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Research article

The theory of cat^1-2-groups among higher categorical models

Sedat Temel*

Department of Mathematics, Recep Tayyip Erdogan University, Rize 53100, TÜRKİYE

* **Correspondence:** Email: sedat.temel@erdogan.edu.tr.

Abstract: In this paper, we introduce the notions of cat^1-2-groups, which are defined as group objects in the category of cat^1-categories (or cat^1-groupoids). We investigated their structure and established fundamental categorical equivalences that connect them with classical algebraic constructs. Central to our work was the introduction of cat^1-crossed modules over groups, which were demonstrated to be an equivalent and more manageable algebraic model for cat^1-2-groups. We established categorical equivalences between cat^1-2-groups, crossed modules over 2-groups, and cat^1-crossed modules. We also obtained cat^1-2-groups as internal cat^1-categories in the category of groups. Furthermore, we explored simplicial 2-groups whose Moore complex was of length one, proving their equivalence with cat^1-2-groups. Our findings were extended to define cat^n-2-groups, generalizing the theory to higher dimensions. These results offer algebraic tractability and deepen the structural understanding within higher-dimensional categorical algebra.

Keywords: 2-group; crossed module; group object; internal category; cat^1-group
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1. Introduction

The study of higher-dimensional algebraic structures has seen remarkable evolution over the past decades, driven by their growing relevance in algebraic topology, category theory, and theoretical physics. Among these, 2-groups, categorical analogues of groups, have emerged as a central notion, formalized by Brown and Spencer in [1] as group objects in the category of all small categories, Cat. Their internal counterpart, following the categorical foundations established by Mac Lane in [2], frames 2-groups as internal categories within the category of groups, establishing a foundational perspective for their structural analysis.

Closely related to 2-groups are crossed modules, initially introduced in the seminal works of Whitehead [3, 4] in the context of homotopy theory. Their categorical equivalence to 2-groups was formally established by Brown and Spencer in [1]. Consisting of a group homomorphism equipped with a compatible action, crossed modules offer an algebraically tractable and conceptually elegant model for the theory of higher groupoids, as further demonstrated by Brown and Spencer in their original treatment [1]. Parallel to these developments, the concept of cat^1-groups, introduced by Loday as referred to in [5] and extensively studied in 2-group theory, provides a rich categorical structure for modeling the internal behaviors of groups and groupoids. We build upon this foundation by defining a 2-dimensional extension of this structure, which gives rise to what we introduce as cat^1-2-groups: A novel class of group objects in the category of cat^1-groupoids. This innovation draws from previous research in categorical algebra, such as the studies in [6, 7], offering new perspectives on the interplay between categorical and algebraic hierarchy.

In this paper, we develop the theory of cat^1-2-groups and establish several categorical equivalences that connect them with classical algebraic constructs. First we investigate the relationships between cat^1-2-groups and structures such as 2-groups, cat^1-groups, precat^1-groupoids, and crossed modules over 2-groups. We introduce the concept of *cat^1-crossed modules over groups*, inspired by the framework of crossed cat^1-modules studied in [8], and demonstrate that they provide a more manageable algebraic model for cat^1-2-groups. We establish an equivalence between these two categories, which is specifically realized as an extension of the Brown and Spencer Theorem, utilizing its original functors to generalize the classical result. Just as a 2-group is an internal category in the category of groups, we establish, in this work, that a cat^1-2-group is precisely an *internal cat^1-category* in the category of groups. Our construction focuses on group objects within cat^1-groupoids (or cat^1-categories). Specifically, we introduce cat^1-group transformations as units acting on objects, providing a distinct structural perspective. Furthermore, we investigate simplicial 2-groups, using the classical notion of simplicial groupoids and prove their categorical equivalence with cat^1-2-groups whose Moore complex is of length one. Finally, we generalize these ideas to define and explore *cat^n-2-groups*, laying the groundwork for future research into multi-level categorical structures.

Our motivation for this research stems from the increasing importance of higher-dimensional algebra in modeling complex symmetries that standard group theory cannot fully capture. While researchers such as those in [9–11] have extensively explored the role of crossed modules in homotopy theory, the transition to *cat^1*-2-groups offers a more refined categorical framework. We aim to fill the gap in the literature regarding the explicit structural equivalences between these higher-dimensional models, thereby providing new computational tools for researchers working in homological algebra and category theory.

This work contributes to a deeper understanding of the interplay between 2-dimensional and higher-dimensional algebraic frameworks, revealing unifying themes across groupoid theory, internal category theory, and simplicial structures. While double groupoids and crossed squares offer alternative models for higher categorical structures, our approach via cat^1-2-groups presents a more algebraically tractable and structurally flexible alternative, particularly within the context of internal category theory.

Throughout this paper, various categorical structures and abbreviations are employed. For the convenience of the reader, a complete list of these notations is provided in Table 1.

Table 1. List of abbreviations and notations.

Abbreviation	Definition
Gp	The category of groups
Cat	The category of all small categories
Gd	The category of groupoids
2Gp	The category of 2-groups
Cm	The category of crossed modules over groups
Cat^1-Gp	The category of *cat*^{1}-groups
CmGd	The category of crossed modules over groupoids
Cat^1-Gd	The category of *cat*^{1}-groupoids
Cat^1-Cat	The category of *cat*^{1}-categories
Cat^1-2Gp	The category of *cat*^{1}-2-groups
Cm2Gp	The category of crossed modules over 2-groups
Simp(2Gp)	The category of simplicial 2-groups
Simp$_{\leq 1}$(2Gp)	The category of simplicial 2-groups whose Moore complex is of length one
Cat^n-Cat	The category of *cat*^{n}-categories
Cat^n-2Gp	The category of *cat*^{n}-2-groups
ker f	Kernel of a group homomorphism f
Ker F	Kernel of a functor (or 2-functor) F

2. Preliminaries

Given a finitely complete category $\mathbb{C}$ and its two objects X,D, an *internal category* $\mathbb{D} = (X, D)$ in $\mathbb{C}$ is defined such that X represents the class of objects and D represents morphisms (i.e., arrows or 1-cells) of $\mathbb{D}$, together with following structure maps as morphisms of $\mathbb{C}$:
– $d_0, d_1 \colon D \to X$ denoting the source and target,
– $\varepsilon\colon X \to D, \varepsilon(x) = 1_x$ assigning identity arrows such that $d_0\varepsilon = d_1\varepsilon = 1_X$,
– $m\colon D \times_X D \to D$ of $\mathbb{C}$ called the composition map expressed as $m(\alpha,\beta) = \beta \circ \alpha$, where the pullback $D \times_X D$ takes over d_0 and d_1 such that $\varepsilon d_0(\alpha) \circ \alpha = \alpha = \alpha \circ \varepsilon d_0(\alpha)$.

$$D \underset{d_1}{\overset{d_0}{\rightrightarrows}} X \quad (\varepsilon \colon X \to D)$$

The internal category $\mathbb{D}$ is called an *internal groupoid* with the following morphism of $\mathbb{C}$.
– $n\colon D \to D,\ n(\alpha) = \overline{\alpha}$ assigns inverse morphism of α such that $\overline{\alpha} \circ \alpha = 1_{d_0(\alpha)}$ and $\alpha \circ \overline{\alpha} = 1_{d_1(\alpha)}$. We denote by $D(x, y)$ the class of arrows from x to y for $x, y \in X$. If $D(x, y) = \emptyset$ for all $x \neq y$, then $\mathbb{D}$ is called a *totally disconnected category*. For further details, see [2, 12].

In this study, the groupoids we consider are defined over the same class of objects, which we will denote by X. Consequently, we will use the notation $\mathbb{G} = (X, G)$ for groupoids throughout the paper.

2-groups (also known as group-groupoids or $\mathcal{G}$*-groupoids* as introduced in [1]) are defined as internal categories within the category Gp of groups (see [2, 12]). Equivalently, such structures could be seen as group objects in the category Cat of all small categories, or, more specifically, in the category Gd of groupoids, as elaborated in [1, 12]. A summary of these categorical structures and

the abbreviations used throughout the manuscript is provided in Table 1. Let $\mathbb{G} = (X, G)$ be a small category in Cat equipped with the following functors:
– A product functor $\mathfrak{m}\colon \mathbb{G} \times \mathbb{G} \to \mathbb{G}$,
– A unit functor $\mathfrak{u}\colon \{*\} \to \mathbb{G}$, where $\{*\}$ is a singleton,
– An inverse functor $\mathfrak{i}\colon \mathbb{G} \to \mathbb{G}$.

