AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 6251–6268.
DOI: 10.3934/math.2026258
Received: 17 January 2026
Revised: 13 February 2026
Accepted: 03 March 2026
Published: 12 March 2026

Research article

On a two-point nonlinear hybrid fractional boundary value problem

Rabab Alghamdi[1,2], **Bashir Ahmad**[1,*] **and Sotiris K. Ntouyas**[3]

[1] Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

[2] College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia

[3] Department of Mathematics, University of Ioannina, 45110, Ioannina, Greece

* **Correspondence:** Email: bahmad@kau.edu.sa, bashirahmad_qau@yahoo.com.

Abstract: We study a nonlinear Caputo type hybrid fractional differential equation subject to a two-point boundary condition. The existence of at least one solution is proven by applying a Burton-type modification of Krasnosel'skii's fixed point theorem. To establish uniqueness, we derive a novel form of Grönwall's inequality and employ it to show that the problem admits a unique solution. In addition, the uniqueness of solutions for the proposed hybrid fractional boundary value problem is further analyzed through Banach's contraction principle. Moreover, the Ulam–Hyers stability of the model is investigated. Finally, illustrative examples are provided to demonstrate the applicability of the theoretical results.

Keywords: fractional hybrid differential equations; existence; Grönwall's inequality; fixed point; Ulam-Hyers stability
Mathematics Subject Classification: 34A08, 34A38, 34B15

1. Introduction

Consider a nonlinear Caputo type hybrid fractional differential equation complemented with a two-point boundary condition given by the following:

$$\begin{cases} {}^cD^{\alpha}(z(t) - p(t, z(t))) = \Phi(t, z(t)), \quad 0 < \alpha \le 1,\ t \in [0, \sigma], \\ z(0) = \zeta\, z(\sigma), \end{cases} \tag{1.1}$$

where $p, \Phi \in C([0, \sigma] \times \mathbb{R}, \mathbb{R})$, $0 < \sigma < \infty$, ${}^cD^{\alpha}$ is the Caputo fractional derivative of order α, and $\zeta \neq 1$ is an unknown real constant.

In the present context, the term hybrid structure refers to a formulation in which a fractional derivative operator acts on a nonlinear expression of the form $(z(t) - p(t, z(t)))$ and yields another nonlinear function $\Phi(t, z(t))$. The expression that appears under the fractional derivative is commonly described as a perturbed term, and differential equations that involve such perturbations are classified as perturbations of the second type, as discussed in [1]. Alternatively, hybrid fractional differential equations may be viewed as models where the fractional derivative of an unknown function is coupled with a nonlinear function, depending on that unknown.

The systematic study of hybrid differential equations was initiated by Dhage and Lakshmikantham in [2], after which the subject attracted significant attention from researchers. For instance, the existence of solutions to fractional hybrid differential inclusions was examined in [3]. Additional developments concerning hybrid fractional integrodifferential equations can be found in [4–6]. Moreover, a nonlocal dual hybrid q-fractional boundary value problem was analyzed in [7], while a hybrid Erdélyi–Kober-type fractional differential equation defined on an unbounded domain was recently investigated in [8]. Applications of fractional hybrid differential equations to problems that arise in physics and engineering—such as electromagnetic wave propagation, gravity-driven flows, and the deformation of curved and multilayer beams—can be found, for example, in [9, 10].

Grönwall's inequality serves as an important tool to find certain estimates in developing the theory of differential equations. In particular, it is extensively applied to establish the uniqueness of solutions to the initial value problems. Several variants of Grönwall's inequality are available in the literature [11], while applications of this inequality in the stability analysis of classical (integer-order) differential equations can be found in the book [12]. Additionally, this inequality is also vital to analyze incommensurate fractional differential systems; for example, see a recent paper [13] and the references cited therein.

The notion of the Ulam–Hyers stability, introduced in [14] and refined in [15], appears in various fields, such as functional equations [16], Black-Scholes equation [17], etc. Recently, in [18], the Ulam-Hyers stability was discussed for an anti-periodic boundary value problem for a family of finitely many nonlinear Riemann–Liouville fractional differential equations. In [19], the authors studied the Ulam-Hyers stability for an incommensurate system of singular integral equations.

Compared with the existing literature and motivated by the foregoing discussion, the present work offers several key aspects. First, we consider a general Caputo-type hybrid fractional differential equation with a two-point boundary condition that allows an unknown scaling constant $\zeta \neq 1$, which introduces additional analytical challenges not addressed in previous works. Second, our existence result employs a Burton-type extension of Krasnosel'skii's fixed point theorem, thus providing a broader framework than classical fixed point approaches used in [3, 4] to deal with more flexible nonlinearities. Third, the uniqueness of solutions is established through a novel form of Grönwall's inequality, thereby complementing standard contraction-based methods and extending earlier results on uniqueness in hybrid fractional systems. Finally, we investigate the Ulam–Hyers stability of Problem (1.1), which, to the best of our knowledge, has not been systematically analyzed for hybrid fractional differential equations with two-point boundary conditions. Thus, our results not only broaden the class of hybrid fractional differential equations that can be rigorously analyzed but also offer improved tools for both existence and stability analyses.

Here, we emphasize that the tools of the fixed point theory provide a useful platform to develop the existence theory for initial and boundary value problems. For applications of a variety of fixed point theorems to fractional differential equations with different boundary conditions, we refer the reader to the text [20] and a recent paper [21].

The complicated structure of the given problem in terms of the nonlinearities present in the perturbed term $(z(t) - p(t, z(t)))$ and the right-hand side of the hybrid equation together with the two-point boundary condition requires the careful construction of bounds to ensure the applicability of the fixed point theorems. Moreover, the failure to apply the classical Grönwall's inequality to the proposed problem has led to the derivation of a novel form of the Grönwall-type inequality applicable to the proposed hybrid fractional boundary value problem.

The structure of the paper is organized as follows. Section 2 introduces essential notions from fractional calculus and establishes an auxiliary lemma that is instrumental for the subsequent analysis. In Section 3, we derive the existence and uniqueness results for Problem (1.1). The existence of solutions is first obtained by employing a Burton-type extension of Krasnosel'skii's fixed point theorem. Subsequently, a new form of Grönwall's inequality is developed and applied to demonstrate the uniqueness of solutions. Additionally, an alternative uniqueness result is derived using Banach's contraction principle. Finally, Section 4 is devoted to the investigation of the Ulam–Hyers stability of Problem (1.1).

2. Preliminaries

First, let us introduce basic definitions related to our work.

Definition 2.1. *([22]) Let $\alpha \in \mathbb{R}^+$ and $v \in L^1[a,b]$, where $a, b \in \mathbb{R}$. The left-sided Riemann–Liouville fractional integral of order α, denoted by $I^{\alpha}_{a^+}v$, is defined as follows:*

$$I^{\alpha}_{a^+}v(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} v(x)dx,$$

where Γ is the Gamma function.

Definition 2.2. *([22]) Let $v \in AC^m[a,b]$, $\alpha \in (m-1, m)$. The left-sided Caputo fractional derivative of order α is given by the following:*

$$^cD^{\alpha}_{a^+}v(t) = \int_a^t \frac{(t-x)^{m-\alpha-1}}{\Gamma(m-\alpha)} v^{(m)}(x)dx.$$

Lemma 2.3. *([22]) The general solution of the fractional differential equation*

$$^cD^{\alpha}_{a^+}v(t) = 0,\ m\text{–}1 < \alpha < m,\ t \in [a,b],$$

is

$$v(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \ldots + c_{m-1}(t-a)^{m-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, \ldots, m-1$, are arbitrary constants. Furthermore,

$$I^{\alpha}_{a^+}\,{}^cD^{\alpha}_{a^+}v(t) = v(t) + \sum_{i=0}^{m-1} c_i(t-a)^i. \tag{2.1}$$

For the sake of simplicity, we represent $I_{a^+}^{\alpha}$ and ${}^cD_{a^+}^{\alpha}$ as I^{α} and ${}^cD^{\alpha}$, respectively, when $a = 0$.

