition (see [6]) if

$$\sum_{m=1}^{\infty} \sup\{\mathrm{d}(\Theta_{m+1}q, \Theta_m q) : q \in \mathfrak{L}\} < \infty, \tag{3.1}$$

for every bounded subset $\mathfrak{L}$ of $\wp$. If $\wp$ is a closed subset and $\{\Theta_m\}$ follows the AKTT condition, then for any $\zeta \in \wp$, we may express a mapping $\Theta : \wp \to \wp$ such that $\Theta\zeta = \lim_{m\to\infty} \Theta_m\zeta$. In this case, we also say that $(\{\Theta_m\}, \Theta)$ fulfills the AKTT condition. Using the same reasoning as in [6, Lemma 3.2], we obtain the following lemma.

**Lemma 3.1.** *Let* $(\{\Theta_m\}, \Theta_\mathrm{m})$ *fulfill the AKTT condition, and then* $\lim_{m\to\infty} \sup\{d(\Theta q, \Theta_m q) : q \in \mathfrak{L}\} = 0$ *for each bounded subset* $\mathfrak{L}$ *of* $\wp$.

**Lemma 3.2.** *([2, Theorem 3.1]) Let* $\Theta : \mathcal{W} \to \mathcal{W}$ *be an enriched nonexpansive mapping and* $\mathcal{W}$ *be the complete* $CAT(0)$ *space. Then, the averaged map* $\Theta_\upsilon : \mathcal{W} \to \mathcal{W}$ *is a nonexpansive mapping.*

**Remark 3.3.** *Let* $(\mathcal{W}, \mathrm{d}, \mathtt{W})$ *be a uniformly convex metric space and let* $\wp \subseteq \mathcal{W}$ *be a nonempty subset. Let* $\{\Theta_m\}$, $\forall m \in \mathbb{N}$, *be a countably infinite family of enriched nonexpansive self-mappings. For* $\upsilon \in (0, 1)$, *define, for each* $m \in \mathbb{N}$, *the mapping* $\Theta_{\upsilon,m}\zeta = W(\zeta, \Theta_m\zeta, \upsilon), \zeta \in \wp$. *Then for every* $m \in \mathbb{N}$, $F_{\Theta_{\upsilon,m}} = F_{\Theta_m}$. *Consequently,* $\bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}} = \bigcap_{m=1}^{\infty} F_{\Theta_m}$.

*Proof.* Step 1: First we will show that $F_{\Theta_m} \subseteq F_{\Theta_{\upsilon,m}}$. For this, let $\zeta \in F_{\Theta_m}$ and then $\Theta_m\zeta = \zeta$. By the definition of $\Theta_{\upsilon,m}\zeta$ and Lemma 2.1(i), we have $\Theta_{\upsilon,m}\zeta = W(\zeta, \Theta_m\zeta, \upsilon) = W(\zeta, \zeta, \upsilon) = \zeta$. Hence $\Theta_{\upsilon,m}\zeta = \zeta$ and therefore $\zeta \in F_{\Theta_{\upsilon,m}}$. This proves $F_{\Theta_m} \subseteq F_{\Theta_{\upsilon,m}}$.
Step 2: Now, we show that $F_{\Theta_{\upsilon,m}} \subseteq F_{\Theta_m}$. Let $\zeta \in F_{\Theta_{\upsilon,m}}$ and then $\zeta = \Theta_{\upsilon,m}\zeta = W(\zeta, \Theta_m\zeta, \upsilon)$. Applying Lemma 2.1(ii) yields $0 = d(\zeta, \zeta) = d(\zeta, W(\zeta, \Theta_m\zeta, \upsilon)) = (1 - \upsilon)d(\zeta, \Theta_m\zeta)$. Hence $0 = (1 - \upsilon)d(\zeta, \Theta_m\zeta)$. Here $1 - \upsilon > 0$, so $d(\zeta, \Theta_m\zeta) = 0$. Therefore $\Theta_m\zeta = \zeta$ so that $\zeta \in F_{\Theta_m}$. Thus $F_{\Theta_{\upsilon,m}} \subseteq F_{\Theta_m}$. From Steps 1 and 2, we obtain $F_{\Theta_{\upsilon,m}} = F_{\Theta_m}$for every $m \in \mathbb{N}$ and hence $\bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}} = \bigcap_{m=1}^{\infty} F_{\Theta_m}$. □

**Lemma 3.4.** *[2, Lemma 2.4] Let* $\wp$ *be a nonempty, closed, and convex subset of the complete* $CAT(0)$ *space* $\mathcal{W}$ *that follows Opial condition. Assume that* $\Theta : \wp \to \wp$ *is an enriched nonexpansive map. Then,* $\Theta\zeta = \zeta$.

**Theorem 3.5.** *Let* $(\mathcal{W}, \mathrm{d}, \mathtt{W})$ *be the complete convex metric space with nonempty, closed, and convex subset* $\wp$ *and satisfy both conditions* $(B)$ *and* $(R)$. *Consider the family of enriched nonexpansive self-mappings of* $\wp$, $\{\Theta_m\}$ *for which* $\bigcap_{m=1}^{\infty} F_{\Theta_m} \neq \emptyset$. *Assume that* $\{\zeta_m\}$ *is the sequence of* $\wp$ *generated by (1.9) and consider* $\{\alpha_m\}$ *and* $\{\beta_m\}$ *as sequences in the interval* $[0, 1]$ *that fulfill the following conditions: (R1)* $0 < \alpha_m < 1$, $\lim_{m\to\infty} \alpha_m = 0$, $\sum_{m=1}^{\infty} \alpha_m = \infty$, *and* $\sum_{m=1}^{\infty} |\alpha_{m+1} - \alpha_m| < \infty$, *(R2)* $\beta_m \in (h, 1]$ *for some* $h \in (0, 1)$ *and* $\sum_{m=1}^{\infty} |\beta_{m+1} - \beta_m| < \infty$. *Assume that* $(\{\Theta_m\}, \Theta)$ *satisfies the AKTT condition. Then* $\lim_{m\to\infty} d(\zeta_{m+1}, \zeta_m) = 0$ *and* $\lim_{m\to\infty} d(\Theta\zeta_m, \zeta_m) = 0$.

*Proof.* As we know that $\Theta_{\upsilon,m}\zeta_m = \mathtt{W}(\zeta_m, \Theta_m\zeta_m, \upsilon)$ and from Remark 3.3, we observe that $F_{\Theta_m} = \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$. Let $\sigma \in \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$. From the definitions of $\{\zeta_m\}$ and $\{\xi_m\}$, we have

$$\begin{aligned}
\mathtt{d}(\zeta_{m+1}, \sigma) &= \mathtt{d}(\mathtt{W}(\xi_m, \Theta_{\upsilon,m}\xi_m, \beta_m), \sigma) \\
&\leq \beta_m \mathtt{d}(\xi_m, \sigma) + (1-\beta_m)\mathtt{d}(\mathtt{W}(\xi_m, \Theta_m\xi_m, \upsilon), \sigma) \\
&\leq \mathtt{d}(\xi_m, \sigma) \\
&= \mathtt{d}(\mathtt{W}(\zeta_0, \Theta_{\upsilon,m}\zeta_m, \alpha_m), \sigma) \\
&\leq \alpha_m \mathtt{d}(\zeta_0, \sigma) + (1-\alpha_m)\mathtt{d}(\mathtt{W}(\zeta_m, \Theta_m\zeta_m, \upsilon), \sigma) \\
&\leq \alpha_m \mathtt{d}(\zeta_0, \sigma) + (1-\alpha_m)\mathtt{d}(\zeta_m, \sigma) \\
&\leq max\{\mathtt{d}(\zeta_0, \sigma), \mathtt{d}(\zeta_m, \sigma)\}.
\end{aligned} \tag{3.2}$$

By induction on $m$, we take $\mathtt{d}(\zeta_m, \sigma) \leq max\{\mathtt{d}(\zeta_0, \sigma), \mathtt{d}(\zeta_1, \sigma)\}$ for all $m \in \mathbb{N}$ and all $\sigma \in \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$. Consequently, the sequence $\{\zeta_m\}$ is bounded, which implies that the sequences $\{\xi_m\}$, $\{\Theta_{\upsilon,m}\zeta_m\}$ and $\{\Theta_{\upsilon,m}\xi_m\}$ are also bounded. It is derived from condition $(R1)$ that

$$\mathtt{d}(\xi_m, \Theta_{\upsilon,m}\zeta_m) = \mathtt{d}(\mathtt{W}(\zeta_0, \Theta_{\upsilon,m}\zeta_m, \alpha_m), (\mathtt{W}(\zeta_m, \Theta_m\zeta_m, \upsilon))) = \alpha_m \mathtt{d}(\zeta_0, (\mathtt{W}(\zeta_m, \Theta_m\zeta_m, \upsilon))) \to 0. \tag{3.3}$$

According to the definitions of $\{\zeta_m\}$ and $\{\xi_m\}$, we obtain

$$\begin{aligned}
\mathtt{d}(\xi_m, \xi_{m-1}) &= \mathtt{d}(\mathtt{W}(\zeta_0, \Theta_{\upsilon,m}\zeta_m, \alpha_m), \mathtt{W}(\zeta_0, \Theta_{\upsilon,m-1}\zeta_{m-1}, \alpha_{m-1})) \\
&\leq \mathtt{d}(\mathtt{W}(\zeta_0, \Theta_{\upsilon,m}\zeta_m, \alpha_m), \mathtt{W}(\zeta_0, \Theta_{\upsilon,m}\zeta_{m-1}, \alpha_m)) \\
&\quad + \mathtt{d}(\mathtt{W}(\zeta_0, \Theta_{\upsilon,m}\zeta_{m-1}, \alpha_m), \mathtt{W}(\zeta_0, \Theta_{\upsilon,m-1}\zeta_{m-1}, \alpha_m)) \\
&\quad + \mathtt{d}(\mathtt{W}(\zeta_0, \Theta_{\upsilon,m}\zeta_{m-1}, \alpha_m), \mathtt{W}(\zeta_0, \Theta_{\upsilon,m-1}\zeta_{m-1}, \alpha_{m-1})) \\
&\leq (1-\alpha_m)\mathtt{d}(\Theta_{\upsilon,m}\zeta_m, \Theta_{\upsilon,m}\zeta_{m-1}) + (1-\alpha_m)\mathtt{d}(\Theta_{\upsilon,m}\zeta_{m-1}, \Theta_{\upsilon,m-1}\zeta_{m-1}) \\
&\quad + |\alpha_m - \alpha_{m-1}|\mathtt{d}(\zeta_0, \Theta_{\upsilon,m-1}\zeta_{m-1}) \\
&\leq (1-\alpha_m)\mathtt{d}(\zeta_m, \zeta_{m-1}) + (1-\alpha_m)\mathtt{d}(\Theta_{\upsilon,m}\zeta_{m-1}, \Theta_{\upsilon,m-1}\zeta_{m-1}) \\
&\quad + |\alpha_m - \alpha_{m-1}|\mathtt{d}(\zeta_0, \Theta_{\upsilon,m-1}\zeta_{m-1}) \\
&\leq (1-\alpha_m)\mathtt{d}(\zeta_m, \zeta_{m-1}) + \mathtt{d}(\Theta_{\upsilon,m}\zeta_{m-1}, \Theta_{\upsilon,m-1}\zeta_{m-1}) \\
&\quad + |\alpha_m - \alpha_{m-1}|\mathtt{d}(\zeta_0, \Theta_{\upsilon,m-1}\zeta_{m-1}).
\end{aligned}$$