These functors satisfy the usual group axioms. Given morphisms $x \xrightarrow{\alpha} y$ and $x' \xrightarrow{\alpha'} y'$, their product is denoted $xx' \xrightarrow{\alpha\alpha'} yy'$. The inverse of $x \xrightarrow{\alpha} y$ is written as $x^{-1} \xrightarrow{\alpha^{-1}} y^{-1}$, and the unit $\mathfrak{u}(*)$ is represented by $e \xrightarrow{1_e} e$. The functor $\mathfrak{m}$ induces the interchange law:

$$(\beta \circ \alpha)(\beta' \circ \alpha') = (\beta\beta') \circ (\alpha\alpha'), \tag{2.1}$$

whenever $d_0(\beta) = d_1(\alpha)$ and $d_0(\beta') = d_1(\alpha')$. Based on this rule, we can express the composition of arrows of $\mathbb{G}$ using the group operation:

$$\beta \circ \alpha = \beta 1_y^{-1} \alpha = \alpha 1_y^{-1} \beta, \tag{2.2}$$

where $d_1(\alpha) = d_0(\beta) = y$. If $y = e$, we get $\beta\alpha = \alpha\beta$. Similarly, the inverse of a morphism α under $\circ$ is given by

$$\overline{\alpha} = 1_x \alpha 1_x^{-1}, \tag{2.3}$$

for $d_0(\alpha) = x$. Thus, category $\mathbb{G}$ becomes a groupoid known as a *2-group* or a *group-groupoid*. Furthermore, if $\alpha, \beta \in \ker d_0$, then

$$\alpha\beta\alpha^{-1} = 1_x \beta 1_x^{-1}, \tag{2.4}$$

where $d_1(\beta) = x$. For further details regarding the equations above, see [1, 2]. Morphisms between 2-groups correspond to groupoid morphisms that respect the underlying group operations. This formalism yields the category of 2-groups, conventionally denoted by 2Gp or GpGd. Comprehensive treatments of 2-groups appear in [13–15].

Following [16], any group G can be endowed with a pair 2-group structure via the direct product $G \times G$. An element $(x, y) \in G \times G$ is a morphism $x \xrightarrow{(x,y)} y$ with source $s(x, y) = x$ and target $t(x, y) = y$. Morphisms compose as $(y, z) \circ (x, y) = (x, z)$, while the group operation is the component-wise product $(x, y)(x', y') = (xx', yy')$.

A *crossed module over groups* $C = (A, B, \mu, \centerdot)$ in the sense of Whitehead's works [3, 4] consists of two groups A and B with a group action $\centerdot\colon B \times A \to A$ and a morphism $\mu\colon A \to B$ of groups called boundary map, which satisfy the following axioms, for all $a, a' \in A$ and $b \in B$:

[CM 1] μ-equivariance condition : $\mu(b \centerdot a) = b\mu(a)b^{-1}$,

[CM 2] Peiffer identity : $\mu(a) \centerdot a' = aa'a^{-1}$.

Following [17], we recall two basic examples of crossed modules:

- Conjugation Crossed Module: For a normal subgroup $N \trianglelefteq G$, the inclusion $\iota : N \hookrightarrow G$ with the conjugation action $g \centerdot n = gng^{-1}$ forms a crossed module.
- Trivial Crossed Module: For any P-module M, the zero map $0 : M \to P$ defines a crossed module.

This trivial case is of fundamental importance as it provides the essential baseline for embedding group actions into higher categorical structures. A concrete and physically significant example of this construction is the trivial crossed module formed by the rotation group SO(3) and its underlying vector space $\mathbb{R}^3$, such that $0 : \mathbb{R}^3 \to \mathrm{SO}(3)$. While this case (where the boundary map is zero) describes basic geometric symmetries, it also serves as the algebraic core for more sophisticated models. Most notably, as discussed by Baez and Lauda in [15], the Poincaré 2-group (also Euclidean 2-group) is constructed as a skeletal 2-group based on the semi-direct product $\mathrm{SO}(3,1)\ltimes\mathbb{R}^4$, which is inherently rooted in this crossed module framework. By establishing this model here, we demonstrate that the cat^1-2-group structures developed in the subsequent sections are natural and higher-dimensional extensions of these pivotal algebraic foundations.

Consider two crossed modules over groups $C = (A, B, \mu, \centerdot)$ and $C' = (A', B', \mu', \centerdot')$. A morphism of crossed modules over groups, represented as $\langle\lambda_1, \lambda_2\rangle\colon C \to C'$, is defined by a pair of group homomorphisms $\lambda_1\colon A \to A', \lambda_2\colon B \to B'$ that preserves the crossed module structure. These maps must satisfy $\lambda_2\mu = \mu'\lambda_1$ and $\lambda_1(b \centerdot a) = \lambda_2(b) \centerdot' \lambda_1(a)$. It follows directly that crossed modules over groups, together with these morphisms, form a category, which we denote by $\mathrm{C{\small M}}$. For further details about crossed modules, see [17].

The Brown and Spencer Theorem, which establishes the categorical equivalence between crossed modules over groups and 2-groups, provides the necessary foundation for the structures developed in Section 3.3. Although this is a classical result established in [1], a sketch of the proof is included below to fix the notation and clarify the functors used throughout this paper.

Theorem 2.1. ***(Brown and Spencer Theorem** [1]) The category* $\mathrm{2G{\small P}}$ *of 2-groups and the category* $\mathrm{C{\small M}}$ *of crossed modules over groups are naturally equivalent.*

Proof. Let $\mathbb{G} = (X, G)$ be a 2-group. The functor $\delta : \mathrm{2G{\small P}} \to \mathrm{C{\small M}}$, $\delta(\mathbb{G}) = (A, B, \mu)$ is defined as an equivalence of categories where $A = \ker d_0$, $B = X$, $\mu = d_1|_{\ker d_0}$ with the action $x \centerdot \alpha = 1_x\alpha 1_x^{-1}$.

Let $F = (f_0, f_1)$ be a morphism of $\mathrm{2G{\small P}}$. Then $\delta(F) = \langle f_1|_{\ker d_0}, f_0\rangle$ is a morphism of $\mathrm{C{\small M}}$.

Now we define a functor $\kappa : \mathrm{C{\small M}} \to \mathrm{2G{\small P}}$ as a weak inverse of δ. If $C = (A, B, \mu)$ is an object of $\mathrm{C{\small M}}$, then $\kappa(C) = \mathbb{G} = (B, B \ltimes A)$ is an object of $\mathrm{2G{\small P}}$ where the structure maps defined by $d_0(b, a) = b$, $d_1(b, a) = \mu(a)b$, $\varepsilon(b) = (b, e_A)$, $n(b, a) = (\mu(a)b, a^{-1})$ for $a \in A, b \in B$. Here, the composition and product of morphisms are defined by $(\mu(a)b, a_1) \circ (b, a) = (b, aa_1)$ and $(b, a) \cdot (b', a') = (bb', a(b \centerdot a'))$.

Let $\langle\lambda_1, \lambda_2\rangle$ be a morphism of $\mathrm{C{\small M}}$. Then $\kappa\langle\lambda_1, \lambda_2\rangle = (\lambda_2, \lambda_2 \times \lambda_1)$ is a morphism of $\mathrm{2G{\small P}}$.

A natural equivalence $\sigma\colon 1_{\mathrm{2G{\small P}}} \to \kappa\delta$ is defined with a map $\sigma_{\mathbb{G}}\colon \mathbb{G} \to \kappa\delta(\mathbb{G})$, which is defined to be an identity on objects, and $\sigma_{\mathbb{G}}(\alpha) = (x, \alpha 1_x^{-1})$ on arrows where $d_0(\alpha) = x$. Thus, $\sigma_{\mathbb{G}}$ is an isomorphism and preserves the group operations and compositions. On the other hand, a natural equivalence $\tau\colon 1_{\mathrm{C{\small M}}} \to \delta\kappa$ is defined via a mapping $\tau_C(b) = b,\ \tau_C(a) = (a, e_B)$ for $a \in A, b \in B$. □

Let Q be a group. Following [5,17], a cat^1-group is defined as a triple $\boldsymbol{Q} = (Q, s, t)$, where $s, t\colon Q \to Q$ are endomorphisms of the group Q, satisfying the following conditions:

[Cat^{1}Gp 1] $st = t$ and $ts = s$,

[Cat^{1}Gp 2] $[\ker s, \ker t] = \{e\}$,

where $[\ker s, \ker t]$ denotes the subgroup generated by the commutators $pqp^{-1}q^{-1}$ for $p \in \ker s$, $q \in \ker t$. These conditions encode a compatibility between the source and target maps, along with commutativity among their respective kernels. The collection of such triples together with morphisms of cat^1-groups; i.e., group homomorphisms $\omega\colon (Q, s, t) \to (Q', s', t')$, satisfying $\omega s = s'\omega$ and $\omega t = t'\omega$, forms the category $\textsc{Cat}^1$-$\textsc{Gp}$ of cat^1-groups.

For any cat^1-group $Q = (Q, s, t)$, the following properties are immediately deduced:

(1) The morphisms s and t restrict the identity map on their common image set $s(Q) = t(Q)$.