Now, we prove an auxiliary lemma that plays a key role in analyzing Problem (1.1).

Lemma 2.4. *Let $h, k \in C([0,\sigma],\mathbb{R})$ with $I^{\alpha}h \in AC[0,\sigma]$, (the space of absolutely continuous functions on $[0,\sigma]$); then, a unique solution $z(t)$ to the fractional hybrid differential equation*

$$ {}^cD^{\alpha}(z(t) - k(t)) = h(t),\ 0 < \alpha \le 1, \tag{2.2} $$

supplemented with the boundary condition $z(0) = \zeta\, z(\sigma)$, $\zeta \neq 1$ is given by

$$ \begin{aligned} z(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ &\quad + \frac{1}{1-\zeta}\Big[\zeta\Big(k(\sigma) + \int_0^{\sigma} \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\Big) - k(0)\Big] + k(t). \end{aligned} \tag{2.3} $$

Proof. Operating the left Riemann–Liouville integral operator I^{α} on (2.2) and using (2.1) with $a = 0$, we get the following:

$$ z(t) - k(t) = I^{\alpha}h(t) + c. \tag{2.4} $$

Inserting (2.4) in the condition $z(0) = \zeta\, z(\sigma)$, we have the following:

$$ k(0) + c = \zeta\,(k(\sigma) + I^{\alpha}h(\sigma) + c)\,. \tag{2.5} $$

Solving (2.5) for c, we get the following:

$$ c = \frac{1}{1-\zeta}\Big[\zeta\Big(k(\sigma) + I^{\alpha}h(\sigma)\Big) - k(0)\Big], $$

which, by inserting into (2.4), yields the solution (2.3). By performing straightforward calculations and invoking the condition $I^{\alpha}h \in AC[0,\sigma]$, the reverse implication of the lemma can be established. □

3. Main results

Let $\mathcal{A} = C([0,\sigma],\mathbb{R})$ be a Banach space of all real continuous functions defined on $[0,\sigma]$ equipped with the norm $\|z\|_{\mathcal{A}} = \max_{t\in[0,\sigma]} |z(t)|$. By Lemma 2.4, we convert Problem (1.1) into a fixed point problem as follows:

$$ z = \mathcal{F}z, $$

where $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is given by

$$ \begin{aligned} (\mathcal{F}z)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Phi(s,z(s))ds + \frac{1}{1-\zeta}\Big[\zeta\Big(p(\sigma,z(\sigma)) + \int_0^{\sigma} \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)} \Phi(s,z(s))ds\Big) \\ &\quad - p(0,z(0))\Big] + p(t,z(t)),\ \ t \in [0,\sigma]. \end{aligned} \tag{3.1} $$

Observe that any fixed point of the operator $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ corresponds to a solution of Problem (1.1).

In the sequel, it is assumed that $p, \Phi \in C([0,\sigma]\times\mathbb{R},\mathbb{R})$ satisfy the Lipschitz condition, that is, there exist positive constants $L_1, L_2 < 1$ such that the following hold:

(A_1) $|p(t,z_1) - p(t,z_2)| \leq L_1|z_1 - z_2|$, for all $z_1, z_2 \in \mathbb{R}$ and $t \in [0,\sigma]$; *and*

(A_2) $|\Phi(t,z_1) - \Phi(t,z_2)| \leq L_2|z_1 - z_2|$, for all $z_1, z_2 \in \mathbb{R}$ and $t \in [0,\sigma]$.

Moreover, we have the following:

(A_3) There exists a nonnegative function $\chi \in C([0,\sigma],\mathbb{R})$ such that $|\Phi(t,z(t))| \leq \chi(t) + b|z(t)|,\ b > 0$.

Remark 3.1. *Substituting $\eta = t - s$ in the integral*

$$\int_0^t (t-s)^{\alpha-1} e^{\xi s} ds,$$

we obtain

$$\begin{aligned}\int_0^t (t-s)^{\alpha-1} e^{\xi s} ds &= e^{\xi t} \int_0^t \eta^{\alpha-1} e^{-\xi\eta} d\eta \\ &= \frac{e^{\xi t}}{\xi^\alpha} \int_0^{\xi t} x^{\alpha-1} e^{-x} dx \quad (x = \xi\eta) \\ &\leq \frac{e^{\xi t}}{\xi^\alpha} \int_0^{\infty} x^{\alpha-1} e^{-x} dx = \frac{e^{\xi t}}{\xi^\alpha}\Gamma(\alpha).\end{aligned}$$

3.1. Existence of solutions by Krasnosel'skiǐ's fixed point theorem

Now, we prove an existence result for Problem (1.1) by utilizing a form of Krasnosel'skiǐ's fixed point theorem proposed by Burton [23], which is stated below.

Theorem 3.2. *Let Ω be a nonempty, closed, bounded and convex subset of a Banach space $(\mathcal{X}, \|.\|)$. Consider two operators, $\Psi_1 : \mathcal{X} \to \mathcal{X}$ and $\Psi_2 : \Omega \to \mathcal{X}$, which satisfy the following conditions:*

(1) Ψ_1 *is a contractive mapping;*

(2) Ψ_2 *is continuous and compact; and*

(3) *whenever $z = \Psi_1 z + \Psi_2\vartheta$ for some $\vartheta \in \Omega$, it follows that $z \in \Omega$.*

Then, the operator equation $\Psi_1 z + \Psi_2 z = z$ admits at least one solution in Ω.

Theorem 3.3. *Assume that Conditions (A_1) and (A_3) are satisfied and that $\Lambda L_1 < 1$, where*

$$\Lambda = \frac{|1-\zeta| + |\zeta| + 1}{|1-\zeta|}. \tag{3.2}$$

Then, at least one solution to Problem (1.1) *exists on* $[0,\sigma]$.

Proof. We begin by defining a closed, bounded, and convex subset S of the space $C([0,\sigma],\mathbb{R})$ as follows:

$$S = \{z \in C([0,\sigma],\mathbb{R}) : |z(t)| \leq \delta e^{\xi t},\ \delta \geq 0,\ \xi > 0,\ t \in [0,\sigma]\}.$$

Define the operators $\mathcal{F}_1 : \mathcal{A} \to \mathcal{A}$ and $\mathcal{F}_2 : S \to \mathcal{A}$ as follows:

$$(\mathcal{F}_1 z)(t) = \frac{1}{1-\zeta}\Big[\zeta\,(p(\sigma, z(\sigma))) - p(0, z(0))\Big] + p(t, z(t)),$$

$$(\mathcal{F}_2 z)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Phi(s,z(s))ds + \frac{\zeta}{1-\zeta}\int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}\Phi(s,z(s))ds.$$

First, let us verify that $\mathcal{F}_1$ is a contraction. Letting z_1 and $z_2 \in \mathcal{A}$, we get the following:

$$\begin{aligned}
|(\mathcal{F}_1 z_1)(t) - (\mathcal{F}_1 z_2)(t)| &= \Big|\frac{1}{1-\zeta}\Big[\zeta\Big(p(\sigma,z_1(\sigma)) - p(\sigma,z_2(\sigma))\Big) - (p(0,z_1(0)) - p(0,z_2(0)))\Big] \\
&\quad + p(t,z_1(t)) - p(t,z_2(t))\Big| \\
&\le \frac{|\zeta|}{|1-\zeta|}|p(\sigma,z_1(\sigma)) - p(\sigma,z_2(\sigma))| + \frac{1}{|1-\zeta|}|p(0,z_1(0)) - p(0,z_2(0))| \\
&\quad + |p(t,z_1(t)) - p(t,z_2(t))| \\
&\le \frac{|\zeta|L_1}{|1-\zeta|}|z_1 - z_2| + \frac{L_1}{|1-\zeta|}|z_1 - z_2| + L_1|z_1 - z_2| \\
&= \Big(\frac{|1-\zeta| + |\zeta| + 1}{|1-\zeta|}\Big)L_1|z_1 - z_2| = \Lambda L_1|z_1 - z_2|.
\end{aligned}$$

Thus, $\mathcal{F}_1$ is a contraction as $\Lambda L_1 < 1$, where Λ is given in (3.2).

