Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 6080–6105.
DOI: 10.3934/math.2026252
Received: 14 December 2025
Revised: 05 February 2026
Accepted: 28 February 2026
Published: 11 March 2026

Research article

Dynamical analysis of an aquaculture management model with stage structure, intraspecific cooperation, refuge effect, and nonlinearly impulsive releasing

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Abstract: In this paper, we propose an aquaculture management model with stage structure, intraspecific cooperation, refuge effect, and nonlinearly impulsive releasing. The global asymptotic stability of the periodic solution for the subsystem of the system was analyzed via the Jury criterion and the Banach contraction mapping principle. Additionally, through the theory of impulsive differential equations, the conditions for the global asymptotic stability of the prey-vanishing periodic solution and for the permanence of the system were acquired. Finally, numerical simulations were utilized to validate the theoretical results. In addition, key parameters affecting the persistence condition B_0 were identified via comparative and sensitivity analysis. Moreover, abundant dynamical behaviors such as period bifurcation and chaos were identified through bifurcation diagrams.

Keywords: stage structure; intraspecific cooperation; refuge effect; nonlinearly impulsive releasing; sensitivity analysis
Mathematics Subject Classification: 34A37, 37M05, 92D25

1. Introduction

Fisheries constitute an indispensable component of the global food supply. Among aquatic products, fish serve as a crucial dietary source of high-quality animal protein and amino acids [1], which can effectively satisfy the daily metabolic requirements of adults and muscle maintenance in the elderly [2, 3]. Moreover, fish are rich in micronutrients, which are of vital importance to human health [4]. However, due to factors such as overfishing and environmental impacts, global fishery resources have shown a declining trend [5]. Therefore, rational fishing and environmental improvement are indispensable requirements for maintaining the sustainable development of fishery resources.

To characterize the dynamic changes of fish populations in fisheries management, many scholars have established corresponding mathematical models using ordinary differential equations. For instance, Das et al. [6] proposed a predator-prey fishery in the presence of toxicity. The predator-prey fishery model developed by Zhang et al. [7] incorporates prey dispersal within a two-patch environment, thereby maintaining the dynamic balance of fishery populations.

The theory of impulsive differential equations has advanced considerably over the past several years. A predator-prey fishery model with birth pulse, impulsive releasing, and prey harvesting was formulated by Quan et al. [8]. Since fish grow from juvenile to adult stages, they exhibit different physiological functions at different stages of their life. Thus a stage structure model was established by Wu et al. [9, 10] to address aquaculture management, making it more practically significant. Li et al. [11] and Sun and Jiao [12] considered nonlinear impulsive releasing and optimized the impulsive control strategy. There are numerous comparable impulsive control models, such as those in [13–15]. In addition, many scholars have considered intraspecific cooperation among predators based on traditional models [16–18]. This cooperation cannot only improve the predation efficiency of predators, but also enhance the stability of the system to a certain extent.

However, the aforementioned researchers considered only a single intraspecific cooperation mode of predators and failed to incorporate the refuge effect of prey as an influencing factor. Actually, there is a close connection between juvenile fish and adult fish. Buckley et al. [19] found that both parents of the Amazonian cichlid (discus fish) provide their offspring with mucus secretions for feeding after hatching. Moreover, fish take shelter in rock crevices, coral reefs, and aquatic plants, using the environment to avoid detection by predators [20]. Many scholars have introduced the refuge effect to characterize this phenomenon, enabling a more accurate reflection of how ecological factors like prey defense strategies affect populations [21–23]. Based on the above research, and in combination with the biological characteristics that juvenile fish have relatively weak independent predation capabilities and rely on adult fish for assistance in completing the predation process, and that prey utilize the environment to avoid predation by predators, we incorporate the aforementioned intraspecific cooperative relationship and the prey refuge effect into the framework to construct a dynamical model of aquaculture management. Furthermore, in the aquaculture management model, incorporating the dependence of nonlinear impulsive stocking of juvenile fish on adult fish density makes the model more consistent with actual aquaculture scenarios. To further advance the research on aquaculture management, we establish an aquaculture management model with stage structure, intraspecific cooperation, refuge effect, and nonlinearly impulsive releasing.

The primary framework of the paper is as follows: Section 2 contains the model formulation. In Section 3, we present several critical lemmas. In the research in Section 4, we address two major analyses: For one, we focus on the global asymptotic stability of the prey-vanishing periodic solution. For the other, we deal with the permanence of system (2.6). In Section 5, we validate our theoretical results through numerical simulations and analyze the effects of some crucial parameters. In Section 6, we present the conclusions.

2. Model formulation

The classic predator-prey model is

$$\begin{cases} \frac{dP(t)}{dt} = rP(t)\left(1 - \frac{P(t)}{K}\right) - \frac{\beta_1 P(t)}{1+\alpha_1 P(t)} P_Z(t), \\ \frac{dP_Z(t)}{dt} = \frac{k_1\beta_1 P(t)}{1+\alpha_1 P(t)} P_Z(t) - d_1 P_Z(t), \end{cases} \tag{2.1}$$

where $P(t)$ and $P_Z(t)$ denote the amount of prey and predator at time t, respectively. Assume that $P(t)$ conforms to the Logistical growth model, $r > 0$ is the intrinsic growth rate, and $K > 0$ is the environmental capacity. $\frac{\beta_1 P(t)}{1+\alpha_1 P(t)}$ is the Holling II functional response, the predation coefficient is denoted by $\beta_1 > 0$, and $\alpha_1 > 0$ is the half-saturation parameter. $k_1 > 0$ denotes the conversion efficiency of prey into predators, while $d_1 > 0$ represents the mortality rate of the predator population. In this paper, fish predators can be divided into adult fish predators $P_A(t)$ and juvenile fish predators $P_J(t)$. Furthermore, juvenile fish will transform into adult fish after a period of growth. Hence, model (2.1) is adjusted to the following form:

$$\begin{cases} \frac{dP(t)}{dt} = rP(t)\left(1 - \frac{P(t)}{K}\right) - \frac{\beta_1 P(t)}{1+\alpha_1 P(t)} P_J(t) - \frac{\beta_2 P(t)}{1+\alpha_2 P(t)} P_A(t), \\ \frac{dP_J(t)}{dt} = -(c_1 + d_1)P_J(t) + \frac{k_1\beta_1 P(t)}{1+\alpha_1 P(t)} P_J(t), \\ \frac{dP_A(t)}{dt} = c_1 P_J(t) - d_2 P_A(t) + \frac{k_2\beta_2 P(t)}{1+\alpha_2 P(t)} P_A(t), \end{cases} \tag{2.2}$$

where $c_1 > 0$ is the conversion rate from juvenile predators to adult predators.

Next, we consider the prey refuge effect. Furthermore, juvenile fish lack the independent foraging ability, so adult predators $P_A(t)$ are required to assist juvenile predators $P_J(t)$ in intraspecific cooperative predation. Here, $p_1, p_2 > 0$ denote the intraspecific cooperation coefficients of predators, and $\theta_1, \theta_2 > 0$ are the prey refuge coefficients. Hence, the functional response terms of the predators can be expressed as follows:

$$\underbrace{\frac{(\beta_1 + p_1 P_A(t))(1-\theta_1)P(t)}{1+\alpha_1(1-\theta_1)P(t)} P_J(t)}_{\substack{\text{The functional response with } P_A(t) \\ \text{assisting } P_J(t) \text{ and refuge effect.}}} \qquad \underbrace{\frac{(\beta_2 + p_2 P_A(t))(1-\theta_2)P(t)}{1+\alpha_2(1-\theta_2)P(t)} P_A(t)}_{\substack{\text{The functional response of intraspecific} \\ \text{cooperation and refuge effect.}}}$$

Therefore, model (2.2) can be modified as follows:

$$\begin{cases} \frac{dP(t)}{dt} = rP(t)\left(1 - \frac{P(t)}{K}\right) - \frac{(\beta_1+p_1P_A(t))(1-\theta_1)P(t)}{1+\alpha_1(1-\theta_1)P(t)} P_J(t) - \frac{(\beta_2+p_2P_A(t))(1-\theta_2)P(t)}{1+\alpha_2(1-\theta_2)P(t)} P_A(t), \\ \frac{dP_J(t)}{dt} = -(c_1 + d_1)P_J(t) + \frac{k_1(\beta_1+p_1P_A(t))(1-\theta_1)P(t)}{1+\alpha_1(1-\theta_1)P(t)} P_J(t), \\ \frac{dP_A(t)}{dt} = c_1 P_J(t) - d_2 P_A(t) + \frac{k_2(\beta_2+p_2P_A(t))(1-\theta_2)P(t)}{1+\alpha_2(1-\theta_2)P(t)} P_A(t), \end{cases} \tag{2.3}$$

Assume $\omega > 0$ denotes the pulse period, and adult predators are harvested at each time $(n+\sigma)\omega, (n \in \mathbb{Z}^+)$ with $0 < \sigma < 1$, and we adopt proportional harvesting, i.e.,

$$\triangle P_A(t) = P_A(t^+) - P_A(t) = -hP_A(t). \tag{2.4}$$

To prevent mutual interference between the harvesting of $P_A(t)$ and the release of $P_J(t)$, we stock juveniles at each time $(n+1)\omega, (n \in \mathbb{Z}^+)$. Moreover, in consideration of the economic value and resource constraints of adult fish, we adopt the following nonlinear stocking strategy, i.e.,

$$\triangle P_J(t) = P_J(t^+) - P_J(t) = \frac{\mu_{max}}{1 + \gamma_1 P_A(t)}, \tag{2.5}$$

the release quantity of juvenile fish depends on the number of adult fish, where $\mu_{max} > 0$ denotes the maximum releasing quantity of predator $P_J(t)$, and $\gamma_1 > 0$ is the half-saturation parameter, as $P_A(t) \to 0, \frac{\mu_{max}}{1+\gamma_1 P_A(t)} \to \mu_{max}$, $P_A(t) \to \infty, \frac{\mu_{max}}{1+\gamma_1 P_A(t)} \to 0$. Based on the above, we establish the following aquaculture management model with stage structure, intraspecific cooperation, refuge effect, and nonlinearly impulsive releasing:

$$\begin{cases} \left.\begin{aligned} \frac{dP(t)}{dt} &= rP(t)\left(1-\frac{P(t)}{K}\right) - \frac{(\beta_1+p_1P_A(t))(1-\theta_1)P(t)}{1+\alpha_1(1-\theta_1)P(t)}P_J(t) - \frac{(\beta_2+p_2P_A(t))(1-\theta_2)P(t)}{1+\alpha_2(1-\theta_2)P(t)}P_A(t),\\ \frac{dP_J(t)}{dt} &= -(c_1+d_1)P_J(t) + \frac{k_1(\beta_1+p_1P_A(t))(1-\theta_1)P(t)}{1+\alpha_1(1-\theta_1)P(t)}P_J(t),\\ \frac{dP_A(t)}{dt} &= c_1P_J(t) - d_2P_A(t) + \frac{k_2(\beta_2+p_2P_A(t))(1-\theta_2)P(t)}{1+\alpha_2(1-\theta_2)P(t)}P_A(t), \end{aligned}\right\} & \begin{aligned} &t \neq (n+\sigma)\omega,\\ &t \neq (n+1)\omega, \end{aligned}\\ \left.\begin{aligned} &\triangle P(t) = 0,\\ &\triangle P_J(t) = 0,\\ &\triangle P_A(t) = -hP_A(t), \end{aligned}\right\} t = (n+\sigma)\omega, & \\ \left.\begin{aligned} &\triangle P(t) = 0,\\ &\triangle P_J(t) = \frac{\mu_{max}}{1+\gamma_1 P_A(t)},\\ &\triangle P_A(t) = 0, \end{aligned}\right\} t = (n+1)\omega. & \end{cases} \tag{2.6}$$

3. Preliminaries and lemmas

Let $S(t) = (P(t), P_J(t), P_A(t))'$ denote the solution of system (2.6), $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}^3_+ = \{S \mid S \geq 0\}$, then $S : \mathbb{R}_+ \to \mathbb{R}^3_+$ is a piecewise continuous function. Let f denote the map defined by the right hand side of system (2.6). Based on [24], system (2.6) has a unique solution defined on $\mathbb{R}^3_+$.

Lemma 3.1. *Suppose*

$$I_1 = \{(P(t), P_J(t), P_A(t)) \mid P(t) \geq 0,\ P_J(t) \geq 0,\ P_A(t) \geq 0\},$$

$$I_2 = \{(P(t), P_J(t), P_A(t)) \mid P(t) > 0,\ P_J(t) > 0,\ P_A(t) > 0\},$$

then the sets I_1 and I_2 form invariant sets of system (2.6).

Proof. For $P(t)$, we have

$$P(t) = P(0^+)e^{\phi(t)},$$

$$\phi(t) = \int_0^t r\left(1-\frac{P(s)}{K}\right) - \frac{(\beta_1+p_1P_A(s))(1-\theta_1)}{1+\alpha_1(1-\theta_1)P(s)}P_J(s) - \frac{(\beta_2+p_2P_A(s))(1-\theta_2)}{1+\alpha_2(1-\theta_2)P(s)}P_A(s)\,ds,$$

given that the exponential function is non-negative, the positivity of $P(t)$ is dictated by $P(0^+)$. Hence, we have $P(t) > 0, \forall t > 0$.

