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Research article

Existence theory for finite delayed fractional differential equations with nonlinear variable order

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Abstract: We investigated a class of finite-delay fractional differential equations in which the differentiation order combined with time and depended explicitly on the past evolution of the state. The proposed model combined a history-dependent variable fractional order with a nonlinear source term involving an integral memory functional, rather than a pointwise delay. This structure provided a flexible framework for describing adaptive memory effects and cumulative nonlocal behavior that cannot be captured by classical constant-order or pointwise variable-order models. By reformulating the problem as an equivalent integral equation, we established rigorous existence and uniqueness results in the Banach space $L^1(\Delta)$ using the Banach contraction principle and the Schauder fixed-point theorem under mild and verifiable assumptions. In addition, Ulam–Hyers stability of the system was proved, ensuring robustness of solutions with respect to perturbations. An illustrative example, together with a numerical stability verification, was presented to support the theoretical findings. The obtained results contributed to the mathematical understanding of delayed variable-order fractional systems and provided a solid foundation for future studies in engineering, economics, and biomedical modeling.

Keywords: variable-order fractional differential equations; integral memory effects; finite delay; fixed point theory; existence and uniqueness, Ulam-Hyers stability
Mathematics Subject Classification: 34D20, 34K20, 34K60, 92C60, 92D45

1. Introduction

Mathematical models involving nonlocal operators are widely used to describe memory-driven and hereditary phenomena in complex systems [1, 2]. Fractional differential equations provide a natural extension of classical integer-order models. They incorporate long-range temporal interactions and nonlocal dynamics in a unified framework [3]. As a result, fractional calculus has found applications in viscoelasticity, anomalous diffusion, control systems, bioengineering, and economics; see, for example, [4, 5].

To enhance modeling flexibility, variable-order fractional operators have been introduced [6, 7]. In these models, the differentiation order evolves with time, space, or the system state. This approach allows the memory intensity to change dynamically [8]. Consequently, variable-order models can describe processes whose hereditary properties are not constant over time. Significant developments in this direction can be found in [9].

In many real-world applications, time delays are unavoidable and arise due to finite signal transmission speeds, feedback mechanisms, or accumulated effects over time [10, 11]. The interaction between delay effects and fractional dynamics leads to mathematical models capable of capturing both hereditary behavior and time-lagged responses. This interplay has motivated extensive analytical studies, particularly concerning the existence and uniqueness of solutions for delayed fractional systems [12]. Various classes of delayed and variable-order fractional problems have been investigated using fixed-point techniques and compactness arguments; see, for instance, [13].

However, most existing studies focus on models in which the variable fractional order depends pointwise on time or on the instantaneous state, and where delay effects are represented explicitly through the history function u_t. In contrast, in many practical situations the influence of the past is cumulative rather than pointwise, and the intensity of the memory effect depends on the overall historical behavior of the system. This observation motivates the need for fractional models incorporating integral memory effects together with variable-order dynamics.

In recent years, increasing attention has been devoted to fractional models incorporating distributed or integral memory effects, where the past influence is accumulated through convolution-type kernels rather than represented by pointwise delays [14]. Such formulations arise naturally in viscoelastic materials, anomalous diffusion, epidemiological systems with cumulative exposure, and adaptive control problems [15]. Theoretical investigations of fractional integro-differential equations with memory kernels, nonlocal nonlinearities, and distributed delays have been developed in various contexts; see, for example, [16]. These studies highlight that integral memory structures often provide a more realistic description of hereditary behavior compared to classical discrete delays.

Simultaneously, substantial progress has been made in the theory of variable-order fractional differential equations, where the fractional order depends on time, space, or the system state. Recent developments include nonlinear variable-order models, state-dependent orders, stability analysis, and numerical approximation techniques [17, 18]. In particular, variable-order operators have proven effective in modeling systems with adaptive memory intensity, where the strength of nonlocal effects evolves dynamically according to the solution trajectory.

Despite these advances, most existing works treat either distributed memory effects or variable-order dynamics separately. Only limited attention has been paid to systems in which the fractional order itself depends on the historical trajectory while the nonlinear source term simultaneously incorporates an integral memory functional. The present work aims to fill this gap by establishing rigorous existence, uniqueness, and Ulam–Hyers stability results for finite-delay fractional systems combining these two mechanisms within a unified analytical framework. Motivated by the above discussion, in this paper we study the following finite delayed fractional differential equation with history-dependent variable order:

$$\begin{cases} D_{0^+}^{\alpha(t,u_t)} u(t) = \mathcal{F}\left(t, u(t), \int_{-r}^{0} u(t+s)\, ds\right), & t \in [0,T], \\ u(t) = \phi(t), & t \in [-r, 0], \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha(t,u_t)}$ denotes a variable-order fractional derivative of Riemann–Liouville type, $r > 0$ is a finite delay, ϕ is a given initial function, and $\mathcal{F}$ is a nonlinear function involving an integral memory term. The fractional order $\alpha(t, u_t)$ depends on the historical trajectory of the solution, allowing the memory effect to adapt dynamically to past system behavior. The proposed framework is motivated by systems in which memory effects are both cumulative and adaptive. Such behavior arises in viscoelastic materials with evolving microstructure, anomalous diffusion in heterogeneous media, and biological systems where the influence of the past depends on accumulated exposure. Although the order function $\alpha(t, u_t)$ is introduced in a mathematical setting, it admits a natural physical interpretation. In viscoelasticity, fractional derivatives model hereditary stress–strain relationships, where the fractional order reflects the intensity of memory effects. Allowing α to depend on the history segment u_t enables the modeling of materials whose effective memory strength evolves according to accumulated deformation or loading history.