By Assumption (A_3), for any $z \in S$, we have the following:

$$\begin{aligned}
\|(\mathcal{F}_2 z)\| &= \max_{t\in[0,\sigma]} |(\mathcal{F}_2 z)(t)| \\
&\le \max_{t\in[0,\sigma]}\Big\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\Phi(s,z(s))|ds + \frac{|\zeta|}{|1-\zeta|}\int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}|\Phi(s,z(s))|ds\Big\} \\
&\le \max_{t\in[0,\sigma]}\Big\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(\chi(s) + b|z(s)|)ds + \frac{|\zeta|}{|1-\zeta|}\int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}(\chi(s) + b|z(s)|)ds\Big\} \\
&\le \frac{\|\chi\|\sigma^\alpha}{\Gamma(\alpha+1)} + b\delta \max_{t\in[0,\sigma]}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{\xi s}ds + \frac{\|\chi\||\zeta|\sigma^\alpha}{|1-\zeta|\Gamma(\alpha+1)} \\
&\quad + \frac{|\zeta|b\,\delta}{|1-\zeta|}\max_{t\in[0,\sigma]}\int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}e^{\xi s}ds \\
&= \frac{\|\chi\|\sigma^\alpha}{\Gamma(\alpha+1)}\Big(\frac{|\zeta| + |1-\zeta|}{|1-\zeta|}\Big) + b\,\delta \max_{t\in[0,\sigma]}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{\xi s}ds \\
&\quad + \frac{b\,|\zeta|\delta}{|1-\zeta|}\max_{t\in[0,\sigma]}\int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}e^{\xi s}ds.
\end{aligned}$$

By Remark 3.1, the above inequality takes the following form:

$$\|(\mathcal{F}_2 z)\| \le \Big(\frac{|\zeta| + |1-\zeta|}{|1-\zeta|}\Big)\Big[\frac{\|\chi\|\sigma^\alpha}{\Gamma(\alpha+1)} + \frac{b\,\delta e^{\xi\sigma}}{\xi^\alpha}\Big].$$

Therefore, $\mathcal{F}_2(S)$ is uniformly bounded.

Note that continuity of $\Phi(t,z(t))$ implies that of the operator $\mathcal{F}_2$.

To demonstrate that the operator $\mathcal{F}_2$ is compact, we first show that the set $\mathcal{F}_2(S)$ constitutes an equicontinuous family of functions. Taking $\max_{(t,z(t))\in[0,\sigma]\times S}|\Phi(t,z(t))| = N < \infty$, $0 < \varrho_1 < \varrho_2 < \sigma$ and $z \in S$, we obtain the following:

$$
\begin{aligned}
|(\mathcal{F}_2z)(\varrho_2)-(\mathcal{F}_2z)(\varrho_1)| &\le \int_0^{\varrho_1}\frac{|(\varrho_2-s)^{\alpha-1}-(\varrho_1-s)^{\alpha-1}|}{\Gamma(\alpha)}|\Phi(s,z(s))|ds\\
&\quad+\int_{\varrho_1}^{\varrho_2}\frac{|\varrho_2-s|^{\alpha-1}}{\Gamma(\alpha)}|\Phi(s,z(s))|ds\\
&\le \frac{N}{\Gamma(\alpha+1)}[2|\varrho_2-\varrho_1|^{\alpha}+|\varrho_2^{\alpha}-\varrho_1^{\alpha}|]\to 0 \text{ as } (\varrho_2-\varrho_1)\to 0,
\end{aligned}
$$

independent of $z\in S$. Thus, $\mathcal{F}_2(S)$ is an equicontinuous set. Since the set $\mathcal{F}_2(S)$ is both uniformly bounded and equicontinuous, it follows from the Arzelá-Ascoli theorem [24] that $\mathcal{F}_2(S)$ is relatively compact. Consequently, $\mathcal{F}_2$ defines a continuous and compact operator on S.

Next, we verify the final condition required by Theorem 3.2. Let $z\in\mathcal{A}$ and $\omega\in S$ such that $z=\mathcal{F}_1z+\mathcal{F}_2\omega$. Then, by Assumption (A_3), we obtain

$$
\begin{aligned}
|z(t)| &\le |(\mathcal{F}_1z)(t)|+|(\mathcal{F}_2\omega)(t)|\\
&\le |p(t,z(t))-p(t,0))|+|p(t,0)|+\frac{1}{|1-\zeta|}\Big|\zeta(p(\sigma,z(\sigma)))-p(0,z(0))\Big|\\
&\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(\chi(s)+b|\omega(s)|)ds+\frac{|\zeta|}{|1-\zeta|}\int_0^{\sigma}\frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}(\chi(s)+b|\omega(s)|)ds\\
&\le L_1|z(t)|+\underbrace{\max_{t\in[0,\sigma]}|p(t,0)|+\frac{1}{|1-\zeta|}\Big|\zeta(p(\sigma,z(\sigma)))-p(0,z(0))\Big|}_{:=q}\\
&\quad+\frac{\sigma^{\alpha}\|\chi\|[|\zeta|+|1-\zeta|]}{\Gamma(\alpha+1)|1-\zeta|}+\frac{b\,\delta}{\xi^{\alpha}}\Big(e^{\xi t}+\frac{|\zeta|e^{\xi\sigma}}{|1-\zeta|}\Big)\\
&= q+L_1|z(t)|+\frac{\sigma^{\alpha}\|\chi\|[|\zeta|+|1-\zeta|]}{\Gamma(\alpha+1)|1-\zeta|}+\frac{b\,\delta}{\xi^{\alpha}}\Big(e^{\xi t}+\frac{e^{\xi\sigma}|\zeta|}{|1-\zeta|}\Big),
\end{aligned}
$$

which can be expressed as

$$
|z(t)| \le \frac{b\,\delta e^{\xi t}}{(1-L_1)\xi^{\alpha}}+\frac{q+\dfrac{\sigma^{\alpha}\|\chi\|[|\zeta|+|1-\zeta|]}{\Gamma(\alpha+1)|1-\zeta|}+\dfrac{b\delta|\zeta|e^{\xi\sigma}}{|1-\zeta|\xi^{\alpha}}}{1-L_1}.
$$

Thus, $z\in S$ if $|z(t)|\le\delta\,e^{\xi t}$ for any $t\in[0,\sigma]$. Therefore, we have

$$
|z(t)| \le \frac{b\,\delta e^{\xi t}}{(1-L_1)\xi^{\alpha}}+\frac{q+\dfrac{\sigma^{\alpha}\|\chi\|[|\zeta|+|1-\zeta|]}{\Gamma(\alpha+1)|1-\zeta|}+\dfrac{b\delta|\zeta|e^{\xi\sigma}}{|1-\zeta|\xi^{\alpha}}}{1-L_1}\le\delta\,e^{\xi t},
$$

which implies that

$$
\delta\,e^{\xi t}\Big(1-\frac{b}{(1-L_1)\xi^{\alpha}}\Big)\ge\frac{q+\dfrac{\sigma^{\alpha}\|\chi\|[|\zeta|+|1-\zeta|]}{\Gamma(\alpha+1)|1-\zeta|}+\dfrac{b\delta|\zeta|e^{\xi\sigma}}{|1-\zeta|\xi^{\alpha}}}{1-L_1}.
$$