For $P_J(t)$, since the impulsive release increases the amount of $P_J(t)$, it suffices to focus on the result without pulse. Then

$$P_J(t) = P_J(0^+)e^{\int_0^t -(c_1+d_1)+\frac{k_1(\beta_1+p_1P_J(s))(1-\theta_1)P(s)}{1+\alpha_1(1-\theta_1)P(s)}\,ds}, \forall t \in (0, +\infty).$$

Thus, the positivity of $P_J(t)$ is determined by $P_J(0^+)$.

For $P_A(t)$, because

$$\begin{cases}\dfrac{dP_A(t)}{dt} \geq -d_2P_A(t) + \dfrac{k_2(\beta_2 + p_2P_A(t))(1-\theta_2)P(t)}{1+\alpha_2(1-\theta_2)P(t)}P_A(t), t \neq (n+\sigma)\omega, \\ \triangle P_A(t) = -hP_A(t), t = (n+\sigma)\omega. \end{cases} \tag{3.1}$$

Hence, we utilize the comparative system [24] as follows:

$$\begin{cases}\dfrac{dq(t)}{dt} = -d_2q(t) + \dfrac{k_2(\beta_2 + p_2q(t))(1-\theta_2)P(t)}{1+\alpha_2(1-\theta_2)P(t)}q(t), t \neq (n+\sigma)\omega, \\ \triangle q(t) = -hq(t), t = (n+\sigma)\omega, \\ q(0^+) = P_A(0^+), \end{cases} \tag{3.2}$$

when $t \in (0, \sigma\omega]$, we can get

$$q(t) = q(0^+)e^{\int_0^t -d_2 + \frac{k_2(\beta_2+p_2q(s))(1-\theta_2)P(s)}{1+\alpha_2(1-\theta_2)P(s)}\, ds}.$$

Then $q(t) > 0, \forall t \in (0, \sigma\omega]$. For all $t \in ((n+\sigma)\omega, (n+\sigma+1)\omega]$, we use the method of mathematical induction as follows:

(1) When $n = 0, \forall t \in (\sigma\omega, (\sigma+1)\omega]$,

$$q(t) = q(\sigma\omega^+)e^{\int_{\sigma\omega}^t -d_2 + \frac{k_2(\beta_2+p_2q(s))(1-\theta_2)P(s)}{1+\alpha_2(1-\theta_2)P(s)}\, ds},$$

$$q(\sigma\omega^+) = (1-h)q(\sigma\omega), 0 < h < 1,$$

thus $q(t) > 0, \forall t \in (\sigma\omega, (\sigma+1)\omega]$.

(2) Suppose the conclusion holds when $n = k$. Next, we prove the case when $n = k+1, t \in ((k+1+\sigma)\omega, (k+2+\sigma)\omega]$,

$$q(t) = q((k+1+\sigma)\omega^+)e^{\int_{(k+1+\sigma)\omega}^t -d_2 + \frac{k_2(\beta_2+p_2q(s))(1-\theta_2)P(s)}{1+\alpha_2(1-\theta_2)P(s)}\, ds},$$

because $q((k+1+\sigma)\omega^+) > 0$, we can get $q(t) > 0, t \in ((k+1+\sigma)\omega, (k+2+\sigma)\omega]$.

Therefore, by mathematical induction, we can obtain $q(t) > 0$ for all $t > 0$. Further, $P_A(t) > 0$ for all $t > 0$. This proof is complete. □

Lemma 3.2. *The solution is uniformly bounded.* $\exists U_0 > 0$, *such that* $P(t) \leq U_0$, $P_J(t) \leq U_0$, *and* $P_A(t) \leq U_0$ *for every solutions of system* (2.6) *when t is taken to be large enough.*

Proof. Denote $Y_L(t) = kP(t) + P_J(t) + P_A(t)$, where $k = \max\{k_1, k_2\}$, thus

$$\begin{aligned} &\frac{dY_L(t)}{dt} + a_0Y_L(t) \\ =&kP'(t) + P_J'(t) + P_A'(t) + a_0[kP(t) + P_J(t) + P_A(t)] \\ \leq&kP(t)\left[r\left(1 - \frac{P(t)}{K}\right) + a_0\right] - (d_1 - a_0)P_J(t) - (d_2 - a_0)P_A(t), t \neq (n+\sigma)\omega, t \neq (n+1)\omega. \end{aligned} \tag{3.3}$$

Obviously, there must be an upper bound for (3.3) when $0 < a_0 < \min\{d_1, d_2\}$. Thus, $\exists G = \frac{K(a_0+r)}{2r} > 0$, then

$$\frac{dY_L(t)}{dt} + a_0Y_L(t) \leq G, t \neq (n+\sigma)\omega, t \neq (n+1)\omega,$$

and

$$\begin{cases}\frac{dY_L(t)}{dt}+a_0Y_L(t)\le G, t\ne(n+\sigma)\omega, t\ne(n+1)\omega,\\ Y_L(t^+)=Y_L(t)-hP_A(t), t=(n+\sigma)\omega,\\ Y_L(t^+)=Y_L(t)+\frac{\mu_{max}}{1+\gamma P_A(t)}, t=(n+1)\omega.\end{cases}\tag{3.4}$$

Further,

$$\left\{\frac{dY_L(t)}{dt}+a_0Y_L(t)\le G, t\ne n\omega, Y_L(t^+)\le Y_L(t)+\mu_{max}, t=n\omega.\right.\tag{3.5}$$

Hence,

$$\begin{aligned}Y_L(t)\le&\left(Y_L(0^+)-\frac{G}{a_0}\right)e^{-a_0t}+\frac{\mu_{max}(1-e^{na_0\omega})}{1-e^{a_0\omega}}e^{-a_0(t-n\omega)}+\frac{G}{a_0}\\ \le&\left(Y_L(0^+)-\frac{G}{a_0}\right)e^{-a_0t}+\frac{u_{max}\left(1-e^{-(na_0\omega)}\right)}{1-e^{-a_0\omega}}+\frac{G}{a_0}\\ \rightarrow&\frac{u_{max}}{1-e^{-a_0\omega}}+\frac{G}{a_0}(t\rightarrow\infty).\end{aligned}$$

Then, Y_L is uniformly bounded and $\exists U_0>0$, such that $P(t)\le U_0$, $P_J(t)\le U_0$, and $P_A(t)\le U_0$ for all solutions of system (2.6) when t is taken to be sufficiently large. The proof is finished. □

Taking into account the subsystem of system (2.6) with $P(t)=0$, we can obtain:

$$\begin{cases}\left.\begin{aligned}\frac{dP_J(t)}{dt}&=-(c_1+d_1)P_J(t),\\ \frac{dP_A(t)}{dt}&=c_1P_J(t)-d_2P_A(t),\end{aligned}\right\} t\ne(n+\sigma)\omega, t\ne(n+1)\omega,\\ \left.\begin{aligned}\triangle P_J(t)&=0,\\ \triangle P_A(t)&=-hP_A(t),\end{aligned}\right\} t=(n+\sigma)\omega,\\ \left.\begin{aligned}\triangle P_J(t)&=\frac{\mu_{max}}{1+\gamma_1P_A(t)},\\ \triangle P_A(t)&=0,\end{aligned}\right\} t=(n+1)\omega.\end{cases}\tag{3.6}$$

To facilitate subsequent analysis, we first state the following two theorems:

Theorem 3.1 ([25])**.** *The fixed point (P_J^*, P_A^*) is locally asymptotically stable if the coefficient matrix M of the linearized system satisfies three Jury conditions:*

$$\begin{cases}1-detM>0,\\ 1-trM+detM>0,\\ 1+trM+detM>0.\end{cases}$$

Theorem 3.2 (Contraction mapping theorem)**.** *Let X be a complete metric space and T a contraction mapping on X, then T has a unique fixed point (i.e., the equation $Tx=x$ has exactly one solution).*

Lemma 3.3. *If $L=max\{\triangle_2+\triangle_3+\triangle_4+\mu_{max}\gamma_1(\triangle_2+\triangle_3),\ \triangle_1+\mu_{max}\gamma_1\triangle_1\}<1$ holds, the positive fixed point (P_J^*, P_A^*) of subsystem* (3.6) *is globally asymptotically stable.*

Proof. We can calculate the analytical solution of subsystem (3.6) as follows:

$$\begin{cases} P_J(t)=\begin{cases} P_J(n\omega^+)e^{-(c_1+d_1)(t-n\omega)}, t\in(n\omega,(n+\sigma)\omega],\\ P_J((n+\sigma)\omega^+)e^{-(c_1+d_1)(t-(n+\sigma)\omega)}, t\in((n+\sigma)\omega,(n+1)\omega],\end{cases}\\ P_A(t)=\begin{cases} e^{-d_2(t-n\omega)}\left[P_A(n\omega^+)+\frac{c_1P_J(n\omega^+)\left(1-e^{-(c_1+d_1-d_2)(t-n\omega)}\right)}{c_1+d_1-d_2}\right], t\in(n\omega,(n+\sigma)\omega],\\ e^{-d_2(t-(n+\sigma)\omega)}\left[P_A((n+\sigma)\omega^+)+\frac{c_1P_J((n+\sigma)\omega^+)\left(1-e^{-(c_1+d_1-d_2)(t-(n+\sigma)\omega)}\right)}{c_1+d_1-d_2}\right], t\in((n+\sigma)\omega,(n+1)\omega],\end{cases}\end{cases} \tag{3.7}$$

and the stroboscopic map of subsystem (3.6):

$$\begin{cases} P_J((n+1)\omega^+)=\triangle_4P_J(n\omega^+)+\frac{\mu_{max}}{1+\gamma_1[\triangle_1P_A(n\omega^+)+\triangle_2P_J(n\omega^+)+\triangle_3P_J(n\omega^+)]},\\ P_A((n+1)\omega^+)=\triangle_1P_A(n\omega^+)+\triangle_2P_J(n\omega^+)+\triangle_3P_J(n\omega^+),\end{cases} \tag{3.8}$$

where

$$\begin{cases} \triangle_1=(1-h)e^{-d_2\omega}<1,\\ \triangle_2=\frac{(1-h)c_1e^{-d_2\omega}\left(1-e^{-(c_1+d_1-d_2)\sigma\omega}\right)}{c_1+d_1-d_2}>0,\\ \triangle_3=\frac{c_1e^{-(c_1+d_1-d_2)\sigma\omega-d_2\omega}\left(1-e^{-(c_1+d_1-d_2)(1-\sigma)\omega}\right)}{c_1+d_1-d_2}>0,\\ \triangle_4=e^{-(c_1+d_1)\omega}<1.\end{cases} \tag{3.9}$$

Further, the fixed point (P_J^*,P_A^*) of subsystem (3.6) is as follows:

$$\begin{cases} P_J^*=\frac{\sqrt{(1-\triangle_1)^2+\frac{4\gamma_1\mu_{max}(\triangle_2+\triangle_3)(1-\triangle_1)}{1-\triangle_4}}-(1-\triangle_1)}{2\gamma_1(\triangle_2+\triangle_3)},\\ P_A^*=\frac{\triangle_2+\triangle_3}{1-\triangle_1}P_J^*.\end{cases} \tag{3.10}$$

Let $(P_J^n,P_A^n)=(P_J(n\omega^+),P_A(n\omega^+))$, and $U^n=P_J^n-P_J^*$, $V^n=P_A^n-P_A^*$. Then, the linear system of (3.8) can be written in the following form:

$$\begin{pmatrix} U^{n+1}\\ V^{n+1}\end{pmatrix}=M\begin{pmatrix} U^n\\ V^n\end{pmatrix}, \tag{3.11}$$

where

$$M=\begin{pmatrix} \triangle_4-\frac{\gamma_1\mu_{max}(\triangle_2+\triangle_3)}{[1+\gamma_1(\triangle_1P_A^*+\triangle_2P_J^*+\triangle_3P_J^*)]^2} & \frac{-\gamma_1\mu_{max}\triangle_1}{[1+\gamma_1(\triangle_1P_A^*+\triangle_2P_J^*+\triangle_3P_J^*)]^2}\\ \triangle_2+\triangle_3 & \triangle_1\end{pmatrix}. \tag{3.12}$$

