Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 6019–6029.
DOI: 10.3934/math.2026249
Received: 08 December 2025
Revised: 13 January 2026
Accepted: 29 January 2026
Published: 10 March 2026

*Research article*

# Structure theory for a characterization of the metric dimension of graphs

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**Abstract:** Characterizing simple connected graphs of order $n$ having metric dimension $n-2$, solved by Chartrand et al. [Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.* **105** (2000), 99–113], is a foundational result in the study of metric dimension. This article presents a refined proof for the non-bipartite case of the original theorem. While this work does not present a new characterization result, its primary contribution is methodological: We reframe the original's lengthy case-by-case elimination argument as a series of standalone lemmas, which we use to formally establish the structural properties that such a graph must satisfy. Building upon these properties, we then provide a direct, constructive proof demonstrating that the graph structure is necessarily the join of a complete and empty graph. This method offers a more elegant argument for this important characterization and also provides a clearer understanding of why this specific graph family emerges.

**Keywords:** graph theory; resolvability; metric dimension; extreme value
**Mathematics Subject Classification:** 05C12

## 1. Introduction

The concept of metric dimension, introduced independently by Slater [1] and Harary and Melter [2], is a fundamental parameter in the study of graph theory. It quantifies the minimum number of vertices required to uniquely identify every other vertex in a graph based on shortest path distances. The problem of characterizing graphs with extremal metric dimensions has been a central theme of research due to its theoretical importance and its applications in diverse fields such as network design, robotics, and chemistry [3], and combinatorial design theory, particularly regarding graph decompositions and orthogonal arrays [4, 5]. In the context of control theory and network systems, the concept of resolvability is closely linked to the problem of "observability", where the goal is to distinguish the state of the entire network using a minimum set of sensor nodes. Bullo et al. [6] discussed how appropriate node selection is critical for establishing such observability in complex networks. Furthermore, recent developments by Meng et al. [7] demonstrate that optimizing sensor configurations for fault-tolerant control relies heavily on the structural identification capabilities of the underlying graph. Consequently, structural characterizations of metric dimension, such as the one refined in this work, provide necessary theoretical foundations for these engineering applications.

In their foundational work, Chartrand et al. [8] determined the necessary and sufficient conditions for a connected graph to have a metric dimension of $n-2$.

This article revisits this foundational theorem with a primary focus on methodology rather than a new theoretical characterization. First, we observe that while the original proof's case-by-case analysis is exhaustive and valid, its pedagogical clarity can be improved. We achieve this by reframing the structural arguments as standalone lemmas.

For clarity, we also provide the proof of "if $|U| = 1$, $G = K_s + (K_1 \cup K_t)$", which is implicitly present in the original literature. Second, we introduce a new, more direct proof method for the theorem's most complex case (the non-bipartite case with $|U| \geq 2$). Our approach is constructive, building the required graph structure directly from the properties established in our lemmas. This method yields a more elegant, rigorous, and clearer argument, providing deeper insight into why the specific graph families in the characterization emerge.

## 2. Preliminaries

In this work, the graphs are finite, simple, and connected. Let $G = (V, E)$ denote a graph with vertex set $V(G)$ and edge set $E(G)$. The order of the graph is given by $n = |V(G)|$. If an edge exists between two distinct vertices $u, v \in V(G)$, we say they are adjacent (denoted $u \sim v$). The open neighborhood of a vertex $v$, denoted $N(v)$, consists of all vertices adjacent to $v$. We refer to vertices that do not share an edge as non-neighbors.

A subgraph consists of subsets of the original vertex and edge sets. If a subgraph contains a subset of vertices $H \subseteq V(G)$ and includes all edges from the original graph that have both endpoints in $H$, we call it the subgraph induced by $H$. We focus on two specific structural subsets: a clique, which is a subset of vertices that are pairwise adjacent, and an independent set, where no two vertices are adjacent. A clique is considered a maximum clique if no other clique in the graph has a larger cardinality.

We utilize standard graph families throughout this paper. The complete graph of order $n$, denoted $K_n$, is a graph where every pair of distinct vertices is adjacent. The empty graph $\overline{K_n}$ consists of $n$ vertices with no edges. Two graphs $G_1$ and $G_2$ are said to be isomorphic, written as $G_1 \cong G_2$, if there is a bijection between their vertex sets that preserves adjacency. Regarding graph operations, the union $G_1 \cup G_2$ is the graph formed by combining the vertex and edge sets of disjoint graphs. The join operation, denoted $G_1 + G_2$, is the union of vertex sets of $G_1$ and $G_2$, with added edges connecting every vertex in $V(G_1)$ to every vertex in $V(G_2)$.