Now, we find

$$\begin{aligned}
\mathtt{d}(\zeta_{m+1}, \zeta_m) &= \mathtt{d}(\mathtt{W}(\xi_m, \Theta_{\upsilon,m}\xi_m, \beta_m), \mathtt{W}(\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1}, \beta_{m-1})) \\
&\leq \mathtt{d}(\mathtt{W}(\xi_m, \Theta_{\upsilon,m}\xi_m, \beta_m), \mathtt{W}(\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1}, \beta_m)) \\
&\quad + \mathtt{d}(\mathtt{W}(\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1}, \beta_m), \mathtt{W}(\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1}, \beta_{m-1}) \\
&\leq \beta_m \mathtt{d}(\xi_m, \xi_{m-1}) + (1-\beta_m)\mathtt{d}(\Theta_{\upsilon,m}\xi_m, \Theta_{\upsilon,m-1}\xi_{m-1}) \\
&\quad + |\beta_m - \beta_{m-1}|\mathtt{d}(\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1}) \\
&\leq \beta_m \mathtt{d}(\xi_m, \xi_{m-1}) + (1-\beta_m)(\mathtt{d}(\Theta_{\upsilon,m}\xi_m, \Theta_{\upsilon,m}\xi_{m-1}) + \mathtt{d}(\Theta_{\upsilon,m}\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1})) \\
&\quad + |\beta_m - \beta_{m-1}|\mathtt{d}(\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1}) \\
&\leq \beta_m \mathtt{d}(\xi_m, \xi_{m-1}) + (1-\beta_m)(\mathtt{d}(\xi_m, \xi_{m-1}) + \mathtt{d}(\Theta_{\upsilon,m}\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1})) \\
&\quad + |\beta_m - \beta_{m-1}|\mathtt{d}(\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1})
\end{aligned} \tag{3.4}$$

$$\leq \mathrm{d}(\xi_m, \xi_{m-1}) + \mathrm{d}(\Theta_{\upsilon,m}\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1}) + |\beta_m - \beta_{m-1}|\mathrm{d}(\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1})$$

implies that

$$\begin{aligned} \mathrm{d}(\zeta_{m+1}, \zeta_m) &\leq (1-\alpha_m)\mathrm{d}(\zeta_m, \zeta_{m-1}) + \mathrm{d}(\Theta_{\upsilon,m}\zeta_{m-1}, \Theta_{\upsilon,m-1}\zeta_{m-1}) \\ &\quad + |\alpha_m - \alpha_{m-1}|\mathrm{d}(\zeta_0, \Theta_{\upsilon,m-1}\zeta_{m-1}) + \mathrm{d}(\Theta_{\upsilon,m}\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1}) \\ &\quad + |\beta_m - \beta_{m-1}|\mathrm{d}(\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1}) \\ &\leq (1-\alpha_m)\mathrm{d}(\zeta_m, \zeta_{m-1}) + (|\alpha_m - \alpha_{m-1}| + |\beta_m - \beta_{m-1}|)L \\ &\quad + \mathrm{d}(\Theta_{\upsilon,m}\zeta_{m-1}, \Theta_{\upsilon,m-1}\zeta_{m-1}) + \mathrm{d}(\Theta_{\upsilon,m}\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1}), \end{aligned}$$

where $L = \max\{\sup_m \mathrm{d}(\zeta_0, \Theta_{\upsilon,m-1}\zeta_{m-1}), \sup_m \mathrm{d}(\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1})\}$.

Putting $\delta_m = (|\alpha_m - \alpha_{m-1}| + |\beta_m - \beta_{m-1}|)L + \mathrm{d}(\Theta_{\upsilon,m}\zeta_{m-1}, \Theta_{\upsilon,m-1}\zeta_{m-1}) + \mathrm{d}(\Theta_{\upsilon,m}\xi_{m-1}, \Theta_{\upsilon,m-1}\xi_{m-1})$, we have

$$\begin{aligned} \sum_{m=2}^{\infty}\delta_m &\leq L\sum_{m=2}^{\infty}(|\alpha_m - \alpha_{m-1}| + |\beta_m - \beta_{m-1}|) + \sum_{m=2}^{\infty}\sup\{\mathrm{d}(\Theta_{\upsilon,m}q, \Theta_{\upsilon,m-1}q) : q \in \{\zeta_k\}\} \\ &\quad + \sum_{m=2}^{\infty}\sup\{\mathrm{d}(\Theta_{\upsilon,m}q, \Theta_{\upsilon,m-1}q) : q \in \{\xi_k\}\}. \end{aligned} \tag{3.5}$$

As a result, Lemma 2.5, the conditions $(R1)$, $(R2)$ and the AKTT condition all imply that

$$\lim_{m\to\infty} \mathrm{d}(\zeta_{m+1}, \zeta_m) = 0. \tag{3.6}$$

Now, we note that

$$\begin{aligned} \mathrm{d}(\zeta_{m+1}, \xi_m) &= \mathrm{d}(\mathtt{W}(\xi_m, \Theta_{\upsilon,m}\xi_m, \beta_m), \xi_m) \\ &= (1-\beta_m)\mathrm{d}(\xi_m, \Theta_{\upsilon,m}\xi_m) \\ &\leq (1-b)(\mathrm{d}(\xi_m, \Theta_{\upsilon,m}\zeta_m) + \mathrm{d}(\Theta_{\upsilon,m}\zeta_m, \Theta_{\upsilon,m}\zeta_{m+1}) \\ &\quad + \mathrm{d}(\Theta_{\upsilon,m}\zeta_{m+1}, \Theta_{\upsilon,m}\xi_m)) \\ &\leq (1-b)(\mathrm{d}(\xi_m, \Theta_{\upsilon,m}\zeta_m) + \mathrm{d}(\zeta_m, \zeta_{m+1}) + \mathrm{d}(\zeta_{m+1}, \xi_m)). \end{aligned} \tag{3.7}$$

We get

$$\mathrm{d}(\zeta_{m+1}, \xi_m) \leq \frac{1-b}{b}(\mathrm{d}(\xi_m, \Theta_{\upsilon,m}\zeta_m) + \mathrm{d}(\zeta_m, \zeta_{m+1})). \tag{3.8}$$

According to (3.3) and (3.6), $\lim_{m\to\infty} \mathrm{d}(\zeta_{m+1}, \xi_m) = 0$. Consequently, it follows that

$$\mathrm{d}(\zeta_m, \xi_m) \leq \mathrm{d}(\zeta_m, \zeta_{m+1}) + \mathrm{d}(\zeta_{m+1}, \xi_m) \to 0. \tag{3.9}$$

Since

$$\mathrm{d}(\Theta_{\upsilon,m}\zeta_m, \zeta_m) \leq \mathrm{d}(\Theta_{\upsilon,m}\zeta_m, \xi_m) + \mathrm{d}(\xi_m, \zeta_m), \tag{3.10}$$

consequently, (3.3) and (3.9) show that

$$\lim_{m\to\infty} \mathrm{d}(\Theta_{\upsilon,m}\zeta_m, \zeta_m) = 0. \tag{3.11}$$

According to (3.11) and Lemma 3.1, we obtain

$$\begin{aligned} \mathrm{d}(\Theta_{\upsilon,m}\zeta_m, \zeta_m) &\leq \mathrm{d}(\Theta_{\upsilon,m}\zeta_m, \Theta_{\upsilon,m}\zeta_m) + \mathrm{d}(\Theta_{\upsilon,m}\zeta_m, \zeta_m) \\ &\leq \sup\{\mathrm{d}(\Theta_{\upsilon,m}q, \Theta_{\upsilon,m}q) : q \in \{\zeta_k\}\} + \mathrm{d}(\Theta_{\upsilon,m}\zeta_m, \zeta_m) \to 0, \\ \mathrm{d}(\Theta_{\upsilon,m}\zeta_m, \zeta_m) &\to 0. \end{aligned} \tag{3.12}$$

By using the definition of $\Theta_{\upsilon,m}$, we get $\upsilon \lim_{m\to\infty} \mathrm{d}(\Theta_\mathrm{m}\zeta_m, \zeta_m) = 0$. Since $\upsilon \neq 0$, then $\lim_{m\to\infty} \mathrm{d}(\Theta_\mathrm{m}\zeta_m, \zeta_m) = 0$. □

We will now show a theorem of convergence in $CAT(0)$ spaces. The following lemmas are proved for an enriched nonexpansive mapping.

