



Research article

Spectral estimates for multiparametric operator products via the \mathbb{A} -Berezin norm in RKHS

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Abstract: This paper addresses the spectral analysis of operators acting on reproducing kernel Hilbert spaces equipped with a semi-inner product induced by a positive operator \mathbb{A} . A fundamental challenge in this setting is the geometric discrepancy between the normalized reproducing kernels and the unit \mathbb{A} -sphere, which renders classical numerical radius techniques inapplicable. By overcoming this structural obstacle, we establish sharp inequalities for the \mathbb{A} -Berezin number and \mathbb{A} -Berezin norm. Our main contribution involves the derivation of multiparametric estimates for triple operator products of the form $P^\alpha X R^\alpha$ involving Schatten-type exponents. These results generalize and refine existing bounds in the literature. Furthermore, we provide a qualitative analysis of the obtained bounds through weighted Toeplitz operators on Hardy spaces and verify the theoretical findings with concrete matrix examples involving the geometric behavior of weight functions.

Keywords: Berezin number; reproducing kernel Hilbert space; semi-inner product; \mathbb{A} -Berezin number; multiparametric inequalities; Schatten-type exponents

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1. Introduction

Reproducing kernel Hilbert spaces (RKHSs) provide a rigorous framework for numerous problems in operator theory, quantum mechanics, and signal processing. A central concept within this framework is the Berezin symbol, originally introduced by Berezin [1]. This function-theoretic representation

maps linear operators to complex-valued functions on the underlying set, facilitating a deeper understanding of operator properties such as boundedness, compactness, and spectral behavior [2, 3]. The supremum of the absolute value of the Berezin symbol, known as the Berezin number, serves as a refined spectral indicator that complements the operator norm and numerical radius. This concept has been extensively studied in the context of analytic function spaces, including Bergman, Fock, and Hardy spaces [4, 5]. Several lines of work provide Berezin-type estimates for RKHS operators, including inequalities for operator matrices [6], approaches based on Kantorovich and Hölder-McCarthy-type tools [7], and norm-based techniques leading to Berezin radius inequalities [8]. Further refinements for Berezin number inequalities in RKHSs can be found in [9, 10], and Davis-Wielandt-Berezin radius inequalities were investigated in [11].

Recent developments in operator inequalities have focused on refining numerical radius bounds using various function properties and generalized convexity arguments [12, 13]. Several refined \mathbb{A} -numerical radius inequalities in semi-Hilbertian spaces have appeared recently, typically built upon extensions of the \mathbb{A} -Buzano inequality and related convexity tools. Such developments provide sharp bounds for $\omega_{\mathbb{A}}(\cdot)$ and for products/sums of operators in the semi-Hilbertian setting. Our work is complementary in nature: The \mathbb{A} -Berezin number involves a supremum restricted to the manifold of \mathbb{A} -normalized reproducing kernels, which in general forms a proper subset of the unit \mathbb{A} -sphere. This geometric restriction prevents a direct transfer of \mathbb{A} -numerical radius techniques, and it motivates the multiparametric estimates developed here for \mathbb{A} -Berezin quantities; see also [14, 15]. In parallel with these advancements, the focus has shifted towards semi-Hilbertian spaces generated by a positive operator \mathbb{A} . The fundamental theory of semi-inner product spaces and related inequalities has been systematically developed in the influential work of Dragomir [16]. Building on these foundations, the semi-inner product defined by $\langle x, y \rangle_{\mathbb{A}} = \langle \mathbb{A}x, y \rangle$ induces a seminorm structure that generalizes the classical Hilbert space geometry. Pioneering works by Zamani [17] and Feki [18, 19] extended numerical radius inequalities to the semi-Hilbertian setting. Bounds based on extensions of the \mathbb{A} -Buzano inequality were obtained in [15, 20], and inequalities for the \mathbb{A} -norm and \mathbb{A} -numerical radius of operator sums were recently studied in [21]. This evolution naturally led to the introduction of the \mathbb{A} -Berezin number, denoted as $\text{ber}_{\mathbb{A}}(P)$ [22]. Subsequent work developed the functional-Hilbert-space framework and refined \mathbb{A} -Berezin radius inequalities [23, 24], along with further RKHS applications in related settings [25].

However, a significant structural gap remains in the existing literature regarding operator combinations in the \mathbb{A} -Berezin framework. The primary technical difficulty arises from the normalization condition. Although the \mathbb{A} -numerical radius is defined as a supremum over the unit \mathbb{A} -sphere, the \mathbb{A} -Berezin number is restricted to the set of normalized reproducing kernels. Crucially, a kernel function \widehat{u}_{ω} that is normalized in the standard Hilbert norm does not necessarily have a unit \mathbb{A} -norm. This geometric mismatch implies that standard proof techniques utilized for the \mathbb{A} -numerical radius cannot be directly transferred to the Berezin setting. Consequently, precise estimates for sums and products of operators in this context require novel approaches that account for the weight operator \mathbb{A} .

Moreover, the study of operators in semi-Hilbertian spaces has gained renewed interest due to its potential applications in quantum information theory and coherent state analysis. In the context of Berezin-Toeplitz quantization, the Berezin symbol serves as a bridge between classical and quantum mechanics. The introduction of the operator \mathbb{A} allows for the modeling of quantum systems with non-

standard metrics or weighted state spaces, where the geometry is distorted by an observable density matrix. Thus, establishing sharp spectral bounds in this setting provides essential tools for stability analysis in quantum control theory.

The primary objective of this paper is to bridge this gap by establishing a comprehensive set of inequalities for the \mathbb{A} -Berezin number and norm. We employ extensions of the Buzano and mixed Schwarz inequalities to derive sharper bounds that elucidate the interplay between the operator structure and the semi-inner product geometry. A key contribution of our work is the development of multiparametric inequalities for triple-operator products, which encompass many existing results as special cases. We further illustrate the relevance of our theoretical findings through applications to weighted Toeplitz operators on Hardy spaces, showing how the weight function explicitly controls the spectral bounds.