The central theme of this study is the metric structure of graphs. The distance between two vertices $u$ and $v$, denoted $d(u, v)$, is defined as the length of a shortest path between them. Two vertices $u, v$

are termed indistinguishable if they share the same distance to every other vertex $z$ in the graph (i.e., $d(u,z) = d(v,z)$ for all $z \in V(G) \setminus \{u,v\}$).

To resolve such vertices, we utilize coordinate representations. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of $V(G)$. The representation of any vertex $v \in V(G)$ induced by the $W$ is the vector:

$$r(v|W) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k)).$$

If distinct vertices $u, v$ always yield distinct representations (i.e., $r(u|W) \neq r(v|W)$), then $W$ is defined as a resolving set. Consequently, if a graph contains a set of $k$ vertices that are mutually indistinguishable, any valid resolving set must include at least $k-1$ of those vertices to ensure uniqueness. For a graph $G$, the metric dimension $\beta(G)$ is the minimum number of vertices forming a resolving set. Such a smallest resolving set is called a metric basis.

Based on these concepts, we revisit the following characterization theorem:

**Theorem 2.1.** *[8] For any connected graph $G$ with $|V(G)| \geq 4$, the equality $\beta(G) = n-2$ holds if and only if $G$ admits the structure of a complete bipartite graph $K_{s,t}$, $s, t \geq 1$, a split graph $K_s + \overline{K_t}$, $s \geq 1$, $t \geq 2$, or the specific join structure $K_s + (K_1 \cup K_t)$, $s, t \geq 1$.*

Our proofs rely on two logical consequences of $\beta(G) = n-2$. First, if a specific structural property forces the metric dimension to be $n-3$ or lower, that structure is forbidden in our target graphs. Second, it is a known bound [8] that for any graph with diameter $d$, $\beta(G) \leq n-d$. Therefore, for a graph to achieve $\beta(G) = n-2$, it must have a diameter of $d = 2$. This condition ensures that any pair of non-adjacent vertices is separated by a distance of exactly 2.

## 3. A new proof for the $n-2$ characterization

In this section, we establish the structural properties required for the refined proof of the main theorem.

**Lemma 3.1.** *If $G$ is a connected bipartite graph with diameter $d = 2$, then $G$ is a complete bipartite graph.*

*Proof.* Let $V_1$ and $V_2$ be the bipartition sets of $G$. Consider an arbitrary pair of vertices $u \in V_1$ and $v \in V_2$. Since $G$ is bipartite, the distance $d(u,v)$ must be odd $(1, 3, 5, \ldots)$. However, since the diameter of $G$ is 2, we must have $d(u,v) \leq 2$. The only odd positive integer less than or equal to 2 is 1. Therefore, $d(u,v) = 1$, which implies $uv \in E(G)$. Since $u$ and $v$ are arbitrary, every vertex in $V_1$ is adjacent to every vertex in $V_2$. Thus, $G$ is the complete bipartite graph $K_{s,t}$, where $s = |V_1|$ and $t = |V_2|$.

**Lemma 3.2.** *A connected bipartite graph $G$ with $n \geq 4$ vertices is isomorphic to a complete bipartite graph, $K_{s,t}$ $(s, t \geq 1)$, if and only if its metric dimension is exactly $n-2$.*

*Proof.* For the forward implication, let $G$ be a connected bipartite graph with $\beta(G) = n-2$. By Theorem 1 in [8], $\beta(G) \leq n - \mathrm{diam}(G)$. Thus, $n-2 \leq n - \mathrm{diam}(G)$, which implies $\mathrm{diam}(G) \leq 2$. Since $n \geq 4$, $G$ cannot be a complete graph (diameter 1), so $\mathrm{diam}(G) = 2$. By Lemma 3.1, a connected bipartite graph with diameter 2 is necessarily a complete bipartite graph $K_{s,t}$.