First, we calculate

$$\begin{aligned} 0&<\frac{\gamma_1\mu_{max}(\triangle_2+\triangle_3)}{[1+\gamma_1(\triangle_1P_A^*+\triangle_2P_J^*+\triangle_3P_J^*)]^2}\\ &=\frac{\gamma_1\mu_{max}(\triangle_2+\triangle_3)}{\left[1+\gamma_1\left(\frac{\triangle_1\left(\sqrt{(1-\triangle_1)^2+\frac{4\gamma_1\mu_{max}(\triangle_2+\triangle_3)(1-\triangle_1)}{1-\triangle_4}}-(1-\triangle_1)\right)}{2\gamma_1(1-\triangle_1)}+\frac{\sqrt{(1-\triangle_1)^2+\frac{4\gamma_1\mu_{max}(\triangle_2+\triangle_3)(1-\triangle_1)}{1-\triangle_4}}-(1-\triangle_1)}{2\gamma_1}\right)\right]^2}\\ &=\frac{4\gamma_1\mu_{max}(\triangle_2+\triangle_3)}{\left[2+\frac{1}{1-\triangle_1}\left(\sqrt{(1-\triangle_1)^2+\frac{4\gamma_1\mu_{max}(\triangle_2+\triangle_3)(1-\triangle_1)}{1-\triangle_4}}-(1-\triangle_1)\right)\right]^2}\\ &=\frac{4\gamma_1\mu_{max}(\triangle_2+\triangle_3)}{\left[1+\sqrt{1+\frac{4\gamma_1\mu_{max}(\triangle_2+\triangle_3)}{(1-\triangle_1)(1-\triangle_4)}}\right]^2}<1,\end{aligned}$$

then

$$\begin{aligned}
&1-detM=1-\triangle_1\triangle_4>0,\\
&1-trM+detM=1-\triangle_1-\triangle_4+\frac{\gamma_1\mu_{max}(\triangle_2+\triangle_3)}{[1+\gamma_1(\triangle_1P_A^*+\triangle_2P_J^*+\triangle_3P_J^*)]^2}+\triangle_1\triangle 4\\
&\qquad=(1-\triangle_1)(1-\triangle_4)+\frac{\gamma_1\mu_{max}(\triangle_2+\triangle_3)}{[1+\gamma_1(\triangle_1P_A^*+\triangle_2P_J^*+\triangle_3P_J^*)]^2}>0,\\
&1+trM+detM=1+\triangle_1+\triangle_4+\triangle_1\triangle_4-\frac{\gamma_1\mu_{max}(\triangle_2+\triangle_3)}{[1+\gamma_1(\triangle_1P_A^*+\triangle_2P_J^*+\triangle_3P_J^*)]^2}\\
&\qquad=(1+\triangle_1)(1+\triangle_4)-\frac{4\gamma_1\mu_{max}(\triangle_2+\triangle_3)}{\left[1+\sqrt{1+\frac{4\gamma_1\mu_{max}(\triangle_2+\triangle_3)}{(1-\triangle_1)(1-\triangle_4)}}\right]^2}>0.
\end{aligned}\tag{3.13}$$

According to Theorem 3.1, (P_J^*,P_A^*) is locally asymptotically stable.

Subsequently, we demonstrate the global attractiveness. Denote

$$F(x,y)=\left(\triangle_4x+\frac{\mu_{max}}{1+\gamma_1(\triangle_1y+\triangle_2x+\triangle_3x)},\triangle_1y+\triangle_2x+\triangle_3x\right):\ \mathbb{R}_+^2\to\mathbb{R}_+^2,$$

and $\|(x,y)\|=|x|+|y|$ for all $(x,y)\in\mathbb{R}_+^2$, then $(\mathbb{R}_+^2,\|\cdot\|)$ is a complete normed linear space. $\forall(x_1,y_1),\ (x_2,y_2)\in\mathbb{R}_+^2$, we have

$$\begin{aligned}
&\|F(x_1,y_1)-F(x_2,y_2)\|\\
=&\left|\triangle_4(x_1-x_2)+\frac{\mu_{max}}{1+\gamma_1(\triangle_1y_1+\triangle_2x_1+\triangle_3x_1)}-\frac{\mu_{max}}{1+\gamma_1(\triangle_1y_2+\triangle_2x_2+\triangle_3x_2)}\right|\\
&+|\triangle_1(y_1-y_2)+\triangle_2(x_1-x_2)+\triangle_3(x_1-x_2)|\\
\le&\triangle_4|x_1-x_2|+\triangle_1|y_1-y_2|+\triangle_2|x_1-x_2|+\triangle_3|x_1-x_2|\\
&+\left|\frac{\mu_{max}}{1+\gamma_1(\triangle_1y_1+\triangle_2x_1+\triangle_3x_1)}-\frac{\mu_{max}}{1+\gamma_1(\triangle_1y_2+\triangle_2x_2+\triangle_3x_2)}\right|\\
\le&\left[\triangle_2+\triangle_3+\triangle_4+\mu_{max}\gamma_1(\triangle_2+\triangle_3)\right]|x_1-x_2|+(\triangle_1+\mu_{max}\gamma_1\triangle_1)|y_1-y_2|\\
\le&L(|x_1-x_2|+|y_1-y_2|)\\
=&L\|(x_1,y_1)-(x_2,y_2)\|,
\end{aligned}$$

because $L<1$, thus F is a contraction mapping. According to Theorem 3.2, $F^n(P_J^0,P_A^0)\to(P_J^*,P_A^*)$, $(n\to\infty)$ for all $(P_J^0,P_A^0)\in\mathbb{R}_+^2$. Hence, the fixed point (P_J^*,P_A^*) is globally asymptotically stable. □

Lemma 3.4. *The subsystem* (3.6) *has the unique positive period solution* $(\widetilde{P_J(t)},\widetilde{P_A(t)})'$*, and is globally asymptotically stable.*

$$\begin{cases}
\widetilde{P_J(t)}=\begin{cases}P_J^*e^{-(c_1+d_1)(t-n\omega)},t\in(n\omega,(n+\sigma)\omega],\\ P_J^*e^{-(c_1+d_1)(t-(n+\sigma)\omega)},t\in((n+\sigma)\omega,(n+1)\omega],\end{cases}\\
\widetilde{P_A(t)}=\begin{cases}e^{-d_2(t-n\omega)}\left[P_A^*+\frac{c_1P_J^*\left(1-e^{-(c_1+d_1-d_2)(t-n\omega)}\right)}{c_1+d_1-d_2}\right],t\in(n\omega,(n+\sigma)\omega],\\ e^{-d_2(t-(n+\sigma)\omega)}\left[P_A^*+\frac{c_1P_J^*\left(1-e^{-(c_1+d_1-d_2)(t-(n+\sigma)\omega)}\right)}{c_1+d_1-d_2}\right],t\in((n+\sigma)\omega,(n+1)\omega].\end{cases}
\end{cases}\tag{3.14}$$

We take series of parameters: $c_1 = 0.7$, $d_1 = 0.9$, $d_2 = 0.5$, $\sigma = 0.5$, $\omega = 2$, $h = 0.5$, $\mu_{max} = 3$, $\gamma_1 = 0.5$, and the period solution is globally asymptotically stable, as shown in Figure 1.

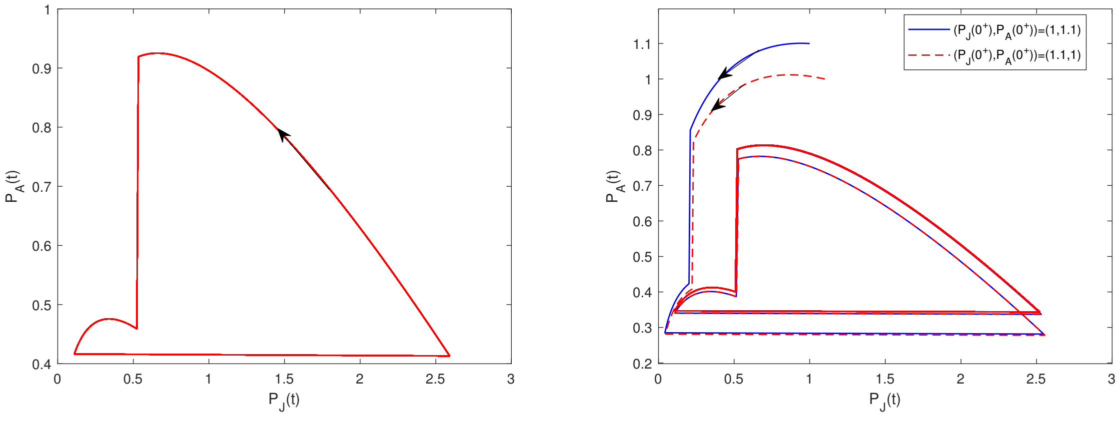


Figure 1. (a) The phase portrait. (b) The stability of the period solution.

In this paper, the supposed conditions are:

$$
\begin{aligned}
&(H_0) \quad max\{\triangle_2+\triangle_3+\triangle_4+\mu_{max}\gamma_1(\triangle_2+\triangle_3),\ \triangle_1+\mu_{max}\gamma_1\triangle_1\}<1,\\
&(H_1) \quad r\omega<\int_0^\omega (1-\theta_1)\left(\beta_1+p_1\widetilde{P_A(t)}\right)\widetilde{P_J(t)}+(1-\theta_2)\left(\beta_2+p_2\widetilde{P_A(t)}\right)\widetilde{P_A(t)}\,dt,\\
&(H_2) \quad (1+\triangle_1)(1+\triangle_4)>\frac{\mu_{max}\gamma_1 c_1 e^{-d_2\omega}}{\left(1+\gamma_1\widetilde{P_A(\omega)}\right)^2(c_1+d_1-d_2)}\left(1-e^{-(c_1+d_1-d_2)\omega}\right),\\
&(H_3) \quad r\omega<\int_0^\omega \frac{\left(\beta_1+p_1\widetilde{P_A(t)}\right)(1-\theta_1)}{1+\alpha_1(1-\theta_1)K}\widetilde{P_J(t)}+\frac{\left(\beta_2+p_2\widetilde{P_A(t)}\right)(1-\theta_2)}{1+\alpha_2(1-\theta_2)K}\widetilde{P_A(t)}\,dt,\\
&(H_4) \quad r\omega>\int_0^\omega (1-\theta_1)\left(\beta_1+p_1\widetilde{P_A(t)}\right)\widetilde{P_J(t)}+(1-\theta_2)\left(\beta_2+p_2\widetilde{P_A(t)}\right)\widetilde{P_A(t)}\,dt,
\end{aligned}
$$

where

$$
\begin{aligned}
Q &= \sqrt{(1-\triangle_1)^2+\frac{4\gamma_1\mu_{max}(\triangle_2+\triangle_3)(1-\triangle_1)}{1-\triangle_4}}-(1-\triangle_1),\\
\widetilde{P_A(\omega)} &= \frac{Q}{2\gamma_1(1-\triangle_1)}e^{-d_2(\omega-\sigma\omega)}+\frac{c_1 Q}{2\gamma_1(\triangle_2+\triangle_3)(c_1+d_1-d_2)}\left(e^{-d_2(\omega-\sigma\omega)}-e^{-(c_1+d_1)(\omega-\sigma\omega)}\right),\\
\int_0^\omega \widetilde{P_J(t)}dt &= \frac{P_J^*}{c_1+d_1}\left(2-e^{-(c_1+d_1)\sigma\omega}-e^{-(c_1+d_1)(\omega-\sigma\omega)}\right),\\
\int_0^\omega \widetilde{P_A(t)}dt &= \left(\frac{P_A^*}{d_2}+\frac{c_1P_J^*}{d_2(c_1+d_1-d_2)}\right)\left(2-e^{-d_2\sigma\omega}-e^{-d_2(\omega-\sigma\omega)}\right)\\
&\quad+\frac{c_1P_J^*}{(c_1+d_1)(c_1+d_1-d_2)}\left(e^{-(c_1+d_1)\sigma\omega}+e^{-(c_1+d_1)(\omega-\sigma\omega)}-2\right).
\end{aligned}
$$

4. The dynamics

Theorem 4.1. *Assuming that conditions (H_1) and (H_2) are satisfied, the periodic solution of prey-vanishing $(0, \widetilde{P_J(t)}, \widetilde{P_A(t)})'$ of system (2.6) is proven to be locally asymptotically stable.*