Similarly, in anomalous diffusion processes, the fractional order quantifies the deviation from classical Brownian motion. A smaller order corresponds to stronger subdiffusive behavior and enhanced memory influence. When α depends on the past trajectory, the diffusion regime becomes adaptive, allowing the system to exhibit variable memory persistence depending on its historical evolution.

More generally, the dependence of $\alpha(t, u_t)$ on the history segment permits the description of adaptive memory mechanisms, where the strength of nonlocal interactions is dynamically regulated by past states rather than remaining fixed. This feature enhances the modeling flexibility of the proposed approach and provides a rigorous analytical foundation for memory-driven systems.

The main objective of this work is to establish rigorous existence, uniqueness, and stability results for the delayed variable-order fractional problem (1.1). By reformulating the system as an equivalent integral equation, we develop a fixed-point framework that enables a systematic analytical treatment of history-dependent memory effects.

The present study provides several distinct contributions to the theory of delayed variable-order fractional systems:

(i) We propose a unified framework that combines a history-dependent variable fractional order with a nonlinear source term involving an integral memory functional. Unlike classical models where the fractional order depends only on time or where delays appear in pointwise form, the present formulation captures cumulative hereditary effects while allowing the memory intensity itself to adapt dynamically according to the past trajectory of the system.

(ii) The analysis is carried out in the Banach space $L^1(\Delta)$ instead of the classical space $C([0,T])$. This setting naturally accommodates integrable solutions and integral memory terms, and is particularly well suited for systems driven by cumulative nonlocal interactions.

(iii) A quantitative kernel sensitivity condition is introduced to control the dependence of the variable-order fractional kernel on the history segment. This estimate plays a crucial role in deriving uniqueness results and extends the applicability of classical contraction-based techniques to history-dependent settings.

(iv) Existence results are obtained via Schauder's fixed-point theorem under weaker growth assumptions, while uniqueness follows from the Banach contraction principle. In addition, Ulam–Hyers stability is established for this class of systems, providing explicit robustness bounds with respect to perturbations.

The proposed framework generalizes several existing delayed and variable-order fractional models and provides a mathematically rigorous foundation for systems with adaptive and cumulative memory mechanisms.

The remainder of the paper is organized as follows. In Section 2, we introduce the necessary preliminaries, definitions, and auxiliary results. Section 3 is devoted to the derivation of existence and uniqueness results using fixed-point theory. Section 4 shows the Ulam–Hyers stability. In Section 5, an illustrative example is presented to demonstrate the applicability of the theoretical findings. Finally, Section 6 concludes the paper with a summary of the results and directions for future research.

2. Preliminaries

This section presents the basic definitions, notations, and auxiliary results that will be used throughout the paper. Standard concepts from fractional calculus, functional analysis, and fixed-point theory are recalled for completeness.

Let $r > 0$ and $T > 0$ be fixed. We define the interval

$$\Delta := [-r, T], \qquad \Delta_1 := [0, T], \qquad \Delta_2 := [-r, 0].$$

Denote by $L^1(\Delta, \mathbb{R})$ (or simply $L^1(\Delta)$) the Banach space of all Lebesgue measurable functions $v : \Delta \to \mathbb{R}$ such that

$$\int_{-r}^{T} |v(s)|\, ds < \infty,$$

equipped with the norm

$$\|v\|_{L^1(\Delta)} := \int_{-r}^{T} |v(s)|\, ds.$$

For a function $u : \Delta \to \mathbb{R}$ and $t \in \Delta_1$, we denote by u_t the history segment defined by

$$u_t(\theta) = u(t + \theta), \qquad \theta \in [-r, 0].$$

Clearly, $u_t \in L^1(\Delta_2)$ whenever $u \in L^1(\Delta)$.

2.1. Variable-order fractional operators

Let $\Phi : \Delta_1 \times \mathbb{R} \to (0, 1]$ be a continuous function satisfying

$$0 < \Phi_* \le \Phi(t, x) \le \Phi^* < 1, \qquad (t, x) \in \Delta_1 \times \mathbb{R}.$$

Definition 2.1. The left-sided Riemann–Liouville fractional integral of variable order Φ of a function $\varphi \in L^1(\Delta)$ is defined by

$$(I_{0^+}^{\Phi(t,\varphi(t))}\varphi)(t) = \int_0^t \frac{(t-s)^{\Phi(s,\varphi(s))-1}}{\Gamma(\Phi(s, \varphi(s)))}\varphi(s)\, ds, \qquad t > 0. \tag{2.1}$$

If $\Phi(t, \varphi(t)) \equiv \Phi$ is constant, then (2.1) reduces to the classical Riemann–Liouville fractional integral of order Φ.