**Lemma 3.6.** *Let $\Theta : \wp \to \wp$ be an enriched nonexpansive mapping and let $\wp$ be a closed and convex subset of a complete $CAT(0)$ space $\mathcal{W}$. Assuming that $\zeta_0 \in \wp$ is fixed, the mapping $\psi_\mathrm{t} : \wp \to \wp$ for each $\mathrm{t} \in (0, 1)$ is defined by*

$$\psi_{\mathrm{t}\zeta} := \mathrm{t}\zeta_0 \oplus (1 - \mathrm{t})\Theta_\upsilon\zeta \text{ for } \zeta \text{ in } \wp. \tag{3.13}$$

*Then, $\psi_t$ has exactly one fixed point $\zeta_\mathrm{t} \in \wp$, which is*

$$\zeta_\mathrm{t} = \psi_\mathrm{t}\zeta_\mathrm{t} = \mathrm{t}\zeta_0 \oplus (1 - \mathrm{t})\Theta_\upsilon\zeta_\mathrm{t}. \tag{3.14}$$

*Proof.* According to Lemma 3.2, $\Theta_\upsilon$ is a nonexpansive map for $\alpha = \frac{1}{1+\mathrm{b}}$. When we take into account the triangle $\triangle(\zeta_0, \Theta_\upsilon\zeta, \Theta_\upsilon\xi)$ and its comparison triangle for $\zeta, \xi \in \wp$, we obtain the following:

$$\begin{aligned} \mathrm{d}(\psi_{\mathrm{t}\zeta}, \psi_{\mathrm{t}\xi}) = \mathrm{d}(\mathrm{t}\zeta_0 \oplus (1-\mathrm{t})\Theta_\upsilon\zeta, \mathrm{t}\zeta_0 \oplus (1-\mathrm{t})\Theta_\upsilon\xi) &\leq \mathrm{d}_{\mathbb{R}^2}(\overline{\mathrm{t}\zeta_0 \oplus (1-\mathrm{t})\Theta_\upsilon\zeta}, \overline{\mathrm{t}\zeta_0 \oplus (1-\mathrm{t})\Theta_\upsilon\xi}) \\ &= (1-\mathrm{t})\mathrm{d}_{\mathbb{R}^2}(\Theta_\upsilon\zeta, \Theta_\upsilon\xi) \\ &= (1-\mathrm{t})\mathrm{d}(\Theta_\upsilon\zeta, \Theta_\upsilon\xi) \\ &\leq (1-\mathrm{t})\mathrm{d}(\zeta, \xi). \end{aligned}$$

This means that $\psi_\mathrm{t}$ constitutes a contraction mapping, hence leading to the conclusion. □

**Lemma 3.7.** *Let $\wp$ and $\Theta$ be defined as in the previous lemma. Then $F_\Theta \neq \emptyset$ if and only if the sequence $\{\zeta_t\}$ is bounded as $t \to 0$. In this instance, the subsequent statements are valid:*

*(1) Convergence of $\{\zeta_t\}$ to the unique fixed point $q$ of $\Theta$ that is closer to $\zeta_0$;*

*(2) For every Banach limit $\mu$ and every bounded sequence $\{\zeta_m\}$ satisfying $\lim_{m\to\infty} d(\zeta_m, \Theta\zeta_m) = 0$, the following inequality holds:*

$$d(\zeta_0, q)^2 \leq \mu_m d(\zeta_0, \zeta_m)^2.$$

*Proof.* It is obvious that $\{\zeta_t\}$ is bounded if $F_\Theta \neq \emptyset$. Conversely, let us suppose that $\{\zeta_t\}$ is bounded. Consider any sequence in the interval $(0, 1)$ such that the limit $\lim_{m\to\infty} t_m = 0$. Then, define $\mathfrak{g} : \wp \to \mathbb{R}$ by

$$\mathfrak{g}(q) = \limsup_{m\to\infty} \mathrm{d}(\zeta_{t_m}, q)^2. \tag{3.15}$$

For every $q \in \wp$, because of the boundedness of $\{\zeta_{t_m}\}$, it follows that $\delta := \inf\{\mathfrak{g}(q) : q \in \wp\} < \infty$. We select a sequence $\{q_m\} \in \wp$ so that $\lim_{m\to\infty} \mathfrak{g}(q_m) = \delta$. According to (2.1) or (2.2), we have

$$\mathrm{d}(\zeta_{t_m}, \frac{1}{2}q_m \oplus \frac{1}{2}q_k)^2 \leq \frac{1}{2}\mathrm{d}(\zeta_{t_m}, q_m)^2 + \frac{1}{2}\mathrm{d}(\zeta_{t_m}, q_k)^2 - \frac{1}{4}\mathrm{d}(q_m, q_k)^2. \tag{3.16}$$

Then, from the convexity of $\wp$, we have

$$\delta \leq \limsup_{m\to\infty} \mathrm{d}(\zeta_{t_m}, \frac{1}{2}q_m \oplus \frac{1}{2}q_k)^2 \leq \frac{1}{2}\mathfrak{g}(q_m) + \frac{1}{2}\mathfrak{g}(q_k) - \frac{1}{4}\mathrm{d}(q_m, q_k)^2. \tag{3.17}$$

Since $\{q_m\}$ is a Cauchy sequence in $\wp$, it converges to a point $q_0 \in \wp$. Let $q'$ be a point in $\wp$ such that $\mathfrak{g}(q') = \delta$. Consequently,

$$\delta \leq \limsup_{m\to\infty} \mathrm{d}(\zeta_{t_m}, \frac{1}{2}q_0 \oplus \frac{1}{2}q')^2 \leq \frac{1}{2}\mathfrak{g}(q_0) + \frac{1}{2}\mathfrak{g}(q') - \frac{1}{4}\mathrm{d}(q_0, q')^2, \tag{3.18}$$

and hence $q' = q_0$. Now from Remark 3.3 (or Lemma 3.6), $F_\Theta = F_{\Theta_\upsilon} \neq \emptyset$. Furthermore, $q_0$ is a fixed point of $\Theta$. To prove this, we consider
$\mathrm{d}(\zeta_{t_m}, \Theta_\upsilon\zeta_{t_m}) = \frac{t_m}{1-t_m}\mathrm{d}(\zeta_0, \zeta_{t_m}) \to 0$ and

$$\begin{aligned}
\limsup_{m\to\infty} \mathrm{d}(\zeta_{t_m}, \Theta_\upsilon q_0)^2 &\leq \limsup_{m\to\infty}(\mathrm{d}(\zeta_{t_m}, \Theta_\upsilon\zeta_{t_m}) + \mathrm{d}(\Theta_\upsilon\zeta_{t_m}, \Theta_\upsilon q_0))^2 \\
&\leq \limsup_{m\to\infty}(\mathrm{d}(\zeta_{t_m}, \Theta_\upsilon\zeta_{t_m}) + \mathrm{d}(\zeta_{t_m}, q_0))^2 \\
&= \limsup_{m\to\infty} \mathrm{d}(\zeta_{t_m}, q_0)^2 = \delta.
\end{aligned}$$

This implies that $q_0 = \Theta_\upsilon q_0$, and $F_{\Theta_\upsilon} \neq \emptyset$ and hence $q_0$ is the fixed point of $\Theta$ that is closer to $\zeta_0$. Now, we establish (2). Let $\{q_{t_m}\}$ be a sequence defined by (3.14), where the sequence $\{t_m\}$ is contained in the interval $(0, 1)$ and satisfies the condition $\lim_{m\to\infty} t_m = 0$.

We further suppose that $q_0 = \lim_{m\to\infty} q_{t_m}$ is the closer point of $F_\Theta$ to $\zeta_0$. From inequality (2.1) and Remark 3.3, we obtain

$$\begin{aligned}
\mathrm{d}(\zeta_m, q_{t_m})^2 &= \mathrm{d}(\zeta_m, t_m\zeta_0 \oplus (1-t_m)\Theta_\upsilon q_{t_m})^2 \\
&\leq t_m\mathrm{d}(\zeta_m, \zeta_0)^2 + (1-t_m)\mathrm{d}(\zeta_m, \Theta_\upsilon q_{t_m})^2 - t_m(1-t_m)\mathrm{d}(\zeta_0, \Theta_\upsilon q_{t_m})^2 \\
&\leq t_m\mathrm{d}(\zeta_m, \zeta_0)^2 + (1-t_m)(\mathrm{d}(\zeta_m, \Theta_\upsilon\zeta_m) + \mathrm{d}(\Theta_\upsilon\zeta_m, \Theta_\upsilon q_{t_m}))^2 \\
&\quad - t_m(1-t_m)\mathrm{d}(\zeta_0, \Theta_\upsilon q_{t_m})^2 \\
&\leq t_m\mathrm{d}(\zeta_m, \zeta_0)^2 + (1-t_m)(\mathrm{d}(\zeta_m, \Theta_\upsilon\zeta_m) + \mathrm{d}(\zeta_m, q_{t_m}))^2 \\
&\quad - t_m(1-t_m)\mathrm{d}(\zeta_0, \Theta_\upsilon q_{t_m})^2.
\end{aligned} \tag{3.19}$$

Let $\mu$ be a Banach limit. Then

$$\mu_m\mathrm{d}(\zeta_m, q_{t_m})^2 \leq t_m\mu_m\mathrm{d}(\zeta_m, \zeta_0)^2 + (1-t_m)\mu_m\mathrm{d}(\zeta_m, q_{t_m})^2 - t_m(1-t_m)\mathrm{d}(\zeta_0, \Theta_\upsilon q_{t_m})^2. \tag{3.20}$$

This implies that

$$\mu_m\mathrm{d}(\zeta_m, q_{t_m})^2 \leq \mu_m\mathrm{d}(\zeta_m, \zeta_0)^2 - (1-t_m)\mathrm{d}(\zeta_0, \Theta_\upsilon q_{t_m})^2. \tag{3.21}$$

Letting $m \to \infty$ gives

$$\mu_m\mathrm{d}(\zeta_m, q)^2 \leq \mu_m\mathrm{d}(\zeta_m, \zeta_0)^2 - \mathrm{d}(\zeta_0, q)^2. \tag{3.22}$$

In particular,

$$\mathrm{d}(\zeta_0, q)^2 \leq \mu_m\mathrm{d}(\zeta_m, \zeta_0)^2, \text{ for each Banach limit } \mu. \tag{3.23}$$

□

*It is known that $CAT(0)$ spaces possess the convex structure given by $\mathtt{W}(\zeta,\xi,\upsilon)=\upsilon\zeta\oplus(1-\upsilon)\xi$, and further, the conditions (A), (B), and (R) hold. Consequently, we obtain the following results.*

**Theorem 3.8.** *Let $\wp$ represent a nonempty, closed, and convex subset of the complete $CAT(0)$ space $\mathcal{W}$. Assume that $\{\Theta_m\}$ denotes a family of enriched nonexpansive self mappings of $\wp$, satisfying the condition that $F_{\Theta_\upsilon}=\bigcap_{m=1}^{\infty}F_{\Theta_{\upsilon,m}}\neq\emptyset$. Let $\zeta_0,\zeta_1\in\wp$ be arbitrarily selected and let $\{\zeta_m\}$ be a sequence in $\wp$ that is generated by*

$$\begin{aligned}\xi_m &= \alpha_m\zeta_0\oplus(1-\alpha_m)\Theta_m\zeta_m,\\ \zeta_{m+1} &= \beta_m\xi_m\oplus(1-\beta_m)\Theta_m\xi_m,\quad \forall m\in\mathbb{N},\end{aligned}\tag{3.24}$$