(2) $ss = s$ and $tt = t$.

The following foundational theorem was established in [5, 17]:

Theorem 2.2. *The categories* $\textsc{Cat}^1$-$\textsc{Gp}$ *of cat^1-groups and the category* $\textsc{Cm}$ *of crossed modules over groups are naturally equivalent.*

This equivalence reflects the deep algebraic parallel between internal groupoid structures and homotopical data in crossed modules.

Remark 1. *These equivalences have been extended to various algebraic settings, including semi-abelian categories, where similar structural correspondences hold in [18]. However, such generalizations do not directly apply to the specific context of our study.*

Let Q be a group, and let $s_i, t_i\colon Q \to Q$ be $2n$ endomorphisms for $i \in \{1, 2, ..., n\}$. Following [5, 19], a *cat^n-group* (also known as *n-cat-group* or *n-categorical group*) (Q, s_i, t_i) is a group satisfying the following conditions for all $i, j \in \{1, 2, ..., n\}$, where $i \neq j$:

[Cat^{n}Gp 1] $s_i t_i = t_i$ and $t_i s_i = s_i$.

[Cat^{n}Gp 2] $s_i s_j = s_j s_i$, $t_i t_j = t_j t_i$ and $s_i t_j = t_j s_i$.

[Cat^{n}Gp 3] $[\ker s_i, \ker t_i] = e$.

The concept of a crossed module over groupoids was established by Brown and Higgins (see [20, 21]), and further explored by Brown and İçen, (see [22]). A *crossed module over groupoids* $\mathfrak{C} = (\mathbb{A}, \mathbb{B}, \mu)$ is defined using two groupoids, $\mathbb{A} = (X, A)$ and $\mathbb{B} = (X, B)$, which share the same set of objects X. A key requirement is that $\mathbb{A}$ must be totally disconnected, meaning the set of arrows $A(x, y)$ is empty for any distinct objects $x, y \in X$. The structure $\mathfrak{C}$ comprises a groupoid morphism $\mu : \mathbb{A} \to \mathbb{B}$ that is the identity on objects, and a groupoid action $\centerdot : B \times A \to A$ of $\mathbb{B}$ on $\mathbb{A}$, which must satisfy the following two axioms for all $x, y \in X$, $\beta \in B(x, y)$, and $\alpha, \alpha' \in A(x, x)$:

[CMGd 1] $\mu(\beta \centerdot \alpha) = \beta \circ \mu(\alpha) \circ \overline{\beta}$;

[CMGd 2] $\mu(\alpha) \centerdot \alpha' = \alpha \circ \alpha' \circ \overline{\alpha}$.

Let $\mathfrak{C} = (\mathbb{A}, \mathbb{B}, \mu)$ and $\mathfrak{C}' = (\mathbb{A}', \mathbb{B}', \mu')$ be two crossed modules over groupoids. A *morphism of crossed modules over groupoids*, denoted $\lambda : \mathfrak{C} \to \mathfrak{C}'$, is a triple $\lambda = \langle \lambda_2, \lambda_1, \lambda_0 \rangle$, where the following conditions hold:
– $(\lambda_0, \lambda_1) : \mathbb{A} \to \mathbb{A}'$ and $(\lambda_0, \lambda_2) : \mathbb{B} \to \mathbb{B}'$ are groupoid morphisms.
– The boundary maps commute as $\lambda_2 \mu = \mu' \lambda_1$.
– The action is preserved: $\lambda_1(\beta \centerdot \alpha) = \lambda_2(\beta) \centerdot' \lambda_1(\alpha)$ for all suitable $\alpha \in A$ and $\beta \in B$.

The collection of crossed modules over groupoids, with these defined triples as morphisms, constitutes a category typically denoted by $\textsc{CmGd}$. The subsequent theorem, established by İçen in [23], is presented below.

Theorem 2.3. *The category of 2-groupoids is equivalent to the category of crossed modules over groupoids.*

Consider a groupoid $\mathbb{Q} = (X, Q)$ with two groupoid morphisms $S, T : \mathbb{Q} \to \mathbb{Q}$ that act as the identity on the objects. Following [6, 24], a *cat*^{1}*-groupoid* is a triple $(\mathbb{Q}, S, T)$ that fulfills the following two conditions:

[Cat^{1}Gd 1] $ST = T$ and $TS = S$.

[Cat^{1}Gd 2] For any $\beta \in \operatorname{Ker} S$ and $\alpha \in \operatorname{Ker} T$ such that $d_0(\beta) = d_0(\alpha)$, the commutator identity $\beta \circ \alpha \circ \overline{\beta} \circ \overline{\alpha} = \varepsilon d_0(\beta)$ must hold.

Here, $\operatorname{Ker} S = \{\beta \in Q \mid S(\beta) = \varepsilon d_0(\beta)\}$ and $\operatorname{Ker} T = \{\alpha \in Q \mid T(\alpha) = \varepsilon d_0(\alpha)\}$ represent the kernels of S and T, respectively.

For any cat^1-groupoid $(\mathbb{Q}, S, T)$ defined on the groupoid $\mathbb{Q} = (X, Q)$, the following properties are immediately deduced, as given in [6].

(1) The functors S and T restrict to the identity map on their common image set $S(Q) = T(Q)$.

(2) $SS = S$ and $TT = T$.

Following [6], a groupoid morphism $F : (\mathbb{Q}, S, T) \to (\mathbb{Q}', S', T')$ is designated a *morphism of cat*^{1}*-groupoids* if it respects the structural functors, specifically if $FS = S'F$ and $FT = T'F$. Consequently, cat^1-groupoids as objects, together with these structure-preserving maps as morphisms, form a category denoted $\textsc{Cat}^1\text{-}\textsc{Gd}$.

The following result, which establishes a fundamental link between these two algebraic structures, was proven in [6]. A sketch proof will be included in Section 3.2 to facilitate necessary details.

Theorem 2.4. *There exists a natural equivalence of categories between the category of cat*^{1}*-groupoids,* $\textsc{Cat}^1\text{-}\textsc{Gd}$*, and the category of crossed modules over groupoids,* $\textsc{CmGd}$*.*

Proof. Let $\mathfrak{C} = (\mathbb{A}, \mathbb{B}, \mu)$ be a crossed module over groupoids $\mathbb{A} = (X, A), \mathbb{B} = (X, B)$. A functor $\psi : \textsc{CmGd} \to \textsc{Cat}^1\text{-}\textsc{Gd}$, $\psi(\mathbb{A}, \mathbb{B}, \mu) = (\mathbb{Q}', S', T')$ can be defined such that $\mathbb{Q}' = (X, B \ltimes A)$ where

$$B \ltimes A = \{(\beta, \alpha) \mid \beta \in B, \alpha \in A, d_1(\beta) = d_0(\alpha) = d_1(\alpha)\}.$$

The structure maps are defined by $S'(\beta, \alpha) = (\beta, \varepsilon d_0(\alpha))$, $T'(\beta, \alpha) = (\mu(\alpha) \circ \beta, \varepsilon d_0(\alpha))$, $d_0(\beta, \alpha) = d_0(\beta)$, $d_1(\beta, \alpha) = d_1(\alpha)$, $\varepsilon(x) = (1_x, 1_x)$ and $n(b, \alpha) = (\overline{\beta}, \overline{\beta} \centerdot \overline{\alpha})$ where the composition is given by

$$(\beta_1, \alpha_1) \circ (\beta, \alpha) = (\beta_1 \circ \beta, \alpha_1 \circ (\beta_1 \centerdot \alpha)).$$

Conversely, given a cat^1-groupoid $(\mathbb{Q}, S, T)$, we define a functor $\gamma : \textsc{Cat}^1\text{-}\textsc{Gd} \to \textsc{CmGd}$, which acts as a weak inverse of ψ, by setting $\gamma(\mathbb{Q}, S, T) = (\operatorname{Ker} S, S(\mathbb{Q}), T|_{\operatorname{Ker} S})$. This forms a crossed module over groupoids with the action $\theta \centerdot \theta' = \theta \circ \theta' \circ \overline{\theta}$, for all $\theta \in S(Q), \theta' \in \operatorname{Ker} S$.

A natural equivalence $\eta : 1_{\textsc{Cat}^1\text{-}\textsc{Gd}} \to \psi\gamma$ is defined via a mapping $\eta(\mathbb{Q}, S, T) = ((X, S(Q) \ltimes \operatorname{Ker} S, S', T')$, $\eta_{\mathbb{Q}}(\theta) = (S(\theta), \theta \circ S(\overline{\theta}))$ for $\theta \in Q$ where $S'(\theta, \theta') = (\theta, \varepsilon d_0(\theta'))$, $T'(\theta, \theta') = (T(\theta') \circ \theta, \varepsilon d_0(\theta'))$. A natural equivalence $\zeta : 1_{\textsc{CmGd}} \to \gamma\psi$ is given by $\zeta_{\mathfrak{C}}(\beta) = (\beta, \varepsilon d_1(\beta))$, $\zeta_{\mathfrak{C}}(\alpha) = (\varepsilon d_1(\alpha), \alpha)$ for $\mathfrak{C} = (\mathbb{A}, \mathbb{B}, \mu), \beta \in B, \alpha \in A$.