Consequently, we obtain the following:

$$\frac{\frac{1}{1-L_1}\left[q+\frac{\sigma^{\alpha}\|\chi\|[|\zeta|+|1-\zeta|]}{\Gamma(\alpha+1)|1-\zeta|}+\frac{b\delta|\zeta|e^{\xi\sigma}}{|1-\zeta|\xi^{\alpha}}\right]}{1-\frac{b}{(1-L_1)\xi^{\alpha}}}\leq \delta\, e^{\xi t}.$$

If $\xi > 0$ and $\delta > 0$ are chosen to be large enough to satisfy the inequalities

$$\xi^{\alpha} > \frac{b}{1-L_1},$$

and

$$\frac{1}{1-L_1}\left[q+\frac{\sigma^{\alpha}\|\chi\|[|\zeta|+|1-\zeta|]}{\Gamma(\alpha+1)|1-\zeta|}+\frac{b\delta|\zeta|e^{\xi\sigma}}{|1-\zeta|\xi^{\alpha}}\right]\leq \min_{t\in[0,\sigma]}\delta e^{\xi t}=\delta,$$

then, we get

$$\frac{1}{1-L_1}\left[q+\frac{\sigma^{\alpha}\|\chi\|[|\zeta|+|1-\zeta|]}{\Gamma(\alpha+1)|1-\zeta|}+\frac{b\delta|\zeta|e^{\xi\sigma}}{|1-\zeta|\xi^{\alpha}}\right]\leq \delta e^{\xi t}.$$

Hence, we deduce that $|z(t)| \leq \delta\, e^{\xi t}$, which means that $z \in S$. Therefore, all conditions of Theorem 3.2 are satisfied, and we conclude that Problem (1.1) admits at least one solution on $[0, T]$. □

Example 3.4. *Consider the fractional hybrid differential equation*

$$^{c}D^{\frac{1}{2}}(z(t)-p(t,z(t)))=\Phi(t,z(t)),\ \ 0<t<2, \tag{3.3}$$

supplemented with the boundary condition $u(0) = 3\,u(2)$, *where*

$$p(t,z(t))=\frac{1}{4}\cos z(t)+\sqrt{t^5+45},\ \ \Phi(t,z(t))=\frac{\tan^{-1}z(t)}{t^2+4}+3\sin z(t).$$

Clearly, $\alpha = \frac{1}{2}, \sigma = 2, \zeta = 3$, $\Lambda = 3$, and p satisfies the Lipschitz condition with $L_1 = \frac{1}{4}$. Moreover, $\chi(t) = \frac{\pi}{2(t^2+4)}, b = 3$ as

$$|\Phi(t,z(t))|\leq \frac{\pi}{2(t^2+4)}+3|z(t)|$$

and $\Lambda L_1 = 0.75 < 1$. Clearly the hypothesis of Theorem 3.3 is satisfied and its conclusion applies to Problem (3.3).

In passing, we remark that we proved the existence of at least one solution to Problem (1.1) in Theorem 3.3, while the uniqueness of its solutions will be established in Theorem 3.8 in the next subsection.

3.2. Grönwall's inequality and uniqueness result

In this section, we prove a new variant of Grönwall's inequality [25, 26], which will be applied to prove a uniqueness result for Problem (1.1). For that, we consider the following integral operator:

$$(\mathcal{N}z)(t)=m\int_0^t (t-s)^{\alpha-1}z(s)ds,\ \ t\geq 0,\ \ m\geq 0,$$

where z is a locally integrable function. First, let us prove the following lemma.

Lemma 3.5. *Let* $m \geq 0, z(t) \geq 0$; *then,*

$$(\mathcal{N}^n z)(t) \leq \int_0^t \frac{(m\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}z(s)ds,\ t \geq 0, \tag{3.4}$$

and $(\mathcal{N}^n z)(t) \to 0$ *as* $n \to +\infty$ *for all* $t \in [0, +\infty)$.

Proof. We complete the proof by mathematical induction. Obviously, Relation (3.4) is true for $n = 1$. Assume that it is true for some $n = k$, that is,

$$(\mathcal{N}^k z)(t) \leq \int_0^t \frac{(m\Gamma(\alpha))^k}{\Gamma(k\alpha)}(t-s)^{k\alpha-1}z(s)ds. \tag{3.5}$$

For $n = k + 1$, we have the following:

$$(\mathcal{N}^{k+1}z)(t) = \mathcal{N}(\mathcal{N}^k z(t)) \leq m \int_0^t (t-s)^{\alpha-1}\Big[\int_0^s \frac{(m\Gamma(\alpha))^k}{\Gamma(k\alpha)}(s-\tau)^{k\alpha-1}z(\tau)d\tau\Big]ds. \tag{3.6}$$

By interchanging the order of integration, we obtain the following:

$$(\mathcal{N}^{k+1}z)(t) \leq m^{k+1}\int_0^t \Big[\int_\tau^t \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)}(t-s)^{\alpha-1}(s-\tau)^{k\alpha-1}ds\Big]z(\tau)d\tau. \tag{3.7}$$

Consider the integral

$$I = \int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{k\alpha-1}ds,$$

which, using the substitution $s = \tau + y(t-\tau)$, yields

$$I = (t-\tau)^{(k+1)\alpha-1}\,\frac{\Gamma(\alpha)\Gamma(k\alpha)}{\Gamma(\alpha(k+1))}. \tag{3.8}$$

Inserting (3.8) in (3.7), we obtain the following:

$$(\mathcal{N}^{k+1}z)(t) \leq m^{k+1}\int_0^t \frac{(\Gamma(\alpha))^{k+1}}{\Gamma(\alpha(k+1))}(t-s)^{(1+k)\alpha-1}z(s)ds.$$

Hence, Relation (3.4) holds true for $n = k+1$. Therefore, Relation (3.4) is true for all $n \in \mathbb{N}$. Moreover, since the denominator tends to infinity faster than numerator in (3.4), we have the following:

$$(\mathcal{N}^n z)(t) \leq \int_0^t \frac{(m\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}z(s)ds \to 0,\ n \to +\infty,\ \ t \in [0, +\infty).$$

The proof is complete. □

Lemma 3.6. *(Grönwall's inequality) Let* $\alpha > 0$ *and* $m \geq 0$. *Let* $z(t)$ *and* $h(t)$ *be nonnegative locally integrable functions on* $[0, T)$ *for some* $T \leq +\infty$, *and satisfy the following inequality:*

$$z(t) \leq h(t) + m\int_0^t (t-s)^{\alpha-1}z(s)ds. \tag{3.9}$$

Then,

$$z(t) \leq h(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(m\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}h(s)\Big]ds,\quad 0 \leq t < T.$$

Proof. From (3.9) and using the recurrence relation, we obtain the following:

$$\begin{aligned} z(t) &\leq h(t) + (\mathcal{N}z)(t) \leq h(t) + (\mathcal{N}h)(t) + (\mathcal{N}^2 z)(t) \leq \dots \\ &\leq h(t) + \sum_{k=1}^{n-1} (\mathcal{N}^k h)(t) + (\mathcal{N}^n z)(t). \end{aligned} \tag{3.10}$$

Taking $n \to \infty$ in (3.10), according to Lemma 3.5, we conclude that $(\mathcal{N}^n z)(t) \to 0$ as $n \to \infty$. Thus, we obtain the following:

$$z(t) \leq h(t) + \sum_{k=1}^{\infty} (\mathcal{N}^k h)(t) \leq h(t) + \sum_{k=1}^{\infty} \int_0^t \frac{(m\Gamma(\alpha))^k}{\Gamma(k\alpha)} (t-s)^{k\alpha-1} h(s) ds.$$