The paper is organized as follows. Section 2 introduces the necessary preliminaries and notation. Section 3 presents our main results, including the multiparametric estimates for operator products and sums. Section 4 provides concrete applications and numerical examples that verify the sharpness of the derived inequalities. Finally, Section 5 concludes the study.

2. Preliminaries

In this section, we establish the notation and recall essential definitions concerning operator theory on semi-Hilbertian spaces. Let \mathcal{H} be a reproducing kernel Hilbert space on a non-empty set F , endowed with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. The space is characterized by the reproducing kernel $k_\omega \in \mathcal{H}$ such that $f(\omega) = \langle f, k_\omega \rangle$ for all $f \in \mathcal{H}$ and $\omega \in F$. We assume $k_\omega \neq 0$ for all $\omega \in F$. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on \mathcal{H} .

We consider a nonzero positive operator $\mathbb{A} \in \mathcal{B}(\mathcal{H})$. This operator induces a semi-inner product defined by $\langle x, y \rangle_{\mathbb{A}} = \langle \mathbb{A}x, y \rangle$ for $x, y \in \mathcal{H}$, with the associated seminorm $\|x\|_{\mathbb{A}} = \langle x, x \rangle_{\mathbb{A}}^{1/2} = \|\mathbb{A}^{1/2}x\|$. The space $(\mathcal{H}, \| \cdot \|_{\mathbb{A}})$ is referred to as a semi-Hilbertian space (see [18, 26]). An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be \mathbb{A} -positive (denoted $P \geq_{\mathbb{A}} 0$) if $\mathbb{A}P \geq 0$.

For an operator $P \in \mathcal{B}(\mathcal{H})$, the \mathbb{A} -adjoint of P , denoted by $P^{\sharp_{\mathbb{A}}}$, is an operator satisfying $\langle Px, y \rangle_{\mathbb{A}} = \langle x, P^{\sharp_{\mathbb{A}}}y \rangle_{\mathbb{A}}$ for all $x, y \in \mathcal{H}$. By Douglas's theorem [27], the existence of $P^{\sharp_{\mathbb{A}}}$ is equivalent to the range inclusion $\mathcal{R}(P^*\mathbb{A}) \subseteq \mathcal{R}(\mathbb{A})$. The set of all operators admitting an \mathbb{A} -adjoint is denoted by $\mathcal{B}_{\mathbb{A}}(\mathcal{H})$. For $P \in \mathcal{B}_{\mathbb{A}}(\mathcal{H})$, the distinguished solution to the adjoint equation is $P^{\sharp_{\mathbb{A}}} = \mathbb{A}^{\dagger}P^*\mathbb{A}$, where \mathbb{A}^{\dagger} denotes the Moore-Penrose inverse. We also define the \mathbb{A} -operator seminorm for $P \in \mathcal{B}_{\mathbb{A}^{1/2}}(\mathcal{H})$ as

$$\|P\|_{\mathbb{A}} = \sup_{\substack{x \in \mathcal{H} \\ \|x\|_{\mathbb{A}}=1}} \|Px\|_{\mathbb{A}}.$$

It is well-known that for $P \in \mathcal{B}_{\mathbb{A}}(\mathcal{H})$, we have $\|P\|_{\mathbb{A}}^2 = \|P^{\sharp_{\mathbb{A}}}P\|_{\mathbb{A}}$.

In the context of RKHSs, the spectral analysis is often conducted via the Berezin symbol. To adapt this to the semi-Hilbertian setting, we introduce the \mathbb{A} -normalized reproducing kernel defined by

$$\widehat{u}_\omega = \frac{k_\omega}{\|k_\omega\|_{\mathbb{A}}}, \quad \text{provided } \|k_\omega\|_{\mathbb{A}} \neq 0.$$

Using this normalization, the \mathbb{A} -Berezin symbol of an operator $P \in \mathcal{B}(\mathcal{H})$ is the function $\widetilde{P}_{\mathbb{A}} : F \rightarrow \mathbb{C}$ given by $\widetilde{P}_{\mathbb{A}}(\omega) = \langle P\widehat{u}_\omega, \widehat{u}_\omega \rangle_{\mathbb{A}}$. Consequently, the \mathbb{A} -Berezin number and \mathbb{A} -Berezin norm are defined,

respectively, as

$$\text{ber}_{\mathbb{A}}(\mathbf{P}) = \sup_{\omega \in F} |\langle \mathbf{P}\widehat{u}_{\omega}, \widehat{u}_{\omega} \rangle_{\mathbb{A}}| \quad \text{and} \quad \|\mathbf{P}\|_{\mathbb{A}\text{-Ber}} = \sup_{\omega \in F} \|\mathbf{P}\widehat{u}_{\omega}\|_{\mathbb{A}}.$$

It is important to emphasize that although $\|\widehat{u}_{\omega}\|_{\mathbb{A}} = 1$ by definition, the set $\{\widehat{u}_{\omega} : \omega \in F\}$ is generally a proper subset of the unit \mathbb{A} -sphere. This distinction prevents the direct application of numerical radius inequalities to the Berezin setting.

To derive our multiparametric estimates, we require the following geometric inequalities extended to the semi-inner product framework.

Lemma 2.1. (*\mathbb{A} -Cauchy-Schwarz*) For any $x, y \in \mathcal{H}$, the inequality $|\langle x, y \rangle_{\mathbb{A}}| \leq \|x\|_{\mathbb{A}}\|y\|_{\mathbb{A}}$ holds.

Lemma 2.2. [28] Let $x, y, e \in \mathcal{H}$ with $\|e\|_{\mathbb{A}} = 1$. Then,

$$|\langle x, e \rangle_{\mathbb{A}}\langle e, y \rangle_{\mathbb{A}}| \leq \frac{1}{2} (\|x\|_{\mathbb{A}}\|y\|_{\mathbb{A}} + |\langle x, y \rangle_{\mathbb{A}}|).$$

The following lemma provides a generalized refinement of the Buzano inequality, which is crucial for our multiparametric results.