Proof. Suppose $E_1 = P(t), E_2 = P_J(t) - \widetilde{P_J(t)}, E_3 = P_A(t) - \widetilde{P_A(t)}$. By Taylor expansion, system (2.6) can be expressed in the form presented below:

$$\begin{cases} \left.\begin{aligned} \frac{dE_1(t)}{dt} &= \left[r - (1-\theta_1)\left(\beta_1 + p_1\widetilde{P_A(t)}\right)\widetilde{P_J(t)} - (1-\theta_2)\left(\beta_2 + p_2\widetilde{P_A(t)}\right)\widetilde{P_A(t)}\right]E_1(t), \\ \frac{dE_2(t)}{dt} &= \left[k_1(1-\theta_1)\left(\beta_1 + p_1\widetilde{P_A(t)}\right)\widetilde{P_J(t)}\right]E_1(t) - (c_1+d_1)E_2(t), \\ \frac{dE_3(t)}{dt} &= \left[k_2(1-\theta_2)\left(\beta_2 + p_2\widetilde{P_A(t)}\right)\widetilde{P_A(t)}\right]E_1(t) + c_1E_2(t) - d_2E_3(t), \end{aligned}\right\} t \neq (n+\sigma)\omega, t \neq (n+1)\omega, \\ \left.\begin{aligned} &\triangle E_1(t) = 0, \\ &\triangle E_2(t) = 0, \\ &\triangle E_3(t) = -hE_3(t), \end{aligned}\right\} t = (n+\sigma)\omega, \\ \left.\begin{aligned} &\triangle E_1(t) = 0, \\ &\triangle E_2(t) = \frac{-\mu_{max}\gamma_1}{\left(1+\gamma_1\widetilde{P_A(t)}\right)^2}E_3(t), \\ &\triangle E_3(t) = 0, \end{aligned}\right\} t = (n+1)\omega, \end{cases} \tag{4.1}$$

the fundamental solution matrix is as follows:

$$\Phi'(t) = \begin{pmatrix} r - (1-\theta_1)\left(\beta_1 + p_1\widetilde{P_A(t)}\right)\widetilde{P_J(t)} - (1-\theta_2)\left(\beta_2 + p_2\widetilde{P_A(t)}\right)\widetilde{P_A(t)} & 0 & 0 \\ k_1(1-\theta_1)\left(\beta_1 + p_1\widetilde{P_A(t)}\right)\widetilde{P_J(t)} & -(c_1+d_1) & 0 \\ k_2(1-\theta_2)\left(\beta_2 + p_2\widetilde{P_A(t)}\right)\widetilde{P_A(t)} & c_1 & -d_2 \end{pmatrix}\Phi(t), \tag{4.2}$$

then

$$\Phi(t) = \begin{pmatrix} e^{\int_0^t r-(1-\theta_1)\left(\beta_1+p_1\widetilde{P_A(s)}\right)\widetilde{P_J(s)}-(1-\theta_2)\left(\beta_2+p_2\widetilde{P_A(s)}\right)\widetilde{P_A(s)}\, ds} & 0 & 0 \\ \bigstar_1 & e^{-(c_1+d_1)t} & 0 \\ \bigstar_2 & \frac{c_1e^{-d_2t}}{c_1+d_1-d_2}\left(1 - e^{-(c_1+d_1-d_2)t}\right) & e^{-d_2t}, \end{pmatrix}, \tag{4.3}$$

and

$$\begin{pmatrix} E_1((n+\sigma)\omega^+) \\ E_2((n+\sigma)\omega^+) \\ E_3((n+\sigma)\omega^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-h \end{pmatrix}\begin{pmatrix} E_1((n+\sigma)\omega) \\ E_2((n+\sigma)\omega) \\ E_3((n+\sigma)\omega) \end{pmatrix}, \tag{4.4}$$

$$\begin{pmatrix} E_1((n+1)\omega^+) \\ E_2((n+1)\omega^+) \\ E_3((n+1)\omega^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{-\mu_{max}\gamma_1}{\left(1+\gamma_1P_A(\widetilde{(n+1)}\omega)\right)^2} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} E_1((n+1)\omega) \\ E_2((n+1)\omega) \\ E_3((n+1)\omega) \end{pmatrix}. \tag{4.5}$$

Further, we can calculate the monodromy matrix as follows:

$$
\begin{aligned}
\Lambda &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-h \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{-\mu_{max}\gamma_1}{\left(1+\gamma_1\widetilde{P_A(\omega)}\right)^2} \\ 0 & 0 & 1 \end{pmatrix}\Phi(\omega) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{-\mu_{max}\gamma_1}{\left(1+\gamma_1\widetilde{P_A(\omega)}\right)^2} \\ 0 & 0 & 1-h \end{pmatrix}\begin{pmatrix} A & 0 & 0 \\ \bigstar_1 & e^{-(c_1+d_1)\omega} & 0 \\ \bigstar_2 & \frac{c_1 e^{-d_2\omega}}{c_1+d_1-d_2}\left(1-e^{-(c_1+d_1-d_2)\omega}\right) & e^{-d_2\omega} \end{pmatrix} \\
&= \begin{pmatrix} A & 0 & 0 \\ \bigstar_1 & e^{-(c_1+d_1)\omega}-\frac{\mu_{max}\gamma_1 c_1 e^{-d_2\omega}}{\left(1+\gamma_1\widetilde{P_A(\omega)}\right)^2(c_1+d_1-d_2)}\left(1-e^{-(c_1+d_1-d_2)\omega}\right) & -\frac{\mu_{max}\gamma_1}{\left(1+\gamma_1\widetilde{P_A(\omega)}\right)^2}e^{-d_2\omega} \\ \bigstar_2 & \frac{(1-h)c_1 e^{-d_2\omega}}{c_1+d_1-d_2}\left(1-e^{-(c_1+d_1-d_2)\omega}\right) & (1-h)e^{-d_2\omega} \end{pmatrix},
\end{aligned}
\tag{4.6}
$$

where

$$A = e^{\int_0^\omega r-(1-\theta_1)\left(\beta_1+p_1\widetilde{P_A(t)}\right)\widetilde{P_J(t)}-(1-\theta_2)\left(\beta_2+p_2\widetilde{P_A(t)}\right)\widetilde{P_A(t)}\,dt}.$$

The eigenvalues λ_1, λ_2, and λ_3 of Λ determine the stability of $(0, \widetilde{P_J(t)}, \widetilde{P_A(t)})'$, and the $\bigstar_1$ and $\bigstar_2$ in matrix Λ cannot affect its result. Based on condition (H_1), we can get $\lambda_1 = A < 1$, and λ_2 and λ_3 are the eigenvalues of J, where

$$J = \begin{pmatrix} e^{-(c_1+d_1)\omega}-\frac{\mu_{max}\gamma_1 c_1 e^{-d_2\omega}}{\left(1+\gamma_1\widetilde{P_A(\omega)}\right)^2(c_1+d_1-d_2)}\left(1-e^{-(c_1+d_1-d_2)\omega}\right) & -\frac{\mu_{max}\gamma_1}{\left(1+\gamma_1\widetilde{P_A(\omega)}\right)^2}e^{-d_2\omega} \\ \frac{(1-h)c_1 e^{-d_2\omega}}{c_1+d_1-d_2}\left(1-e^{-(c_1+d_1-d_2)\omega}\right) & (1-h)e^{-d_2\omega} \end{pmatrix}, \tag{4.7}$$

according to the condition (H_2), we have

$$
\begin{aligned}
1-detJ &= 1-(1-h)e^{-(c_1+d_1+d_2)\omega} = 1-\triangle_1\triangle_4 > 0, \\
1-trJ+detJ &= [1-(1-h)e^{-d_2\omega}][1-e^{-(c_1+d_1)\omega}] + \frac{\mu_{max}\gamma_1 c_1 e^{-d_2\omega}}{\left(1+\gamma_1\widetilde{P_A(\omega)}\right)^2(c_1+d_1-d_2)}\left(1-e^{-(c_1+d_1-d_2)\omega}\right) \\
&= (1-\triangle_1)(1-\triangle_4) + \frac{\mu_{max}\gamma_1 c_1 e^{-d_2\omega}}{\left(1+\gamma_1\widetilde{P_A(\omega)}\right)^2(c_1+d_1-d_2)}\left(1-e^{-(c_1+d_1-d_2)\omega}\right) > 0, \\
1+trJ+detJ &= [1+(1-h)e^{-d_2\omega}][1+e^{-(c_1+d_1)\omega}] - \frac{\mu_{max}\gamma_1 c_1 e^{-d_2\omega}}{\left(1+\gamma_1\widetilde{P_A(\omega)}\right)^2(c_1+d_1-d_2)}\left(1-e^{-(c_1+d_1-d_2)\omega}\right) \\
&= (1+\triangle_1)(1+\triangle_4) - \frac{\mu_{max}\gamma_1 c_1 e^{-d_2\omega}}{\left(1+\gamma_1\widetilde{P_A(\omega)}\right)^2(c_1+d_1-d_2)}\left(1-e^{-(c_1+d_1-d_2)\omega}\right) > 0.
\end{aligned}
$$

Hence, we can get $\lambda_2 < 1$ and $\lambda_3 < 1$. Using Floquet theory [24], $(0, \widetilde{P_J(t)}, \widetilde{P_A(t)})'$ is locally asymptotically stable. □

Theorem 4.2. *If condition* (H_0) *and* (H_3) *hold, the periodic solution of prey-vanishing* $(0, \widetilde{P_J(t)}, \widetilde{P_A(t)}'$ *of system* (2.6) *is globally attractive.*

Proof. First, for $P(t)$, according to system (2.6), we have

$$\begin{cases}\frac{dP(t)}{dt} \le rP(t)\left(1-\frac{P(t)}{K}\right), t \ne n\omega,\\ \triangle P(t)=0, t=n\omega,\end{cases} \tag{4.8}$$

through simple calculation, we can get

$$P(t) \le \frac{1}{\frac{1}{K}(1-e^{-rt})+\frac{1}{P(0^+)}e^{-rt}}.$$

Hence, $P(t) \le K$ as $t \to \infty$. Using Lemma 3.4, it follows that $|P_J(t)-\widetilde{P_J(t)}| \to 0$, and $|P_A(t)-\widetilde{P_A(t)}| \to 0$. $\forall \epsilon_1 > 0$, $\exists T > 0$, and when $t > T$, it can be expressed in the form presented below:

$$\begin{cases}P_J(t) \ge \widetilde{P_J(t)} - \epsilon_1,\\ P_A(t) \ge \widetilde{P_A(t)} - \epsilon_1,\end{cases} \tag{4.9}$$

based on condition (H_3), by applying the local sign-preserving theorem for limits, we can choose the aforementioned case $\epsilon_1 > 0$, such that

$$\begin{aligned} r\omega < \int_{n\omega}^{(n+1)\omega} & \frac{\left(\beta_1 + p_1(\widetilde{P_A(t)}-\epsilon_1)\right)(1-\theta_1)}{1+\alpha_1(1-\theta_1)K}(\widetilde{P_J(t)}-\epsilon_1) \\ & + \frac{\left(\beta_2 + p_2(\widetilde{P_A(t)}-\epsilon_1)\right)(1-\theta_2)}{1+\alpha_2(1-\theta_2)K}(\widetilde{P_A(t)}-\epsilon_1)dt, \end{aligned}$$

according to the first equation within system (2.6), when $t \in (n\omega, (n+1)\omega]$, we have

$$\begin{cases}\frac{dP(t)}{dt} \le P(t)\left[r - \frac{\left(\beta_1+p_1(\widetilde{P_A(t)}-\epsilon_1)\right)(1-\theta_1)}{1+\alpha_1(1-\theta_1)K}(\widetilde{P_J(t)}-\epsilon_1) - \frac{\left(\beta_2+p_2(\widetilde{P_A(t)}-\epsilon_1)\right)(1-\theta_2)}{1+\alpha_2(1-\theta_2)K}(\widetilde{P_A(t)}-\epsilon_1)\right], t \ne n\omega,\\ P(t^+) = P(t), t = n\omega.\end{cases} \tag{4.10}$$

Let $n_0 \in Z^+$ and $n_0\omega > T$, then $\forall n > n_0$, we have

$$\begin{aligned} P(t) \le P(n\omega^+)e^{\int_{n\omega}^{t} r - \frac{\left(\beta_1+p_1(\widetilde{P_A(s)}-\epsilon_1)\right)(1-\theta_1)}{1+\alpha_1(1-\theta_1)K}(\widetilde{P_J(s)}-\epsilon_1) - \frac{\left(\beta_2+p_2(\widetilde{P_A(s)}-\epsilon_1)\right)(1-\theta_2)}{1+\alpha_2(1-\theta_2)K}(\widetilde{P_A(s)}-\epsilon_1)\,ds}, \\ t \in (n\omega, (n+1)\omega]. \end{aligned}$$

Further,

$$\begin{aligned} P((n+1)\omega^+) &\le P(n\omega^+)e^{r\omega - \int_{n\omega}^{(n+1)\omega} \frac{\left(\beta_1+p_1(\widetilde{P_J(t)}+\epsilon_1)\right)(1-\theta_1)}{1+\alpha_1(1-\theta_1)K}(\widetilde{P_J(t)}-\epsilon_1) + \frac{\left(\beta_2+p_2(\widetilde{P_A(t)}-\epsilon_1)\right)(1-\theta_2)}{1+\alpha_2(1-\theta_2)K}(\widetilde{P_A(t)}-\epsilon_1)dt} \\ &\triangleq P(n\omega^+)e^{\rho}, \end{aligned}$$

based on condition (H_3), $\lim\limits_{\epsilon_1 \to 0} \rho < 0$, then $\forall N_1 \in Z^+$, we have

$$P((n_0+N_1)\omega^+) \le P(n_0\omega^+)e^{N_1\rho} \to 0\ (N_1 \to \infty),$$

thus, $\lim\limits_{t \to \infty} P(t) = 0$.