*where $\{\alpha_m\}$ and $\{\beta_m\}$ are sequences in the closed interval $[0,1]$ that fulfill the conditions $(R1)$ and $(R2)$ as stated in Theorem 3.5. Assume that $(\{\Theta_m\},\Theta)$ satisfies the AKTT condition. T*

*Then, $\lim_{m\to\infty} d(\zeta_{m+1},\zeta_m)=0$ and $\lim_{m\to\infty} d(\Theta\zeta_m,\zeta_m)=0$.*

**Theorem 3.9.** *Let $\wp$ represent a nonempty, closed, and convex subset of the complete $CAT(0)$ space $\mathcal{W}$. Let $\{\Theta_m\}$ be a family of enriched nonexpansive self mappings of $\wp$ such that $F_{\Theta_m}=\bigcap_{m=1}^{\infty}F_{\Theta_{\upsilon,m}}\neq\emptyset$. Let $\{\zeta_m\}$ be a sequence of $\wp$ generated by (3.24), where $\{\alpha_m\}$ and $\{\beta_m\}$ are sequences in $[0,1]$, which satisfy the conditions $(R1)$ and $(R2)$ as stated in Theorem 3.5. Assume that $(\{\Theta_m\},\Theta)$ fulfills the AKTT condition. Then the sequence $\{\zeta_m\}$ converges strongly to the common fixed point of $\{\Theta_{\upsilon,m}\}$ that is closer to $\zeta_0$.*

*Proof.* According to Theorem 3.8, it follows that $\lim_{m\to\infty}\mathrm{d}(\Theta\zeta_m,\zeta_m)=0$. Let $q_t$ represent the unique fixed point in $\wp$ that fulfills the condition $q_t=t\zeta_0\oplus(1-t)\Theta_\upsilon q_t$. According to Lemma 3.7, the sequence $\{q_t\}$ converges to the point $q\in F_{\Theta_\upsilon}$ that is closer to $\zeta_0$ and

$$\mathrm{d}(\zeta_0,q)^2\leq\mu_m\mathrm{d}(\zeta_0,\zeta_m)^2)\text{ for every Banach limit }\mu,\tag{3.25}$$

that is, $\mu_m(\mathrm{d}(\zeta_0,q)^2-\mathrm{d}(\zeta_0,\zeta_m)^2\leq 0$.
Furthermore, by using Theorem 3.8, we get $\lim_{m\to\infty}\mathrm{d}(\zeta_{m+1},\zeta_m)=0$.
Consequently,

$$\limsup_{m\to\infty}((\mathrm{d}(\zeta_0,q)^2-\mathrm{d}(\zeta_0,\zeta_{m+1})^2)-(\mathrm{d}(\zeta_0,q)^2-\mathrm{d}(\zeta_0,\zeta_m)^2))=0.\tag{3.26}$$

By $\lim_{m\to\infty}\mathrm{d}(\Theta_{\upsilon,m}\zeta_m,\zeta_m)=0$ and using Lemma 2.4, we get

$$\limsup_{m\to\infty}(\mathrm{d}(\zeta_0,q)^2-(1-\alpha_m)\mathrm{d}(\zeta_0,\Theta_{\upsilon,m}\zeta_m)^2)=\limsup_{m\to\infty}(\mathrm{d}(\zeta_0,q)^2-\mathrm{d}(\zeta_0,\zeta_m)^2)\leq 0.\tag{3.27}$$

Lastly, we prove that $\lim_{m\to\infty}\mathrm{d}(\zeta_m,q)=0$. Now, by using the definitions of $\{\zeta_m\}$ and $\{\xi_m\}$, we obtain

$$\begin{aligned}\mathrm{d}(\zeta_{m+1},q)^2 &= \mathrm{d}(\beta_m\xi_m\oplus(1-\beta_m)\Theta_{\upsilon,m}\xi_m,q)^2\\ &\leq (\beta_m\mathrm{d}(\xi_m,q)+(1-\beta_m)\mathrm{d}(\Theta_{\upsilon,m}\xi_m,q))^2\\ &\leq \mathrm{d}(\xi_m,q)^2=\mathrm{d}(\alpha_m\zeta_0\oplus(1-\alpha_m)\Theta_{\upsilon,m}\zeta_m,q)^2\\ &\leq \alpha_m\mathrm{d}(\zeta_0,q)^2+(1-\alpha_m)\mathrm{d}(\Theta_{\upsilon,m}\zeta_m,q)^2\\ &\quad-\alpha_m(1-\alpha_m)\mathrm{d}(\zeta_0,\Theta_{\upsilon,m}\zeta_m)^2\\ &\leq \alpha_m\mathrm{d}(\zeta_0,q)^2+(1-\alpha_m)\mathrm{d}(\zeta_m,q)^2-\alpha_m(1-\alpha_m)\mathrm{d}(\zeta_0,\Theta_{\upsilon,m}\zeta_m)^2\end{aligned}\tag{3.28}$$

$$= (1-\alpha_m)\mathsf{d}(\zeta_m, q)^2 + \alpha_m(\mathsf{d}(\zeta_0, q)^2 - (1-\alpha_m)\mathsf{d}(\zeta_0, \Theta_{\upsilon,m}\zeta_m)^2).$$

Inequality (3.27), Lemma 2.5, and $\sum_{m=1}^{\infty}\alpha_m = \infty$ imply that $\lim_{m\to\infty} \mathsf{d}(\zeta_m, q)^2 = 0$.
Therefore, $\{\zeta_m\}$ converges to $q \in F_{\Theta_\upsilon} = \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$ that is closer to $\zeta_0$. □

**Corollary 3.10.** *Suppose that $\wp$ is a nonempty, closed, and convex subset of $\mathcal{W}$, a complete $CAT(0)$ space. Assume that $\{\Theta_m\}$ is a family of enriched nonexpansive self-mappings of $\wp$ such that $F_{\Theta_\upsilon} = \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}} \neq \emptyset$. Let $\zeta_0, \zeta_1 \in \wp$ be arbitrary selected and let $\{\zeta_m\}$ denote a sequence in $\wp$ generated by*

$$\zeta_{m+1} = \alpha_m\zeta_0 \oplus (1-\alpha_m)\Theta_m\zeta_m, \quad \forall m \in \mathbb{N}, \tag{3.29}$$

*where, according to Theorem 3.5, $\{\alpha_m\}$ is a sequence in interval $[0, 1]$ that fulfills the condition (R1). Assume that $F_{\Theta_\upsilon} = \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$ and that $(\{\Theta_m\}, \Theta)$ fulfills the AKTT condition. Then the sequence $\{\zeta_m\}$ converges strongly to the common fixed point of $\{\Theta_{\upsilon,m}\}$ that is closer to $\zeta_0$.*

*Proof.* By taking $\beta_m = 1$ for any $m \in \mathbb{N}$ in Theorem 3.9, we get the required result. □

A condition in Banach spaces for the countably infinite family of nonexpansive mappings that differs from the AKTT condition was presented by Song and Zheng [38] in 2009. They also gave certain examples of a family of mappings that fulfill this condition. This condition is now expressed in $CAT(0)$ spaces and is known as the SZ condition in the following sense. We shall express it here for enriched nonexpansive mappings.
Suppose $\wp$ represents a nonempty closed and convex subset of $\mathcal{W}$, a complete $CAT(0)$ space. Let $\{\Theta_m\}$ be a family of enriched nonexpansive self mappings of $\wp$ such that $F_{\Theta_\upsilon} = \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}} \neq \emptyset$. The $\{\Theta_m\}$ is said to satisfy the SZ condition if for each bounded subset $\mathfrak{L}$ of $\wp$, there exists an enriched nonexpansive self mapping $\Theta$ of $\wp$ such that

$$\lim_{m\to\infty} \sup\{\mathsf{d}(\Theta(\Theta_m\zeta), \Theta_m\zeta) : \zeta \in \mathfrak{L}\} = 0, \quad F_{\Theta_\upsilon} = \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}. \tag{3.30}$$

**Theorem 3.11.** *Let $\wp$ be a nonempty, closed, and convex subset of the complete $CAT(0)$ space $\mathcal{W}$. Given a family of enriched nonexpansive self mappings of $\wp$ into itself, let $\{\Theta_m\}$ be such that $F_{\Theta_\upsilon} = \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}} \neq \emptyset$ and let it fulfill the SZ condition. Assume that $\{\zeta_m\}$ is a sequence of $\wp$ defined by (3.24), where $\lim_{m\to\infty} \mathsf{d}(\zeta_{m+1}, \zeta_m) = 0$. Assume that the sequences $\{\alpha_m\}$ and $\{\beta_m\}$ in $[0, 1]$ fulfill the following conditions:*

*$(H1)$ $\sum_{m=1}^{\infty}\alpha_m = \infty$, $0 < \alpha_m < 1$, and $\lim_{m\to\infty} \alpha_m = 0$;*

*$(H2)$ The limit of $\beta_m$ as m tends to infinity is 1.*

*Then, the sequence $\{\zeta_m\}$ converges strongly to the common fixed point of $\{\Theta_{\upsilon,m}\}$ that is closer to $\zeta_0$.*