Lemma 2.3. [21] Let $a, b, e \in \mathcal{H}$ with $\|e\|_{\mathbb{A}} = 1$. For any $\alpha \in [0, 1]$ and $r \geq 1$, the following inequality holds:

$$|\langle a, e \rangle_{\mathbb{A}}\langle e, b \rangle_{\mathbb{A}}|^r \leq \left(\frac{1+\alpha}{2}\right) \|a\|_{\mathbb{A}}^r \|b\|_{\mathbb{A}}^r + \left(\frac{1-\alpha}{2}\right) |\langle a, b \rangle_{\mathbb{A}}|^r.$$

We also recall the following convexity inequalities for scalar arguments.

Lemma 2.4. [29] Let $a, b \geq 0$, $\alpha \in [0, 1]$, and $p, q > 1$ such that $1/p + 1/q = 1$. Then:

- (i) $a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b \leq (\alpha a^r + (1-\alpha)b^r)^{1/r}$ for any $r \geq 1$.
- (ii) (Young's inequality)

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq \left(\frac{a^p}{p} + \frac{b^q}{q}\right)^{1/r},$$

for any $r \geq 1$.

3. Main results

In this section, we present our primary findings concerning the spectral estimation of operators in semi-Hilbertian spaces. We deviate from the traditional approach of starting with elementary sum inequalities. Instead, we first establish general multiparametric estimates for operator products involving Schatten-type exponents. These results provide a unifying framework that encompasses several existing inequalities as special cases. Subsequently, we derive refined bounds for operator sums and the \mathbb{A} -Berezin radius by utilizing the geometric properties of the \mathbb{A} -normalized reproducing kernels.

Throughout Section 3, unless otherwise stated, we work with operators admitting an \mathbb{A} -adjoint, that is, $T \in \mathcal{B}_{\mathbb{A}}(\mathcal{H})$, and with $\mathbb{A}^{1/2}$ -bounded operators $X \in \mathcal{B}_{\mathbb{A}^{1/2}}(\mathcal{H})$ so that $\|Xx\|_{\mathbb{A}} \leq \|X\|_{\mathbb{A}}\|x\|_{\mathbb{A}}$ for all $x \in \mathcal{H}$. Whenever we say that an operator is positive, it is meant in the usual Hilbert space order.

3.1. Multiparametric estimates for operator products

Our first major result establishes a weighted estimate for the triple product of operators. This theorem addresses the spectral behavior of the composite operator sum $\sum P_j^\alpha X_j R_j^\alpha$ and highlights the role of the weight operator \mathbb{A} in controlling the Berezin number.

Theorem 3.1. *Let $P_j, R_j \in \mathcal{B}_{\mathbb{A}}(\mathcal{H})$ be positive (self-adjoint) operators, and let $X_j \in \mathcal{B}_{\mathbb{A}^{1/2}}(\mathcal{H})$ for $j = 1, \dots, n$. Let $r \geq 1$ and $\alpha \in [0, 1]$. If $p_j, q_j > 1$ are real numbers satisfying $1/p_j + 1/q_j = 1$, then the inequality*

$$\text{ber}_{\mathbb{A}}^r \left(\sum_{j=1}^n P_j^\alpha X_j R_j^\alpha \right) \leq n^{r-1} \|X\|_{\mathbb{A}}^r \sum_{j=1}^n \left[\frac{1}{p_j} \|P_j^\alpha\|_{\mathbb{A}\text{-Ber}}^{p_j r} + \frac{1}{q_j} \|R_j^\alpha\|_{\mathbb{A}\text{-Ber}}^{q_j r} \right] \quad (3.1)$$

holds, where $\|X\|_{\mathbb{A}} = \max_{1 \leq j \leq n} \|X_j\|_{\mathbb{A}}$.

Proof. Fix $\omega \in F$ such that $\|\widehat{u}_\omega\|_{\mathbb{A}} = 1$. Because P_j and R_j are positive operators, their fractional powers are well-defined via the continuous functional calculus. We begin by estimating the inner product of the operator sum:

$$\left| \left\langle \left(\sum_{j=1}^n P_j^\alpha X_j R_j^\alpha \right) \widehat{u}_\omega, \widehat{u}_\omega \right\rangle_{\mathbb{A}} \right| = \left| \sum_{j=1}^n \langle X_j R_j^\alpha \widehat{u}_\omega, P_j^\alpha \widehat{u}_\omega \rangle_{\mathbb{A}} \right|.$$

Using the definition of the \mathbb{A} -adjoint and the generalized Cauchy-Schwarz inequality, followed by the property of the \mathbb{A} -operator seminorm (valid since $X_j \in \mathcal{B}_{\mathbb{A}^{1/2}}(\mathcal{H})$), we obtain

$$\left| \sum_{j=1}^n \langle X_j R_j^\alpha \widehat{u}_\omega, P_j^\alpha \widehat{u}_\omega \rangle_{\mathbb{A}} \right| \leq \sum_{j=1}^n \|X_j\|_{\mathbb{A}} \|R_j^\alpha \widehat{u}_\omega\|_{\mathbb{A}} \|P_j^\alpha \widehat{u}_\omega\|_{\mathbb{A}}.$$

Factorizing the maximum norm $\|X\|_{\mathbb{A}}$, we arrive at

$$\left| \left\langle \sum_{j=1}^n P_j^\alpha X_j R_j^\alpha \widehat{u}_\omega, \widehat{u}_\omega \right\rangle_{\mathbb{A}} \right| \leq \|X\|_{\mathbb{A}} \sum_{j=1}^n \|P_j^\alpha \widehat{u}_\omega\|_{\mathbb{A}} \|R_j^\alpha \widehat{u}_\omega\|_{\mathbb{A}}.$$

Let $a_j = \|P_j^\alpha \widehat{u}_\omega\|_{\mathbb{A}}$ and $b_j = \|R_j^\alpha \widehat{u}_\omega\|_{\mathbb{A}}$. Taking the r th power of both sides and applying the convexity inequality $(\sum_{j=1}^n x_j)^r \leq n^{r-1} \sum_{j=1}^n x_j^r$ (for $x_j \geq 0, r \geq 1$), we get

$$\left(\sum_{j=1}^n a_j b_j \right)^r \leq n^{r-1} \sum_{j=1}^n (a_j b_j)^r = n^{r-1} \sum_{j=1}^n a_j^r b_j^r.$$