Subsequently, for $P_J(t)$ and $P_A(t)$, $\forall 0 < \epsilon_2 < 0 < m_2 < min\left\{\frac{d_1}{k_1(\beta_1+p_1N)(1-\theta_1)}, \frac{d_2}{k_2(\beta_2+p_2N)(1-\theta_2)}\right\}$, $\exists n_1 \in Z^+$ when $t > n_1\omega$, such that $P(t) < \epsilon_2$, denote $(\Gamma_1(n_1\omega), \Gamma_2(n_1\omega)) = (P_J(n_1\omega), P_A(n_1\omega))$, we consider the following comparison system:

$$\begin{cases} \left.\begin{aligned} \frac{d\Gamma_1(t)}{dt} &= -(c_1 + d_1)\Gamma_1(t) + \frac{k_1\left(\beta_1 + p_1\Gamma_1(t)\right)(1-\theta_1)\epsilon_2}{1+\alpha_1(1-\theta_1)\epsilon_2}\Gamma_1(t), \\ \frac{d\Gamma_2(t)}{dt} &= c_1\Gamma_1(t) - d_2\Gamma_2(t) + \frac{k_2\left(\beta_2 + p_2\Gamma_2(t)\right)(1-\theta_2)\epsilon_2}{1+\alpha_2(1-\theta_2)\epsilon_2}\Gamma_2(t), \end{aligned}\right\} t \neq (n+\sigma)\omega, t \neq (n+1)\omega, \\ \left.\begin{aligned} \triangle\Gamma_1(t) &= 0, \\ \triangle\Gamma_2(t) &= -h\Gamma_2(t), \end{aligned}\right\} t = (n+\sigma)\omega, \\ \left.\begin{aligned} \triangle\Gamma_1(t) &= \frac{\mu_{max}}{1+\gamma_1\Gamma_2(t)}, \\ \triangle\Gamma_2(t) &= 0, \end{aligned}\right\} t = (n+1)\omega, \end{cases} \tag{4.11}$$

based on Lemma 3.4 and the comparison theorem, the corresponding result can be obtained:

$$\begin{cases} \widetilde{P_J(t)} - \epsilon \leq P_J(t) \leq \Gamma_1(t) \leq \widetilde{\Gamma_1(t)} + \epsilon, \\ \widetilde{P_A(t)} - \epsilon \leq P_A(t) \leq \Gamma_2(t) \leq \widetilde{\Gamma_2(t)} + \epsilon, \end{cases} \tag{4.12}$$

where $(\widetilde{\Gamma_1(t)}, \widetilde{\Gamma_2(t)})$ stands for the positive periodic solution of system (4.11). Let $\epsilon_2 \to 0$, and we can get

$$\begin{cases} \widetilde{P_J(t)} - \epsilon \leq P_J(t) \leq \widetilde{P_J(t)} + \epsilon, \\ \widetilde{P_A(t)} - \epsilon \leq P_A(t) \leq \widetilde{P_A(t)} + \epsilon, \end{cases} \tag{4.13}$$

then $P_J(t) \to \widetilde{P_J(t)}, P_A(t) \to \widetilde{P_A(t)}$. This proof is complete. □

Theorem 4.3. *If condition* (H_0) *and* (H_4) *hold, system* (2.6) *is permanent.*

Proof. Let $(P(t), P_J(t), P_A(t))'$ stand for a solution to system (2.6), satisfying $P(0^+) > 0$, $P_J(0^+) > 0$, and $P_A(0^+) > 0$. According to Lemma 3.2, it has been proven that $\exists U_0 > 0$, such that $P(t) \leqslant U_0$, $P_J(t) \leqslant U_0$, and $P_A(t) \leqslant U_0$ $(t \to \infty)$. Based on (4.9), as t becomes sufficiently large, it holds that $P_J(t) \geq \widetilde{P_J(t)} - \epsilon$ and $P_A(t) \geq \widetilde{P_A(t)} - \epsilon$ with $\epsilon > 0$. Thus,

$$P_J(t) \geq \min_{[0,\omega]} \widetilde{P_J(t)} = m_1, P_A(t) \geq \min_{[0,\omega]} \widetilde{P_A(t)} = \widehat{m}_1,$$

when t is sufficiently large. Thus, it suffices to investigate the lower bound of $P(t)$. Next, we will conduct the argument in two steps.

Step 1. $\exists m_2 > 0, t_1 \geq 0$, such that $P(t) \geq m_2$, we adopt the proof by contradiction, that is for all $0 < m_2 < min\left\{\frac{d_1}{k_1(\beta_1+p_1U_0)(1-\theta_1)}, \frac{d_2}{k_2(\beta_2+p_2U_0)(1-\theta_2)}\right\}$, $P(t) < m_2$, $\forall t \geq 0$. Then,

$$\begin{cases} \frac{dP_J(t)}{dt} \leq -(c_1 + d_1)P_J(t) + \frac{k_1(\beta_1+p_1U_0)(1-\theta_1)m_2}{1+\alpha_1(1-\theta_1)m_2}P_J(t), \\ \frac{dP_A(t)}{dt} \leq c_1P_J(t) - d_2P_A(t) + \frac{k_2(\beta_2+p_2U_0)(1-\theta_2m_2)}{1+\alpha_2(1-\theta_2)m_2}P_A(t), \end{cases} \tag{4.14}$$

consider the following comparison system with $(\psi_1(0), \psi_2(0)) = (P_J(0), P_A(0))$:

$$\begin{cases} \left.\begin{aligned} \frac{d\psi_1(t)}{dt} &= -(c_1+d_1)\psi_1(t) + \frac{k_1\,(\beta_1 + p_1 U_0)\,(1-\theta_1)m_2}{1+\alpha_1(1-\theta_1)m_2}\psi_1(t), \\ \frac{d\psi_2(t)}{dt} &= c_1\psi_1(t) - d_2\psi_2(t) + \frac{k_2\,(\beta_2 + p_2 U_0)\,(1-\theta_2)m_2}{1+\alpha_2(1-\theta_2)m_2}\psi_2(t), \end{aligned}\right\} t \neq (n+\sigma)\omega, t \neq (n+1)\omega, \\ \left.\begin{aligned} &\triangle\psi_1(t) = 0, \\ &\triangle\psi_2(t) = -h\psi_2(t), \end{aligned}\right\} t = (n+\sigma)\omega, \\ \left.\begin{aligned} &\triangle\psi_1(t) = \frac{\mu_{max}}{1+\gamma_1\psi_2(t)}, \\ &\triangle\psi_2(t) = 0, \end{aligned}\right\} t = (n+1)\omega, \end{cases} \tag{4.15}$$

based on Lemma 3.4 and the comparison theorem, $\forall\epsilon > 0, \exists N_0 \in Z^+$, when $t > N_0\omega$, we have

$$\begin{cases} P_J(t) \le \psi_1(t) \le \widetilde{\psi_1(t)} + \epsilon, \\ P_A(t) \le \psi_2(t) \le \widetilde{\psi_2(t)} + \epsilon, \end{cases} \tag{4.16}$$

where $(\widetilde{\psi_1(t)}, \widetilde{\psi_2(t)})$ stands for the positive periodic solution of the system (4.15). For the aforementioned $t > N_0\omega$, we have

$$\begin{cases} \frac{dP(t)}{dt} \ge P(t)\left[r\left(1-\frac{m_2}{K}\right) - (1-\theta_1)(\beta_1 + p_1(\widetilde{\psi_2(t)}+\epsilon))(\widetilde{\psi_1(t)}+\epsilon) - (1-\theta_2)(\beta_2 + p_2(\widetilde{\psi_2(t)}+\epsilon))(\widetilde{\psi_2(t)}+\epsilon)\right], t \neq n\omega, \\ \triangle P(t) = 0, t = n\omega, \end{cases} \tag{4.17}$$

then

$$P(t) \ge P(n\omega^+)e^{r\omega - \frac{rm_2\omega}{K} - \int_{n\omega}^{t} (1-\theta_1)\left(\beta_1 + p_1(\widetilde{\psi_2(s)}+\epsilon)\right)(\widetilde{\psi_1(s)}+\epsilon) + (1-\theta_2)\left(\beta_2 + p_2(\widetilde{\psi_2(s)}+\epsilon)\right)(\widetilde{\psi_2(s)}+\epsilon)\ ds}. \tag{4.18}$$

Further,

$$\begin{aligned} P((n+1)\omega^+) &\ge P(n\omega^+)e^{r\omega - \frac{rm_2\omega}{K} - \int_{n\omega}^{(n+1)\omega} (1-\theta_1)\left(\beta_1 + p_1(\widetilde{\psi_2(t)}+\epsilon)\right)(\widetilde{\psi_1(t)}+\epsilon) + (1-\theta_2)\left(\beta_2 + p_2(\widetilde{\psi_2(t)}+\epsilon)\right)(\widetilde{\psi_2(t)}+\epsilon)\ dt} \\ &\triangleq P(n\omega^+)e^{\eta}, \end{aligned} \tag{4.19}$$

based on condition (H_4), $\lim\limits_{\substack{m_2\to 0 \\ \epsilon\to 0}} \eta > 0$, then $\forall N_1 \in Z^+$, we have

$$P((N_0+N_1)\omega^+) \ge P(N_0\omega^+)e^{N_1\eta} \to \infty \ (N_1 \to \infty),$$

this is contrary to the boundness of $P(t)$. Hence, $\exists m_2 > 0, t_1 > 0$, such that $P(t_1) > m_2$.

Step 2. If $\forall t > t_1$, $P(t) \ge m_2$, then we finish this proof. If $\exists \hat{t_1} > t_1$, such that $P(\hat{t_1}) < m_2$, denote $D = \{t \mid t > t_1, P(t) < m_2\}$. Then $\exists t^* = inf\ D$. Based on the continuity of $P(t)$, $\forall t \in [t_1, t^*)$, $P(t) \ge m_2$, we have $P(t^*) = m_2$. Further, $\exists\varrho_1 \in Z^+$, such that $t^* \in (\varrho_1\omega, (\varrho_1+1)\omega]$. For the above ϱ_1, $\exists\varrho_2$, $\varrho_3 \in Z^+$, such that

$$\varrho_1 + \varrho_2 + \varrho_3 \ge N_0, \ e^{\eta_1(\varrho_2+1)\omega}e^{\varrho_3\eta} > 1,$$

where

$$\eta_1 = r\left(1 - \frac{m_2}{K}\right) - (\beta_1 + p_1 U_0)U_0 - (\beta_2 + p_2 U_0)U_0 < 0,$$

then $t_2 \in (t^*, (\varrho_1 + \varrho_2 + \varrho_3 + 1)\omega]$ exists, such that $P(t_2) \geq m_2$. Otherwise $P(t) < m_2$, $\forall t \in (t^*, (\varrho_1 + \varrho_2 + \varrho_3 + 1)\omega]$. Let the initial value $\psi_1((\varrho_1 + 1)\omega) = P_J(0), \psi_2((\varrho_1 + 1)\omega) = P_A(0)$. Based on (4.19), we have

$$P((\varrho_1 + \varrho_2 + \varrho_3 + 1)\omega^+) \geq P((\varrho_1 + \varrho_2 + 1)\omega^+)e^{\varrho_3\eta},$$

then the first equation within system (2.6) shows that

$$\begin{aligned}\frac{dP(t)}{dt} &\geq P(t)\left[r\left(1 - \frac{m_2}{K}\right) - (\beta_1 + p_1U_0)U_0 - (\beta_2 + p_2U_0)U_0\right] \\ &\triangleq P(t)\eta_1,\end{aligned} \tag{4.20}$$

integrating (4.20) on $(t^*, (\varrho_1 + \varrho_2 + 1)\omega]$, we have

$$P((\varrho_1 + \varrho_2 + \varrho_3 + 1)\omega^+) \geq P(t^*)e^{\eta_1(t-t^*)} \geq m_2e^{\eta_1(\varrho_2+1)\omega}.$$

Further,

$$P((\varrho_1 + \varrho_2 + \varrho_3 + 1)\omega^+) \geq m_2e^{\eta_1(\varrho_2+1)\omega}e^{\varrho_3\eta} > m_2,$$

this is contrary to the boundness of $P(t)$. Hence, $\exists t_2 \in (t^*, ((\varrho_1 + \varrho_2 + \varrho_3 + 1)\omega)]$, such that $P(t_2) \geq m_2$. Set $D_1 = \{t \mid t > t^*, P(t) > m_2\}$ and $\hat{t}^* = inf\ D_1$ exists. $\forall \hat{t}^* \in (t^*, \hat{t}^*]$, $P(t) \leq m_2$ and $P(\hat{t}^*) \geq m_2$. Then $\forall t \in (t^*, \hat{t}^*]$, and integrating (4.20) on $(t^*, \hat{t}^*]$, we have

$$P(t) \geq P(t^*)e^{\eta_1(t-t^*)} \geq m_2e^{\eta_1(\varrho_2+\varrho_3+1)\omega} \triangleq m_3.$$

Thus, $P(t) \geq m_3$, for $t \in [t^*, \hat{t}^*]$. Given that $P(\hat{t}^*) \geq m_3$, we can proceed with the same process. Hence, there exists a global attractor in system (2.6),

$$Q = \left\{(P, P_J, P_A) \mid P + P_J + P_A \leq U_0, P \geq m_3, P_J \geq m_1, P_A \geq \widehat{m}_1\right\}.$$

This proof is complete. □

Remark 4.1. *Denote*

$$\begin{aligned}B_0 &= min\left\{\frac{r\omega}{\int_0^\omega (1-\theta_1)\left(\beta_1 + p_1\widetilde{P_A(t)}\right)\widetilde{P_J(t)} + (1-\theta_2)\left(\beta_2 + p_2\widetilde{P_A(t)}\right)\widetilde{P_A(t)}dt},\right.\\ &\qquad\left.\frac{1}{max\left\{\triangle_2 + \triangle_3 + \triangle_4 + \mu_{max}\gamma_1(\triangle_2 + \triangle_3),\ \triangle_1 + \mu_{max}\gamma_1\triangle_1\right\}}\right\},\\ B_1 &= \frac{\mu_{max}\gamma_1c_1e^{-d_2\omega}}{(1-\triangle_1)(1-\triangle_4)\left(1+\gamma_1\widetilde{y_2(\omega)}\right)^2(c_1+d_1-d_2)}\left(1 - e^{-(c_1+d_1-d_2)\omega}\right),\\ B_2 &= \frac{r\omega}{\int_0^\omega \frac{\left(\beta_1 + p_1\widetilde{P_A(t)}\right)(1-\theta_1)}{1+\alpha_1(1-\theta_1)K}\widetilde{P_J(t)} + \frac{\left(\beta_2 + p_2\widetilde{P_A(t)}\right)(1-\theta_2)}{1+\alpha_2(1-\theta_2)K}\widetilde{P_A(t)}dt},\end{aligned}$$