□

3. Structural characterizations and equivalences of Cat^1-2-groups

3.1. Group objects in the category of cat^1-categories

A 2-group is traditionally understood as a group object within the category of small categories, Cat, as defined by Brown and Spencer in [1]. In this section, we obtain the notions of cat^1-2-groups establishing that a group object within the category of cat^1-groupoids, as introduced in [6]. Similarly, to achieve greater generality, we can first define a cat^1-category and subsequently assert that a group object derived therefrom is inherently a groupoid without requiring further proof.

Definition 3.1. *Let $\mathbb{C}$ be a category and $S, T\colon \mathbb{C} \to \mathbb{C}$ be functors that are identities on objects. If the following conditions are satisfied, the triple $(\mathbb{C}, S, T)$ is called a cat^1-category.*

[C1Cat 1] $ST = T$ and $TS = S$,

[C1Cat 2] $\beta \circ \alpha = \alpha \circ \beta$, for all $\beta \in \operatorname{Ker} S,\ \alpha \in \operatorname{Ker} T$, where $d_0(\beta) = d_0(\alpha)$.

Example 3.1. *A cat^1-category with a unique object is a a cat^1-monoid as defined in [25].*

Definition 3.2. *Let $F\colon (\mathbb{C}, S, T) \to (\mathbb{C}', S', T')$ be a functor such that $FS = S'F$ and $FT = T'F$. Then, we form the category* Cat^1-Cat *of cat^1-categories together with these structure-preserving maps as morphisms.*

Remark 2. *The category* Cat *can be considered a full subcategory of* Cat^1-Cat *using the following inclusion functor:*

$$I : \text{Cat} \to \text{Cat}^1\text{-Cat}$$

given by $I(\mathbb{C}) = (\mathbb{C}, 1, 1)$.

Given that group objects in Cat are known to be 2-groups, the group objects within the encompassing category Cat^1-Cat must analogously possess a groupoid structure. We recall the result from [1] that any group object in the category Cat is a groupoid, often termed a "2-group" or a "group-groupoid". This motivates the following definition:

Definition 3.3. *A group object in* Cat^1-Cat *is necessarily a cat^1-groupoid. By extending the terminology from* Cat*, this group object is identified as a cat^1-2-group or cat^1-group-groupoid.*

Example 3.2. *Let $\mathbb{G}$ be a 2-group. Since $(\mathbb{G}, 1, 1)$ is a cat^1-groupoid, then $\mathbb{G}$ is a cat^1-2-group.*

Example 3.3. *Given a cat^1-group (Q, s, t), one can construct a pair 2-group structure $(s(Q), s(Q) \times s(Q))$ that constitutes a cat^1-2-group.*

Example 3.4. *Let X, Q be two groups such that Q is abelian. If (Q, s, t) is a cat^1-group, then, we can obtain the trivial cat^1-2-group $\mathbb{Q} = (X, X \times Q \times X)$, where $S(x, q, y) = (x, s(q), y)$ and $T(x, q, y) = (x, t(q), y)$. For further details, see [6, 26].*

Example 3.5. *A cat^1-2-group with one object is a cat^1-group whose underlying group is necessarily abelian.*

The cat^1-2-group structure carries some fundamental structural components: The underlying groupoid structure, the 2-group structure, the precat^1-groupoid structure, and the cat^1-group structure. The following propositions formalize the connections between these structures, particularly focusing on the role of the functors S and T.

Proposition 3.1. *Let $\mathbb{G}$ be a 2-group and $S, T : \mathbb{G} \to \mathbb{G}$ be 2-group morphisms, such that $ST = T$ and $TS = S$, i.e., satisfying the condition [C1Cat 1]. Then $\mathbb{G}$ is a cat^1-2-group.*

Proof. Due to Eq (2.2), condition [C1Cat 2] is satisfied. □

This proposition enables us to regard a cat^1-2-group as a group object within the category of *precat^1-categories* (or *precat^1-groupoids*), defined as categories (or groupoids) that fulfill only the condition [C1Cat 1].

Proposition 3.2. *Every cat^1-group can be constructed as a cat^1-2-group, and vice-a-versa.*

Proof. Let (Q, s, t) be a cat^1-group. We know from [5] that $\mathbb{Q} = (s(Q), Q)$ is a 2-group whose morphisms are elements of Q as follows:

$$s(q) \xrightarrow{q} t(q) .$$

The morphisms $q, p \in Q$ are composable iff $t(q) = s(p)$ and their composition is defined as

$$p \circ q = ps(p)^{-1}q.$$

To construct a cat^1-groupoid, let $S = s$, $T = t$. Since s and t are identities on $s(Q) = t(Q)$, we can consider these maps as cat^1-structure functors of $\mathbb{Q}$. Let $p_1 \in \operatorname{Ker} S$, $q_1 \in \operatorname{Ker} T$, i.e., $s(q_1) = t(q_1) = s(p_1)$. Since $p_1 s(p_1)^{-1} \in \ker s$ and $p_1 t(p_1)^{-1} \in \ker t$, they commute and we get

$$p_1 \circ q_1 = p_1 s(p_1)^{-1} q_1 = p_1 s(p_1)^{-1} q_1 t(q_1)^{-1} t(q_1) = q_1 t(q_1)^{-1} p_1 s(p_1)^{-1} t(q_1) = q_1 s(q_1)^{-1} p_1 = q_1 \circ p_1.$$

Hence, $\mathbb{Q}$ becomes a cat^1-groupoid. Let $t(q') = s(p')$. Since

$$(p \circ q)(p' \circ q') = pt(q)^{-1}qp's(p')^{-1}q' = pp's(p')^{-1}t(q)^{-1}qq' = pp's(p')^{-1}s(p)^{-1}qq' = pp's(pp')^{-1}qq' = pp' \circ qq',$$

the interchange rule holds and so $\mathbb{Q}$ becomes a cat^1-2-group.

Let $(\mathbb{G}, S, T)$ be a cat^1-2-group whose underlying groupoid $\mathbb{G} = (X, G)$. Then $(X * G, s, t)$ is a cat^1-group where $X * G = \{(x, g) | d_0(g) = x\}$, $s(x, g) = (S(x), S(g)) = (x, S(g))$ and $t(x, g) = (x, T(g))$ with the direct product of groups. □

Remark 3. *Although reciprocal functors exist between cat^1-groups and cat^1-2-groups, they do not form an equivalence, as any potential natural transformation fails to be one-to-one on the class of objects X of the cat^1-2-group (X, G).*

Proposition 3.3. *Let $(\mathbb{G}, S, T)$ be a cat^1-2-group whose underlying groupoid $\mathbb{G} = (X, G)$. Then, (G, S, T) is a cat^1-group.*

Proof. Since S and T are identities on objects, then $\ker S = \{\alpha \in G | S(\alpha) = 1_e\}$ and $d_0(\alpha) = d_1(\alpha) = e$. If $\alpha \in \ker S$, $\beta \in \ker T$, then $\alpha \in KerS$ and $\beta \in KerT$. Hence, we write $\alpha \circ \beta \circ \overline{\alpha} \circ \overline{\beta} = 1_e$. Using Eqs (2.1)–(2.3), we get $\overline{\alpha} = \alpha^{-1}$ and

$$\alpha \circ \beta \circ \overline{\alpha} \circ \overline{\beta} = (\alpha 1_e^{-1} \beta) \circ (\alpha^{-1} 1_e^{-1} \beta^{-1}) = (\alpha\beta) \circ (\alpha^{-1}\beta^{-1}) = (\alpha\beta) 1_e^{-1} (\alpha^{-1}\beta^{-1}) = \alpha\beta\alpha^{-1}\beta^{-1} = 1_e.$$

This means that (G, S, T) is a cat^1-group. □

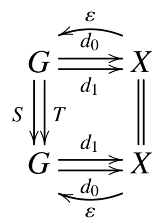
The category Cat^1-2Gp is defined as having cat^1-2-groups, denoted by $(\mathbb{G}, S, T)$, as its objects. A map $F \colon \mathbb{G} \to \mathbb{G}'$ is a morphism in Cat^1-2Gp if it simultaneously serves as a 2-group morphism and a cat^1-groupoid morphism. This ensures that the maps preserve the complete cat^1-2-group structure.

By Example 3.2, the category 2Gp of 2-groups can be considered as a full subcategory of the category Cat^1-2Gp.