Now, we apply the scalar Young's inequality to the product $a_j^r b_j^r$ with conjugate exponents p_j and q_j :

$$a_j^r b_j^r \leq \frac{1}{p_j} (a_j^r)^{p_j} + \frac{1}{q_j} (b_j^r)^{q_j} = \frac{1}{p_j} a_j^{p_j r} + \frac{1}{q_j} b_j^{q_j r}.$$

Combining these steps yields

$$\left| \left\langle \sum_{j=1}^n P_j^\alpha X_j R_j^\alpha \widehat{u}_\omega, \widehat{u}_\omega \right\rangle_{\mathbb{A}} \right|^r \leq n^{r-1} \|X\|_{\mathbb{A}}^r \sum_{j=1}^n \left(\frac{1}{p_j} \|P_j^\alpha \widehat{u}_\omega\|_{\mathbb{A}}^{p_j r} + \frac{1}{q_j} \|R_j^\alpha \widehat{u}_\omega\|_{\mathbb{A}}^{q_j r} \right).$$

Finally, using the definition of the \mathbb{A} -Berezin norm, that is, $\|\widehat{T u}_\omega\|_{\mathbb{A}} \leq \|T\|_{\mathbb{A}\text{-Ber}}$, and taking the supremum over $\omega \in F$, we confirm the inequality (3.1).

Remark 3.1. It is important to highlight the generality of Theorem 3.1. In the specific case where $\mathbb{A} = I$ (the identity operator), our result recovers the classical Berezin number estimates for operator products in standard RKHSs. Furthermore, if we restrict the exponents to $p = q = 2$ and $\alpha = 1$, inequality (3.1) yields a weighted analogue of the Hilbert-Schmidt norm bounds. Thus, Theorem 3.1 not only extends existing results to the semi-Hilbertian setting but also refines them by introducing the flexibility of Schatten-type exponents p and q .

An important consequence of Theorem 3.1 arises in the symmetric case where the Schatten exponents are equal.

Corollary 3.1. Under the assumptions of Theorem 3.1, if we take $r = 1$, $\alpha = 1$, and $p_j = q_j = 2$ for all $j = 1, \dots, n$, then the inequality

$$\text{ber}_{\mathbb{A}} \left(\sum_{j=1}^n P_j X_j R_j \right) \leq \|X\|_{\mathbb{A}} \sum_{j=1}^n \left(\frac{1}{2} \|P_j\|_{\mathbb{A}\text{-Ber}}^2 + \frac{1}{2} \|R_j\|_{\mathbb{A}\text{-Ber}}^2 \right)$$

holds.

We next investigate the mixed-power case, which is particularly relevant for analyzing the asymmetry between left and right multiplication operators.

Theorem 3.2. Let $P_j, X_j, R_j \in \mathcal{B}_{\mathbb{A}}(\mathcal{H})$ for $j = 1, \dots, n$. For any $\alpha \in [0, 1]$ and conjugate exponents $p, q > 1$ (i.e., $1/p + 1/q = 1$), the inequality

$$\text{ber}_{\mathbb{A}} \left(\sum_{j=1}^n P_j X_j R_j^{1-\alpha} \right) \leq n^{1-\frac{1}{p}} \|X\|_{\mathbb{A}} \sum_{j=1}^n \left[\frac{1}{p} \|P_j^{\sharp_{\mathbb{A}}}\|_{\mathbb{A}\text{-Ber}}^p + \frac{1}{q} \|R_j^{1-\alpha}\|_{\mathbb{A}\text{-Ber}}^q \right] \quad (3.2)$$

holds, where $\|X\|_{\mathbb{A}} = \max_{1 \leq j \leq n} \|X_j\|_{\mathbb{A}}$.

Proof. Let $\omega \in F$ with $\|\widehat{u}_{\omega}\|_{\mathbb{A}} = 1$. By the definition of the \mathbb{A} -adjoint and the triangle inequality, we write

$$\left| \left\langle \left(\sum_{j=1}^n P_j X_j R_j^{1-\alpha} \right) \widehat{u}_{\omega}, \widehat{u}_{\omega} \right\rangle_{\mathbb{A}} \right| \leq \sum_{j=1}^n \left| \langle X_j R_j^{1-\alpha} \widehat{u}_{\omega}, P_j^{\sharp_{\mathbb{A}}} \widehat{u}_{\omega} \rangle_{\mathbb{A}} \right|.$$

Using the \mathbb{A} -Cauchy-Schwarz inequality and the operator norm property $\|X_j y\|_{\mathbb{A}} \leq \|X_j\|_{\mathbb{A}} \|y\|_{\mathbb{A}}$, we have

$$\left| \langle X_j R_j^{1-\alpha} \widehat{u}_{\omega}, P_j^{\sharp_{\mathbb{A}}} \widehat{u}_{\omega} \rangle_{\mathbb{A}} \right| \leq \|X_j\|_{\mathbb{A}} \|R_j^{1-\alpha} \widehat{u}_{\omega}\|_{\mathbb{A}} \|P_j^{\sharp_{\mathbb{A}}} \widehat{u}_{\omega}\|_{\mathbb{A}}.$$

Factoring out $\|X\|_{\mathbb{A}}$ and applying the scalar Young's inequality $ab \leq a^p/p + b^q/q$, we obtain

$$\|P_j^{\sharp_{\mathbb{A}}} \widehat{u}_{\omega}\|_{\mathbb{A}} \|R_j^{1-\alpha} \widehat{u}_{\omega}\|_{\mathbb{A}} \leq \frac{1}{p} \|P_j^{\sharp_{\mathbb{A}}} \widehat{u}_{\omega}\|_{\mathbb{A}}^p + \frac{1}{q} \|R_j^{1-\alpha} \widehat{u}_{\omega}\|_{\mathbb{A}}^q.$$

Applying the Hölder inequality for sums (introducing the factor $n^{1-1/p}$) and using the definition of the \mathbb{A} -Berezin norm (i.e., $\|\widehat{T u}_{\omega}\|_{\mathbb{A}} \leq \|T\|_{\mathbb{A}\text{-Ber}}$), we derive

$$\text{ber}_{\mathbb{A}} \left(\sum_{j=1}^n P_j X_j R_j^{1-\alpha} \right) \leq n^{1-\frac{1}{p}} \|X\|_{\mathbb{A}} \sum_{j=1}^n \left(\frac{1}{p} \|P_j^{\sharp_{\mathbb{A}}}\|_{\mathbb{A}\text{-Ber}}^p + \frac{1}{q} \|R_j^{1-\alpha}\|_{\mathbb{A}\text{-Ber}}^q \right).$$

This completes the proof.