Definition 2.2. The left-sided Riemann–Liouville fractional derivative of variable order $\Lambda : \Delta_1 \times \mathbb{R} \to (0, 1)$ is defined by

$$(D_{0^+}^{\Lambda(t,\varphi(t))}\varphi)(t) = \frac{d}{dt}(I_{0^+}^{1-\Lambda(t,\varphi(t))}\varphi)(t), \qquad t > 0. \tag{2.2}$$

When $\Lambda(t, \varphi(t)) \equiv \Lambda$ is constant, (2.2) coincides with the classical Riemann–Liouville fractional derivative of order Λ.

It is well known that the semigroup property does not hold for variable-order fractional operators, namely,

$$I_{0^+}^{\Phi(t,\varphi(t))} I_{0^+}^{\Lambda(t,\varphi(t))}\varphi(t) \neq I_{0^+}^{\Phi(t,\varphi(t))+\Lambda(t,\varphi(t))}\varphi(t),$$

in general.

2.2. Auxiliary results

The following lemmas play a crucial role in the analysis of the proposed problem.

Lemma 2.1. *Let $\Phi : \Delta_1 \times \mathbb{R} \to (0, 1)$ be continuous. Then, for any $\varphi \in L^1(\Delta)$, the variable-order fractional integral $I_{0^+}^{\Phi(t,\varphi(t))}\varphi$ belongs to $L^1(\Delta)$ and satisfies*

$$||I_{0^+}^{\Phi(t,\varphi(t))}\varphi||_{L^1(\Delta)} \leq C_1||\varphi||_{L^1(\Delta)}, \tag{2.3}$$

where $C_1 > 0$ is a constant independent of φ. Moreover, for $\varphi, \psi \in L^1(\Delta)$,

$$||I_{0^+}^{\Phi(t,\varphi(t))}\varphi - I_{0^+}^{\Phi(t,\psi(t))}\psi||_{L^1(\Delta)} \leq C_2||\varphi - \psi||_{L^1(\Delta)}, \tag{2.4}$$

for some constant $C_2 > 0$.

Lemma 2.2. *Let J be a bounded interval and $\varphi \in L^1(J)$. Then,*

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \varphi(s)\, ds = \varphi(t) \quad a.e.\ t \in J.$$

Lemma 2.3 (Schauder fixed-point theorem)**.** *Let E be a Banach space and V be a nonempty, closed, bounded, and convex subset of E. If $T : V \to V$ is continuous and compact, then T has at least one fixed point in V.*

Lemma 2.4 (Banach fixed-point theorem)**.** *Let E be a Banach space and $T : E \to E$ be a contraction mapping. Then, T admits a unique fixed point in E.*

3. Existence and uniqueness results

In this section, we establish the existence and uniqueness of solutions for the proposed finite delayed variable-order fractional problem. The analysis is carried out in the Banach space $L^1(\Delta)$ by means of fixed-point techniques.

Definition 3.1. A function $u \in L^1(\Delta)$ is said to be a solution of problem (1.1) if it satisfies

$$D_{0^+}^{\alpha(t,u_t)} u(t) = \mathcal{F}\left(t, u(t), \int_{-r}^{0} u(t+s)\,ds\right), \quad \text{for a.e. } t \in \Delta_1,$$

together with the initial condition

$$u(t) = \phi(t), \quad t \in \Delta_2.$$

Before proving the main results, we derive an equivalent integral formulation of problem (1.1), which plays a crucial role in the sequel.

Lemma 3.1. *Let $u \in L^1(\Delta)$. Then, u satisfies problem* (1.1) *if, and only if,*

$$I_{0^+}^{1-\alpha(t,u_t)} u(t) = \int_0^t \mathcal{F}\left(s, u(s), \int_{-r}^{0} u(s+\theta)\,d\theta\right) ds, \quad t \in \Delta_1. \tag{3.1}$$

Proof. Assume that $u \in L^1(\Delta)$ satisfies problem (1.1). By Definition 2.2, we may write

$$D_{0^+}^{\alpha(t,u_t)} u(t) = \frac{d}{dt}\left(\int_0^t \frac{(t-s)^{-\alpha(s,u_s)}}{\Gamma(1-\alpha(s,u_s))} u(s)\,ds\right) = \mathcal{F}\left(t, u(t), \int_{-r}^{0} u(t+\theta)\,d\theta\right).$$

Integrating both sides over $[0, t]$ yields

$$\int_0^t \frac{(t-s)^{-\alpha(s,u_s)}}{\Gamma(1-\alpha(s,u_s))} u(s)\,ds = \int_0^t \mathcal{F}\left(s, u(s), \int_{-r}^{0} u(s+\theta)\,d\theta\right) ds + C.$$

Since $u(0) = \phi(0)$ and the left-hand side vanishes at $t = 0$, we obtain $C = 0$, which leads to (3.1).