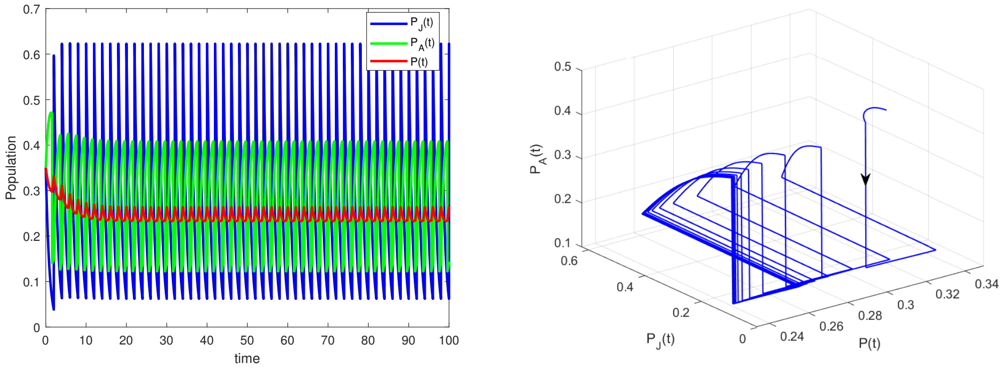
system (2.6) *is permanent when* $B_0 > 1$*, and if* $B_0 < 1$*,* $B_1 < 1$*, and* $B_2 < 1$*, then the periodic solution of prey-vanishing* $(0, \widetilde{P_J(t)}, \widetilde{P_A(t)})'$ *is globally asymptotically stable.*

5. Numerical simulations

In this section, we utilize numerical simulations to verify the aforementioned conclusions and further investigate the influence patterns of key parameters on the persistence condition B_0.

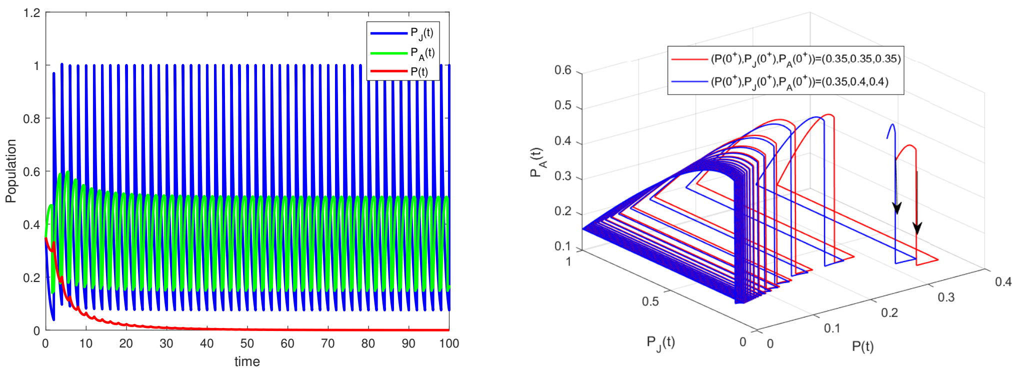
(1) Consider $(P(0^+), P_J(0^+), P_A(0^+)) = (0.35, 0.35, 0.35)$, $r = 0.6$, $K = 1$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\beta_1 = 1$, $p_1 = 0.4$, $\alpha_1 = 0.1$, $\beta_2 = 0.8$, $p_2 = 0.5$, $\alpha_2 = 0.2$, $c_1 = 0.8$, $d_1 = 0.5$, $k_1 = 0.6$, $d_2 = 0.3$, $k_2 = 0.7$, $h = 0.7$, $\omega = 2$, $\mu_{max} = 0.6$, $\gamma_1 = 1.5$, $\sigma = 0.8$, and it can be know through calculation that $B_0 = min\left\{\frac{1.2}{0.9594}, \frac{1}{0.2489}\right\} > 1$, so system (2.6) is permanent. The results of the numerical simulation are shown in Figure 2.

(2) Consider $r = 0.6$, $K = 1$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\beta_1 = 1$, $p_1 = 0.4$, $\alpha_1 = 0.1$, $\beta_2 = 0.8$, $p_2 = 0.5$, $\alpha_2 = 0.2$, $c_1 = 0.8$, $d_1 = 0.5$, $k_1 = 0.6$, $d_2 = 0.3$, $k_2 = 0.7$, $h = 0.7$, $\omega = 2$, $\mu_{max} = 1$, $\gamma_1 = 1.5$, $\sigma = 0.8$, and it can be know through calculation that $B_0 = min\left\{\frac{1.2}{1.6515}, \frac{1}{0.2758}\right\} < 1$, $B_1 = 0.1738 < 1$, $B_2 = \frac{1.2}{1.4795} < 1$, meaning the periodic solution of prey-vanishing $(0, \widetilde{P_J(t)}, \widetilde{P_A(t)})'$ is globally asymptotically stable. The results of the numerical simulation are shown in Figure 3.



(a) (b)

Figure 2. The permanence of system (2.6). (a) Time series of populations $(P(t), P_J(t), P_A(t))$. (b) Phase portrait of system (2.6).



(a) (b)

Figure 3. Global asymptotic stability of the prey-vanishing periodic solution in system (2.6). (a) Time series of populations $(P(t), P_J(t), P_A(t))$. (b) Phase portrait of system (2.6).

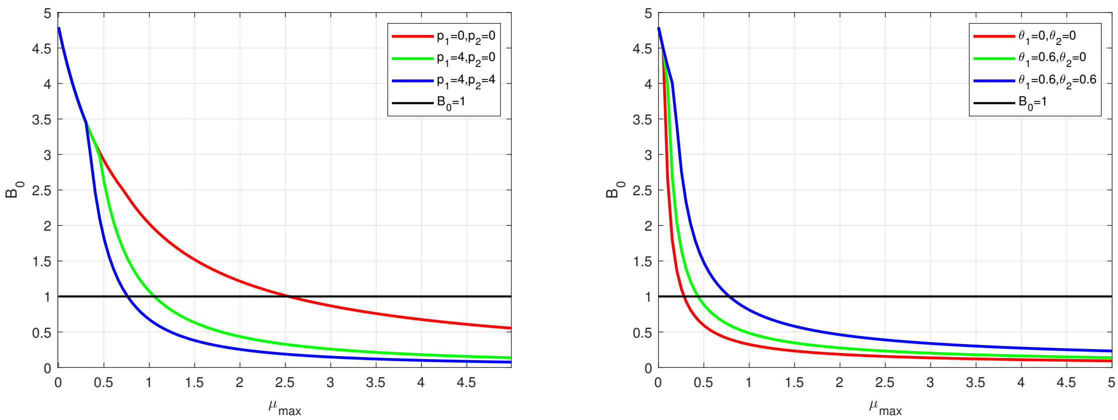
5.1. Analysis charts of persistence condition B_0

To explore the correlation between key parameters and the persistence condition B_0, consider the following two sets of parameters:

(1) $r = 0.8$, $K = 0.1$, $\theta_1 = 0.1$, $\theta_2 = 0.1$, $\beta_1 = 0.5$, $\alpha_1 = 0.1$, $\beta_2 = 0.4$, $\alpha_2 = 0.2$, $c_1 = 0.8$, $d_1 = 0.5$, $k_1 = 0.7$, $d_2 = 0.3$, $k_2 = 0.7$, $h = 0.7$, $\omega = 2$, $\gamma_1 = 2$, $\sigma = 0.8$,

(2) $r = 2$, $K = 0.1$, $\beta_1 = 8$, $p_1 = 0.5$, $\alpha_1 = 0.1$, $\beta_2 = 8$, $p_2 = 0.6$, $\alpha_2 = 0.2$, $c_1 = 0.8$, $d_1 = 0.5$, $k_1 = 0.7$, $d_2 = 0.3$, $k_2 = 0.7$, $h = 0.7$, $\omega = 2$, $\gamma_1 = 2$, $\sigma = 0.8$.

Then, the corresponding phase diagrams are shown in Figure 4. As illustrated in Figure 4(a), as the release amount of predator $P_J(t)$ increases, B_0 gradually decreases to a value less than 1. Therefore, an appropriate amount of releasing is beneficial for maintaining the balance of the system. Further, under the same value of μ_{max}, as the cooperation coefficient increases, B_0 gradually decreases, which indicates that intraspecific cooperation can effectively improve the predation rate. From Figure 4(b), as the prey $P(t)$ refuge coefficient increases, B_0 also increases accordingly, which indicates that the prey refuge coefficient can play a certain defensive role in the predation process of predators.



(a) **(b)**

Figure 4. Phase diagrams of changes in B_0 and μ_{max} under different cooperation and refuge coefficients. (a) Altered values of p_1 and p_2. (b) Altered values of θ_1 and θ_2.

For a more intuitive illustration of the association between the key parameters and B_0, we consider the following four sets of parameters:

(1) $r = 0.5$, $K = 0.1$, $\theta_1 = 0.8$, $\theta_2 = 0.8$, $\beta_1 = 1$, $p_1 = 0.8$, $\alpha_1 = 0.3$, $\beta_2 = 0.5$, $p_2 = 0.7$, $\alpha_2 = 0.6$, $c_1 = 0.8$, $d_1 = 0.5$, $k_1 = 0.6$, $d_2 = 0.3$, $k_2 = 0.7$, $h = 0.7$, $\gamma_1 = 1$, $\sigma = 0.8$,

(2) $r = 3$, $K = 0.1$, $\theta_1 = 0.2$, $\theta_2 = 0.1$, $\beta_1 = 8$, $\alpha_1 = 0.3$, $\beta_2 = 6$, $p_2 = 0.5$, $\alpha_2 = 0.6$, $c_1 = 0.8$, $d_1 = 0.5$, $k_1 = 0.6$, $d_2 = 0.3$, $k_2 = 0.7$, $h = 0.7$, $\mu_{max} = 0.6$, $\gamma_1 = 0.5$, $\sigma = 0.8$,

(3) $r = 3$, $K = 0.1$, $\theta_2 = 0.1$, $\beta_1 = 8$, $p_1 = 1.4$, $\alpha_1 = 0.3$, $\beta_2 = 6$, $p_2 = 0.5$, $\alpha_2 = 0.6$, $c_1 = 0.8$, $d_1 = 0.5$, $k_1 = 0.6$, $d_2 = 0.3$, $k_2 = 0.7$, $h = 0.7$, $\mu_{max} = 0.6$, $\gamma_1 = 0.5$, $\sigma = 0.8$,

(4) $r = 3$, $K = 0.1$, $\theta_2 = 0.1$, $\beta_1 = 8$, $\alpha_1 = 0.3$, $\beta_2 = 6$, $p_2 = 0.5$, $\alpha_2 = 0.6$, $c_1 = 0.8$, $d_1 = 0.5$, $k_1 = 0.6$, $d_2 = 0.3$, $k_2 = 0.7$, $h = 0.7$, $\omega = 2$, $\mu_{max} = 0.6$, $\gamma_1 = 0.6$, $\sigma = 0.8$,

Subsequently, the corresponding two-parameter analysis plots are presented in Figure 5. From Figure 5(a), when μ_{max} is small, ω exhibits a higher sensitivity to the persistence of system (2.6). From Figure 5(b), ω has a more significant effect on system persistence than p_1. From Figure 5(c), as ω

increases, system (2.6) is ultimately persistent regardless of the value of θ_1. From Figure 5(d), θ_1 has a more significant effect on system persistence than p_1. In conclusion, ω is relatively sensitive to the persistence of system (2.6). Therefore, in aquaculture practice, an appropriate pulse period is critical for maintaining the sustainable development of fishery resources.