Conversely, let $G \cong K_{s,t}$ with partitions $V_1$ and $V_2$ of sizes $s$ and $t$, respectively $(s+t=n)$. We claim the vertices within $V_1$ are mutually indistinguishable. For any distinct $u, w \in V_1$, they share the same open neighborhood (all of $V_2$). Thus, for any $z \in V_2$, $d(u,z) = 1 = d(w,z)$. For any $z \in V_1 \setminus \{u,w\}$, $d(u,z) = 2 = d(w,z)$. Since no vertex in $G$ can distinguish $u$ from $w$, any resolving set must contain at least $|V_1| - 1$ vertices from $V_1$ and similarly $|V_2| - 1$ vertices from $V_2$.

Therefore, $\beta(G) \geq (s-1) + (t-1) = s + t - 2 = n - 2$. Since we can easily construct a resolving set of this size (by excluding one vertex from each partition), $\beta(G) = n - 2$.

**Lemma 3.3.** *For $s \geq 1, t \geq 2$, $\beta\left(K_s + \overline{K_t}\right) = n - 2$.*

*Proof.* Let $G \cong K_s + \overline{K_t}$. The vertex set of $G$ is partitioned into the clique $V(K_s)$ of size $s$ and the independent set $V(\overline{K_t})$ of size $t$. The vertices within the clique $V(K_s)$ are mutually indistinguishable, as are the vertices within the independent set $V(\overline{K_t})$. Any resolving set for $G$ must therefore contain at least $(s-1)$ vertices from $V(K_s)$ and $(t-1)$ vertices from $V(\overline{K_t})$. This establishes the lower bound: $\beta(G) \geq (s-1) + (t-1) = s + t - 2 = n - 2$.

To show sufficiency, consider the set $W = V(K_s \setminus \{v_a\}) \cup V(\overline{K_t} \setminus \{v_b\})$, where $v_a \in V(K_s)$ and $v_b \in V(\overline{K_t})$. The size of $W$ is $(s-1) + (t-1) = n - 2$. The only vertices not in $W$ are $v_a$ and $v_b$. All vertices within $W$ are resolved by their own "0" entry in their respective representations. We only need to check the representations of $v_a$ and $v_b$ with respect to $W$.

- For $v_a \in K_s$, its distance to every vertex in $W$ is 1. Thus, $r(v_a|W)$ is an all-1s vector.
- For $v_b \in \overline{K_t}$, its distance to any vertex in $V(K_s \setminus \{v_a\}) \subseteq W$ is 1, but its distance to any vertex in $V(\overline{K_t} \setminus \{v_b\}) \subseteq W$ is 2. Thus, $r(v_b|W)$ contains both 1s and 2s.

Since $r(v_a|W) \neq r(v_b|W)$, and neither representation is an all-0s vector, the set $W$ is a valid resolving set. Thus, $\beta(G) = n - 2$.

**Lemma 3.4.** *Let $G$ be a connected graph of order $n$ with a maximum clique $Q$ such that $|Q| \geq 3$ and $|V(G) \setminus Q| = 1$. Then $\beta(G) = n - 2$ if and only if $G \cong K_s + (K_1 \cup K_t)$ for some integers $s, t \geq 1$.*

*Proof.* For the forward proof, let $G$ be a graph on $n$ vertices with the maximum clique $Q$, with $|Q| \geq 3$. Let $U = V \setminus Q$, then $|U| = 1$. Let $U = \{u\}$, then, since $G$ is connected, $u$ must be adjacent to at least one vertex in $Q$. Moreover, since $Q$ is a maximum clique, $u$ cannot be adjacent to all vertices in $Q$ (otherwise, $V(G)$ would form a $K_n$, for which $\beta(G) = n - 1$). Thus, the neighborhood of $u$, $N(u)$, is a non-empty, proper subset of $Q$.

Let $Q_A = N(u)$, the set of $s$ vertices adjacent to $u$, and $Q_N = Q \setminus N(u)$, the set of vertices non-adjacent to $u$, then these define a partition for $Q$. By definition, all vertices within $Q_A$ and $Q_N$ are mutually adjacent. The graph $G$ is thus formed by three sets of vertices: the clique $Q_A \cong K_s$, the clique $Q_N \cong K_t$, and the single vertex $\{u\} \cong K_1$. Every vertex in $Q_A$ is adjacent to every vertex in $\{u\} \cup Q_N$. This is precisely the definition of the graph join $K_s + (K_1 \cup K_t)$, as illustrated in Figure 1.