Let $\mathbb{G} = (X, G)$ be a 2-group and $(\mathbb{G}, S, T)$ be a cat^1-2-group. Since S, T are identities on objects, then

$$d_0 S = d_0, \quad d_1 S = d_1, \quad d_0 T = d_0, \quad d_1 T = d_1. \tag{3.1}$$

Since S, T are functors, $S(1_x) = 1_{S(x)} = 1_x$ and $T(1_x) = 1_{T(x)} = 1_x$. Hence, the following diagram is commutative:



3.2. Crossed modules over 2-groups

In this section, we begin by recalling the definition of crossed modules over 2-groups, as introduced in [7]. This sets the foundation for our primary result. The established equivalence between crossed modules over groupoids and cat^1-groupoids (as proved in [6]) serves as the crucial link in our argument. Leveraging this fundamental correspondence, we then proceed to demonstrate the existence of a categorical equivalence between the category of cat^1-2-groups and the category of crossed modules over 2-groups.

Definition 3.4. *Let $\mathbb{A} = (X, A)$ and $\mathbb{B} = (X, B)$ be 2-groups over the same object set X, $\mathbb{A}$ be totally disconnected, and $\mathfrak{C} = (\mathbb{A}, \mathbb{B}, \mu)$ be a crossed module over underlying groupoids of $\mathbb{A}$ and $\mathbb{B}$. If the following conditions are satisfied, the triple $\mathfrak{C} = (\mathbb{A}, \mathbb{B}, \mu)$ is called a crossed module over 2-groups.*

[Cm2Gp 1] μ is a morphism of 2-groups,

[Cm2Gp 2] the following interchange rule holds where $\beta, \beta' \in B, \alpha, \alpha' \in A$,

$$(\beta \centerdot \alpha)(\beta' \centerdot \alpha') = (\beta\beta') \centerdot (\alpha\alpha'). \tag{3.2}$$

Example 3.6. *Let $\mathbb{G}$ be a 2-group and $\mathbb{N}$ be a normal sub-2-group of $\mathbb{G}$ in the sense of [16]. Since $\mathbb{N}$ is totally disconnected, we can construct a crossed module over 2-groups $\mathfrak{C} = (\mathbb{N}, \mathbb{G}, I)$, where $I : \mathbb{N} \to \mathbb{G}$*

is the natural inclusion functor. The action of $\mathbb{G}$ *on* $\mathbb{N}$ *is defined by the conjugation* $\beta \centerdot \alpha = \beta \circ \alpha \circ \overline{\beta}$ *for* $\beta \in G(x,y)$ *and* $\alpha \in N(x,x)$*. In this context, the condition [Cm2Gp 1] is naturally satisfied as the inclusion is a 2-group morphism. The interchange rule [Cm2Gp 2] holds as a direct consequence of the interchange law of the 2-group* $\mathbb{G}$*.*

This example is crucial as it demonstrates that the abstract definition of a crossed module over 2-groups is a natural extension of the classical "normal subgroup" relation. It shows that the structural properties of 2-groups, specifically the interplay between their group operations and their categorical compositions, are perfectly captured by the interchange rule (3.2). By utilizing the definition of normal sub-2-groups from [16], we provide a concrete realization of our theory that is grounded in the literature.

A morphism of crossed modules over 2-groups is defined as a morphism of crossed modules over groupoids that, in addition, preserves the underlying group structures. The category whose objects are the crossed modules over 2-groups and whose morphisms are these specialized structure-preserving maps is denoted by Cm2Gp.

Theorem 3.1. *There exists a natural equivalence of categories between* Cat^1-2Gp *and* Cm2Gp.

Proof. The objective of this proof is to demonstrate that the functors established in Theorem 2.4 admit an extension to the setting of 2-group concepts. Let $\mathfrak{C} = (\mathbb{A}, \mathbb{B}, \mu)$ be a crossed module over 2-groups. Then, $\psi(\mathbb{A}, \mathbb{B}, \mu)$ is a cat^1-2-group where the group product of morphisms is defined by $(\beta,\alpha)\cdot(\beta',\alpha') = (\beta\beta', \alpha\alpha')$, where $\beta, \beta' \in B, \alpha, \alpha' \in A$. We will verify that the composition of morphisms and the group multiplication satisfy the interchange rule using Eqs (2.1) and (3.2), for $\beta_1, \beta'_1 \in B, \alpha_1, \alpha'_1 \in A$.

$$\begin{aligned}
\Big[(\beta_1,\alpha_1)\circ(\beta,\alpha)\Big]\cdot\Big[(\beta'_1,\alpha'_1)\circ(\beta',\alpha')\Big] &= \Big[\beta_1\circ\beta, \alpha_1\circ(\beta_1 \centerdot \alpha)\Big]\cdot\Big[\beta'_1\circ\beta', \alpha'_1\circ(\beta'_1 \centerdot \alpha')\Big] \\
&= \Big((\beta_1\circ\beta)(\beta'_1\circ\beta'), (\alpha_1\circ(\beta_1 \centerdot \alpha))(\alpha'_1\circ(\beta'_1 \centerdot \alpha'))\Big) \\
&= \Big((\beta_1\beta'_1)\circ(\beta\beta'), (\alpha_1\alpha'_1)\circ((\beta_1 \centerdot \alpha)(\beta'_1 \centerdot \alpha'))\Big) \\
&= \Big((\beta_1\beta'_1)\circ(\beta\beta'), (\alpha_1\alpha'_1)\circ((\beta_1\beta'_1) \centerdot (\alpha\alpha'))\Big) \\
&= (\beta_1\beta'_1, \alpha_1 a'_1)\circ(\beta\beta', \alpha\alpha') \\
&= \Big[(\beta_1,\alpha_1)\cdot(\beta'_1,\alpha'_1)\Big]\circ\Big[(\beta,\alpha)\cdot(\beta',\alpha')\Big].
\end{aligned}$$

Now we will check that S, T and n are homomorphisms of groups using Eqs (2.1) and (3.2).

$$S((\beta,\alpha)\cdot(\beta',\alpha')) = S(\beta\beta',\alpha\alpha') = (\beta\beta', \varepsilon d_0(\alpha\alpha')) = (\beta, \varepsilon d_0(\alpha))\cdot(\beta', \varepsilon d_0(\alpha')) = S(\beta,\alpha)\cdot S(\beta',\alpha').$$

$$\begin{aligned}
T((\beta,\alpha)\cdot(\beta',\alpha')) &= T(\beta\beta',\alpha\alpha') \\
&= (\mu(\alpha\alpha')\circ(\beta\beta'), \varepsilon d_0(\alpha\alpha')) \\
&= ((\mu(\alpha)\mu(\alpha'))\circ(\beta\beta'), \varepsilon d_0(\alpha)\varepsilon d_0(\alpha')) \\
&= ((\mu(\alpha)\circ\beta)(\mu(\alpha')\circ\beta'), \varepsilon d_0(\alpha)\varepsilon d_0(\alpha')) \\
&= (\mu(\alpha)\circ\beta, \varepsilon d_0(\alpha))\cdot(\mu(\alpha')\circ\beta', \varepsilon d_0(\alpha')) \\
&= T(\beta,\alpha)\cdot T(\beta',\alpha').
\end{aligned}$$

$$n((\beta,\alpha)\cdot(\beta',\alpha')) = n(\beta\beta',\alpha\alpha') = (\overline{\beta\beta'}, \overline{\beta\beta'} \centerdot \overline{\alpha\alpha'}) = (\overline{\beta\beta'}, (\overline{\beta} \centerdot \overline{\alpha})(\overline{\beta'} \centerdot \overline{\alpha'})) = (\overline{\beta}, \overline{\beta} \centerdot \overline{\alpha})(\overline{\beta'}, \overline{\beta'} \centerdot \overline{\alpha'}) = n(\beta,a)\cdot n(\beta',\alpha').$$

Let $(\mathbb{Q}, S, T)$ be a cat^1-2-group. Then $\gamma(\mathbb{Q}, S, T) = (\mathrm{Ker}\, S, S(\mathbb{Q}), T|_{\mathrm{Ker}\, S})$ is a crossed module over 2-groups. We will now verify that the interchange rule holds:

$$(\theta \centerdot \theta_1) \cdot (\theta' \centerdot \theta_1') = (\theta \circ \theta_1 \circ \overline{\theta})(\theta' \circ \theta_1' \circ \overline{\theta'}) = (\theta\theta') \circ (\theta_1\theta_1') \circ (\overline{\theta\theta'}) = (\theta\theta') \centerdot (\theta_1\theta_1').$$

Finally we prove that $\eta_{\mathbb{Q}}(\theta) = (S(\theta), \theta \circ S(\overline{\theta}))$ preserves the group multiplication by (2.1).

$$\begin{aligned}
\eta_{\mathbb{Q}}(\theta\theta') &= (S(\theta\theta'), (\theta\theta') \circ S(\overline{\theta\theta'})) \\
&= (S(\theta)S(\theta'), (\theta\theta') \circ S(\overline{\theta})S(\overline{\theta'})) \\
&= (S(\theta)S(\theta'), (\theta \circ S(\overline{\theta}))(\theta' \circ S(\overline{\theta'}))) \\
&= (S(\theta), \theta \circ S(\overline{\theta})) \cdot (S(\theta'), \theta' \circ S(\overline{\theta'})) \\
&= \eta_{\mathbb{Q}}(\theta) \cdot \eta_{\mathbb{Q}}(\theta').
\end{aligned}$$

Other details are straightforward and are omitted. □

This equivalence highlights a significant distinction of our work: It bridges the gap between group-2-groupoids (group objects in the category of 2-categories) and crossed modules over 2-groups by representing these higher-dimensional and relatively complex structures through more accessible and simpler algebraic frameworks. Hence, we should mention that cat^1-2-groups (crossed modules over 2-groups) are categorically equivalent to group-2-groupoids, as explored in [7, 27], double group-groupoids with thin structure, as presented by [28], and special internal categories within crossed modules over groups following the constructions in [7, 29]. Unlike the multifaceted nature of group-2-groupoids, our approach provides a more streamlined and intuitive description of these symmetries through cat^1-structures.