□

Lemma 3.7. *If the hypothesis of Lemma 3.6 holds and* $h(t)$ *is a nondecreasing function for* $t \in [0, \sigma)$*, then* $z(t) \leq h(t) E_\alpha(m\Gamma(\alpha)t^\alpha)$, $\alpha > 0$, *where* E_α *denotes the Mittag-Leffler function defined by the following:*

$$E_\alpha(y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(n\alpha + 1)}.$$

Proof. From the hypotheses of Lemma 3.6, we have the following:

$$\begin{aligned} z(t) &\leq h(t) + \sum_{n=1}^{\infty} \int_0^t \frac{(m\Gamma(\alpha))^n}{\Gamma(k\alpha)} (t-s)^{n\alpha-1} h(s) ds \\ &= h(t)\Big[1 + \sum_{n=1}^{\infty} \frac{m^n (\Gamma(\alpha))^n}{\Gamma(n\alpha)} \int_0^t (t-s)^{n\alpha-1} ds\Big] \\ &= h(t)\Big[1 + \sum_{n=1}^{\infty} \frac{m^n (\Gamma(\alpha))^n}{\Gamma(n\alpha)} \frac{t^{n\alpha}}{n\alpha}\Big] \\ &= h(t)\Big[1 + \sum_{n=1}^{\infty} \frac{(m\Gamma(\alpha)t^\alpha)^n}{\Gamma(n\alpha + 1)}\Big] \\ &= h(t) \sum_{n=0}^{\infty} \frac{(m\Gamma(\alpha)t^\alpha)^n}{\Gamma(n\alpha + 1)} \\ &= h(t) E_\alpha[m\Gamma(\alpha)t^\alpha]. \end{aligned}$$

□

3.3. Uniqueness of solutions by Grönwall's inequality

Here, we prove the uniqueness of solutions to Problem (1.1) via the Grönwall's inequality given in Lemma 3.6.

Theorem 3.8. *If Conditions* (A_1) *and* (A_2) *hold, then Problem* (1.1) *has a unique solution on* $[0, \sigma]$.

Proof. Let $z(t)$ and $\widehat{z}(t)$ be two solutions of Problem (1.1). Then, we have

$$\begin{aligned} z(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Phi(s,z(s))ds + \frac{1}{1-\zeta}\Big[\zeta\Big(p(\sigma,z(\sigma)) + \int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}\Phi(s,z(s))ds\Big) \\ &\quad -p(0,z(0))\Big] + p(t,z(t)), \end{aligned}$$

and

$$\begin{aligned} \widehat{z}(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Phi(s,\widehat{z}(s))ds + \frac{1}{1-\zeta}\Big[\zeta\Big(p(\sigma,\widehat{z}(\sigma)) + \int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}\Phi(s,\widehat{z}(s))ds\Big) \\ &\quad -p(0,\widehat{z}(0))\Big] + p(t,\widehat{z}(t)), \end{aligned}$$

with $z(0) = \zeta\, z(\sigma)$ and $\widehat{z}(0) = \zeta\, \widehat{z}(\sigma)$. In consequence, we get

$$\begin{aligned} &|z(t)-\widehat{z}(t)| \\ \leq\ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\Phi(s,z(s))-\Phi(s,\widehat{z}(s))|ds + \frac{|\zeta|}{|1-\zeta|}|p(\sigma,z(\sigma))-p(\sigma,\widehat{z}(\sigma))| \\ & +\frac{1}{|1-\zeta|}|p(0,z(0))-p(0,\widehat{z}(0))| + |p(t,z(t))-p(t,\widehat{z}(t))| \\ & +\frac{|\zeta|}{|1-\zeta|\Gamma(\alpha)}\int_0^\sigma (\sigma-s)^{\alpha-1}|\Phi(s,z(s))-\Phi(s,\widehat{z}(s))|ds \\ \leq\ & \frac{L_2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s)-\widehat{z}(s)|ds + \frac{|\zeta|L_1}{|1-\zeta|}|z(\sigma)-\widehat{z}(\sigma)| \\ & +\frac{L_1}{|1-\zeta|}|z(0)-\widehat{z}(0)| + L_1|z(t)-\widehat{z}(t)| + \frac{L_2|\zeta|}{|1-\zeta|\Gamma(\alpha)}\int_0^\sigma (\sigma-s)^{\alpha-1}|z(s)-\widehat{z}(s))|ds, \end{aligned}$$

which can be expressed as

$$\begin{aligned} |z(t)-\widehat{z}(t)| &\leq \frac{L_1}{(1-L_1)|1-\zeta|}\Big(|\zeta||z(\sigma)-\widehat{z}(\sigma)| + |z(0)-\widehat{z}(0)|\Big) \\ &\quad +\frac{L_2}{(1-L_1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s)-\widehat{z}(s)|ds \\ &\quad +\frac{L_2|\zeta|}{(1-L_1)|1-\zeta|\Gamma(\alpha)}\int_0^\sigma (\sigma-s)^{\alpha-1}|z(s)-\widehat{z}(s)|ds \\ &\leq \frac{L_1}{(1-L_1)|1-\zeta|}\Big(|\zeta||z(\sigma)-\widehat{z}(\sigma)| + |z(0)-\widehat{z}(0)|\Big) \\ &\quad +\frac{L_2}{(1-L_1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s)-\widehat{z}(s)|ds \\ &\quad +\frac{L_2\sigma^\alpha|\zeta|}{(1-L_1)|1-\zeta|\Gamma(\alpha+1)}|z(\sigma)-\widehat{z}(\sigma)| \\ &\leq \frac{1}{(1-L_1)|1-\zeta|}\Big(|\zeta|\Big[L_1+\frac{L_2\sigma^\alpha}{\Gamma(\alpha+1)}\Big]|z(\sigma)-\widehat{z}(\sigma)| + L_1|z(0)-\widehat{z}(0)|\Big) \\ &\quad +\frac{L_2}{(1-L_1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s)-\widehat{z}(s)|ds. \end{aligned}$$

By Lemmas 3.6 and 3.7, we obtain the following:

$$\begin{aligned}
|z(t)-\widehat{z}(t)| &\leq \frac{1}{(1-L_1)|1-\zeta|}\Big(|\zeta|\Big[L_1+\frac{L_2\sigma^\alpha}{\Gamma(\alpha+1)}\Big]|z(\sigma)-\widehat{z}(\sigma)|+L_1|z(0)-\widehat{z}(0)|\Big)\\
&\quad\times\Big[1+\int_0^t\sum_{n=1}^{\infty}\Big(\frac{L_2}{1-L_1}\Big)^n\frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)}ds\Big]\\
&= \frac{1}{(1-L_1)|1-\zeta|}\Big(|\zeta|\Big[L_1+\frac{L_2\sigma^\alpha}{\Gamma(\alpha+1)}\Big]|z(\sigma)-\widehat{z}(\sigma)|+L_1|z(0)-\widehat{z}(0)|\Big)\\
&\quad\times\Big[1+\sum_{n=1}^{\infty}\Big(\frac{L_2}{1-L_1}\Big)^n\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\Big]\\
&\leq \frac{1}{(1-L_1)|1-\zeta|}\Big(|\zeta|\Big[L_1+\frac{L_2\sigma^\alpha}{\Gamma(\alpha+1)}\Big]|z(\sigma)-\widehat{z}(\sigma)|+L_1|z(0)-\widehat{z}(0)|\Big)\\
&\quad\times\Big[1+\sum_{n=1}^{\infty}\Big(\frac{L_2}{1-L_1}\Big)^n\frac{\sigma^{n\alpha}}{\Gamma(n\alpha+1)}\Big].
\end{aligned}$$