For the converse part, let $G \cong K_s + (K_1 \cup K_t)$ with $s, t \geq 1$. The order of the graph is $n = s + t + 1$, and the vertex set is partitioned as $V(K_s) \cup \{u\} \cup V(K_t)$. The $s$ vertices of $K_s$ are mutually indistinguishable, as are the $t$ vertices of $K_t$, hence a resolving set $W$ of $G$ must contain, at the very minimum, $(s-1) + (t-1) = s + t - 2 = n - 3$ vertices. However, a set of this size, for instance $W = V(K_s \setminus \{v_a\}) \cup V(K_t \setminus \{v_b\})$, fails to resolve the pair $\{v_a, v_b\}$, as $d(v_a, w) = 1$ and $d(v_b, w) = 1$ for all $w \in W$. Thus, $\beta(G) \geq n - 2$.

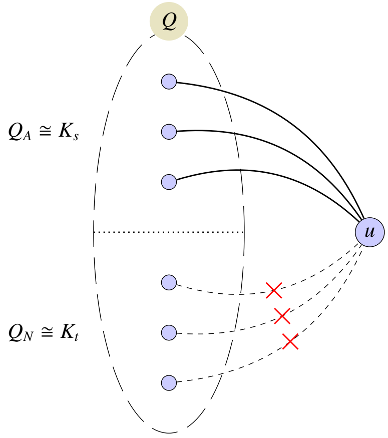


**Figure 1.** The structure of a graph with $|U| = 1$, showing the partition of the maximum clique $Q$ into vertices adjacent ($Q_A$) and non-adjacent ($Q_N$) to the single vertex $u \in U$.

To show sufficiency, consider the set $W' = V(K_s) \cup V(K_t \setminus \{v_b\})$ of size $s + (t-1) = n-2$. The only vertices not in $W'$ are $u$ and $v_b$. Since $W' \subset Q$, $y \in K_t$ has a distance vector containing all 1s. On the other hand, $u$ is adjacent to vertices of $K_s$ but non-adjacent to vertices of $K_t$, implying that $r(u|W')$ contains both 1s and 2s. Finally, all vertices in $W'$ are resolved by their own "0" entry in their respective representations. Hence, $W'$ is a valid resolving set, and $\beta(G) = n-2$.

The following lemmas establish the necessary structural properties of $G$ by contradiction. In each case, we assume the graph lacks a specific structural uniformity (e.g., independence of $U$ or uniform neighborhoods). We then demonstrate that this lack of uniformity allows us to distinguish the vertices more efficiently than expected, enabling the construction of a resolving set of size $n-3$. This contradicts the condition $\beta(G) = n-2$.

**Lemma 3.5.** *For a connected graph $G$ with $\beta(G) = n-2$, let the vertex set of the maximum clique in $G$ be denoted by $Q$, with $|Q| \geq 3$, and let $U = V(G) \setminus Q$. If $|U| > 1$, then $U$ is an independent set.*

*Proof.* On the contrary, we assume that $U$ is not an independent set, then there exist at least two adjacent vertices, $u, v \in U$. Since $Q$ is a maximum clique, neither $u$ nor $v$ can be adjacent to all vertices in $Q$. Therefore, both $u$ and $v$ must have non-neighbors within $Q$. We consider two exhaustive cases based on the relationship between these non-neighbors.

**Case 1:** If there exists a vertex $q \in Q$ that is not adjacent to both $u$ and $v$.

Let $q \in Q$ be such that $uq \notin E(G)$ and $vq \notin E(G)$. Since $G$ is connected, $u$ and $v$ must have neighbors in $Q$. As $q$ is not adjacent to both $u$ and $v$, there must exist another vertex $v_a \in Q$, which has an edge to at least one of them, say, $uv_a \in E(G)$. Since $|Q| \geq 3$, there is also $v_b \in Q$, different from $q, v_a$. This scenario is represented in Figure 2(a).

Now, consider the set of three vertices $\{u, v, v_b\}$. We will show that the set $W = V(G) \setminus \{u, v, v_b\}$ is a resolving set. Note that $q, v_a \in W$. The distance vectors of the three removed vertices induced by $\{q, v_a\} \subseteq W$ are

- $r(u|\{q, v_a\} \cup W) = (2, 1, \dots)$,
- $r(v|\{q, v_a\} \cup W) = (2, 2, \dots)$,

- $r(v_b|\{q, v_a\} \cup W) = (1, 1, \ldots)$.