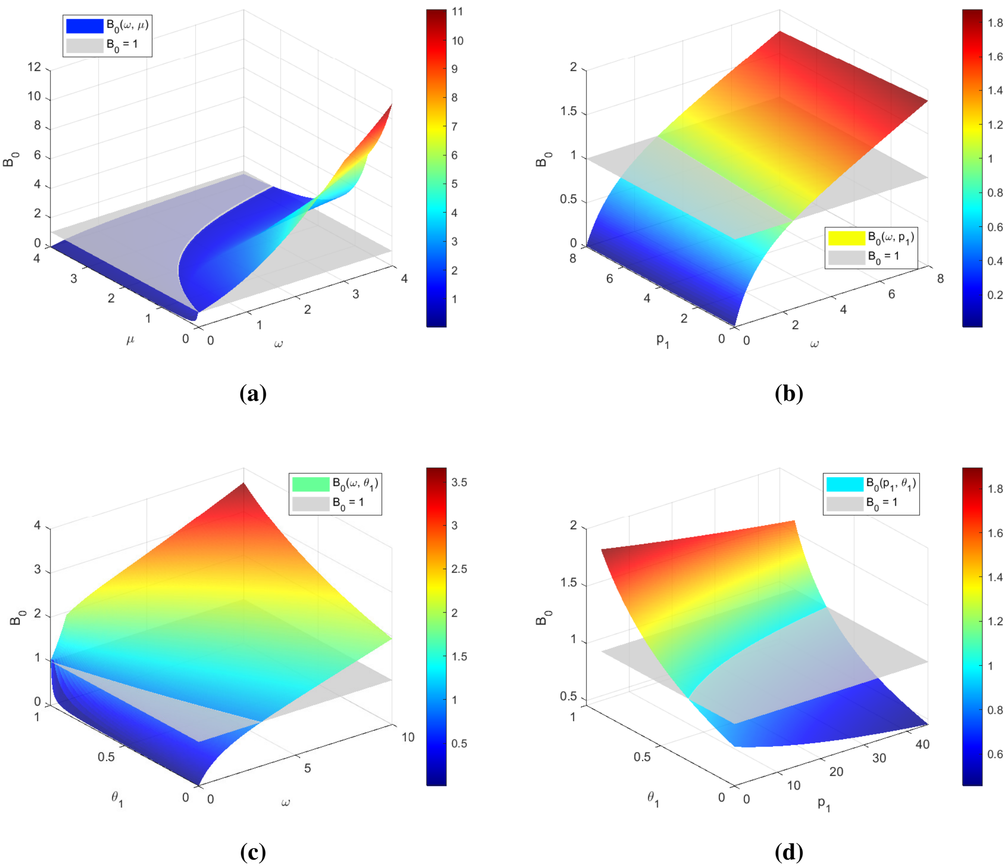


Figure 5. Phase diagram of persistence condition B_0 and crucial parameters. (a) Persistent condition B_0 with parameters ω and μ. (b) Persistent condition B_0 with parameters ω and p_1. (c) Persistent condition B_0 with parameters ω and θ_1. (d) Persistent condition B_0 with parameters p_1 and θ_1.

To identify the key parameters affecting the persistence condition B_0, we calculate the partial rank correlation coefficients (PRCC) of each parameter with B_0 and analyze their sensitivity to B_0. Additionally, the following rules are applied:

(1) A positive PRCC value indicates that the parameter is positively correlated with B_0, where an increase in the parameter leads to an increase in B_0.

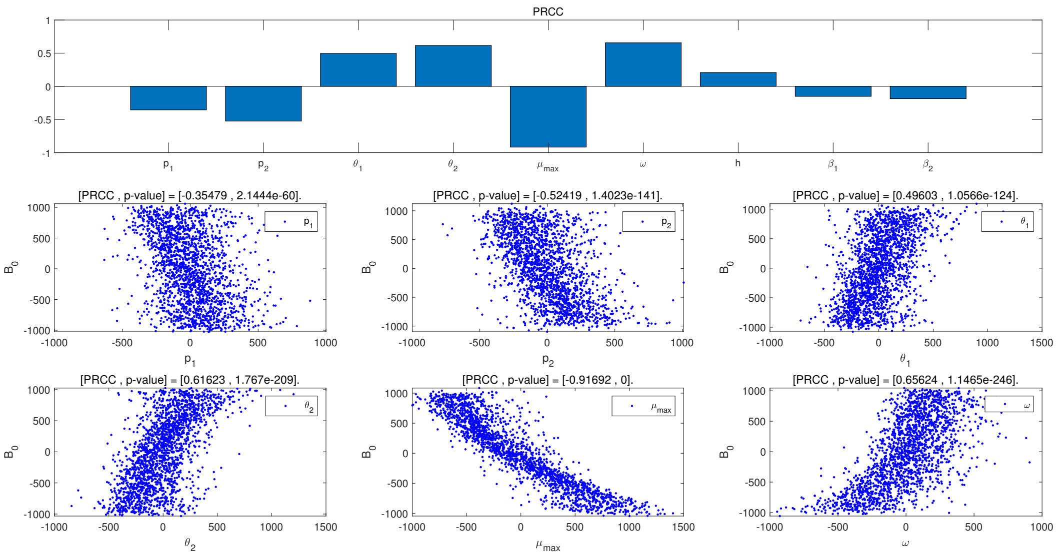
(2) A negative PRCC value indicates that the parameter is negatively correlated with B_0, where an increase in the parameter leads to a decrease in B_0.

(3) If $|PRCC| > 0.4$, the parameter exhibits a strong correlation with B_0, indicating it exerts a more significant influence on B_0.

(4) If $0.2 < |PRCC| < 0.4$, the parameter has a moderate impact on B_0.

(5) If $|PRCC| < 0.2$, the parameter has a weak impact on B_0.

Consider $r = 0.5$, $K = 0.5$, $\theta_1 = 0.5$, $\theta_2 = 0.3$, $\beta_1 = 0.6$, $p_1 = 5$, $\alpha_1 = 2$, $\beta_2 = 0.8$, $p_2 = 4$, $\alpha_2 = 1.2$, $k_1 = 0.5$, $k_2 = 0.6$, $c_1 = 0.7$, $d_1 = 0.2$, $d_2 = 0.3$, $\mu_{max} = 1.5$, $\gamma_1 = 1.5$, $\sigma = 0.8$, $\omega = 0.5$, $h = 0.7$, then the diagrams are shown in Figure 6. It can be concluded from Figure 6 that θ_1, θ_2, and ω are positively correlated with the persistence condition B_0, implying that increasing these parameters is beneficial for maintaining the permanence of system (2.6). In contrast, p_1, p_2, and μ_{max} are negatively correlated with the persistence condition B_0, and excessive increases in these parameters will cause the system to become imbalanced. Given that the PRCC values of p_2, θ_1, θ_2, μ_{max}, and ω are each greater than 0.4, they hold a vital role in the system. Further, the PRCC value of p_2 is higher than that of p_1, which implies that the cooperative predation efficiency among adult fish predators is higher than the efficiency of adult fish assisting juvenile fish in predation (see Table 1).



(a)

Figure 6. The PRCC values of the persistence condition B_0 regarding key parameters and the scatter plots of Parameters p_1, p_2, θ_1, θ_2, μ_{max}, and ω.

Table 1. The influence of key parameters on the persistence condition B_0.

Parameter	PRCC value	Correlation	Influence intensity
p_1	-0.35	Negative	Moderate
p_2	-0.52	Negative	Stronger
θ_1	0.49	Positive	Stronger
θ_2	0.61	Positive	Strong
$\mu_{\max}$	-0.91	Negative	Strong
ω	0.65	Positive	Strong
h	0.24	Positive	Weak
β_1	-0.21	Negative	Weak
β_2	-0.22	Negative	Weak

In fishery aquaculture management, to ensure the stable operation of the aquaculture system and the sustainable development of fishery resources, it is necessary not only to strictly regulate the release amount of juvenile fish in combination with the carrying capacity of aquaculture water and the growth characteristics of fish populations, but also to rationally plan the impulsive release period based on the dynamic characteristics of the system to avoid the impairment of the aquaculture system's stability caused by excessively frequent releases.

5.2. *Bifurcation analysis*

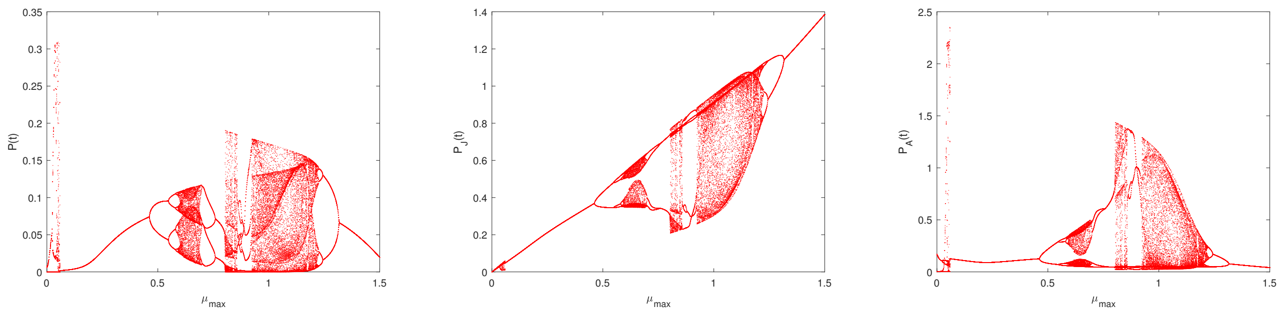
To reflect the rich dynamical properties of the system, such as periodic bifurcation and chaos, we select μ_{max}, h, and ω as the bifurcation parameters. Consider the following three sets of parameters:

(1) $r = 2$, $K = 1$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\beta_1 = 5$, $p_1 = 0.4$, $\alpha_1 = 0.1$, $\beta_2 = 5$, $p_2 = 0.5$, $\alpha_2 = 0.2$, $c_1 = 0.8$, $d_1 = 0.5$, $k_1 = 1$, $d_2 = 0.3$, $k_2 = 5$, $h = 0.7$, $\omega = 8$, $\gamma_1 = 2$, $\sigma = 0.8$, $P(0^+) = P_J(0^+) = P_A(0^+) = 0.5$, adjusting μ_{max} from 0 to 1.5.

(2) $r = 3$, $K = 0.5$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\beta_1 = 5$, $p_1 = 0.4$, $\alpha_1 = 0.1$, $\beta_2 = 3$, $p_2 = 0.5$, $\alpha_2 = 0.2$, $c_1 = 0.6$, $d_1 = 0.5$, $k_1 = 1$, $d_2 = 0.2$, $k_2 = 5$, $\omega = 8$, $\mu_{max} = 0.8$, $\gamma_1 = 1$, $\sigma = 0.8$, $P(0^+) = P_J(0^+) = P_A(0^+) = 0.5$, adjusting h from 0 to 1.

(3) $r = 3$, $K = 0.5$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\beta_1 = 5$, $p_1 = 0.4$, $\alpha_1 = 0.1$, $\beta_2 = 6$, $p_2 = 0.5$, $\alpha_2 = 0.2$, $c_1 = 0.6$, $d_1 = 0.5$, $k_1 = 2$, $d_2 = 0.2$, $k_2 = 4$, $h = 0.7$, $\mu_{max} = 0.8$ $\gamma_1 = 1$, $\sigma = 0.8$, $P(0^+) = P_J(0^+) = P_A(0^+) = 0.5$, adjusting ω from 3 to 7.

Subsequently, the bifurcation parameter diagrams corresponding to μ_{max}, h, and ω are shown in Figures 7, 9, and 11, respectively. Moreover, Figures 8, 10, and 12 show the phase portraits of system (2.6) corresponding to different values of μ_{max}, h, and ω in sequence, including period-one solution, period-two solution, period-four solution, and chaos. In aquaculture management, our primary objective is to maintain the dynamic balance of fish population abundance. The occurrence of chaotic behavior tends to induce drastic fluctuations in fish population abundance, and thus targeted measures should be taken to avoid such phenomena. Results from the bifurcation diagram demonstrate that period-doubling bifurcation is a critical precursor to the emergence of chaotic behavior. Therefore, dynamic monitoring of fish population abundance should be strengthened; once abnormal fluctuations in population abundance are detected, regulatory intervention measures should be implemented in a timely manner to prevent the aquaculture system from entering a chaotic state.



(a) (b) (c)

Figure 7. Bifurcation parameter diagram for μ_{max}. (a) Bifurcation diagram of $P(t)$. (b) Bifurcation diagram of $P_J(t)$. (c) Bifurcation diagram of $P_A(t)$.

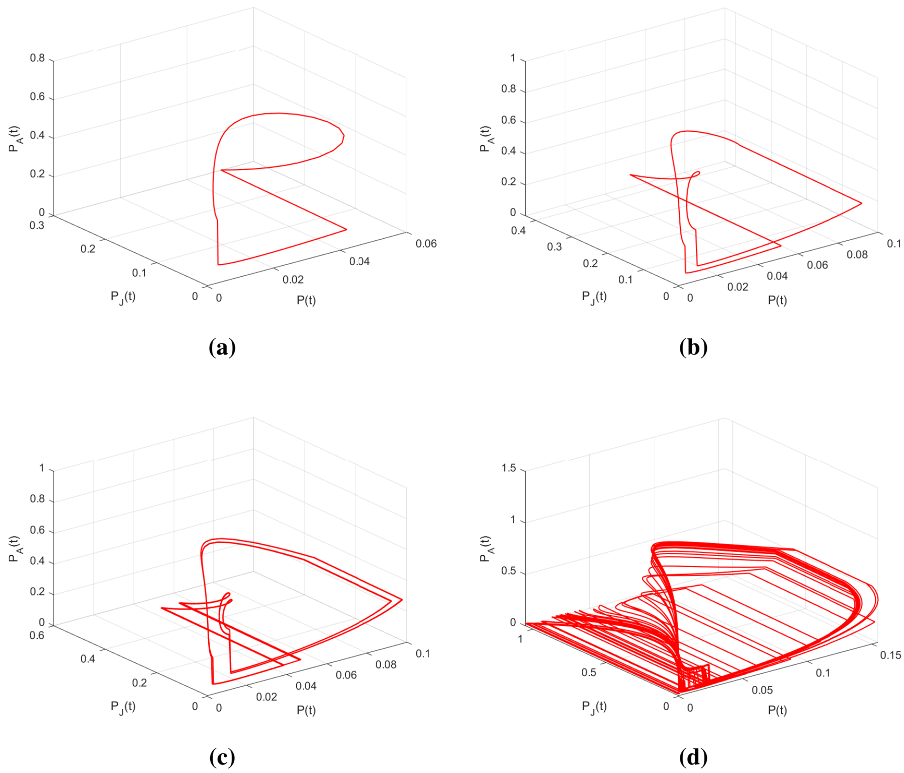


Figure 8. Phase diagram of system (2.6) with different μ_{max} values. (a) $\mu_{max} = 0.3$, period-one solution. (b) $\mu_{max} = 0.5$, period-two solution (c) $\mu_{max} = 0.55$, period-four solution. (d) $\mu_{max} = 1.1$, chaos.