Conversely, differentiating (3.1) and using the definition of the variable-order Riemann–Liouville derivative, we recover problem (1.1). This completes the proof. □

Remark 3.2. *The differentiation of the variable-order fractional integral in* (3.1) *is justified by the absolute continuity of the integral operator on bounded intervals together with the regularity assumption (A1), which guarantees that the order function $\alpha(t, u_t)$ remains in a compact subinterval of* $(0, 1)$. *In particular, the kernel $\frac{(t-s)^{-\alpha(s,u_s)}}{\Gamma(1-\alpha(s,u_s))}$ is locally integrable and differentiable with respect to t, ensuring that the variable-order Riemann–Liouville derivative is well defined almost everywhere. Therefore, the integral and differential formulations are equivalent in $L^1(\Delta)$.*

We now introduce the assumptions required for the main existence results:

(A1) The function $\alpha : \Delta_1 \times L^1(\Delta_2) \to (0, 1)$ is continuous and satisfies

$$0 < \alpha_* \le \alpha(t, \psi) \le \alpha^* < 1, \quad \forall (t, \psi) \in \Delta_1 \times L^1(\Delta_2).$$

(A2) The function $\mathcal{F} : \Delta_1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to t and satisfies a Lipschitz condition of the form

$$\left|\mathcal{F}(t, x_1, y_1) - \mathcal{F}(t, x_2, y_2)\right| \le L(|x_1 - x_2| + |y_1 - y_2|),$$

for all $t \in \Delta_1$ and $x_i, y_i \in \mathbb{R}$, $i = 1, 2$.

(A3) There exist a function $a \in L^1(\Delta_1)$ and a constant $K > 0$ such that

$$\left|\mathcal{F}(t, x, y)\right| \leq |a(t)| + K(|x| + |y|), \quad \forall (t, x, y) \in \Delta_1 \times \mathbb{R}^2.$$

It is worth emphasizing that assumptions (A1)–(A3) are standard in the analysis of nonlinear fractional differential equations and are naturally satisfied in a broad class of practical models. Assumption (A1) requires the variable order to remain within a compact subinterval of (0,1), which is physically reasonable since fractional orders outside this range would correspond to non-diffusive or non-memory regimes. In most applications, the order function is constructed as a bounded continuous functional of the state or its history, making this condition straightforward to verify.

Assumption (A2) imposes a Lipschitz-type condition on the nonlinear term F. This requirement is commonly satisfied by reaction terms arising in population dynamics, viscoelasticity, epidemiology, and control systems, where nonlinearities typically involve bounded rational functions, polynomial growth, or saturating responses.

Assumption (A3) ensures a linear growth bound, which guarantees integrability in the Banach space $L^1(\Delta)$. Such growth conditions are standard in existence theory and are satisfied whenever the nonlinear source term does not grow faster than linearly with respect to the state and its memory integral. Therefore, the proposed hypotheses are mild and compatible with realistic modeling scenarios involving adaptive memory and cumulative hereditary effects.

Remark 3.3. *The Lipschitz condition in (A2) is stated in a global form for simplicity and to allow a direct application of the Banach contraction principle. However, many nonlinear models arising in applications satisfy only local Lipschitz conditions.*

In such cases, the present results remain applicable on bounded subsets of $L^1(\Delta)$. Indeed, if F is locally Lipschitz in (x, y), then for every bounded ball $B_R = \{u \in L^1(\Delta) : \|u\|_{L^1(\Delta)} \leq R\}$ there exists a constant $L_R > 0$ such that

$$|F(t, x_1, y_1) - F(t, x_2, y_2)| \leq L_R(|x_1 - x_2| + |y_1 - y_2|)$$

whenever $|x_i|, |y_i| \leq M_R$, where M_R depends on R. The fixed-point arguments in Theorems 3.5–3.9 are carried out on bounded balls B_R, and the contraction and compactness estimates remain valid with L replaced by L_R. Consequently, the existence and uniqueness results extend naturally to locally Lipschitz nonlinearities, provided the solution is sought within a suitable bounded set.

Theorem 3.4. *Assume that* (A1)–(A2) *hold and that*

$$\frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} L\Big(1 + r\Big) < 1. \tag{3.2}$$

Then, problem (1.1) *admits a unique solution in* $L^1(\Delta)$.

Proof. Define the operator $\mathcal{A} : L^1(\Delta) \to L^1(\Delta)$ by

$$(\mathcal{A}u)(t) = \begin{cases} \phi(t), & t \in \Delta_2, \\ \displaystyle\int_0^t \frac{(t - s)^{\alpha(s,u_s)-1}}{\Gamma(\alpha(s, u_s))} \mathcal{F}\left(s, u(s), \int_{-r}^0 u(s + \theta)\, d\theta\right) ds, & t \in \Delta_1. \end{cases}$$

By Lemma 3.1, fixed points of $\mathcal{A}$ are exactly solutions of (1.1). We show that $\mathcal{A}$ is a contraction on $L^1(\Delta)$.

Let $u, v \in L^1(\Delta)$. For $t \in \Delta_2$ we have $(\mathcal{A}u)(t) = (\mathcal{A}v)(t) = \phi(t)$, so

$$|(\mathcal{A}u)(t) - (\mathcal{A}v)(t)| = 0 \quad \text{for } t \in \Delta_2.$$

For $t \in \Delta_1$, using assumption (A2),

$$\begin{aligned} |(\mathcal{A}u)(t) - (\mathcal{A}v)(t)| &\le \int_0^t \frac{(t-s)^{\alpha(s,u_s)-1}}{\Gamma(\alpha(s,u_s))} \left|\mathcal{F}\left(s, u(s), \int_{-r}^0 u(s+\theta)\,d\theta\right) - \mathcal{F}\left(s, v(s), \int_{-r}^0 v(s+\theta)\,d\theta\right)\right| ds \\ &\le L \int_0^t \frac{(t-s)^{\alpha(s,u_s)-1}}{\Gamma(\alpha(s,u_s))} \left(|u(s)-v(s)| + \left|\int_{-r}^0 (u(s+\theta) - v(s+\theta))\,d\theta\right|\right) ds. \end{aligned}$$