*Proof.* As we know that $T_{\upsilon,m}\zeta = (1-\upsilon)\zeta + \upsilon T_m\zeta$ and according to Theorem 3.5, the sequences $\{\zeta_m\}$ and $\{\Theta_{\upsilon,m}\zeta_m\}$ are bounded. Assume that $\{\Theta_{\upsilon,m}\}$ follows the SZ condition and there exists an averaged nonexpansive mapping $\Theta_\upsilon$ of $\wp$ into itself such that $\lim_{m\to\infty} \sup\{\mathsf{d}(\Theta_\upsilon(\Theta_{\upsilon,m}\zeta), \Theta_{\upsilon,m}\zeta) : \zeta \in \{\zeta_k\}\} = 0$ with $F_{\Theta_\upsilon} = \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$. Using the definitions of $\{\zeta_m\}$ and $\{\xi_m\}$, we obtain

$$\begin{aligned}\mathsf{d}(\zeta_{m+1}, \Theta_{\upsilon,m}\zeta_m) &= \mathsf{d}(\beta_m\xi_m \oplus (1-\beta_m)\Theta_{\upsilon,m}\xi_m, \Theta_{\upsilon,m}\zeta_m)\\ &\leq \beta_m\mathsf{d}(\xi_m, \Theta_{\upsilon,m}\zeta_m) + (1-\beta_m)\mathsf{d}(\Theta_{\upsilon,m}\xi_m, \Theta_{\upsilon,m}\zeta_m)\end{aligned}$$

$$\leq \beta_m \mathrm{d}(\xi_m, \Theta_{\upsilon,m}\zeta_m) + (1-\beta_m)\mathrm{d}(\xi_m, \zeta_m) \tag{3.31}$$
$$= \beta_m \mathrm{d}(\alpha_m\zeta_0 \oplus (1-\alpha_m)\Theta_{\upsilon,m}\zeta_m, \Theta_{\upsilon,m}\zeta_m)$$
$$+ (1-\beta_m)\mathrm{d}(\alpha_m\zeta_0 \oplus (1-\alpha_m)\Theta_{\upsilon,m}\zeta_m, \zeta_m)$$
$$\leq \beta_m\alpha_m \mathrm{d}(\zeta_0, \Theta_{\upsilon,m}\zeta_m) + (1-\beta_m)(\alpha_m \mathrm{d}(\zeta_0, \zeta_m) + (1-\alpha_m)\mathrm{d}(\Theta_{\upsilon,m}\zeta_m, \zeta_m)).$$

From conditions $(H1)$ and $(H2)$, we get

$$\lim_{m\to\infty} \mathrm{d}(\zeta_{m+1}, \Theta_{\upsilon,m}\zeta_m) = 0. \tag{3.32}$$

Since

$$\begin{aligned} \mathrm{d}(\zeta_{m+1}, \Theta_\upsilon\zeta_{m+1}) &\leq \mathrm{d}(\zeta_{m+1}, \Theta_{\upsilon,m}\zeta_m) + \mathrm{d}(\Theta_{\upsilon,m}\zeta_m, \Theta_\upsilon(\Theta_{\upsilon,m}\zeta_m) \\ &\quad + \mathrm{d}(\Theta_\upsilon(\Theta_{\upsilon,m}\zeta_m), \Theta_\upsilon\zeta_{m+1}) \\ &\leq 2\mathrm{d}(\zeta_{m+1}, \Theta_{\upsilon,m}\zeta_m) \\ &\quad + \sup\{\mathrm{d}(\Theta_\upsilon(\Theta_{\upsilon,m}\zeta), \Theta_{\upsilon,m}\zeta) : \zeta \in \{\zeta_k\}\}, \end{aligned} \tag{3.33}$$

according to (3.32) and the SZ condition, we obtain

$$\lim_{m\to\infty} \mathrm{d}(\zeta_m, \Theta_\upsilon\zeta_m) = 0. \tag{3.34}$$

Taking $\lim_{m\to\infty} \mathrm{d}(\zeta_{m+1}, \zeta_m) = 0$ and

$$\mathrm{d}(\zeta_m, \Theta_{\upsilon,m}\zeta_m) \leq \mathrm{d}(\zeta_m, \zeta_{m+1}) + \mathrm{d}(\zeta_{m+1}, \Theta_{\upsilon,m}\zeta_m), \tag{3.35}$$

consequently,

$$\lim_{m\to\infty} \mathrm{d}(\zeta_m, \Theta_{\upsilon,m}\zeta_m) = 0. \tag{3.36}$$

Applying the same methods and reasoning as in Theorem 3.9, we can prove that the sequence $\{\zeta_m\}$ converges to the common fixed point of $\{\Theta_{\upsilon,m}\}$ that is closer to $\zeta_0$. □

**Corollary 3.12.** *Let $\mathcal{W}$ be the complete CAT$(0)$ space and let $\wp$ be a nonempty, closed, and convex subset of it. We are given a family of enriched nonexpansive self-mappings of $\wp$ and let $\{\Theta_m\}$ be such that $F_{\Theta_\upsilon} = \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}} \neq \emptyset$ and fulfills the SZ condition. Assume that $\{\zeta_m\}$ is a sequence in $\wp$ defined by (3.29) with $\lim_{m\to\infty} \mathrm{d}(\zeta_{m+1}, \zeta_m) = 0$. Consider the sequence $\{\alpha_m\}$ in $[0,1]$ that fulfills condition $(H1)$ according to Theorem 3.11. Then, the sequence $\{\zeta_m\}$ converges strongly to the common fixed point of $\{\Theta_{\upsilon,m}\}$ that is closer to $\zeta_0$.*

*Proof.* By setting $\beta_m = 1$ for every $m \in \mathbb{N}$ in Theorem 3.11, we obtain the desired result. □

## 4. W-mapping in convex metric spaces

We must suppose that $(\{\Theta_m\}, \Theta)$ fulfills the AKTT condition in order to obtain a convergence result in Theorems 3.5, 3.8, 3.9 and Corollary 3.10. Generally, these results cannot be applied to a sequence of enriched nonexpansive mappings. We provide an example of a sequence $\{\Theta_m\}$ of enriched nonexpansive mappings that fulfill the AKTT condition.

In a convex structure, the enriched nonexpansive mapping can be defined as: $\Theta : \wp \to \wp$ if

$$\mathtt{d}(\mathtt{W}(\zeta, \Theta\zeta; \upsilon), \mathtt{W}(\xi, \Theta\xi, \upsilon)) \leq \mathtt{d}(\zeta, \xi),$$

where $\upsilon \in [0, 1)$. Setting $\Theta_\upsilon\zeta = \mathtt{W}(\zeta, \Theta\zeta, \upsilon)$ and $\Theta_\upsilon\xi = \mathtt{W}(\xi, \Theta\xi, \upsilon)$ in the above inequality, we have $\mathtt{d}(\Theta_\upsilon\zeta, \Theta_\upsilon\xi) \leq \mathtt{d}(\zeta, \xi)$. Suppose that $\{\Theta_m\}$ denotes a family of enriched nonexpansive mappings from $\wp$ to itself, where $\wp$ is the convex subset of convex metric space $(\mathcal{W}, \mathtt{d}, \mathtt{W})$. In this section, we use the notation $\Theta_{\upsilon,m} = \mathtt{W}(\Theta_m, \zeta, \upsilon)$ for a countably infinite family of averaged nonexpansive mappings. Now we define the mappings $\mathtt{U}_{m;1}, \mathtt{U}_{m;2}, ..., \mathtt{U}_{m;m}$ and $\psi_m$ as follows. For a sequence $\{\vartheta_m\} \in [0, 1]$ and $\zeta \in \mathcal{W}$,

$$\begin{array}{rcl}
\mathtt{U}_{m;m}\zeta &=& \mathtt{W}(\Theta_{\upsilon,m}\zeta, \zeta, \vartheta_m), \\
\mathtt{U}_{m;m-1}\zeta &=& \mathtt{W}(\Theta_{\upsilon,m-1}\mathtt{U}_{m;m}\zeta, \zeta, \vartheta_{m-1}), \\
\mathtt{U}_{m;m-2}\zeta &=& \mathtt{W}(\Theta_{\upsilon,m-2}\mathtt{U}_{m;m-1}\zeta, \zeta, \vartheta_{m-2}), \\
&\vdots& \\
\mathtt{U}_{m;k}\zeta &=& \mathtt{W}(\Theta_{\upsilon,k}\mathtt{U}_{m;k+1}\zeta, \zeta, \vartheta_k), \\
\mathtt{U}_{m;k-1}\zeta &=& \mathtt{W}(\Theta_{\upsilon,k-1}\mathtt{U}_{m;k}\zeta, \zeta, \vartheta_{k-1}), \\
&\vdots& \\
\mathtt{U}_{m;2}\zeta &=& \mathtt{W}(\Theta_{\upsilon,2}\mathtt{U}_{m;3}\zeta, \zeta, \vartheta_2), \\
\psi_m\zeta = \mathtt{U}_{m;1}\zeta &=& \mathtt{W}(\Theta_{\upsilon,1}\mathtt{U}_{m;2}\zeta, \zeta, \vartheta_1).
\end{array} \tag{4.1}$$

In a mapping of that kind, $\psi_m$ is referred to as the W-mapping defined by $\Theta_{\upsilon,1}, \Theta_{\upsilon,2}, \cdots, \Theta_{\upsilon,m}$ and $\vartheta_1, \vartheta_2, \cdots, \vartheta_m$.
Takahashi [41] presented the W-mapping from Banach spaces to convex metric spaces, which was generalized in 2007 by Shimizu [31]. The following result is established with the same proof as in [31, Lemma 2].

**Lemma 4.1.** *Suppose that $\wp$ is a nonempty, closed, and convex subset of a uniformly convex metric space $(\mathcal{W}, \mathtt{d}, \mathtt{W})$, which has the continuous structure $\mathtt{W} : \mathcal{W} \times \mathcal{W} \times [0, 1] \to \mathcal{W}$. Assume that $\Theta_1, \Theta_2, \cdots, \Theta_N$ are enriched nonexpansive self-mappings of $\wp$ such that $F_{\Theta_\upsilon} = \bigcap_{m=1}^N F_{\Theta_{\upsilon,m}} \neq \emptyset$. For each $m = 1, 2, \cdots, N$, let $\vartheta_1, \vartheta_1, \cdots, \vartheta_N$ be real values such that $0 < \vartheta_m < 1$. Assume that $\psi_N$ is the W-mapping of $\wp$ into itself generated by $\Theta_1, \Theta_2, \cdots, \Theta_N$. Then $F_{\psi_N} = \bigcap_{m=1}^N F_{\Theta_{\upsilon,m}}$.*

Subsequently, we examine the W-mapping defined by a countably infinite family of enriched nonexpansive mappings within a uniformly convex metric space.