3.2. Refined estimates via geometric inequalities

Whereas the previous subsection focused on operator products, we now turn our attention to the structural relationship between the \mathbb{A} -Berezin number and the \mathbb{A} -Berezin norm. A key tool in this analysis is the extension of the Buzano inequality, which allows us to derive tighter bounds than those obtained via the standard Cauchy-Schwarz inequality.

The following theorem provides a sharp estimate for the square of the \mathbb{A} -Berezin number, incorporating a parameter α that refines the bound.

Theorem 3.3. *Let $P \in \mathcal{B}_{\mathbb{A}}(\mathcal{H})$. Then, for every $\alpha \in [0, 1]$, the inequality*

$$\text{ber}_{\mathbb{A}}^2(P) \leq \frac{1 + \alpha}{4} \left(\|P\|_{\mathbb{A}\text{-Ber}}^2 + \|P^{\sharp_{\mathbb{A}}}\|_{\mathbb{A}\text{-Ber}}^2 \right) + \frac{1 - \alpha}{2} \text{ber}_{\mathbb{A}}(P^2) \quad (3.3)$$

holds. In particular, taking the maximum over the norms yields

$$\text{ber}_{\mathbb{A}}^2(P) \leq \frac{1 + \alpha}{2} \max \{ \|P\|_{\mathbb{A}\text{-Ber}}^2, \|P^{\sharp_{\mathbb{A}}}\|_{\mathbb{A}\text{-Ber}}^2 \} + \frac{1 - \alpha}{2} \text{ber}_{\mathbb{A}}(P^2).$$

Proof. Fix $\omega \in F$ with $\|\widehat{u}_{\omega}\|_{\mathbb{A}} = 1$. We apply Lemma 2.3 (the generalized Buzano inequality) with $a = P\widehat{u}_{\omega}$, $b = P^{\sharp_{\mathbb{A}}}\widehat{u}_{\omega}$, and $e = \widehat{u}_{\omega}$. Observe that

$$\langle a, e \rangle_{\mathbb{A}} \langle e, b \rangle_{\mathbb{A}} = \langle P\widehat{u}_{\omega}, \widehat{u}_{\omega} \rangle_{\mathbb{A}} \langle \widehat{u}_{\omega}, P^{\sharp_{\mathbb{A}}}\widehat{u}_{\omega} \rangle_{\mathbb{A}}.$$

Using the definition of the adjoint, $\langle \widehat{u}_{\omega}, P^{\sharp_{\mathbb{A}}}\widehat{u}_{\omega} \rangle_{\mathbb{A}} = \langle P\widehat{u}_{\omega}, \widehat{u}_{\omega} \rangle_{\mathbb{A}}$. Thus, the left-hand side becomes $|\langle P\widehat{u}_{\omega}, \widehat{u}_{\omega} \rangle_{\mathbb{A}}|^2$.

For the cross term $|\langle a, b \rangle_{\mathbb{A}}|$, we utilize the adjoint property again:

$$|\langle a, b \rangle_{\mathbb{A}}| = |\langle P\widehat{u}_{\omega}, P^{\sharp_{\mathbb{A}}}\widehat{u}_{\omega} \rangle_{\mathbb{A}}| = |\langle P(P\widehat{u}_{\omega}), \widehat{u}_{\omega} \rangle_{\mathbb{A}}| = |\langle P^2\widehat{u}_{\omega}, \widehat{u}_{\omega} \rangle_{\mathbb{A}}|.$$

By definition, this term is bounded by the Berezin number of the square of the operator:

$$|\langle P^2\widehat{u}_{\omega}, \widehat{u}_{\omega} \rangle_{\mathbb{A}}| \leq \text{ber}_{\mathbb{A}}(P^2).$$

Applying Lemma 2.3, we obtain

$$|\langle P\widehat{u}_{\omega}, \widehat{u}_{\omega} \rangle_{\mathbb{A}}|^2 \leq \frac{1 + \alpha}{2} \|P\widehat{u}_{\omega}\|_{\mathbb{A}} \|P^{\sharp_{\mathbb{A}}}\widehat{u}_{\omega}\|_{\mathbb{A}} + \frac{1 - \alpha}{2} \text{ber}_{\mathbb{A}}(P^2).$$

Using the elementary inequality $xy \leq (x^2 + y^2)/2$ for the product of norms and noting that $\|\widehat{T}\widehat{u}_{\omega}\|_{\mathbb{A}} \leq \|T\|_{\mathbb{A}\text{-Ber}}$, we arrive at

$$\|P\widehat{u}_{\omega}\|_{\mathbb{A}} \|P^{\sharp_{\mathbb{A}}}\widehat{u}_{\omega}\|_{\mathbb{A}} \leq \frac{1}{2} \left(\|P\|_{\mathbb{A}\text{-Ber}}^2 + \|P^{\sharp_{\mathbb{A}}}\|_{\mathbb{A}\text{-Ber}}^2 \right).$$

Substituting this back and taking the supremum over $\omega \in F$ yields (3.3).

We now derive upper bounds for the \mathbb{A} -Berezin norm of operator sums. These inequalities generalize the classical Euclidean norm bounds to the semi-Hilbertian RKHS setting.