These representations are distinct, and it can be verified that $W$ resolves all pairs of vertices in $G$. Thus, $W$ is a resolving set of size $n-3$, which implies $\beta(G) \leq n-3$. This contradicts our premise that $\beta(G) = n-2$.


**(a)** Vertex $q \in Q$ not adjacent to both $u, v \in U$

**(b)** All vertices in $Q$ adjacent to at least one of $u, v \in U$

**Figure 2.** Structural scenarios for the proof of Lemma 3.5, assuming $U$ is not an independent set.

**Case 2:** If every vertex in $Q$ is adjacent to at least one of $u$ or $v$.

This case covers the logical complement of Case 1. Now, since $Q$ is a maximum clique, there must exist distinct vertices $v_a, v_b \in Q$ such that $uv_a \notin E(G)$ and $vv_b \notin E(G)$. Since every vertex of $Q$ is adjacent to at least one of $u, v$, we get $vv_a \in E(G)$ and $uv_b \in E(G)$. Since $|Q| \geq 3$, there exists another vertex $q \in Q$ (distinct from $v_a, v_b$), which must also be adjacent to $u$ or $v$, say, $uq \in E(G)$. The structural setup for this case is illustrated in Figure 2(b).

Consider the set of three vertices $\{u, v, q\}$. Let $W = V(G) \setminus \{u, v, q\}$, which contains $v_a$ and $v_b$. The distance vectors of $\{u, v, q\}$ induced by the ordered subset $\{v_a, v_b\} \subseteq W$ are

- $r(u|\{v_a, v_b\} \cup W) = (2, 1, \ldots)$,
- $r(v|\{v_a, v_b\} \cup W) = (1, 2, \ldots)$,
- $r(q|\{v_a, v_b\} \cup W) = (1, 1, \ldots)$.

Again, these representations are distinct, and $W$ forms a resolving set of size $n-3$, leading to a contradiction.

Since both cases lead to a contradiction, our assumption must be false. Therefore, $U$ is an independent set.

In short, Lemma 3.5 states that if $U$ is not independent, there exists an edge $uv$ within it. This edge breaks the symmetry between $u$ and $v$ relative to the clique $Q$. By selecting a resolving set that exploits this specific adjacency, we can uniquely identify $u$ and $v$ without needing to include as many clique vertices in the basis, leading to a dimension of $n-3$.

**Lemma 3.6.** *Let $\beta(G) = n-2$, where the vertex set of the maximum clique in $G$ is $Q$, with $|Q| \geq 3$, and let $U = V(G) \setminus Q$. If $|U| \geq 2$, then all vertices in $U$ have the same neighborhood.*

*Proof.* From Lemma 3.5, we know that $U$ is an independent set. We proceed by contradiction. Assume that not all vertices in $U$ have the same neighborhood. This implies there must exist two vertices, $u, v \in U$, and a vertex $v_b \in Q$ such that $uv_b \in E(G)$ but $vv_b \notin E(G)$. Since the diameter of $G$ is 2, there must be a path of length 2 between $v$ and $v_b$. Since $U$ is independent, this path can not pass through a vertex of $U$, hence it must pass through $v_a \in Q$.

Since $Q$ is a maximum clique, there exists at least one vertex which is non-adjacent to $u$. We discuss the following two exhaustive cases.

**Case 1:** If $u$ is not adjacent to $v_a$.

This scenario is depicted in Figure 3(a). This means $u$ is adjacent to every vertex in $Q$, including $v_b$, except for $v_a$. Since $|Q| \geq 3$, there must be another vertex $q \in Q$ distinct from $v_a$ and $v_b$. By our premise, $uq \in E(G)$.

Let $W = V(G) \setminus \{u, v, q\}$, which is of size $n-3$ and contains both $v_a$ and $v_b$. The distance vectors of these three vertices induced by the ordered subset $\{v_a, v_b\} \subseteq W$ are

- $r(u|\{v_a, v_b\} \cup W) = (2, 1, \dots)$,
- $r(v|\{v_a, v_b\} \cup W) = (1, 2, \dots)$,
- $r(q|\{v_a, v_b\} \cup W) = (1, 1, \dots)$.

These representations are distinct, and $W$ is a resolving set, leading to a contradiction.