**Lemma 4.2.** *Suppose that $\wp$ is a nonempty, closed, and convex subset of a complete uniformly convex metric space $(\mathcal{W}, \mathtt{d}, \mathtt{W})$ that has the condition $(E)$. Consider $\{\Theta_m\}$ the family of enriched nonexpansive self-mappings of $\wp$ such that $\bigcap_{m=1}^\infty F_{\Theta_{\upsilon,m}} \neq \emptyset$ and let $\vartheta_1, \vartheta_2, \cdots$ be real numbers such that $0 < \vartheta_m \leq b < 1$ for each $m \in \mathbb{N}$. Then for each $\zeta \in \wp$ and $k \in \mathbb{N}$, $\lim_{m\to\infty} U_{m;k}\zeta$ exists.*

*Proof.* We know that $\Theta_{\upsilon,m} = \mathtt{W}(\Theta_m, \zeta, \upsilon)$ and, from Remark 3.3, we observe that $F_{\Theta_\upsilon} = \bigcap_{m=1}^\infty F_{\upsilon,m} \neq \emptyset$. Now, let $\zeta \in \wp$ and $\varpi \in \bigcap_{m=1}^\infty F_{\Theta_{\upsilon,m}}$. Let $k \in \mathbb{N}$ be fixed. For each $m \in \mathbb{N}$ with $m > k$, we obtain

$$\mathtt{d}(\mathtt{U}_{m+1;k}\zeta, \mathtt{U}_{m;k}\zeta) = \mathtt{d}(\mathtt{W}(\Theta_{\upsilon,k}\mathtt{U}_{m+1;k+1}\zeta, \zeta, \vartheta_k), \mathtt{W}(\Theta_{\upsilon,k}\mathtt{U}_{m;k+1}\zeta, \zeta, \vartheta_k))$$

$$
\begin{aligned}
&\leq \vartheta_k \mathtt{d}(\Theta_{\upsilon,k}\mathtt{U}_{m+1;k+1}\zeta, \Theta_{\upsilon,k}\mathtt{U}_{m;k+1}\zeta) \\
&\leq \vartheta_k \mathtt{d}(\mathtt{U}_{m+1;k+1}\zeta, \mathtt{U}_{m;k+1}\zeta) \\
&= \vartheta_k \mathtt{d}(\mathtt{W}(\Theta_{\upsilon,k+1}\mathtt{U}_{m+1;k+2}\zeta, \zeta, \vartheta_{k+1}), \mathtt{W}(\Theta_{\upsilon,k+1}\mathtt{U}_{m;k+2}\zeta, \zeta, \vartheta_{k+1})) \\
&\leq \vartheta_k \vartheta_{k+1} \mathtt{d}(\mathtt{U}_{m+1;k+2}\zeta, \mathtt{U}_{m;k+2}\zeta) \\
&\vdots \\
&\leq \vartheta_k \vartheta_{k+1} \cdots \vartheta_{m-1} \mathtt{d}(\mathtt{U}_{m+1;m}\zeta, \mathtt{U}_{m;m}\zeta) \\
&= \vartheta_k \vartheta_{k+1} \cdots \vartheta_{m-1} \mathtt{d}(\mathtt{W}(\Theta_{\upsilon,m}\mathtt{U}_{m+1;m+1}\zeta, \zeta, \vartheta_m), \mathtt{W}(\Theta_{\upsilon,m}\zeta, \zeta, \vartheta_m)) \\
&\leq \vartheta_k \vartheta_{k+1} \cdots \vartheta_m \mathtt{d}(\Theta_{\upsilon,m}\mathtt{U}_{m+1;m+1}\zeta, \Theta_{\upsilon,m}\zeta) \\
&\leq \vartheta_k \vartheta_{k+1} \cdots \vartheta_m \mathtt{d}(\mathtt{U}_{m+1;m+1}\zeta, \zeta) \\
&= \vartheta_k \vartheta_{k+1} \cdots \vartheta_m \mathtt{d}(\mathtt{W}(\Theta_{\upsilon,m+1}\zeta, \zeta, \vartheta_{m+1}), \zeta) \\
&= \vartheta_k \vartheta_{k+1} \cdots \vartheta_{m+1} \mathtt{d}(\Theta_{\upsilon,m+1}\zeta, \zeta) \\
&\leq \vartheta_k \vartheta_{k+1} \cdots \vartheta_{m+1} (\mathtt{d}(\Theta_{\upsilon,m+1}\zeta, \varpi) + \mathtt{d}(\varpi, \zeta)) \\
&\leq 2\mathtt{d}(\varpi, \zeta) b^{m-k+2}.
\end{aligned}
\tag{4.2}
$$

Thus for $\mathfrak{l} > m$,

$$
\begin{aligned}
\mathtt{d}(\mathtt{U}_{\mathfrak{l};k}\zeta, \mathtt{U}_{m;k}\zeta) &\leq \mathtt{d}(\mathtt{U}_{\mathfrak{l};k}\zeta, \mathtt{U}_{\mathfrak{l}-1;k}\zeta) + \mathtt{d}(\mathtt{U}_{\mathfrak{l}-1;k}\zeta, \mathtt{U}_{\mathfrak{l}-2;k}\zeta) + \cdots + \mathtt{d}(\mathtt{U}_{m+1;k}\zeta, \mathtt{U}_{m;k}\zeta) \\
&\leq 2\mathtt{d}(\varpi, \zeta) b^{(\mathfrak{l}-1)-k+2} + 2\mathtt{d}(\varpi, \zeta) b^{(\mathfrak{l}-2)-k+2} \\
&\quad + \cdots + 2\mathtt{d}(\varpi, \zeta) b^{m-k+2} \\
&= 2\mathtt{d}(\varpi, \zeta) \sum_{i=m}^{\mathfrak{l}-1} b^{i-k+2}.
\end{aligned}
\tag{4.3}
$$

Consequently, the sequence $\{\mathtt{U}_{m;k}\}$ is a Cauchy sequence. Thus, $\lim_{m\to\infty} \mathtt{U}_{m;k}\zeta$ exists. □

By using the previous lemma, the mappings $\mathtt{U}_{\infty;k}$ and $\psi$ from $\wp$ to itself may be defined as follows.

$$
\mathtt{U}_{\infty;k}\zeta = \lim_{m\to\infty} \mathtt{U}_{m;k}\zeta, \qquad \psi\zeta = \lim_{m\to\infty} \psi_m\zeta = \lim_{m\to\infty} \mathtt{U}_{m;1}\zeta, \tag{4.4}
$$

for every $\zeta \in \wp$. Such a mapping $\psi$ is known as a W-mapping, which is defined by $\Theta_{\upsilon,1}, \Theta_{\upsilon,2}, \cdots$ and $\vartheta_1, \vartheta_2, \cdots$.

**Lemma 4.3.** *Suppose that $\wp$ represents a nonempty, closed, and convex subset of a complete uniformly convex metric space $(\mathcal{W}, \mathtt{d}, \mathtt{W})$ with property $(E)$. We are given a family of enriched nonexpansive self-mappings $\{\Theta_m\}$ of $\wp$ such that $F_{\Theta_\upsilon} = \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}} \neq \emptyset$. Let $\vartheta_1, \vartheta_2, \cdots$ be real values and let $0 < \vartheta_m \leq b < 1$ for each $m \in \mathbb{N}$. Moreover, $\psi$ is the W-mapping defined by $\Theta_{\upsilon,1}, \Theta_{\upsilon,2}, \cdots$ and $\vartheta_1, \vartheta_2, \cdots$. Then, $F_\psi = \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$ and $\psi$ is a nonexpansive mapping.*

*Proof.* First, we will show that $\psi$ is an enriched nonexpansive mapping. For $\zeta, \xi \in \wp$, we have

$$
\begin{aligned}
\mathtt{d}(\psi_m\zeta, \psi_m\xi) &= \mathtt{d}(\mathtt{W}(\Theta_{\upsilon,1}\mathtt{U}_{m;2}\zeta, \zeta, \vartheta_1), \mathtt{W}(\Theta_{\upsilon,1}\mathtt{U}_{m;2}\xi, \xi, \vartheta_1)) \\
&\leq \vartheta_1 \mathtt{d}(\Theta_{\upsilon,1}\mathtt{U}_{m;2}\zeta, \Theta_{\upsilon,1}\mathtt{U}_{m;2}\xi) + (1 - \vartheta_1)\mathtt{d}(\zeta, \xi) \\
&\leq \vartheta_1 \mathtt{d}(\mathtt{U}_{m;2}\zeta, \mathtt{U}_{m;2}\xi) + (1 - \vartheta_1)\mathtt{d}(\zeta, \xi) \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
&\leq \vartheta_1\vartheta_2\cdots\vartheta_{m-1}\mathtt{d}(\mathtt{U}_{m;m}\zeta,\mathtt{U}_{m;m}\xi) + (1-\vartheta_1\vartheta_2\cdots\vartheta_{m-1})\mathtt{d}(\zeta,\xi)\\
&= \vartheta_1\vartheta_2\cdots\vartheta_{m-1}\mathtt{d}(\mathtt{W}(\Theta_{\upsilon,m}\zeta,\zeta,\vartheta_m),\mathtt{W}(\Theta_{\upsilon,m}\xi,\xi,\vartheta_m))\\
&\quad + (1-\vartheta_1\vartheta_2\cdots\vartheta_{m-1})\mathtt{d}(\zeta,\xi)\\
&\leq \vartheta_1\vartheta_2\cdots\vartheta_{m-1}\vartheta_m\mathtt{d}(\Theta_{\upsilon,m}\zeta,\Theta_m\xi) + \vartheta_1\vartheta_2\cdots\vartheta_{m-1}(1-\vartheta_m)\mathtt{d}(\zeta,\xi)\\
&\quad + (1-\vartheta_1\vartheta_2\cdots\vartheta_{m-1})\mathtt{d}(\zeta,\xi)\\
&\leq \mathtt{d}(\zeta,\xi). \qquad (4.5)
\end{aligned}
$$

Therefore, $\psi_m$ is a nonexpansive mapping and $\mathtt{d}(\psi\zeta,\psi\xi) = \lim_{m\to\infty}\mathtt{d}(\psi_m\zeta,\psi_m\xi) \leq \mathtt{d}(\zeta,\xi)$. Thus, $\psi$ is also a nonexpansive mapping.