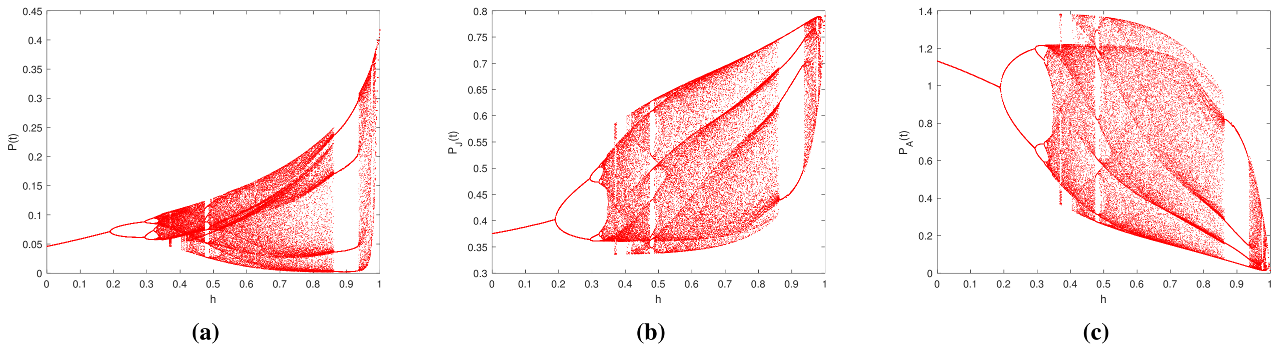


Figure 9. Bifurcation parameter diagram for h. (a) Bifurcation diagram of $P(t)$. (b) Bifurcation diagram of $P_J(t)$. (c) Bifurcation diagram of $P_A(t)$.

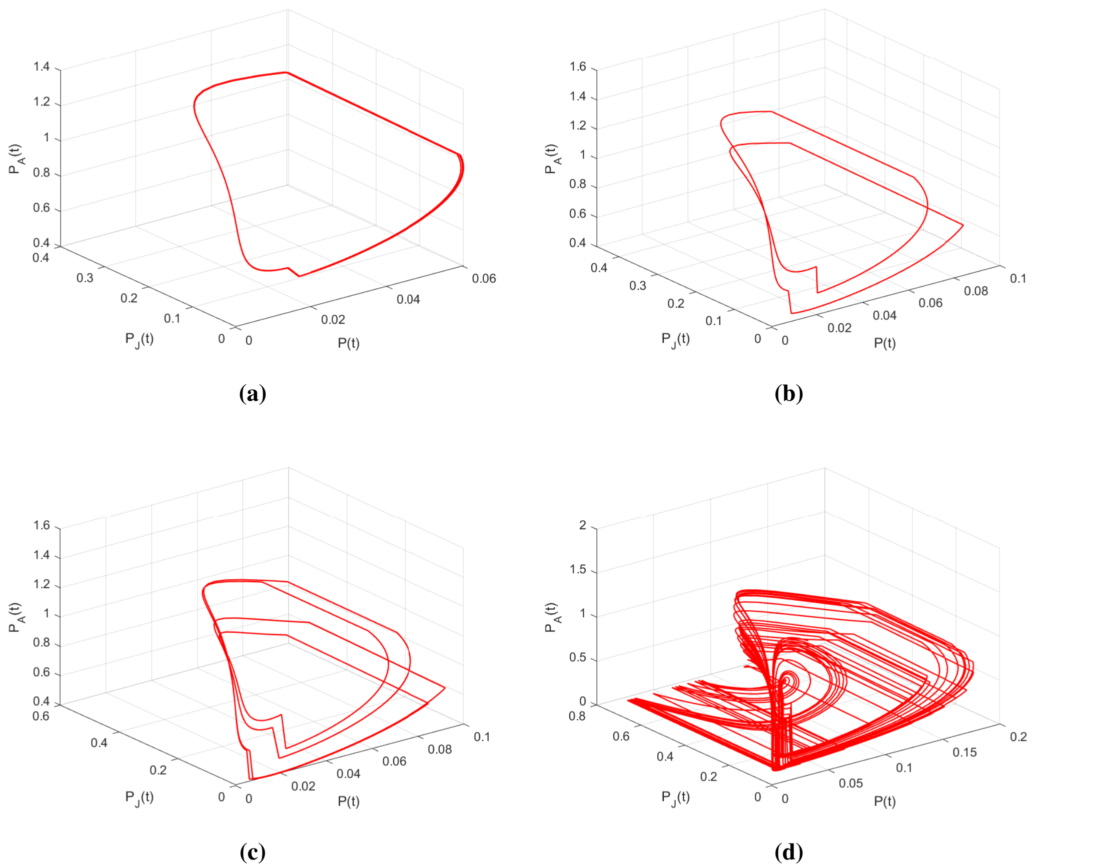


Figure 10. Phase diagram of system (2.6) with different h values. (a) $h = 0.1$, period-one solution. (b) $h = 0.25$, period-two solution. (c) $h = 0.31$, period-four solution. (d) $h = 0.7$, chaos.

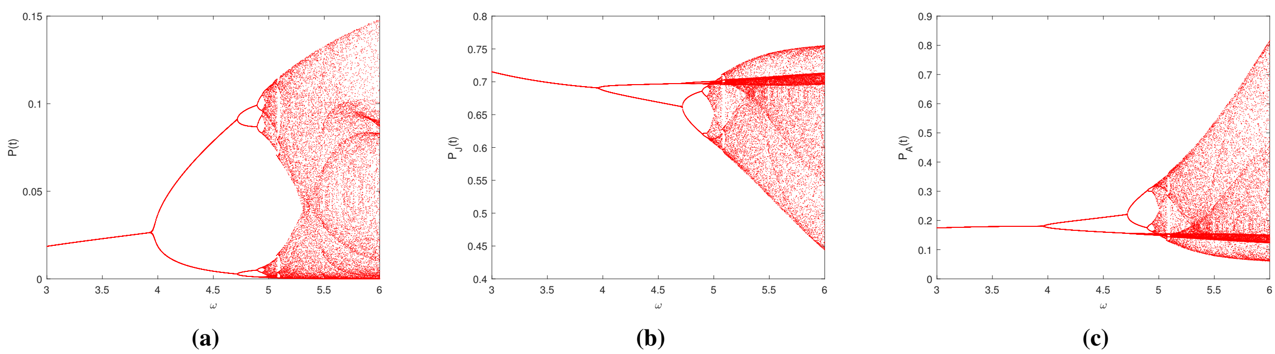


Figure 11. Bifurcation parameter diagram for ω. (a) Bifurcation diagram of $P(t)$. (b) Bifurcation diagram of $P_J(t)$. (c) Bifurcation diagram of $P_A(t)$.

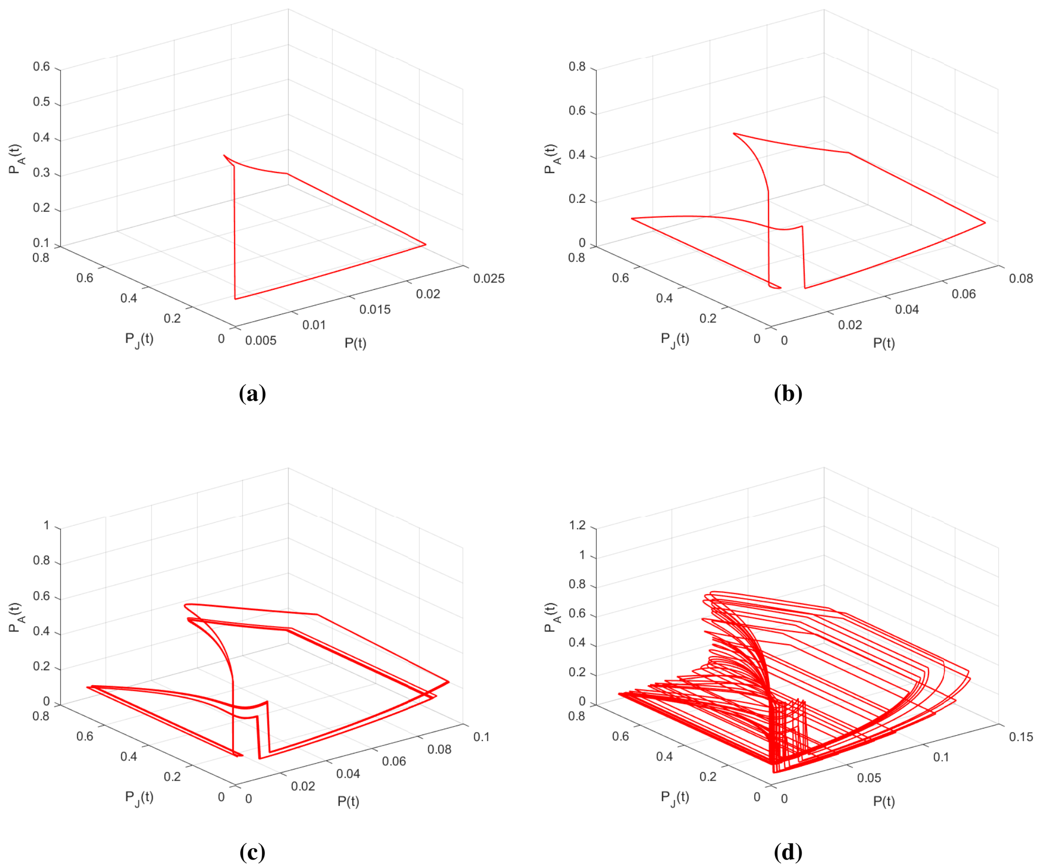


Figure 12. Phase diagram of system (2.6) with different ω values. (a) $\omega = 3.5$, period-one solution. (b) $\omega = 4.5$, period-two solution. (c) $\omega = 4.8$, period-four solution. (d) $\omega = 5.5$, chaos.

6. Conclusions

In this paper, an aquaculture management model with stage structure, intraspecific cooperation, refuge effect, and nonlinearly impulsive releasing is established. The major results of the research are as follows:

(1) The solution $(P(t), P_J(t), P_A(t))$ of system (2.6) is uniformly bounded.

(2) Through the Banach contraction mapping principle, we concluded that the periodic solutions of the subsystem are globally asymptotically stable.

(3) If $B_0 > 1$, system (2.6) is permanent.

(4) If $B_0 < 1, B_1 < 1, B_2 < 1$, the periodic solution of prey-vanishing $(0, \widetilde{P_J(t)}, \widetilde{P_A(t)}'$ is globally asymptotically stable.

The above conclusions were obtained via numerical simulations (see Figures 1–3). For the purpose of examining the effects of multiple crucial parameters on system (2.6), we conducted comparative analysis diagrams (see Figures 4 and 5). Further, to identify the key parameters affecting the persistence condition B_0, we calculated the partial rank correlation coefficients (PRCC) of each parameter with B_0 and analyzed their sensitivity to B_0 (see Figure 6). Moreover, a bifurcation analysis was performed on the parameters ω, μ, and h, revealing abundant dynamical properties of system (2.6), such as periodic bifurcations and chaos (see Figures 7–12).

The above research conclusions provide a scientific reference for aquaculture management. As indicated by the analysis results in Fig. 6, the release amount of juvenile fish and the impulsive release period are the core key factors determining the persistence of this aquacultural dynamical system. This implies that in aquaculture management, the coordinated regulation of juvenile fish release amount and impulsive period should be regarded as a core control priority, and a suitable balance between the two should be achieved through quantitative analysis and dynamic optimization. On this basis, scientific and practicable fishing strategies and impulsive juvenile fish release strategies can be formulated in combination with the carrying capacity of aquaculture water, the growth characteristics of fish populations, and the differentiated demands of actual aquaculture scenarios to ensure the stable operation of the aquacultural system and the efficient utilization of fishery resources.

However, the established model has room for further expansion and refinement. The living environment of populations is frequently perturbed by stochastic factors, and stochasticity within the system weakens the deterministic correlations identified in this study between key parameters and system stability. Stochastic disturbances may cause the system to deviate from the stable equilibrium state predicted by the deterministic model; in extreme cases, they may even drive the system into an unstable state. Furthermore, the fishing of adult fish involves a relevant decision-making process. Moreover, all these aspects represent key issues to be addressed in our subsequent research.

Author contributions

Xueying Zhu: Conceptualization, formal analysis, writing–original draft; Jianjun Jiao, Lin Wu and Die Li: Writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by National Natural Science Foundation of China (No. 12261018), Guizhou University of Finance and Economics Graduate Program (No. 2025BAZYSY221).

Conflict of interest

The authors declare no conflicts of interest.

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