Therefore, since $|z(0)-\widehat{z}(0)|=|\zeta||z(\sigma)-\widehat{z}(\sigma)|$, we have the following:

$$|z(t)-\widehat{z}(t)| \leq \frac{1}{(1-L_1)|1-\zeta|}\Big(2L_1+\frac{L_2\sigma^\alpha}{\Gamma(\alpha+1)}\Big)E_\alpha\Big(\frac{L_2\sigma^\alpha}{1-L_1}\Big)|z(0)-\widehat{z}(0)|,$$

where E_α is the Mittag-Leffler function. Thus, the uniqueness of solutions for Problem (1.1) follows when $z(0)=\widehat{z}(0)$. □

3.4. Uniqueness result via Banach's fixed point theorem

Theorem 3.9. *If Assumptions* (A_1) *and* (A_2) *are satisfied, then Problem* (1.1) *admits a unique solution on* $[0,\sigma]$ *if* $\Omega_1<1$, *where*

$$\Omega_1=\frac{L_1(|\zeta|+|1-\zeta|+1)}{|1-\zeta|}+\frac{L_2\sigma^\alpha(|1-\zeta|+|\zeta|)}{|1-\zeta|\Gamma(\alpha+1)}. \tag{3.11}$$

Proof. Let us begin by verifying the conditions of Banach's fixed point theorem. To this end, we consider a closed and bounded ball of radius λ in the Banach space $\mathcal{A}$ defined by $B_\lambda=\{z\in\mathcal{A}:\|z\|\leq\lambda\}$, where $\lambda\geq\dfrac{\Omega_2}{1-\Omega_1}$, Ω_1 is defined in (3.11) and

$$\Omega_2=\frac{K_1(|\zeta|+|1-\zeta|+1)}{|1-\zeta|}+\frac{K_2\sigma^\alpha(|1-\zeta|+|\zeta|)}{|1-\zeta|\Gamma(\alpha+1)}, \tag{3.12}$$

with $K_1=\max\limits_{t\in[0,\sigma]}|p(t,0)|, K_2=\max\limits_{t\in[0,\sigma]}|\Phi(t,0)|$. Letting $z\in B_\lambda$, it follows by Assumptions (A_1) and (A_2) that

$$|p(t,z(t))|=|p(t,z(t))-p(t,0)+p(t,0)|\leq|p(t,z(t))-p(t,0)|+|p(t,0)|\leq L_1\|z\|+K_1\leq L_1\lambda+K_1,$$

$$|\Phi(t,z(t))|=|\Phi(t,z(t))-\Phi(t,0)+\Phi(t,0)|\leq|\Phi(t,z(t))-\Phi(t,0)|+|\Phi(t,0)|\leq L_2\|z\|+K_2\leq L_2\lambda+K_2.$$

For any $z \in B_\lambda, t \in [0,\sigma]$, we have the following:

$$\begin{aligned}
\|(\mathcal{F}z)\| &= \max_{t\in[0,\sigma]} |(\mathcal{F}z)(t)| \\
&\leq \max_{t\in[0,\sigma]} \Big\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\Phi(s,z(s))| ds \\
&\quad + \frac{1}{|1-\zeta|}\Big[|\zeta|\Big(|p(\sigma,z(\sigma))| + \int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}|\Phi(s,z(s))|ds\Big) + |p(0,z(0))|\Big] + |p(t,z(t))|\Big\} \\
&\leq (L_2\lambda + K_2) \max_{t\in[0,\sigma]} \Big\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds\Big\} \\
&\quad + \frac{1}{|1-\zeta|}\Big[|\zeta|\Big((L_1\lambda + K_1) + \int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}(L_2\lambda + K_2)ds\Big) + L_1\lambda + K_1\Big] + L_1\lambda + K_1 \\
&= (L_1\lambda + K_1)\Big[\frac{|\zeta| + |1-\zeta| + 1}{|1-\zeta|}\Big] + \frac{(L_2\lambda + K_2)\sigma^\alpha}{\Gamma(\alpha+1)}\Big[\frac{|1-\zeta| + |\zeta|}{|1-\zeta|}\Big] \\
&= \lambda\Big\{\frac{L_1(|\zeta| + |1-\zeta| + 1)}{|1-\zeta|} + \frac{L_2\sigma^\alpha(|1-\zeta| + |\zeta|)}{|1-\zeta|\Gamma(\alpha+1)}\Big\} \\
&\quad + \frac{K_1(|\zeta| + |1-\zeta| + 1)}{|1-\zeta|} + \frac{K_2\sigma^\alpha(|1-\zeta| + |\zeta|)}{|1-\zeta|\Gamma(\alpha+1)} \\
&= \lambda\Omega_1 + \Omega_2 \leq \lambda,
\end{aligned}$$

which implies that $\mathcal{F}z \in B_\lambda$. Therefore, since $z \in B_\lambda$ is an arbitrary element, $\mathcal{F}B_\lambda \subset B_\lambda$.

Next, we take $z_1, z_2 \in B_\lambda, t \in [0,\sigma]$ and find that

$$\begin{aligned}
&\|(\mathcal{F}z_1) - (\mathcal{F}z_2)\| \\
= &\max_{t\in[0,\sigma]} |(\mathcal{F}z_1)(t) - (\mathcal{F}z_2)(t)| \\
\leq &\max_{t\in[0,\sigma]} \Big\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\Phi(s,z_1(s)) - \Phi(s,z_2(s))| ds \\
&+ \frac{1}{|1-\zeta|}\Big[|\zeta|\Big(|p(\sigma,z_1(\sigma)) - p(\sigma,z_2(\sigma))| + \int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}|\Phi(s,z_1(s)) - \Phi(s,z_2(s))|ds\Big) \\
&+ |p(0,z_1(0)) - p(0,z_2(0))|\Big] + |p(t,z_1(t)) - p(t,z_2(t))|\Big\} \\
\leq &L_1\|z_1 - z_2\|\Big[\frac{|\zeta| + |1-\zeta| + 1}{|1-\zeta|}\Big] + L_2\|z_1 - z_2\|\Big[\frac{\sigma^\alpha(|1-\zeta| + |\zeta|)}{|1-\zeta|\Gamma(\alpha+1)}\Big] \\
= &\Big[\frac{L_1(|\zeta| + |1-\zeta| + 1)}{|1-\zeta|} + \frac{L_2\sigma^\alpha(|1-\zeta| + |\zeta|)}{|1-\zeta|\Gamma(\alpha+1)}\Big]\|z_1 - z_2\| \\
= &\Omega_1\|z_1 - z_2\|,
\end{aligned}$$

where Ω_1 is defined by (3.11). Thus, $\mathcal{F}$ is a contraction since $\Omega_1 < 1$. Therefore, operator $\mathcal{F}$ has a unique fixed in B_λ by Banach's fixed point theorem. Consequently, Problem (1.1) admits a unique solution on $[0,\sigma]$. □

Example 3.10. *Consider Problem* (3.3) *with*

$$p(t,z(t)) = \frac{1}{25}\tan^{-1} z(t) + \frac{\cos t}{\sqrt{t^2+4}}, \quad \Phi(t,z(t)) = \frac{1}{36}\sin z(t) + 5\sqrt[3]{t^7 + 20}. \tag{3.13}$$

It is easy to verify that p and Φ satisfy the Lipschitz condition with $L_1 = \frac{1}{25}$ and $L_2 = \frac{1}{36}$, respectively, and $\Omega_1 \simeq 0.23081730$. Therefore, since the hypotheses of Theorem 3.9 is satisfied, its conclusion implies that Problem (3.3) with p and Φ given in (3.13) admits a unique solution on $[0, 2]$.