**(a)** Case 1: The only non-neighbor of $u$ is $v_a$.

**(b)** Case 2: The non-neighbor of $u$ is $q \neq v_a$.

**Figure 3.** Structural scenarios for the proof of Lemma 3.6.

**Case 2:** If $u$ is adjacent to $v_a$.

This scenario is shown in Figure 3(b). Since $u$ must be non-adjacent to some vertex in $Q$, and since it is adjacent to $v_a$, its non-adjacent vertex must be some other vertex, say, $q \in Q$, where $q \neq v_a$.

Consider the set of three vertices $\{u, q, v\}$. The set $W = V(G) \setminus \{u, q, v\}$ is of size $n-3$ and contains both $v_a$ and $v_b$. The distance vector of these vertices induced by the ordered subset $\{v_a, v_b, \dots\} \subseteq W$ are

- $r(u|\{v_a, v_b\} \cup W) = (1, 1, \dots)$,
- $r(v|\{v_a, v_b\} \cup W) = (1, 2, \dots)$,
- $r(q|\{v_a, v_b\} \cup W) = (1, 1, \dots)$.

These representations are distinct, and it can be verified that $W$ is a resolving set, which again leads to a contradiction.

Since both cases lead to a contradiction, our assumption that vertices in $U$ have different neighborhoods must be false. Therefore, all vertices in $U$ must have the same neighborhood.

Intuitively, if two vertices in $U$ connect to different subsets of the clique $Q$, this difference in connectivity acts as a "fingerprint". We can use the specific clique vertices that differ between their neighborhoods to distinguish them from a distance. This reduces the burden on the resolving set, again allowing us to form a basis of size $n - 3$.

**Lemma 3.7.** *Let $G$ be a connected graph with $\beta(G) = n - 2$. Let $Q$ be the vertex set of a maximum clique in $G$, with $|Q| \geq 3$, and let $U = V(G) \setminus Q$ be an independent set of size $|U| \geq 2$. Then, $N(u) = Q$ or $N(u) = Q - v_b$, for all $u \in U$, and for some $v_b \in Q$.*

*Proof.* From Lemmas 3.5 and 3.6, we know that $U$ is an independent set and all its vertices share a common neighborhood, which we denote as $N(U)$. We proceed by contradiction. Assume the conclusion is false, i.e., $\exists u \in U$, which is non-adjacent to at least two distinct vertices in the clique, say, $v_a, q \in Q$.

Since $G$ is connected, there exists at least one $v_b \in Q$ such that $u \sim v_b$. Note that $v_b$ is distinct from $v_a$ and $q$. Also, since $|U| \geq 2$, let $v$ be another vertex in $U$. By Lemma 3.6, $v$ has the same neighborhood as $u$, so $vv_b \in E(G)$, while $vv_a \notin E(G)$ and $vw \notin E(G)$. This structural setup is illustrated in Figure 4.

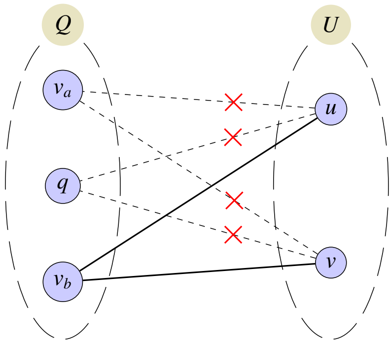


**Figure 4.** Structural scenario for the proof of Lemma 3.7.

Consider the set of three vertices $\{v, q, v_b\}$. Let $W = V(G) \setminus \{v, q, v_b\}$, which is of size $n - 3$ and contains both $u$ and $v_a$. The distance vectors of the removed vertices induced by the ordered subset $\{u, v_a, \ldots\} \subseteq W$ are

- $r(v|\{u, v_a\} \cup W) = (2, 2, \ldots)$,
- $r(q|\{u, v_a\} \cup W) = (2, 1, \ldots)$,
- $r(v_b|\{u, v_a\} \cup W) = (1, 1, \ldots)$.

The representations are all distinct, and hence $W$ resolves $G$. This implies that $\beta(G) \leq n - 3$, which contradicts our initial premise.

Therefore, our assumption that a vertex in $U$ can be non-adjacent to two or more vertices in $Q$ must be false. This completes the proof.

With the necessary structural properties established in the preceding lemmas, we now present the proof of the main theorem.

**Theorem 3.1.** *[8] Let G be a connected graph of order $n \geq 4$. Then $\beta(G) = n - 2$ if and only if G is isomorphic to one of the following:*

*(i)* $K_{s,t}$ *for* $s, t \geq 1$,

*(ii)* $K_s + (K_1 \cup K_t)$ *for* $s, t \geq 1$,

*(iii)* $K_s + \overline{K_t}$ *for* $s \geq 1, t \geq 2$.