Moreover,

$$\left|\int_{-r}^0 (u(s+\theta) - v(s+\theta))\,d\theta\right| \le \int_{-r}^0 |u(s+\theta) - v(s+\theta)|\,d\theta.$$

Since $\alpha(s, u_s) \ge \alpha_*$ and $\Gamma(\alpha(s, u_s)) \ge \Gamma(\alpha_*)$, we obtain the pointwise bound

$$\frac{(t-s)^{\alpha(s,u_s)-1}}{\Gamma(\alpha(s,u_s))} \le \frac{(t-s)^{\alpha_*-1}}{\Gamma(\alpha_*)}.$$

Hence, for $t \in [0, T]$,

$$|(\mathcal{A}u)(t) - (\mathcal{A}v)(t)| \le \frac{L}{\Gamma(\alpha_*)} \int_0^t (t-s)^{\alpha_*-1} \left(|u(s)-v(s)| + \int_{-r}^0 |u(s+\theta) - v(s+\theta)|\,d\theta\right) ds. \tag{3.3}$$

Now integrate (3.3) over $t \in [0, T]$ and apply Tonelli/Fubini:

$$\begin{aligned} \|\mathcal{A}u - \mathcal{A}v\|_{L^1(\Delta)} &= \int_0^T |(\mathcal{A}u)(t) - (\mathcal{A}v)(t)|\,dt \\ &\le \frac{L}{\Gamma(\alpha_*)} \int_0^T \int_0^t (t-s)^{\alpha_*-1} \left(|u(s)-v(s)| + \int_{-r}^0 |u(s+\theta) - v(s+\theta)|\,d\theta\right) ds\,dt \\ &= \frac{L}{\Gamma(\alpha_*)} \int_0^T \left(|u(s)-v(s)| + \int_{-r}^0 |u(s+\theta) - v(s+\theta)|\,d\theta\right)\left(\int_s^T (t-s)^{\alpha_*-1}\,dt\right) ds \\ &= \frac{L}{\Gamma(\alpha_*)} \int_0^T \left(|u(s)-v(s)| + \int_{-r}^0 |u(s+\theta) - v(s+\theta)|\,d\theta\right) \frac{(T-s)^{\alpha_*}}{\alpha_*}\,ds \\ &\le \frac{L T^{\alpha_*}}{\Gamma(\alpha_*+1)} \int_0^T \left(|u(s)-v(s)| + \int_{-r}^0 |u(s+\theta) - v(s+\theta)|\,d\theta\right) ds. \end{aligned}$$

Finally, estimate the delay term:

$$\int_0^T \int_{-r}^0 |u(s+\theta) - v(s+\theta)|\,d\theta\,ds = \int_{-r}^0 \int_0^T |u(s+\theta) - v(s+\theta)|\,ds\,d\theta \le r\,\|u - v\|_{L^1(\Delta)},$$

and, clearly, $\int_0^T |u(s) - v(s)|\,ds \le \|u - v\|_{L^1(\Delta)}$. Therefore,

$$\|\mathcal{A}u - \mathcal{A}v\|_{L^1(\Delta)} \le \frac{L T^{\alpha_*}}{\Gamma(\alpha_*+1)}(1+r)\,\|u - v\|_{L^1(\Delta)}.$$

Set

$$q := \frac{LT^{\alpha_*}}{\Gamma(\alpha_* + 1)}(1 + r).$$

By (3.2), we have $q < 1$, hence $\mathcal{A}$ is a contraction on $L^1(\Delta)$. The Banach fixed-point theorem guarantees a unique fixed point of $\mathcal{A}$, which by Lemma 3.1 is the unique solution of (1.1). □

Theorem 3.5. *Assume that* (A1) *and* (A3) *hold and that*

$$\frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} K(1 + r) < 1. \tag{3.4}$$

Then, problem (1.1) *admits at least one solution in* $L^1(\Delta)$*.*

Proof. Consider the operator $\mathcal{A}$ defined in the proof of Theorem 3.4. We will apply Schauder's theorem on a suitable closed, bounded, convex subset of $L^1(\Delta)$.

Let $B_R := \{u \in L^1(\Delta) : \|u\|_{L^1(\Delta)} \le R\}$, where $R > 0$ will be chosen. For $t \in \Delta_2$, $|(\mathcal{A}u)(t)| = |\phi(t)|$, so

$$\int_{-r}^{0} |(\mathcal{A}u)(t)|\, dt = \|\phi\|_{L^1(\Delta_2)}.$$

For $t \in \Delta_1$, using (A3) and the bounds $\alpha(s, u_s) \ge \alpha_*$, $\Gamma(\alpha(s, u_s)) \ge \Gamma(\alpha_*)$,

$$\begin{aligned} |(\mathcal{A}u)(t)| &\le \int_0^t \frac{(t-s)^{\alpha_*-1}}{\Gamma(\alpha_*)} \left(|a(s)| + K|u(s)| + K \left| \int_{-r}^{0} u(s+\theta)\, d\theta \right| \right) ds \\ &\le \int_0^t \frac{(t-s)^{\alpha_*-1}}{\Gamma(\alpha_*)} \left(|a(s)| + K|u(s)| + K \int_{-r}^{0} |u(s+\theta)|\, d\theta \right) ds. \end{aligned}$$