4. Ulam-Hyers stability

Let us begin this section with concepts of the Ulam-Hyers stability related to Problem (1.1).

Definition 4.1. *Consider the inequality*

$$|^cD^\alpha(w(t) - p(t, w(t))) - \Phi(t, w(t))| \leq \epsilon, \quad t \in [0, \sigma], \quad \epsilon > 0, \tag{4.1}$$

with the boundary condition

$$w(0) = \zeta w(\sigma), \ \zeta \neq 1. \tag{4.2}$$

Then, Problem (1.1) *is called Ulam-Hyers stable if there exists* $\upsilon > 0$ *such that, for any solution* w *in* $C([0, \sigma], \mathbb{R})$ *of the fractional hybrid differential inequality* (4.1) *with the boundary condition* (4.2)*, there exists a unique solution* $z \in C([0, \sigma], \mathbb{R})$ *of Problem* (1.1) *that satisfies the following:*

$$\|w - z\| \leq \upsilon\epsilon, \ t \in [0, \sigma].$$

Definition 4.2. *Problem* (1.1) *is called generalized Ulam-Hyer stable if there exists a continuous function* $\psi : [0, \infty) \to [0, \infty)$ *with* $\psi(0) = 0$ *such that, for any solution* $w \in C([0, \sigma], \mathbb{R})$ *of the fractional hybrid differential inequality* (4.1) *with the boundary condition* (4.2)*, there exists a unique solution* $z \in C([0, \sigma], \mathbb{R})$ *of Problem* (1.1) *that satisfies the following:*

$$\|w - z\| \leq \psi(\epsilon), \ t \in [0, \sigma], \quad \epsilon > 0.$$

Remark 4.3. *A function* $w \in C([0, \sigma], \mathbb{R})$ *satisfies the fractional hybrid differential inequality*

$$|^cD^\alpha(w(t) - p(t, w(t))) - \Phi(t, w(t))| \leq \epsilon, \quad t \in [0, \sigma], \quad \epsilon > 0,$$

with the boundary condition $w(0) = \zeta w(\sigma)$ *if and only if there exists a function* $\mathcal{M} \in C([0, \sigma], \mathbb{R})$ *such that* $|\mathcal{M}(t)| \leq \epsilon, \ \epsilon > 0$ *and* ${}^cD^\alpha(w(t) - p(t, w(t))) = \Phi(t, w(t)) + \mathcal{M}(t), t \in [0, \sigma]$ *with the boundary condition* $w(0) = \zeta w(\sigma), \ \ \zeta \neq 1$.

Theorem 4.4. *Let the assumptions of Theorem 3.9 be satisfied. Then, Problem* (1.1) *is Ulam-Hyers stable and generalized Ulam-Hyers stable in* $C([0, \sigma], \mathbb{R})$.

Proof. By Theorem 3.9, Problem (1.1) has a unique solution $z = \mathcal{F}z$. Let w be a solution to the fractional hybrid differential inequality (4.1) with the boundary condition in (4.2). Then, by Remark 4.3, there is a function $\mathcal{M} \in C([0, \sigma])$ with $|\mathcal{M}(t)| \leq \epsilon, \ \epsilon > 0, \ t \in [0, \sigma]$ such that

$${}^cD^\alpha(w(t) - p(t, w(t))) = \Phi(t, w(t)) + \mathcal{M}(t), t \in [0, \sigma], \ w(0) = \zeta w(\sigma), \ \ \zeta \neq 1. \tag{4.3}$$

By Lemma 2.4, the solution of (4.3) can be written as follows:

$$
\begin{aligned}
w(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(\Phi(s,w(s))+\mathcal{M}(s))ds \\
&+\frac{1}{1-\zeta}\Big[\zeta\Big(p(\sigma,w(\sigma))+\int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}(\Phi(s,w(s))+\mathcal{M}(s))ds\Big)-p(0,w(0))\Big] \\
&+p(t,w(t)).
\end{aligned}
$$

In view of $|\mathcal{M}|\leq\epsilon$, we have the following:

$$
\begin{aligned}
\|w-(\mathcal{F}w)\| &= \max_{t\in[0,\sigma]}|w(t)-(\mathcal{F}w)(t)| \\
&= \max_{t\in[0,\sigma]}\Big\{\Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(\Phi(s,w(s))+\mathcal{M}(s))ds \\
&\quad+\frac{1}{1-\zeta}\Big[\zeta\Big(p(\sigma,w(\sigma))+\int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}(\Phi(s,w(s))+\mathcal{M}(s))ds\Big)-p(0,w(0))\Big] \\
&\quad+p(t,w(t))-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Phi(s,w(s))ds \\
&\quad-\frac{1}{1-\zeta}\Big[\zeta\Big(p(\sigma,w(\sigma))+\int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}\Phi(s,w(s))ds\Big)-p(0,w(0))\Big] \\
&\quad-p(t,w(t))\Big|\Big\} \\
&\leq \max_{t\in[0,\sigma]}\Big\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\mathcal{M}(s)|ds+\frac{|\zeta|}{|1-\zeta|}\int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}|\mathcal{M}(s)|ds\Big\} \\
&\leq \epsilon\max_{t\in[0,\sigma]}\Big\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\frac{|\zeta|}{|1-\zeta|}\int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Big\} \\
&= \frac{\epsilon\,\sigma^\alpha(|1-\zeta|+|\zeta|)}{\Gamma(\alpha+1)|1-\zeta|}=\epsilon\gamma, \qquad (4.4)
\end{aligned}
$$

where

$$
\gamma=\frac{\sigma^\alpha(|1-\zeta|+|\zeta|)}{\Gamma(\alpha+1)|1-\zeta|}. \qquad (4.5)
$$

Next, by Assumptions (A_1) and (A_2), we have the following:

$$
\begin{aligned}
&\|(\mathcal{F}w)-(\mathcal{F}z)\|=\max_{t\in[0,\sigma]}|(\mathcal{F}w)(t)-(\mathcal{F}z)(t)| \\
\leq&\ \max_{t\in[0,\sigma]}\Big\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\Phi(s,w(s))-\Phi(s,z(s))|ds \\
&+\frac{1}{|1-\zeta|}\Big[|\zeta|\Big(|p(\sigma,w(\sigma))-p(\sigma,z(\sigma))|+\int_0^\sigma \frac{(\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}|\Phi(s,w(s))-\Phi(s,z(s))|ds\Big) \\
&+|p(0,w(0))-p(0,z(0))|\Big]+|p(t,w(t))-p(t,z(t))|\Big\} \\
\leq&\ L_1\|w-z\|\Big[\frac{|\zeta|+|1-\zeta|+1}{|1-\zeta|}\Big]+L_2\|w-z\|\Big[\frac{\sigma^\alpha(|1-\zeta|+|\zeta|)}{|1-\zeta|\Gamma(\alpha+1)}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\frac{L_1(|\zeta|+|1-\zeta|+1)}{|1-\zeta|}+\frac{L_2\sigma^{\alpha}(|1-\zeta|+|\zeta|)}{|1-\zeta|\Gamma(\alpha+1)}\right]\|w-z\| \\
&= \Omega_1\|w-z\|, \qquad (4.6)
\end{aligned}
$$

where Ω_1 is defined by (3.11). By (4.4) and (4.6), we find that

$$
\begin{aligned}
\|w-z\| &= \|w-(\mathcal{F}z)\| \\
&\leq \|w-(\mathcal{F}w)\|+\|(\mathcal{F}w)-(\mathcal{F}z)\| \\
&\leq \epsilon\gamma+\Omega_1\|w-z\|,
\end{aligned}
$$

which can be written as

$$
\|w-z\| \leq \frac{\epsilon\gamma}{1-\Omega_1},
$$

where Ω_1 is given by (3.11). Letting $\nu = \dfrac{\gamma}{1-\Omega_1}$, we get the following:

$$
\|w-z\| \leq \nu\epsilon,
$$

which implies that Problem (1.1) is Ulam-Hyers stable. Moreover, it is generalized Ulam-Hyers stable in $C([0,\sigma],\mathbb{R})$ if we take $\psi = \dfrac{\epsilon\gamma}{1-\Omega_1} = \nu\epsilon$ with $\psi(0)=0$, that is,

$$
\|w-z\| \leq \psi(\epsilon).
$$

This complete the proof. □

Example. Problem (3.3) with p and Φ given in (3.13) is Ulam-Hyers stable as $\Omega_1 \simeq 0.23081730, \gamma = 5\sqrt{2/\pi}$ and $\nu \simeq 5.18657375$; hence, it is generalized Ulam-Hyers stable with $\psi(\epsilon) \simeq 5.18657375\epsilon$.