3.3. *Cat^1-crossed modules over groups*

The inherent complexity of cat^1-2-groups necessitates a more tractable approach for their study. Fortunately, a well-established categorical equivalence exists between 2-groups and crossed modules, which are algebraically simpler than 2-groups. Crossed modules, characterized by a homomorphism between two groups satisfying compatibility conditions, provide a powerful tool for understanding 2-group structures. Building upon this fundamental equivalence, our aim of this section is to investigate and establish a novel equivalence between cat^1-2-groups and cat^1-crossed modules that inherently incorporate crossed modules over groups and cat^1-group structures. The definition of cat^1-crossed modules builds upon the structure "crossed cat^1-module" defined in [8], presenting a particular specialization of it. While the categorical equivalence with crossed squares is demonstrated in [8], we investigate the equivalence with cat^1-2-groups, a distinct yet related categorical structure. This exploration seeks to leverage the simplicity and established properties of crossed modules to shed light on the deeper algebraic and categorical properties of cat^1-2-groups.

Definition 3.5. *Let $C = (A, B, \mu)$ be a crossed module over groups and $\mathcal{A} = (A, s, t)$ be a cat^1-group. If $\langle s, 1_B\rangle$ and $\langle t, 1_B\rangle : C \to C$ are morphisms of the crossed module C; i.e.,*

[C1Cm 1] $\mu s = \mu$, $\mu t = \mu$,

[C1Cm 2] $s(b \centerdot a) = b \centerdot s(a)$ *and* $t(b \centerdot a) = b \centerdot t(a)$,

then, $C = (A, B, \mu, s, t)$ is called a cat^1-crossed modules over groups.

Example 3.7. *Let $C = (A, B, \mu)$ be a crossed module. Since $(A, 1, 1)$ is a cat^1-group, then, $(A, B, \mu, 1, 1)$ is a cat^1-crossed module.*

Example 3.8. *Let $C = (A, B, \mu)$ be a crossed module such that A is abelian. Since (A, s, t) is a cat^1-group with trivial structure maps $s(a) = t(a) = 0$, it follows that (A, B, μ, s, t) is a cat^1-crossed module.*

Example 3.9. *Let $\mathcal{A} = (A, s, t)$ be a cat^1-group. Since $(A, A, 1)$ is a crossed module, $(A, A, 1, s, t)$ is a cat^1-crossed module.*

Example 3.10. *Let $\mathcal{A} = (A, s, t)$ be a cat^1-group. Since $s(A) \trianglelefteq A$, the inclusion map $\iota : s(A) \hookrightarrow A$ with the conjugation action of A on $s(A)$ defines a crossed module in the classical sense. Furthermore, by considering the induced cat^1-structures, the triple $(s(A), A, \iota, s, t)$ naturally forms a cat^1-crossed module.*

Example 3.11. *Let M be a P-module and let $s, t : M \to M$ be P-module endomorphisms, satisfying the condition [Cat^{1}Gp 1]. Since the zero map $0 : M \to P$ defines a trivial crossed module, the structure $(M, P, 0)$ can be naturally enriched. Hence, the quintuple $(M, P, 0, s, t)$ constitutes a cat^1-crossed module.*

Example 3.12. *Let M be a P-module. Building on the trivial crossed module structure, we can define a crossed module $(M \times M, P, 0)$, where the action is given by $p \centerdot (m, m') = (p \centerdot m, p \centerdot m')$. This structure can be enriched into a cat^1-crossed module $(M \times M, P, 0, s, t)$, where the source and target endomorphisms $s, t : M \times M \to M \times M$ are defined as $s(m, m') = (m, m)$ and $t(m, m') = (m', m')$. These maps satisfy the [Cat^{1}Gp 1] condition, effectively encoding the transition between pairs of elements in M under the action of P.*

Let $C = (A, B, \mu, s, t), C' = (A', B', \mu', s', t')$ be cat^1-crossed modules and $\langle \lambda_1, \lambda_2 \rangle : C \to C'$ be a morphism of crossed modules such that $\lambda_1 s = s' \lambda_1$ and $\lambda_1 t = t' \lambda_1$, i.e, λ_1 is a morphism of cat^1-groups. Then, we form the category $\textsc{Cat}^1$-$\textsc{Cm}$ of cat^1-crossed modules. By Example 3.7, the category $\textsc{Cm}$ of crossed modules over groups can be considered a full subcategory of $\textsc{Cat}^1$-$\textsc{Cm}$ using the inclusion functor

$$I : \textsc{Cm} \to \textsc{Cat}^1\text{-}\textsc{Cm}$$

given by $I(A, B, \mu) = (A, B, \mu, 1, 1)$.

Theorem 3.2. *There exists a natural equivalence of categories between* $\textsc{Cat}^1$-2$\textsc{Gp}$ *and* $\textsc{Cat}^1$-$\textsc{Cm}$.

Proof. Since the fundamental idea of this proof is based on the Brown and Spencer Theorem, we extend the functors δ and κ (established in Theorem 2.1) to the required setting. Let $(\mathbb{Q}, S, T)$ be a cat^1-2-group whose underlying 2-group is (X, Q). Then, the functor

$$\delta : \textsc{Cat}^1\text{-}2\textsc{Gp} \to \textsc{Cat}^1\text{-}\textsc{Cm}, \quad \delta(\mathbb{Q}, S, T) = (A, B, \mu, s, t)$$

is extended as an equivalence of categories where $s = S|_{\ker d_0}$ and $t = T|_{\ker d_0}$. The crossed module $C = (A, B, \mu)$ here is derived from the underlying 2-group via Brown and Spencer Theorem. Since S, T are identities on objects, then $s(\theta), t(\theta) \in \ker d_0$ for all $\theta \in \ker d_0$. Hence, (A, s, t) is a cat^1-group via Proposition 3.3. By (3.1), the condition [C1Cm 1] is satisfied. Let $x \in X, \theta \in Q$. Since

$$s(x \centerdot \theta) = S(1_x \theta 1_x^{-1}) = S(1_x) S(\theta) S(1_x)^{-1} = 1_x S(\theta) 1_x^{-1} = x \centerdot s(\theta),$$

and similarly $t(x \centerdot \theta) = x \centerdot t(\theta)$, the condition [C1Cm 2] is satisfied.

Given a cat^1-crossed module (A, B, μ, s, t), a 2-group $\mathbb{Q} = (B, B \ltimes A)$ can be obtained from the underlying crossed module via the process of the Brown and Spencer Theorem. Now we extend the functor

$$\kappa : \textsc{Cat}^1\text{-}\textsc{Cm} \to \textsc{Cat}^1\text{-}2\textsc{Gp},$$

such that $\kappa(A, B, \mu, s, t) = (\mathbb{Q}', S', T')$ is an object of $\textsc{Cat}^1$-2$\textsc{Gp}$ where $S'(b, a) = (b, s(a))$ and $T'(b, a) = (b, t(a))$. By [C1Cm 1], S' and T' are identities on objects.

[C1Cat 1] $S'T'(b, a) = S'(b, t(a)) = (b, st(a)) = (b, t(a)) = T'(b, a)$ and similarly $T'S' = S'$.

[C1Cat 2] Let $(b, a) \in \operatorname{Ker} S' = B \times \ker s$, $(b, a') \in \operatorname{Ker} T' = B \times \ker t$ and $\mu(a) = \mu(a') = e_B$. Since $a \in \ker s, a' \in \ker t$, we write $aa' = a'a$. Hence

$$(b, a) \circ (b, a') = (b, aa') = (b, a'a) = (b, a') \circ (b, a).$$

Since $S'\sigma_{\mathbb{Q}}(\theta) = S'(x, \theta 1_x^{-1}) = (x, S(\theta 1_x^{-1})) = (x, S(\theta)1_x^{-1}) = \sigma_{\mathbb{Q}} S(\theta)$ and similarly $T'\sigma_{\mathbb{Q}}(\theta) = \sigma_{\mathbb{Q}} T(\theta)$, the natural equivalence map $\sigma_{\mathbb{Q}}$ is a morphism of $\textsc{Cat}^1$-$\textsc{Cm}$.