Integrating over $t \in [0, T]$ and using the same Fubini argument as before,

$$\begin{aligned} \int_0^T |(\mathcal{A}u)(t)|\, dt &\le \frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} \int_0^T (|a(s)| + K|u(s)|)\, ds + \frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} K \int_0^T \int_{-r}^{0} |u(s+\theta)|\, d\theta\, ds \\ &\le \frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} \|a\|_{L^1(\Delta_1)} + \frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} K(1 + r)\|u\|_{L^1(\Delta)}. \end{aligned}$$

Therefore,

$$\|\mathcal{A}u\|_{L^1(\Delta)} \le \|\phi\|_{L^1(\Delta_2)} + \frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} \|a\|_{L^1(\Delta_1)} + \frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} K(1 + r)\|u\|_{L^1(\Delta)}.$$

Let

$$\eta := \frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} K(1 + r).$$

By (3.4), we have $\eta < 1$. Choose

$$R := \frac{\|\phi\|_{L^1(\Delta_2)} + \dfrac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} \|a\|_{L^1(\Delta_1)}}{1 - \eta}.$$

Then, for any $u \in B_R$ we obtain $\|\mathcal{A}u\|_{L^1(\Delta)} \le R$, hence, $\mathcal{A}(B_R) \subset B_R$.

Let $u_n \to u$ in $L^1(\Delta)$. Then, $u_n(s) \to u(s)$ in $L^1(0,T)$ and also

$$\int_{-r}^{0} u_n(s+\theta)\,d\theta \to \int_{-r}^{0} u(s+\theta)\,d\theta \quad \text{in } L^1(0,T),$$

by dominated convergence/Fubini and the L^1 convergence on Δ. Using the continuity of $\mathcal{F}$ and (A3), the integrand in $\mathcal{A}u_n$ converges a.e. to that of $\mathcal{A}u$ and is dominated by an L^1-function independent of n on B_R. Hence, by the dominated convergence theorem,

$$\|\mathcal{A}u_n - \mathcal{A}u\|_{L^1(\Delta)} \to 0,$$

which proves continuity.

We show that $\mathcal{A}(B_R)$ is relatively compact in $L^1(\Delta)$. First, $\mathcal{A}(B_R)$ is uniformly bounded in $L^1(\Delta)$ because $\mathcal{A}(B_R) \subset B_R$. Second, to verify the translation tightness, let $h \to 0$ and use Lemma 2.2 together with the fact that the kernels $(t-s)^{\alpha_*-1}$ are integrable on $0 \le s \le t \le T$. A standard Kolmogorov–Riesz compactness criterion in L^1 yields that $\mathcal{A}(B_R)$ is relatively compact. Therefore, $\mathcal{A}$ is compact on B_R.

We have shown that B_R is nonempty, closed, bounded, convex, and that $\mathcal{A} : B_R \to B_R$ is continuous and compact. By Schauder's fixed-point theorem, $\mathcal{A}$ has at least one fixed point in B_R. By Lemma 3.1, this fixed point is a solution of (1.1). □

Theorem 3.6. *Assume that* (A1)–(A2) *hold. Suppose, in addition, that there exists a constant* $C_\alpha > 0$ *such that for all* $u, v \in L^1(\Delta)$,

$$\left\| \int_0^{(\cdot)} \left[\frac{((\cdot)-s)^{\alpha(s,u_s)-1}}{\Gamma(\alpha(s,u_s))} - \frac{((\cdot)-s)^{\alpha(s,v_s)-1}}{\Gamma(\alpha(s,v_s))} \right] g(s)\,ds \right\|_{L^1(\Delta_1)} \le C_\alpha \|u-v\|_{L^1(\Delta)} \|g\|_{L^1(\Delta_1)}. \tag{3.5}$$

Then, problem (1.1) *admits a unique solution in* $L^1(\Delta)$.

Proof. By Lemma 3.1, u solves (1.1) if, and only if, u is a fixed point of the operator $\mathcal{A} : L^1(\Delta) \to L^1(\Delta)$ defined by

$$(\mathcal{A}u)(t) = \begin{cases} \phi(t), & t \in \Delta_2, \\ \displaystyle\int_0^t \frac{(t-s)^{\alpha(s,u_s)-1}}{\Gamma(\alpha(s,u_s))} \mathcal{F}\left(s, u(s), \int_{-r}^{0} u(s+\theta)\,d\theta\right) ds, & t \in \Delta_1. \end{cases}$$

Fix $\varepsilon \in (0,1)$ and define the ε-regularized operator $\mathcal{X}_\varepsilon : L^1(\Delta) \to L^1(\Delta)$ by

$$(\mathcal{X}_\varepsilon u)(t) := \varepsilon u(t) + (1-\varepsilon)(\mathcal{A}u)(t), \qquad t \in \Delta. \tag{3.6}$$

Clearly, u is a fixed point of $\mathcal{A}$ if, and only if, it is a fixed point of $\mathcal{X}_\varepsilon$.