Finally, we establish $F_\psi = \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$. Let $\varpi \in \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$. Then, it is obviously that $\mathtt{U}_{m;k}\varpi = \varpi$ for each $m,k \in \mathbb{N}$ with $m > k$. Therefore, we obtain $\mathtt{U}_{\infty;k}\varpi = \varpi$ for each $k \in \mathbb{N}$. Therefore, we have $\psi\varpi = \mathtt{U}_{\infty;1}\varpi = \varpi$ and hence, $\bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}} \subseteq F_\psi$. We now show that $F_\psi \subseteq \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$. Let $\zeta \in F_s$ and $\varpi \in \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$. Then, we have

$$
\begin{aligned}
\mathtt{d}(\psi_m\varpi,\psi_m\zeta) &= \mathtt{d}(\mathtt{U}_{m;1}\varpi,\mathtt{U}_{m;1}\zeta)\\
&= \mathtt{d}(\varpi,\mathtt{W}(\Theta_{\upsilon,1}\mathtt{U}_{m;2}\zeta,\zeta,\vartheta_1))\\
&\leq \vartheta_1\mathtt{d}(\varpi,\Theta_{\upsilon,1}\mathtt{U}_{m;2}\zeta) + (1-\vartheta_1)\mathtt{d}(\varpi,\zeta)\\
&\leq \vartheta_1\mathtt{d}(\varpi,\mathtt{U}_{m;2}\zeta) + (1-\vartheta_1)\mathtt{d}(\varpi,\zeta)\\
&\;\vdots\\
&\leq \vartheta_1\vartheta_2\cdots\vartheta_{k-1}\mathtt{d}(\varpi,\mathtt{U}_{m;k}\zeta) + (1-\vartheta_1\vartheta_2\cdots\vartheta_{k-1})\mathtt{d}(\varpi,\zeta)\\
&= \vartheta_1\vartheta_2\cdots\vartheta_{k-1}\mathtt{d}(\varpi,\mathtt{W}(\Theta_{\upsilon,K}\mathtt{U}_{m;k+1}\zeta,\zeta,\vartheta_k)) + (1-\vartheta_1\vartheta_2\cdots\vartheta_{k-1})\mathtt{d}(\varpi,\zeta)\\
&\leq \vartheta_1\vartheta_2\cdots\vartheta_{k-1}\vartheta_k\mathtt{d}(\varpi,\Theta_{\upsilon;k}\mathtt{U}_{m;k+1}\zeta) + \vartheta_1\vartheta_2\cdots\vartheta_{k-1}(1-\vartheta_k)\mathtt{d}(\varpi,\zeta)\\
&\quad + (1-\vartheta_1\vartheta_2\cdots\vartheta_{k-1})\mathtt{d}(\varpi,\zeta)\\
&= \vartheta_1\vartheta_2\cdots\vartheta_k\mathtt{d}(\varpi,\Theta_{\upsilon,k}\mathtt{U}_{m;k+1}\zeta) + (1-\vartheta_1\vartheta_2\cdots\vartheta_k)\mathtt{d}(\varpi,\zeta),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\mathtt{d}(\psi_m\varpi,\psi_m\zeta) &\leq \vartheta_1\vartheta_2\cdots\vartheta_k\mathtt{d}(\varpi,\mathtt{U}_{m;k+1}\zeta) + (1-\vartheta_1\vartheta_2\cdots\vartheta_k)\mathtt{d}(\varpi,\zeta)\\
&\;\vdots\\
&\leq \vartheta_1\vartheta_2\cdots\vartheta_{m-1}\mathtt{d}(\varpi,\mathtt{U}_{m;m}\zeta) + (1-\vartheta_1\vartheta_2\cdots\vartheta_{m-1})\mathtt{d}(\varpi,\zeta)\\
&= \vartheta_1\vartheta_2\cdots\vartheta_{m-1}\mathtt{d}(\varpi,\mathtt{W}(\Theta_{\upsilon,m}\zeta,\zeta,\vartheta_m)) + (1-\vartheta_1\vartheta_2\cdots\vartheta_{m-1}\mathtt{d}(\varpi,\zeta)\\
&\leq \vartheta_1\vartheta_2\cdots\vartheta_{m-1}\vartheta_m\mathtt{d}(\varpi,\Theta_{\upsilon,m}\zeta) + \vartheta_1\vartheta_2\cdots\vartheta_{m-1}(1-\vartheta_m)\mathtt{d}(\varpi,\zeta)\\
&\quad + (1-\vartheta_1\vartheta_2\cdots\vartheta_{m-1})\mathtt{d}(\varpi,\zeta)\\
&= \vartheta_1\vartheta_2\cdots\vartheta_m\mathtt{d}(\varpi,\Theta_{\upsilon,m}\zeta) + (1-\vartheta_1\vartheta_2\cdots\vartheta_m)\mathtt{d}(\varpi,\zeta)\\
&\leq \mathtt{d}(\varpi,\zeta). \qquad (4.6)
\end{aligned}
$$

Applying $m \to \infty$, we obtain

$$
\begin{aligned}
\mathtt{d}(\psi\varpi,\psi\zeta) &\leq \vartheta_1\vartheta_2\cdots\vartheta_{k-1}\mathtt{d}(\varpi,\mathtt{W}(\Theta_{\upsilon,k}\mathtt{U}_{\infty;k+1}\zeta,\zeta,\vartheta_k)) + (1-\vartheta_1\vartheta_2\cdots\vartheta_{k-1})\mathtt{d}(\varpi,\zeta)\\
&\leq \vartheta_1\vartheta_2\cdots\vartheta_{k-1}\vartheta_k\mathtt{d}(\varpi,\Theta_{\upsilon,k}\mathtt{U}_{\infty;k+1}\zeta) + \vartheta_1\vartheta_2\cdots\vartheta_{k-1}(1-\vartheta_k)\mathtt{d}(\varpi,\zeta)
\end{aligned}
$$

$$
\begin{aligned}
& + (1-\vartheta_1\vartheta_2\cdots\vartheta_{k-1})\mathrm{d}(\varpi,\zeta) \quad (4.7)\\
= \;& \vartheta_1\vartheta_2\cdots\vartheta_k\mathrm{d}(\varpi,\Theta_{\upsilon,k}\mathtt{U}_{\infty;k+1}\zeta) + (1-\vartheta_1\vartheta_2\cdots\vartheta_k)\mathrm{d}(\varpi,\zeta)\\
\leq \;& d(\varpi,\zeta).
\end{aligned}
$$

Since $\varpi \in \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}} \subseteq F_{\psi}$, we have $\mathrm{d}(\psi\varpi,\psi\zeta) = \mathrm{d}(\varpi,\zeta)$. Then, for $\vartheta_m \in (0,1) \in \mathbb{N}$, we have

$$
\mathrm{d}(\varpi,\Theta_{\upsilon,k}\mathtt{U}_{\infty;k+1}\zeta) = \mathrm{d}(\varpi,\zeta), \qquad \mathrm{d}(\varpi,\mathtt{W}(\Theta_{\upsilon,k}\mathtt{U}_{\infty;k+1}\zeta,\zeta,\vartheta_k)) = \mathrm{d}(\varpi,\zeta), \tag{4.8}
$$

for each $k \in \mathbb{N}$. Assume that $\Theta_{\upsilon,k}\mathtt{U}_{\infty;k+1}\zeta \neq \zeta$. Then $\mathrm{d}(\Theta_{\upsilon,k}\mathtt{U}_{\infty;k+1}\zeta,\zeta) > 0$. By applying Lemma 2.6, we obtain the following result:

$$
\mathrm{d}(\varpi,\mathtt{W}(\Theta_{\upsilon,k}\mathtt{U}_{\infty;k+1}\zeta,\zeta,\vartheta_k)) < \mathrm{d}(\varpi,\zeta). \tag{4.9}
$$

This is a contradiction. Hence, $\Theta_{\upsilon,k}\mathtt{U}_{\infty;k+1}\zeta = \zeta$.
Since $\mathtt{U}_{m;k+1}\zeta = \mathtt{W}(\Theta_{\upsilon,k+1}\mathtt{U}_{m;k+2}\zeta,\zeta,\vartheta_{k+1})$, we take

$$
\mathtt{U}_{\infty;k+1}\zeta = \lim_{m\to\infty}\mathtt{U}_{m;k+1}\zeta = \mathtt{W}(\Theta_{\upsilon,k+1}\mathtt{U}_{\infty;k+2}\zeta,\zeta,\vartheta_{k+1}) = \zeta. \tag{4.10}
$$

So, we have $\zeta = \Theta_{\upsilon,k}\mathtt{U}_{\infty;k+1}\zeta = \Theta_{\upsilon,k}\zeta$ for each $k \in \mathbb{N}$.
Thus $\zeta \in \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$. Consequently, we obtain $F_{\psi} \subseteq \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$. □

**Lemma 4.4.** *Suppose that $\mathcal{W}, \wp, \{\Theta_m\}, \{\vartheta_m\}$ are defined as in the above Lemma* 4.3. *Let $\psi_m$ and $\psi$ be the* W*-mappings defined by $\Theta_{\upsilon,1},\Theta_{\upsilon,2},\cdots,\Theta_{\upsilon,m}$ and $\vartheta_1,\vartheta_2,\cdots,\vartheta_m$, and $\Theta_{\upsilon,1},\Theta_{\upsilon,2},\cdots$ and $\vartheta_1,\vartheta_2,\cdots$, respectively. Consequently, $(\{\psi_m\},\psi)$ fulfills the AKTT condition and $F_{\psi} = \bigcap_{m=1}^{\infty} F_{\psi_m}$.*

*Proof.* Let $\mathfrak{L}$ be a bounded subset of $\wp$ and $\zeta \in \mathfrak{L}$. For $\varpi \in \bigcap_{m=1}^{\infty} F_{\Theta_{\upsilon,m}}$, we obtain

$$
\begin{aligned}
\mathrm{d}(\psi_{m+1}\zeta,\psi_m\zeta) &= \mathrm{d}(\mathtt{U}_{m+1;1}\zeta,\mathtt{U}_{m;1}\zeta)\\
&= \mathrm{d}(\mathtt{W}(\Theta_{\upsilon,1}\mathtt{U}_{m+1;2}\zeta,\zeta,\vartheta_1),\mathtt{W}(\Theta_{\upsilon,1}\mathtt{U}_{m;2}\zeta,\zeta,\vartheta_1))\\
&\leq \vartheta_1\mathrm{d}(\mathtt{U}_{m+1;2}\zeta,\mathtt{U}_{m;2}\zeta)\\
&\;\;\vdots\\
&\leq \vartheta_1\vartheta_2\cdots\vartheta_{m-1}\mathrm{d}(\mathtt{U}_{m+1;m}\zeta,\mathtt{U}_{m;m}\zeta)\\
&= \vartheta_1\vartheta_2\cdots\vartheta_{m-1}\mathrm{d}(\mathtt{W}(\Theta_{\upsilon,m}\mathtt{U}_{m+1;m+1}\zeta,\zeta,\vartheta_m),\mathtt{W}(\Theta_{\upsilon,m}\zeta,\zeta,\vartheta_m)) \qquad (4.11)\\
&\leq \vartheta_1\vartheta_2\cdots\vartheta_m\mathrm{d}(\mathtt{U}_{m+1;m+1}\zeta,\zeta)\\
&= \vartheta_1\vartheta_2\cdots\vartheta_m\mathrm{d}(\mathtt{W}(\Theta_{\upsilon,m+1}\zeta,\zeta,\vartheta_{m+1}),\zeta)\\
&\leq \vartheta_1\vartheta_2\cdots\vartheta_{m+1}\mathrm{d}(\Theta_{\upsilon,m+1}\zeta,\zeta)\\
&\leq \vartheta_1\vartheta_2\cdots\vartheta_{m+1}(\mathrm{d}(\Theta_{\upsilon,m+1}\zeta,\varpi)+\mathrm{d}(\varpi,\zeta))\\
&\leq 2\vartheta_1\vartheta_2\cdots\vartheta_{m+1}\mathrm{d}(\varpi,\zeta)\\
&\leq 2b^{m+1}\mathrm{d}(\varpi,\zeta).
\end{aligned}
$$