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\sigma_{\mathbb{Q}}} & \kappa\delta(\mathbb{Q}) \\ {\scriptstyle S}\downdownarrows{\scriptstyle T} & & {\scriptstyle T'}\downdownarrows{\scriptstyle S'} \\ \mathbb{Q} & \xrightarrow[\sigma_{\mathbb{Q}}]{} & \kappa\delta(\mathbb{Q}) \end{array}$$

Other details are straightforward and are omitted. □

Remark 4. *The equivalence established in this theorem can also be straightforwardly obtained by considering the structure within the cat^1-crossed module as a precat^1-group, i.e., a group satisfying only condition [C1G 1]. For details about the precat^1-group, see [17].*

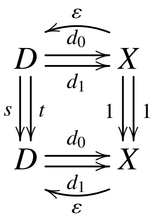
This equivalence is significant as it provides a concrete algebraic representation for the more abstract cat^1-2-group structures. By mapping cat^1-2-groups to cat^1-crossed modules, we transition from a structure involving group objects in groupoids to a more manageable model consisting of group homomorphisms and actions. This simplification is crucial for practical computations, as it enables one to verify higher categorical identities through basic group-theoretic calculations rather than complex diagram chasing in 2-groups.

3.4. Internal categories

A 2-group is also defined in [2] as an internal category in the category $\textsc{Gp}$ of groups. Using similar methods, internal categories in the category $\textsc{Cat}^1$-$\textsc{Gp}$ of cat^1-groups were characterized in [10, 30] and compared to the corresponding structures related to crossed squares and crossed modules. Let $\mathbb{D} = (X, D)$ be an internal category in $\textsc{Cat}^1$-$\textsc{Gp}$. Then, $\mathbb{D}$ consists of two cat^1-groups (D, s, t) and (X, s', t'). Here the grupoid structure maps of $\mathbb{D}$ are cat^1-group morphisms, i.e., the following diagram is commutative:

$$\begin{array}{ccc} D & \overset{\varepsilon}{\underset{d_1}{\overset{d_0}{\rightrightarrows}}} & X \\ {\scriptstyle s}\downdownarrows{\scriptstyle t} & & {\scriptstyle t'}\downdownarrows{\scriptstyle s'} \\ D & \underset{\varepsilon}{\underset{d_1}{\overset{d_0}{\rightrightarrows}}} & X \end{array}$$

These structures encode the group-theoretic and categorical features of 2-groups in terms of cat^1-group data, enabling homotopical and algebraic reasoning to interact. Since the structure maps S and T in the definition of a cat^1-category are identity morphisms on objects, the resulting internal category admits a cat^1-2-group structure only if the structure maps of the cat^1-group (X, s', t') are the identity on objects, i.e., $s' = t' = 1_X$.



Then, $\mathbb{D}$ is a cat^1-2-group such that S and T are identities on objects and $S = s$, $T = t$ on morphisms.

In [10], it is proved that the category of internal categories within the category of cat^1-groups is naturally equivalent to the category of internal categories within the category of crossed modules over groups. There is also a natural equivalence between internal categories within the category of crossed squares and crossed modules over groups (see [29]). Using these equivalences, categorical correspondences can be readily established between the category of cat^1-2-groups and certain special cases of crossed squares and internal categories within crossed modules over groups.

Our approach in this paper, utilizing a common object set, namely X, for the underlying groupoids within the construction of cat^1-2-groups, is consistent with established methodologies in higher-dimensional categorical algebra. This practice is notably evident in the foundational papers [20, 21] of Brown and Higgins on crossed complexes and other higher structures, as well as in Brown and İçen [22]'s treatment of homotopies and automorphisms of crossed modules over groupoids. The use of a shared object set X is crucial to ensure the coherence and compatibility required when defining multi-dimensional algebraic structures. It naturally facilitates the composition and interaction of higher-order morphisms and ensures that the domains and codomains of the components align correctly. This common base is fundamental for establishing well-defined operations, such as homotopies and automorphisms, across the layers of these complex algebraic models, enabling a consistent and robust theoretical framework for cat^1-2-groups.

As seen above, to obtain a cat^1-2-group in the category of cat^1-groups using the conventional internal category structure, we must select the identity morphisms as the cat^1-structure transformations on objects. To resolve this ambiguity, we may define an *internal cat^1-category* as an internal category, satisfying only the condition [C1Cat 1]. With this definition, we can assert the following theorem without proof.

Theorem 3.3. *An internal cat^1-category in the category* Gp *of groups is a cat^1-2-group.*

3.5. Simplicial 2-groups

In this section, we extend the well-established notion of simplicial groupoids to the notions of 2-groups. Following Dwyer and Kan [31], we inherently assume that a simplicial groupoid (or a simplicial category), and consequently the underlying groupoids in our construction of simplicial 2-groups and cat^1-2-groups, are defined over the same object class X in each dimension (see also [32]). We present the definition of a simplicial 2-group as a simplicial object in the category 2Gp of 2-groups. This construction naturally generalizes the interplay between simplicial structures and categorical algebra. A key contribution of this section is the demonstration of a natural categorical equivalence between simplicial 2-groups whose Moore complex is of length one and cat^1-2-groups. This equivalence not only provides a deeper understanding of the underlying structures but also opens avenues for translating results and techniques between these seemingly distinct mathematical objects.

Let Δ be the category of finite ordinals. A *simplicial 2-group* is a sequence of 2-groups $\mathfrak{G} = \{\mathbb{G}_n\}_{n\geq 0}$, such that each $\mathbb{G}_n = (X, G_n)$ is a 2-group defined over the same object set X. Formally, $\mathfrak{G}$ is a contravariant functor from the opposite category Δ^{op} to the category of 2-groups, 2Gp. This implies a sequence of 2-groups connected by 2-group morphisms, represented by the following diagram:

$$G_0 \;\underset{\xleftarrow[\rho_1]{}}{\overset{\xleftarrow{\rho_0}}{\xrightarrow{\nu_0}}}\; G_1 \;\begin{smallmatrix}\xleftarrow{\rho_0}\\ \xrightarrow{\nu_0}\\ \xleftarrow{\rho_1}\\ \xrightarrow{\nu_1}\\ \xleftarrow{\rho_2}\end{smallmatrix}\; G_2 \ldots G_{n-1} \;\begin{smallmatrix}\xleftarrow{\rho_0}\\ \xrightarrow{\nu_0}\\ \xleftarrow{\rho_1}\\ \vdots\\ \xrightarrow{\nu_{n-1}}\\ \xleftarrow{\rho_n}\end{smallmatrix}\; G_n \ldots$$

Here, the face operators (ϱ_i) and degeneracy operators (s_i) are 2-group morphisms that act as identities on the common object set X. These operators must also satisfy the following usual simplicial identities, as detailed in [33, 34]:

$$\rho_i\rho_j = \rho_{j-1}\rho_i, \quad if \;\; i < j,$$

$$\rho_i\nu_j = \begin{cases} \nu_{j-1}\rho_i, & i < j, \\ 1, & i \in \{j, j+1\}, \\ \nu_j\rho_{i-1}, & i > j+1, \end{cases}$$

$$\nu_j\nu_i = \nu_i\nu_{j-1}, \quad if \;\; i < j.$$

Let $\mathfrak{G}, \mathfrak{G}'$ be simplicial 2-groups. A simplicial 2-group morphism $f : \mathfrak{G} \to \mathfrak{G}'$ is a family of 2-group morphisms $\{f_n : G_n \to G'_n\}$, such that $\rho'_i f_n = f_{n-1}\rho_i$ and $f_n\nu_j = \nu'_j f_{n-1}$, for $i, j \in \{0, 1, 2, ...\}$. Then, the category of simplicial 2-groups will be denoted by Simp(2Gp).

The Moore complex $N\mathfrak{G}$ of a simplicial object $\mathfrak{G}$ in Simp2Gp is the sequence

$$N\mathfrak{G} : \qquad NG_0 \xleftarrow{\rho_1} NG_1 \xleftarrow{\rho_2} NG_2 \ldots NG_{n-1} \xleftarrow{\rho_n} NG_n \ldots,$$

where $NG_0 = G_0$ and $NG_n = \bigcap_{i=0}^{n} \mathrm{Ker}\,\rho'_i$, where ρ'_i is the restriction of ρ_i to NG_i. We say that the Moore complex $N\mathfrak{G}$ of a simplicial object $\mathfrak{G}$ is of length k if $NG_n = \{1_e\}$, for all $n \geq k+1$, where 1_e is the morphism from e to e. Now we define the category Simp$_{\leq k}$(2Gp) whose objects are simplicial objects possessing a Moore complex of length k. The morphisms in this category consist of families of 2-group morphisms that maintain compatibility with the respective face and degeneracy maps.