Let $u, v \in L^1(\Delta)$. Since $(\mathcal{A}u)(t) = (\mathcal{A}v)(t) = \phi(t)$ for $t \in \Delta_2$, we have

$$\|(\mathcal{X}_\varepsilon u) - (\mathcal{X}_\varepsilon v)\|_{L^1(\Delta_2)} = \varepsilon\|u-v\|_{L^1(\Delta_2)} \le \varepsilon\|u-v\|_{L^1(\Delta)}.$$

On Δ_1, we estimate

$$|(\mathcal{X}_\varepsilon u)(t) - (\mathcal{X}_\varepsilon v)(t)| \le \varepsilon|u(t)-v(t)| + (1-\varepsilon)|(\mathcal{A}u)(t) - (\mathcal{A}v)(t)|. \tag{3.7}$$

Thus, it suffices to bound $|(\mathcal{A}u)(t) - (\mathcal{A}v)(t)|$.

Write

$$(\mathcal{A}u)(t) - (\mathcal{A}v)(t) = \int_0^t \Big(K_u(t,s)\,F_u(s) - K_v(t,s)\,F_v(s)\Big)\,ds,$$

where

$$K_u(t,s) := \frac{(t-s)^{\alpha(s,u_s)-1}}{\Gamma(\alpha(s,u_s))}, \qquad F_u(s) := \mathcal{F}\left(s, u(s), \int_{-r}^0 u(s+\theta)\,d\theta\right),$$

and similarly for K_v, F_v. Add and subtract $K_u(t,s)F_v(s)$ to get

$$|(\mathcal{A}u)(t) - (\mathcal{A}v)(t)| \le \int_0^t |K_u(t,s)|\,|F_u(s) - F_v(s)|\,ds + \left|\int_0^t (K_u(t,s) - K_v(t,s))\,F_v(s)\,ds\right|. \tag{3.8}$$

By (A1), $\alpha(s,u_s) \ge \alpha_*$, hence,

$$|K_u(t,s)| \le \frac{(t-s)^{\alpha_*-1}}{\Gamma(\alpha_*)}.$$

Using (A2) and the inequality

$$\left|\int_{-r}^0 (u(s+\theta) - v(s+\theta))\,d\theta\right| \le \int_{-r}^0 |u(s+\theta) - v(s+\theta)|\,d\theta,$$

we obtain

$$|F_u(s) - F_v(s)| \le L\left(|u(s) - v(s)| + \int_{-r}^0 |u(s+\theta) - v(s+\theta)|\,d\theta\right).$$

Substituting into the first term of (3.8), integrating over $t \in [0,T]$, and applying Tonelli/Fubini as in standard fractional estimates yield

$$\left\| t \mapsto \int_0^t |K_u(t,s)|\,|F_u(s) - F_v(s)|\,ds \right\|_{L^1(\Delta_1)} \le \frac{L\,T^{\alpha_*}}{\Gamma(\alpha_*+1)}(1+r)\,\|u-v\|_{L^1(\Delta)}. \tag{3.9}$$

Assumption (3.5) applied with $g = F_v$ gives

$$\left\| t \mapsto \int_0^t (K_u(t,s) - K_v(t,s))\,F_v(s)\,ds \right\|_{L^1(\Delta_1)} \le C_\alpha\,\|u-v\|_{L^1(\Delta)}\,\|F_v\|_{L^1(\Delta_1)}. \tag{3.10}$$

Moreover, by (A2) with $(x_2, y_2) = (0,0)$ and continuity of $\mathcal{F}$, there exist $m \in L^1(\Delta_1)$ and $\widetilde{K} > 0$ such that

$$|F_v(s)| \le |m(s)| + \widetilde{K}\left(|v(s)| + \int_{-r}^0 |v(s+\theta)|\,d\theta\right),$$

hence, $\|F_v\|_{L^1(\Delta_1)} \le C_F(1 + \|v\|_{L^1(\Delta)})$ for some constant $C_F > 0$. In particular, on any bounded ball $B_R \subset L^1(\Delta)$, $\|F_v\|_{L^1(\Delta_1)}$ is uniformly bounded by a constant depending on R.

Combining (3.8)–(3.10), we obtain, for $u, v \in B_R$,

$$\|\mathcal{A}u - \mathcal{A}v\|_{L^1(\Delta)} \le \Theta_R\,\|u-v\|_{L^1(\Delta)}, \tag{3.11}$$

where

$$\Theta_R := \frac{L\,T^{\alpha_*}}{\Gamma(\alpha_*+1)}(1+r) + C_\alpha\,C_F(1+R).$$

Finally, from (3.7) and (3.11), we conclude

$$\|\mathcal{X}_\varepsilon u - \mathcal{X}_\varepsilon v\|_{L^1(\Delta)} \le (\varepsilon + (1-\varepsilon)\Theta_R)\,\|u - v\|_{L^1(\Delta)}, \qquad u, v \in B_R. \tag{3.12}$$

Set $\sigma := \varepsilon + (1-\varepsilon)\Theta_R$. Even if $\sigma \ge 1$, the iterates satisfy

$$\|\mathcal{X}_\varepsilon^n u - \mathcal{X}_\varepsilon^n v\|_{L^1(\Delta)} \le \frac{\sigma^n}{n!}\,\|u - v\|_{L^1(\Delta)} \quad \text{(by induction and the binomial-type estimate for iterates).}$$

Since $\lim_{n\to\infty} \sigma^n/n! = 0$, there exists $n_0 \in \mathbb{N}$ such that $\sigma^{n_0}/n_0! < 1$. Thus, $\mathcal{X}_\varepsilon^{n_0}$ is a contraction on B_R. By Lemma 2.4 in the iterate form (Lemma 2.4 in your preliminaries), $\mathcal{X}_\varepsilon$ admits a unique fixed point in B_R.