*Proof.* We prove both implications of the characterization.

First the converse part, if $G$ is isomorphic to one of the three listed families, by Lemmas 3.2–3.4, $\beta(G) = n - 2$.

For the forward implication, let $\beta(G) = n - 2$. Assuming $G$ to be a bipartite graph, Lemma 3.2 implies that it must be a complete bipartite graph, $G \cong K_{s,t}$.

If $G$ is non-bipartite, it must contain an odd cycle. Let us assume that $G$ contains an induced cycle $C_k$ of length $k \geq 5$. Let the vertices of this cycle be $v_1, v_2, \ldots, v_k$ in order. Let $W = V(G) \setminus \{v_2, v_3, v_4\}$, then it contains both $v_1$ and $v_5$. Now,

- $r(v_2|\{v_1, v_5\} \cup W) = (1, 2, \ldots)$,
- $r(v_3|\{v_1, v_5\} \cup W) = (2, 2, \ldots)$,
- $r(v_4|\{v_1, v_5\} \cup W) = (2, 1, \ldots)$.

Hence, $W$ is a resolving set of size $n - 3$, which is a contradiction. Therefore, the shortest odd cycle of $G$ must be $C_3$. Let the vertex set of the maximum clique of $G$ be $Q$, then $|Q| \geq 3$. Let $U = V(G) \setminus Q$, then, since $G$ is not a complete graph ($\beta(K_n) = n - 1$), the set $U$ must be non-empty.

Now, if $|U| = 1$, by Lemma 3.4, $G \simeq K_s + (K_1 \cup K_t)$. On the other hand, if $|U| \geq 2$, by Lemma 3.5, $U$ must be an independent set. Also, by Lemma 3.6, vertices of $U$ share a common neighborhood, say, $N(U)$. Lastly, by Lemma 3.7, $N(U) = Q$ or $N(U) = Q - v_b$ for some $v_b \in Q$.

- If $N(U) = Q$, the graph is, by definition, $K_s + \overline{K_t}$, where $s = |Q|$ and $t = |U|$.
- If $N(U) = Q - v_b$, we can re-partition the vertices of $G$ into two new sets. Let $Q' = Q - v_b$ and $U' = U \cup \{v_b\}$. The set $Q'$ induces a clique $K_{s-1}$ (where $s = |Q|$), and the set $U'$ induces an independent set $\overline{K_{t+1}}$ (where $t = |U|$). Every vertex in $Q'$ is adjacent to every vertex in $U'$, implying the structure $K_{s-1} + \overline{K_{t+1}}$.

In both scenarios, the graph is shown to be the join of a complete graph and an empty graph.

Since all cases have been exhaustively analyzed, the characterization is complete.

## 4. Conclusions

In this paper, we revisited the classical characterization of connected graphs with metric dimension $\beta(G) = n - 2$. Our contribution focuses on refining the proof methodology rather than introducing a new characterization. By reformulating the original case-based arguments as a sequence of structured lemmas, we established the necessary structural properties of such graphs in a clearer and more systematic manner.

Using these structural insights, we presented a direct and constructive argument for the non-bipartite case and showed that graphs satisfying $\beta(G) = n-2$ must belong to one of the well-known families $K_{s,t}$, $K_s + K_t$, or $K_s + (K_1 \cup K_t)$. The lemma-based framework improves the logical organization of the proof and provides clearer insight into why these specific graph structures arise. This refined presentation may also serve as a useful reference for future work on extremal metric parameters and structural characterizations in graph theory.

## Author contributions

H. Luo and G. Haidar: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing–original draft preparation, Writing-review and editing, Visualization; M. I. Khan and S. Hayat: Formal analysis, Investigation, Writing–original draft preparation, and editing; M. J. F. Alenazi: Methodology, Validation, Resources, Writing–original draft preparation. All authors have read and agreed to the published version of the manuscript.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

S. Hayat is supported by UBD Faculty Research Grants (No. UBD/RSCH/1.4/FICBF/2025/011). M. J. F. Alenazi was supported by Ongoing Research Funding program (ORF-2025-582), King Saud University, Riyadh, Saudi Arabia.

## Conflict of interest

The authors declare that they have no known competing financial interests.

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