For a simplicial 2-group $\mathfrak{G}$ in Simp(2Gp), the Moore complex of length 1 can be pictured as follows:

$$G_0 \;\underset{\xleftarrow[\rho_1]{}}{\overset{\xleftarrow{\rho_0}}{\xrightarrow{\nu_0}}}\; \mathrm{Ker}\,\rho_0 \longleftarrow \{1_e\} \ldots \{1_e\} \longleftarrow \{1_e\} \ldots$$

Let Simp$_{\leq 1}$(2Gp) be the category of simplicial 2-groups whose Moore complex is of length 1.

Theorem 3.4. *The category* Simp$_{\leq 1}$(2Gp) *is naturally equivalent to the category* Cat^1-2Gp.

Proof. We define a functor $\vartheta\colon \textsc{Cat}^1\text{-2Gp} \to \textsc{Simp}_{\leq 1}(\text{2Gp})$. Given a cat^1-2-group $(\mathbb{Q}, S, T)$ whose underlying 2-group is $\mathbb{Q} = (X, Q)$, then $\vartheta(\mathbb{Q}) = \mathfrak{G}$ is an object of $\textsc{Simp}_{\leq 1}(\text{2Gp})$, where $G_0 = S(Q)$, $G_1 = Q$, $\rho_0 = S$, $\rho_1 = T$, and $\nu_0 = \imath$ is the inclusion map. The objects of $\mathfrak{G}$ are the same as $\mathbb{Q}$. Since S, T are identities on $S(Q) = T(Q)$, we get

$$\rho_0\nu_0(\theta) = S(\theta) = \theta, \ \ \rho_1\nu_0(\theta) = T(\theta) = \theta.$$

Given a morphism F of cat^1-2-groups, $\{F|_{S(Q)}, F\}$ is a morphism of $\textsc{Simp}_{\leq 1}(\text{2Gp})$.

Now we define a functor $\phi\colon \textsc{Simp}_{\leq 1}(\text{2Gp}) \to \textsc{Cat}^1\text{-2Gp}$ as a weak inverse of ϑ. Given an object $\mathfrak{G} = \{\mathbb{G}_0, \mathbb{G}_1\}$ of $\textsc{Simp}_{\leq 1}(\text{2Gp})$, then $(\mathbb{G}_1, S, T)$ is a cat^1-2-group where $S = \nu_0\rho_0$ and $T = \nu_0\rho_1$. Now, let us verify that the triple $(\mathbb{G}_1, S, T)$ satisfies the cat^1-2-group conditions. By the simplicial identity $\rho_0\nu_0 = 1$ and $\rho_1\nu_0 = 1$, we have $ST(\theta) = \nu_0\rho_0\nu_0\rho_1(\theta) = \nu_0\rho_1(\theta) = T(\theta)$ and, similarly, $TS(\theta) = S(\theta)$, for $\theta \in G_1$. Therefore, the condition [C1Cat 1] is satisfied. Since ν_0 is an injection, we have $\text{Ker}\, S = \text{Ker}(\nu_0\rho_0) = \text{Ker}\,\rho_0$. Similarly, $\text{Ker}\, T = \text{Ker}(\nu_0\rho_1) = \text{Ker}\,\rho_1$. Hence, the condition [C1Cat 2] automatically follows from the structure of a simplicial 2-group.

Other details are straightforward and are omitted. □

This result is significant as it proves that the intricate structure of simplicial 2-groups can be represented by the much simpler and more manageable framework of cat^1-2-groups and, consequently, cat^1-crossed modules. By reducing the infinite complexity of simplicial maps to these finite categorical models, we provide a more accessible tool for computing higher-dimensional symmetries.

4. Higher dimensional extensions: Cat^n-2-groups and their properties

In this final section, we provide a study on the generalization of the cat^1-2-group definition by introducing the broader concept of a cat^n-2-group. This extension enables a more comprehensive understanding of higher-dimensional categorical structures. Following this new definition, we aim to demonstrate that results analogous to those presented in Section 3.3 of this paper, originally derived for cat^1-2-groups, hold true for this newly generalized cat^n-2-group structure.

Definition 4.1. *Let $\mathbb{C}$ be a category and $S_i, T_i\colon \mathbb{C} \to \mathbb{C}$ be 2n functors that are identities on objects, for $i = 1, 2, ..., n$. If the following conditions are satisfied, then $(\mathbb{C}, S_i, T_i)$ is called a cat^n-category.*

[CnCat 1] $S_iT_i = T_i$ and $T_iS_i = S_i$,

[CnCat 2] $S_iS_j = S_jS_i$, $T_iT_j = T_jT_i$ and $S_iT_j = T_jS_i$,

[CnCat 3] $\beta_i \circ \alpha_i = \alpha_i \circ \beta_i$, for all $\beta_i \in \text{Ker}S_i$ and $\alpha_i \in \text{Ker}T_i$ where $d_0(\beta_i) = d_0(\alpha_i)$.

Let $F\colon (\mathbb{C}, S_i, T_i) \to (\mathbb{C}', S'_i, T'_i)$ be a functor such that $FS_i = S'_iF$ and $FT_i = T'_iF$, for $i = 1, 2, ..., n$. Then, we form the category $\textsc{Cat}^n\text{-}\textsc{Cat}$ of cat^n-categories. The definition of cat^n-groupoid was given in [35]. Hence, we define the category $\textsc{Cat}^n\text{-}\textsc{Gpd}$.

Definition 4.2. *A group object in* $\textsc{Cat}^n\text{-}\textsc{Cat}$ *(or in* $\textsc{Cat}^n\text{-}\textsc{Gpd}$*) is called cat^n-2-group or cat^n-group-groupoid.*

Definition 4.3. *Let $C = (A, B, \mu)$ be a crossed module over groups and $Q = (Q, s_i, t_i)$ be a cat^n-group, for $i \in \{1, 2, ..., n\}$. If $\langle s_i, 1_B\rangle$ and $\langle t_i, 1_B\rangle : C \to C$ are morphisms of the crossed module C; i.e.,* *[C1Cm 1]* $\mu s_i = \mu$, $\mu t_i = \mu$,

[C1Cm 2] $s_i(n \centerdot m) = n \centerdot s_i(m)$ *and* $t_i(n \centerdot m) = n \centerdot t_i(m)$,

then, $C = (A, B, \mu, s_i, t_i)$ *is called a cat*^{n}*-crossed modules over groups.*

The examples and propositions presented in Sections 3.1 and 3.3 can be readily extended to cat^n-2-groups notions. Hence, we introduce the following theorem without proof:

Theorem 4.1. *The category* Cat^n-2Gp *of cat*^{n}*-2-groups and the category* Cat^n-Cm *of cat*^{n}*-crossed modules are naturally equivalent.*

This result demonstrates that the complexity of cat^n-2-groups can be reduced to the more manageable cat^n-crossed module structure. Since a cat^n-crossed module is essentially built upon a single crossed module and a cat^n-group, it offers a much simpler algebraic framework for computations than the n-fold categorical structure of 2-groups. This reduction provides a significant advantage in terms of algebraic tractability in higher dimensions.

5. Conclusions

In this study, we introduced the concept of cat^1-2-groups as group objects in the category of cat^1-groupoids, establishing several categorical equivalences that connect them with well-known algebraic frameworks. By leveraging crossed modules, simplicial 2-groups, and internal categories, we demonstrated that cat^1-2-groups provide a versatile and algebraically tractable model for exploring higher-dimensional group structures. Furthermore, the development of cat^1-crossed modules over groups and their equivalence with cat^1-2-groups offers a promising approach to bridging abstract categorical concepts with concrete group-theoretic tools. The generalization to cat^n-2-groups underscores the depth and extensibility of the proposed framework, paving the way for future exploration in higher categorical algebra and its applications.

We extend the Brown and Spencer equivalence to higher categorical structures by incorporating cat^1-behavior. In particular, the definition and equivalence of cat^1-crossed modules provides an algebraic replacement for complex categorical models.

In the future, researchers may investigate homotopical interpretations of cat^1-2-groups, as well as applications to the cohomology theory of higher groupoids. Moreover, an extension to Lie 2-groups (as given in [15]) could open pathways into differential geometric contexts. One may consider developing a similar framework to that presented in [36] by Porter, where extensions, crossed modules, and internal categories are studied within categories of groups with operations. Such an approach could enrich the structural understanding of cat^1-2-groups and their transformation theory. The results obtained in this paper can potentially be generalized to monoids, in a manner similar to the approach presented in [25], where crossed semimodules and cat^1-monoids are studied.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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