Theorem 3.4. *For any $P_1, \dots, P_n \in \mathcal{B}_{\mathbb{A}}(\mathcal{H})$, we have*

$$\left\| \sum_{k=1}^n P_k \right\|_{\mathbb{A}\text{-Ber}}^2 \leq \left\| \sum_{k=1}^n P_k^{\sharp_{\mathbb{A}}} P_k \right\|_{\mathbb{A}\text{-Ber}} + \frac{1}{2} \left\| (n-2) \sum_{k=1}^n P_k^{\sharp_{\mathbb{A}}} P_k + \left(\sum_{k=1}^n P_k^{\sharp_{\mathbb{A}}} \right) \left(\sum_{k=1}^n P_k \right) \right\|_{\mathbb{A}\text{-Ber}}.$$

Proof. Let $\omega \in F$ with $\|\widehat{u}_\omega\|_{\mathbb{A}} = 1$. We start with the identity

$$\left\| \sum_{k=1}^n P_k \widehat{u}_\omega \right\|_{\mathbb{A}}^2 = \sum_{k=1}^n \|P_k \widehat{u}_\omega\|_{\mathbb{A}}^2 + \sum_{j \neq k} \operatorname{Re} \langle P_k \widehat{u}_\omega, P_j \widehat{u}_\omega \rangle_{\mathbb{A}}.$$

Utilizing the inequality $\operatorname{Re} \langle a, b \rangle_{\mathbb{A}} \leq \|a + b\|_{\mathbb{A}}^2 / 4$, we bound the cross terms by

$$\sum_{j \neq k} \operatorname{Re} \langle P_k \widehat{u}_\omega, P_j \widehat{u}_\omega \rangle_{\mathbb{A}} \leq \frac{1}{4} \sum_{j \neq k} \langle (P_k^{\sharp_{\mathbb{A}}} + P_j^{\sharp_{\mathbb{A}}})(P_k + P_j) \widehat{u}_\omega, \widehat{u}_\omega \rangle_{\mathbb{A}}.$$

Algebraic expansion of the double sum leads to the operator combination

$$(n-2) \sum_{k=1}^n P_k^{\sharp_{\mathbb{A}}} P_k + \left(\sum_{k=1}^n P_k^{\sharp_{\mathbb{A}}} \right) \left(\sum_{k=1}^n P_k \right).$$

Combining the diagonal and off-diagonal estimates and taking the supremum over ω completes the proof.

Finally, we establish a fundamental inequality relating the square of the \mathbb{A} -Berezin norm to the \mathbb{A} -Berezin number of the operator's \mathbb{A} -modulus.

Theorem 3.5. *Let $P \in \mathcal{B}_{\mathbb{A}}(\mathcal{H})$. Then, the inequality*

$$\|P\|_{\mathbb{A}\text{-Ber}}^2 \leq \operatorname{ber}_{\mathbb{A}}(P^{\sharp_{\mathbb{A}}} P)$$

holds. Consequently, if P is \mathbb{A} -normal (i.e., $P^{\sharp_{\mathbb{A}}} P = P P^{\sharp_{\mathbb{A}}}$), we have

$$\|P\|_{\mathbb{A}\text{-Ber}}^2 \leq \|P^{\sharp_{\mathbb{A}}} P\|_{\mathbb{A}\text{-Ber}}.$$

Proof. Let $\omega \in F$ with $\|\widehat{u}_\omega\|_{\mathbb{A}} = 1$. Using the definition of the \mathbb{A} -norm and the \mathbb{A} -adjoint, we have

$$\|P \widehat{u}_\omega\|_{\mathbb{A}}^2 = \langle P \widehat{u}_\omega, P \widehat{u}_\omega \rangle_{\mathbb{A}} = \langle P^{\sharp_{\mathbb{A}}} P \widehat{u}_\omega, \widehat{u}_\omega \rangle_{\mathbb{A}}.$$

Taking the modulus, we obtain

$$\|P \widehat{u}_\omega\|_{\mathbb{A}}^2 = |\langle P^{\sharp_{\mathbb{A}}} P \widehat{u}_\omega, \widehat{u}_\omega \rangle_{\mathbb{A}}|.$$

By the definition of the \mathbb{A} -Berezin number, the right-hand side is bounded by $\operatorname{ber}_{\mathbb{A}}(P^{\sharp_{\mathbb{A}}} P)$. Thus,

$$\|P \widehat{u}_\omega\|_{\mathbb{A}}^2 \leq \operatorname{ber}_{\mathbb{A}}(P^{\sharp_{\mathbb{A}}} P).$$

Taking the supremum over all $\omega \in F$ on the left-hand side yields the desired result,

$$\|P\|_{\mathbb{A}\text{-Ber}}^2 \leq \operatorname{ber}_{\mathbb{A}}(P^{\sharp_{\mathbb{A}}} P).$$

Remark 3.2. *It should be noted that although the classical numerical radius satisfies the norm inequality $\|T\| \leq 2w(T)$, a similar universal lower bound of the form $\|T\|_{\mathbb{A}\text{-Ber}} \leq C \operatorname{ber}_{\mathbb{A}}(T)$ does not generally hold in the RKHS setting. This is due to the fact that the set of normalized reproducing kernels is generally a proper subset of the unit sphere, and polarization identities cannot be restricted to this manifold. Theorem 3.5 provides a rigorous alternative by bounding the Berezin norm in terms of the Berezin number of the operator's modulus.*

4. Applications and numerical examples

In this section, we apply the theoretical results established in Section 3 to concrete operator classes acting on reproducing kernel Hilbert spaces. We specifically focus on weighted Hardy and Bergman spaces to illustrate how the weight parameter influences the \mathbb{A} -Berezin bounds. Furthermore, we provide matrix examples to numerically verify the multiparametric inequalities.

4.1. Weighted shifts on Bergman spaces

Consider the standard Bergman space $\mathcal{H} = L_a^2(\mathbb{D})$ on the unit disk, equipped with the reproducing kernel $k_\omega(z) = (1 - \bar{\omega}z)^{-2}$. Let the weight operator \mathbb{A} be the Toeplitz operator T_g with symbol $g(z) = (1 - |z|^2)^\beta$ where $\beta \geq 0$. Because g is nonnegative, T_g is a positive operator, providing a valid semi-inner product framework.