5. Conclusions

In this work, we analyzed a hybrid fractional two-point boundary value problem with respect to the existence, uniqueness, and Ulam–Hyers stability of solutions. The analysis relies on a Burton-type extension of Krasnosel'skii's fixed point theorem, a newly derived version of Grönwall's inequality introduced herein, and Banach's contraction principle. The theoretical findings are supported by illustrative examples and represent original contribution to the current literature. Moreover, by setting $\zeta = -1$, the obtained results reduce to the case of hybrid fractional differential equations subject to anti-periodic boundary conditions, which, to the best of our knowledge, have not been previously reported.

Author contributions

Rabab Alghamdi: Methodology, formal analysis, investigation, writing-original draft preparation. Bashir Ahmad: Conceptualization, Methodology, formal analysis, investigation, writing-original draft preparation. Sotiris K. Ntouyas: Conceptualization, Methodology, formal analysis, investigation, writing-original draft preparation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The authors thank King Abdulaziz University, Jeddah, Saudi Arabia for technical support.

Conflict of interest

Prof. Sotiris K. Ntouyas is the Guest Editor of special issue "Advances in Fractional Boundary Value Problems and Their Applications" for AIMS Mathematics. Prof. Sotiris K. Ntouyas was not involved in the editorial review and the decision to publish this article.

All authors declare no conflicts of interest in this paper.

References

1. B. C. Dhage, Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations, *Differ. Equ. Appl.*, **2** (2010), 465–486. https://doi.org/10.7153/dea-02-28
2. B. C. Dhage, V. Lakshmikantham, Basic results on hybrid differential equations, *Nonlinear Anal.: Hybrid Syst.*, **4** (2010), 414–424. https://doi.org/10.1016/j.nahs.2009.10.005
3. B. C. Dhage, S. K. Ntouyas, Existence results for boundary value problems for fractional hybrid differential inclusions, *Topol. Methods Nonlinear Anal.*, **44** (2014), 229–238.
4. B. C. Dhage, Existence and attractivity theorems for nonlinear hybrid fractional integrodifferential equations with anticipation and retardation, *Cubo*, **22** (2020), 325–350. https://doi.org/10.4067/s0719-06462020000300325
5. C. Kiataramkul, S. K. Ntouyas, J. Tariboon, Existence results for ψ-Hilfer fractional integro-differential hybrid boundary value problems for differential equations and inclusions, *Adv. Math. Phys.*, **2021** (2021), 9044313. https://doi.org/10.1155/2021/9044313
6. B. Ahmad, S. K. Ntouyas, J. Tariboon, A nonlocal hybrid boundary value problem of Caputo fractional integro-differential equations, *Acta Math. Sci.*, **36** (2016), 1631–1640. https://doi.org/10.1016/s0252-9602(16)30095-9
7. A. Alsaedi, M. Bohner, B. Ahmad, B. Alharbi, On a fully coupled nonlocal multipoint boundary value problem for a dual hybrid system of nonlinear q-fractional differential equations, *Bull. Math. Sci.*, **14** (2024), 2450006. https://doi.org/10.1142/s1664360724500061
8. N. Nyamoradi, B. Ahmad, Positive solutions for hybrid Erdélyi–Kober-type fractional differential equations on an infinite interval via fixed point theory, *Bol. Soc. Mat. Mex.*, **31** (2025), 141. https://doi.org/10.1007/s40590-025-00823-8
9. K. Diethelm, N. J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.*, **265** (2002), 229–248. https://doi.org/10.1006/jmaa.2000.7194
10. K. Diethelm, N. J. Ford, Multi-order fractional differential equations and their numerical solution, *Appl. Math. Comput.*, **154** (2004), 621–640. https://doi.org/10.1016/s0096-3003(03)00739-2

11. B. G. Pachpatte, *Inequalities for differential and integral equations*, Mathematics in Science and Engineering, San Diego: Academic Press, 1998.
12. V. Lakshmikantham, S. Leela, A. A. Martynyuk, *Stability analysis of nonlinear systems (Systems & control: foundations & applications)*, 2 Eds., Birkhäuser, Cham, 2015.
13. D. Baleanu, J. J. Nieto, B. Shiri, Grönwall inequality for incommensurate systems of weakly singular integral equations and its application to FDEs, *Differ. Integral Equ.*, **39** (2026), 1–14. https://doi.org/10.57262/die039-0102-1
14. S. M. Ulam, *A collection of mathematical problems*, Interscience Tracts in Pure and Applied Mathematics, New York-London: Interscience Publishers, 1960.
15. T. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297–300. https://doi.org/10.2307/2042795
16. S. M. Jung, *Hyers-Ulam-Rassias stability of functional equations in mathematical analysis*, Palm Harbor: Hadronic Press, 2011. https://doi.org/10.1007/978-1-4419-9637-4
17. N. Lungu, S. A. Ciplea, Ulam-Hyers stability of Black-Scholes equation, *Stud. Univ. Babes-Bolyai Math.*, **61** (2016), 467–472.
18. B. Ahmad, H. A. Saeed, A. Alsaedi, R. P. Agarwal, On a system of finitely many nonlinear Riemann–Liouville fractional differential equations with fractional anti-periodic boundary conditions, *Georgian Math. J.*, 2025. https://doi.org/10.1515/gmj-2025-2086
19. B. Shiri, Y. G Shi, D. Baleanu, Ulam-Hyers stability of incommensurate systems for weakly singular integral equations, *J. Comput. Appl. Math.*, **474** (2026), 116920. https://doi.org/10.1016/j.cam.2025.116920
20. B. Ahmad, S. K. Ntouyas, *Nonlocal nonlinear fractional-order boundary value problems*, Singapore: World Scientific, 2021. https://doi.org/10.1142/12102
21. B. Shiri, Y. G Shi, D. Baleanu, The Well-Posedness of Incommensurate FDEs in the space of continuous functions, *Symmetry*, **16** (2024), 1058. https://doi.org/10.3390/sym16081058
22. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Amsterdam: Elsevier Science B.V., 2006.
23. T. A. Burton, A fixed-point theorem of Krasnoselskii, *Appl. Math. Lett.*, **11** (1998), 85–88. https://doi.org/10.1016/s0893-9659(97)00138-9
24. A. Granas, J. Dugundji, *Fixed point theory*, New York: Springer-Verlag, 2003. https://doi.org/10.1007/978-0-387-21593-8
25. D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, Berlin-New York: Springer Verlang, 1981. https://doi.org/10.1007/bfb0089647
26. H. Ye, J. Gao, Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.*, **328** (2007), 1075–1081. https://doi.org/10.1016/j.jmaa.2006.05.061

AIMS Press