This implies

$$\sum\nolimits_{m=1}^{\infty} \sup\{\mathsf{d}(\psi_{m+1}\zeta, \psi_m\zeta) : \zeta \in \mathcal{B}\} < \infty. \tag{4.12}$$

Consequently, $(\{\psi_m\}, \psi)$ fulfills the AKTT requirement. Furthermore, from Lemmas 4.1–4.3, we deduce that $F_\psi = \bigcap_{m=1}^{\infty} F_{\psi_m}$. □

**Remark 4.5.** *Suppose that $\mathcal{W}, \wp, \{\Theta_m\}, \{\vartheta_m\}$ are defined as in Lemma 4.3. Let $\psi_m$ and $\psi$ denote the $W$-mappings generated via $\Theta_{\upsilon,1}, \Theta_{\upsilon,2}, \ldots, \Theta_{\upsilon,m}$ and $\vartheta_1, \vartheta_2, \ldots, \vartheta_m$ as well as by $\Theta_{\upsilon,1}, \Theta_{\upsilon,2}, \ldots$ and $\vartheta_1, \vartheta_2, \ldots$, respectively. According to Lemma 4.4, the sequence $\{\psi_m\}$ and $\psi$ fulfill the AKTT condition and it ensures that $F_\psi = \bigcap_{m=1}^{\infty} F_{\psi_m}$. Consequently, in Theorems 3.5, 3.8, and 3.9, as well as Corollary 3.10, the mapping $\Theta_{\upsilon,m}$may be substituted with $\psi_m$ without the necessity of the AKTT condition and it holds that $F_\psi = \bigcap_{m=1}^{\infty} F_{\psi_m}$.*

## 5. Numerical results

In this section, we provide a numerical example to demonstrate the effectiveness of Algorithm 1.9 and to further support the main theoretical results obtained in this paper.

**Example 5.1.** Let $W = \mathbb{R}^2$ be a complete convex metric space endowed with the usual Euclidean metric. Moreover, we consider the nonempty, closed, and convex subset $\wp = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$. For each $m \in \mathbb{N}$, define the mapping $\Theta_m : \wp \to \wp$ by $\Theta_m(\zeta_1, \zeta_2) = (\zeta_1 - c_m\zeta_1^3,\ \zeta_2 - c_m\zeta_2^3)$, where $c_m := 1 - \frac{1}{(m+1)^2} \in (0, 1)$.

Thus, $\{\Theta_m\}_{m=1}^{\infty}$ forms a countably infinite family of nonlinear self-mappings on $\wp$. Furthermore, one can verify that there exists a constant $b > 0$ (for instance, $b \geq \frac{1}{2}$) such that each $\Theta_m$ satisfies the enriched nonexpansive inequality. Hence, every $\Theta_m$ belongs to the class of enriched nonexpansive mappings on $\wp$. We note that $(0, 0)$ is a fixed point of every mapping $\Theta_m$ and $\bigcap_{m=1}^{\infty} F_{\Theta_m} = \{(0, 0)\} \neq \emptyset$. Furthermore, we also choose $\alpha_m = \frac{1}{m+2}$ and $\beta_m = \frac{m+1}{m+2}$ for all $m \in \mathbb{N}$.

Now, we consider the following four cases and obtain the numerical results for Algorithms 1.9 and 2.2 of Berinde [8] as shown by the following graphs:

Case 1: $\zeta_0 = (0.60,\ 0.40)$, $\zeta_1 = (0.40,\ -0.30)$,

Case 2: $\zeta_0 = (0.75,\ -0.20)$, $\zeta_1 = (0.55,\ 0.10)$,

Case 3: $\zeta_0 = (-0.70,\ 0.50)$, $\zeta_1 = (-0.40,\ 0.30)$,

Case 4: $\zeta_0 = (0.25,\ -0.90)$, $\zeta_1 = (0.10,\ -0.60)$.

The numerical computations shown in Figure 1 (cases (a) and (b)) and Figure 2 (cases (c) and (d)) illustrate the convergence behavior of Algorithms 1.9 and 2.2 of Berinde [8]. In each case, the error sequence $\|\zeta_m - \zeta_{m-1}\|$ of our algorithm as compared to the Brinde algorithm decreases rapidly as the iteration progresses and approaches zero. This observation confirms the fast and stable convergence of the proposed algorithm toward the common fixed point under different initializations.

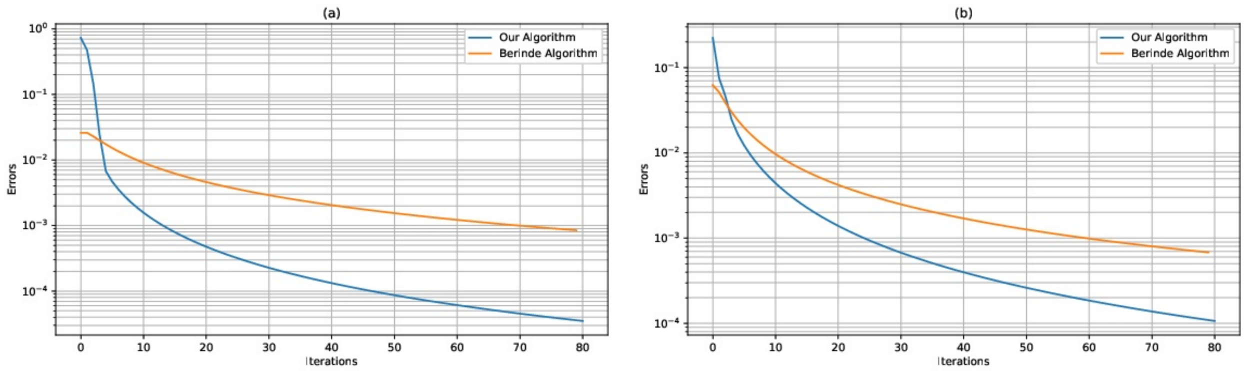


**Figure 1.** The convergence performance of Algorithms 1.9 and 2.2 of Berinde [8] for the cases (a) and (b). In particular, the plots illustrate the decay of $\|\zeta_m - \zeta_{m-1}\|$ with respect to the iteration index $m$, confirming the stable convergence of the proposed algorithm under different initial values.

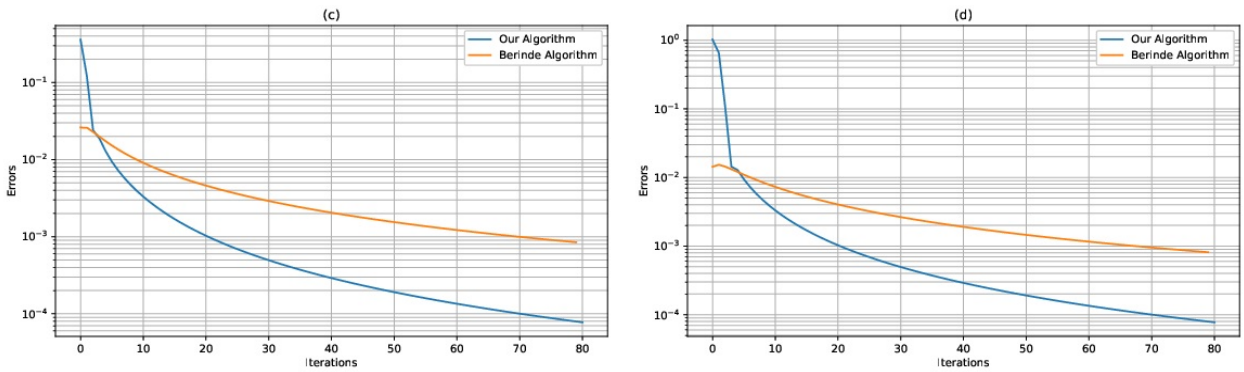


**Figure 2.** The error convergence behavior of Algorithms 1.9 and 2.2 of Berinde [8] for the cases (c) and (d), showing the decrease of $\|\zeta_m - \zeta_{m-1}\|$ as the iterations proceed.

## 6. Conclusions

In this paper, we have presented a new modified Halpern iterate for a countably infinite family of enriched nonexpansive mappings in convex metric spaces. Under the AKTT condition, we have demonstrated that the generated sequence approximates common fixed points of enriched nonexpansive mappings. Furthermore, strong convergence results have been established, ensuring that the iterative sequence converges to a common fixed point in $CAT(0)$ spaces, provided that the AKTT and SZ conditions hold. We have provided a numerical example for a countably infinite family of enriched nonexpansive mappings using Algorithms 1.9 and 2.2 of Berinde [8], showing that our Algorithm 1.9 converges faster than Algorithm 2.2 of Berinde to the common fixed point $(0, 0)$. Additionally, the concept of $\mathtt{W}$-mappings has been extended from Banach spaces to convex metric spaces, thereby broadening the applicability of existing results.

## Author contributions

Muhammad Jabir Khan: Writing review and editing, and funding acquisition; Somayya Komal: Supervision, investigation, and data curation; Athar Abbas: Conceptualization, writing–original draft, methodology, formal analysis and validation.

## Use of Generative-AI tools declaration

The authors declare that generative Artificial Intelligence (AI) tools were used only for minor language editing and grammatical improvement during the preparation of this manuscript. The AI tools were not used for the development of the mathematical results, proofs, algorithms, or scientific analysis presented in this article.

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## Conflict of interest

The authors declare no conflicts of interest in this paper.

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