We analyze the spectral behavior of the forward shift operator $P = M_z$. Although the exact computation of the \mathbb{A} -Berezin number in this context involves complex hypergeometric series, the qualitative impact of the weight parameter β can be effectively analyzed through a simplified radial model. The interplay between the growth of the shift operator (proportional to $r = |z|$) and the boundary decay of the weight suggests the study of the spectral profile function:

$$h_\beta(r) = r(1 - r^2)^\beta, \quad r \in [0, 1).$$

This function serves as a heuristic proxy for the \mathbb{A} -Berezin integrand, capturing the geometric distortion introduced by the metric. For the unweighted case ($\beta = 0$), the model yields $h_0(r) = r$, consistent with the fact that the shift operator M_z has spectral radius 1, and its spectral profile is boundary dominated. For $\beta > 0$, elementary calculus shows that the maximum occurs at the critical radius $r^* = (1 + 2\beta)^{-1/2}$. Substituting this back yields the maximal value:

$$\max_{r \in [0, 1)} h_\beta(r) = \frac{1}{\sqrt{1 + 2\beta}} \left(\frac{2\beta}{1 + 2\beta} \right)^\beta.$$

Note that the expression is continuous in the limit $\beta \rightarrow 0^+$, consistently yielding $\max h_0 = 1$. This result provides qualitative evidence for the damping effect of the weight: as β increases, the maximum value decreases and shifts towards the origin. This suggests that the operator \mathbb{A} acts as a geometric filter, restricting the effective spectral radius of the shift operator.

To visualize this phenomenon, we plot the profile $h_\beta(r)$ for various values of β in Figure 1.

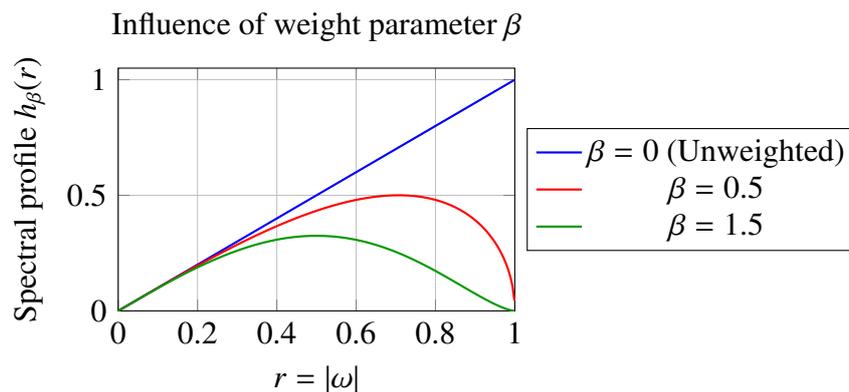


Figure 1. The behavior of the heuristic spectral function $h_\beta(r) = r(1-r^2)^\beta$ for different weight parameters. In the unweighted case ($\beta = 0$), the profile exhibits linear growth approaching the boundary, whereas for $\beta > 0$, the peak shifts inward, illustrating the geometric distortion.

4.2. Verification of multiparametric inequalities

We now verify the sharpness of the inequality derived in Corollary 3.1 using a finite-dimensional model. Let $\mathcal{H} = \mathbb{C}^2$, and consider the positive operators:

$$P_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbb{A} = I.$$

We test the inequality for the sum $S = \sum_{j=1}^2 P_j X P_j$ (setting $R_j = P_j$). Here, $\|X\|_{\mathbb{A}} = 1$.

For the left-hand side, the operator sum is $S = P_1^2 + P_2^2$. Computing the squares, we get $P_1^2 = \text{diag}(4, 1)$ and $P_2^2 = \text{diag}(1, 9)$. Thus, $S = \text{diag}(5, 10)$, and consequently $\text{ber}_{\mathbb{A}}(S) = 10$.

Turning to the right-hand side estimation provided by Corollary 3.1, the bound becomes $\sum_{j=1}^2 \|P_j\|_{\mathbb{A}\text{-Ber}}^2$. Because the matrices are diagonal and positive, $\|P_1\|_{\mathbb{A}\text{-Ber}} = 2$, and $\|P_2\|_{\mathbb{A}\text{-Ber}} = 3$. Therefore, the upper bound is

$$2^2 + 3^2 = 4 + 9 = 13.$$

The inequality $10 \leq 13$ is thus verified. The observed gap between the exact value (10) and the upper bound (13) quantifies the geometric distortion introduced by the component-wise estimation. Although the bound is naturally conservative for commuting diagonal matrices, it provides a necessary uniform upper estimate for the general class of noncommuting operators where exact spectral computation is intractable.

A noncommuting 2×2 example.

Consider again the discrete model $\Omega = \{1, 2\}$ with $k_1 = e_1$ and $k_2 = e_2$ in $\mathcal{H} = \mathbb{C}^2$, and take $\mathbb{A} = I$. Let

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, P and R are positive, and moreover, $PX \neq XP$. A direct computation gives

$$PXR = \begin{bmatrix} 1 & 6 \\ 2 & 3 \end{bmatrix}, \quad \text{ber}_{\mathbb{A}}(PXR) = \max\{|1|, |3|\} = 3 \quad (\mathbb{A} = I).$$

Furthermore,

$$\begin{aligned}\|P\|_{\mathbb{A}\text{-Ber}} &= \max\{\|(2, 1)^T\|, \|(1, 2)^T\|\} = \sqrt{5}, \\ \|R\|_{\mathbb{A}\text{-Ber}} &= \max\{\|(1, 0)^T\|, \|(0, 3)^T\|\} = 3.\end{aligned}$$

Therefore,

$$\frac{1}{2}\|P\|_{\mathbb{A}\text{-Ber}}^2 + \frac{1}{2}\|R\|_{\mathbb{A}\text{-Ber}}^2 = \frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 9 = 7.$$

Because $\|X\| = 1$, Corollary 3.1 yields

$$\text{ber}(\text{PXR}) \leq \|X\| \left(\frac{1}{2}\|P\|_{\mathbb{B}\text{er}}^2 + \frac{1}{2}\|R\|_{\mathbb{B}\text{er}}^2 \right) = 7,$$

which confirms the bound in a genuinely noncommuting setting.

4.3. Geometric discrepancy in matrix weights

Finally, to illustrate the structural difference between the classical adjoint and the \mathbb{A} -adjoint which motivates our use of the Buzano inequality, we consider the matrices

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbb{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

A direct computation of the \mathbb{A} -adjoint yields $T^{\sharp\mathbb{A}} = \mathbb{A}^{-1}T^*\mathbb{A} = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}$. The stark contrast between the nilpotent operator T and the resulting $T^{\sharp\mathbb{A}}$ confirms that the geometry of the \mathbb{A} -space drastically alters the operator's properties. This transformation is visualized in the heatmaps presented in Figure 2.

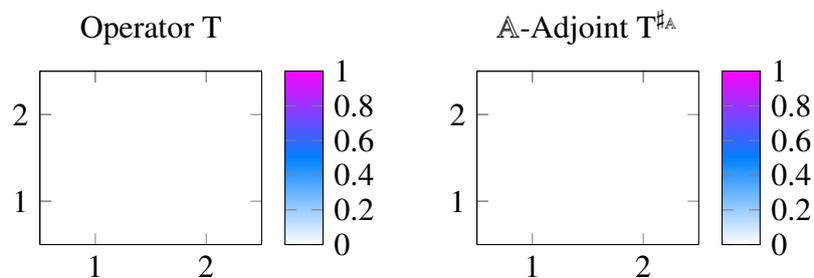


Figure 2. Comparison of the operator T and its \mathbb{A} -adjoint. The nontrivial structure of $T^{\sharp\mathbb{A}}$ necessitates the advanced estimation techniques developed in Section 3.

To provide a quantitative verification of Theorem 3.3 in a finite-dimensional RKHS model, consider $\Omega = \{1, 2\}$ with reproducing kernels $k_1 = e_1$ and $k_2 = e_2$ in $\mathcal{H} = \mathbb{C}^2$. Then, $\|e_1\|_{\mathbb{A}}^2 = \langle \mathbb{A}e_1, e_1 \rangle = 2$, and $\|e_2\|_{\mathbb{A}}^2 = \langle \mathbb{A}e_2, e_2 \rangle = 1$; hence, $\widehat{u}_1 = e_1/\sqrt{2}$, and $\widehat{u}_2 = e_2$. Moreover, because $T\widehat{u}_1 = 0$, and $T\widehat{u}_2 = e_1$, we obtain

$$\langle T\widehat{u}_1, \widehat{u}_1 \rangle_{\mathbb{A}} = 0, \quad \langle T\widehat{u}_2, \widehat{u}_2 \rangle_{\mathbb{A}} = \langle e_1, e_2 \rangle_{\mathbb{A}} = \langle \mathbb{A}e_1, e_2 \rangle = 1,$$

and hence, $\text{ber}_{\mathbb{A}}(T) = 1$. Furthermore,

$$\|T\|_{\mathbb{A}\text{-Ber}} = \max\{\|T\widehat{u}_1\|_{\mathbb{A}}, \|T\widehat{u}_2\|_{\mathbb{A}}\} = \|e_1\|_{\mathbb{A}} = \sqrt{2}.$$

Using $T^{\sharp_{\mathbb{A}}} = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}$, we also have $T^{\sharp_{\mathbb{A}}}\widehat{u}_2 = (-1, 2)^{\top}$ with $\|(-1, 2)\|_{\mathbb{A}} = \sqrt{2}$ and $T^{\sharp_{\mathbb{A}}}\widehat{u}_1 = (-2, 4)^{\top} / \sqrt{2}$ with $\|(-2, 4) / \sqrt{2}\|_{\mathbb{A}} = 2$, so $\|T^{\sharp_{\mathbb{A}}}\|_{\mathbb{A}\text{-Ber}} = 2$. Finally, $T^2 = 0$ yields $\text{ber}_{\mathbb{A}}(T^2) = 0$. Thus, Theorem 3.3 gives

$$\text{ber}_{\mathbb{A}}^2(T) = 1 \leq \frac{1 + \alpha}{4} (\|T\|_{\mathbb{A}\text{-Ber}}^2 + \|T^{\sharp_{\mathbb{A}}}\|_{\mathbb{A}\text{-Ber}}^2) = \frac{1 + \alpha}{4} (2 + 4),$$

which holds for all $\alpha \in [0, 1]$ and illustrates the quantitative behavior of the bound.

5. Conclusions

In this study, we have successfully addressed the spectral estimation problem for operators acting on reproducing kernel Hilbert spaces equipped with a semi-inner product. By identifying and overcoming the geometric discrepancy between the normalized reproducing kernels and the unit \mathbb{A} -sphere, we established a robust framework for \mathbb{A} -Berezin analysis. Our primary contributions include the derivation of general multiparametric inequalities for triple operator products involving Schatten-type exponents, which unify and refine several existing results in the literature. Furthermore, utilizing the generalized Buzano inequality, we obtained sharper estimates for the \mathbb{A} -Berezin norm of operator sums and established tight bounds relating the \mathbb{A} -Berezin number to the Cartesian decomposition of operators. These theoretical findings were substantiated by concrete applications to weighted Toeplitz operators on Hardy spaces and verified through numerical matrix examples, demonstrating the practical utility of our bounds in analyzing operator combinations under distorted metrics. Future research will focus on extending these multiparametric bounds to the context of \mathbb{A} -numerical ranges and exploring their potential applications in quantum information theory where such weighted metrics naturally arise.

Author contributions

Xiu-Liang Qiu, Mehmet Gürdal, Selim Çetin, Ömer Kişi and Qing-Bo Cai: Conceptualization, Methodology, Formal analysis, Writing—original draft, Writing—review and editing; Xiu-Liang Qiu and Qing-Bo Cai: Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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