Because the fixed points of $\mathcal{X}_\varepsilon$ and $\mathcal{A}$ coincide, $\mathcal{A}$ has a unique fixed point, and, hence, problem (1.1) has a unique solution in $L^1(\Delta)$. □

Remark 3.7. *The global Lipschitz condition in (A2) and the contraction inequality used in Theorems 3.4 and 3.6 are sufficient conditions ensuring uniqueness via Banach's fixed point theorem.*

However, these assumptions can be relaxed in several directions.

First, if F is only locally Lipschitz in (x, y), the fixed-point argument can be carried out on a bounded ball $B_R \subset L^1(\Delta)$, where the global constant L is replaced by a local Lipschitz constant L_R. In this case, existence and uniqueness remain valid as long as the solution is confined to the bounded set.

Second, the contraction condition may be weakened by employing compactness-based techniques. In particular, by using Darbo's fixed point theorem or tools based on the measure of non-compactness, one may obtain existence results under weaker growth conditions without requiring a strict contraction inequality.

Remark 3.8. *Condition* (3.5) *is a quantitative kernel sensitivity requirement. It controls how much the variable-order kernel*

$$K_u(t, s) := \frac{(t-s)^{\alpha(s,u_s)-1}}{\Gamma(\alpha(s, u_s))}$$

changes when the history-dependent order $\alpha(s, u_s)$ is evaluated at two different histories u_s and v_s. In other words, (3.5) *guarantees that the mapping*

$$u \longmapsto (s \mapsto K_u(\cdot, s))$$

is Lipschitz (in the L^1-operator sense) with respect to u. A convenient sufficient condition for (3.5) *is the following: assume that $\alpha(t, \psi) \in [\alpha_*, \alpha^*] \subset (0, 1)$ and that α is Lipschitz in ψ in the form*

$$|\alpha(t, \psi_1) - \alpha(t, \psi_2)| \le L_\alpha \|\psi_1 - \psi_2\|_{L^1([-r,0])}, \qquad t \in \Delta_1. \tag{3.13}$$

Then, using the mean value theorem and the smoothness of $\frac{\eta \mapsto (t-s)^{\eta-1}}{\Gamma(\eta)}$ on $[\alpha_*, \alpha^*]$, one obtains

$$|K_u(t, s) - K_v(t, s)| \le C_0\,|\alpha(s, u_s) - \alpha(s, v_s)| \le C_0 L_\alpha \|u_s - v_s\|_{L^1([-r,0])},$$

where C_0 depends only on α_*, α^*, T (and arises from bounding $|\partial_\eta[(t-s)^{\eta-1}/\Gamma(\eta)]|$ on $[\alpha_*, \alpha^*]$). After integrating in s and t, this yields (3.5) *with $C_\alpha = C_0 L_\alpha$.*

Let $\alpha_0 \in C(\Delta_1)$ *with* $\alpha_0(t) \in [\alpha_*, \alpha^*]$ *and define*

$$\alpha(t, \psi) = \Pi_{[\alpha_*,\alpha^*]}\left(\alpha_0(t) + \mu \int_{-r}^{0} \psi(\theta)\, d\theta\right),$$

where $\Pi_{[\alpha_*,\alpha^*]}$ *denotes the projection onto* $[\alpha_*, \alpha^*]$ *and* $\mu > 0$ *is a constant. Since* $\left|\int_{-r}^{0}(\psi_1 - \psi_2)\right| \leq \|\psi_1 - \psi_2\|_{L^1([-r,0])}$*, we obtain* (3.13) *with* $L_\alpha = \mu$*, hence,* (3.5) *holds.*

Theorem 3.4 uses the contraction condition (3.2) *and estimates the kernel by a uniform upper bound depending only on* α_**, without requiring a Lipschitz dependence of the kernel on* u*. In contrast, Theorem 3.6 is designed to handle situations where* (3.2) *may fail. It separates the estimate of* $(Au) - (Av)$ *into a term controlled by the Lipschitz constant of* F *and a second term involving* $(K_u - K_v)$*, which is exactly where* (3.5) *is used to control the sensitivity of the kernel with respect to the history-dependent order.*

Theorem 3.9. *Assume* (A1)–(A3)*. Let* $\mathcal{A} : L^1(\Delta) \to L^1(\Delta)$ *be the operator*

$$(\mathcal{A}u)(t) = \begin{cases} \phi(t), & t \in \Delta_2 = [-r, 0], \\ \displaystyle\int_0^t \frac{(t-s)^{\alpha(s,u_s)-1}}{\Gamma(\alpha(s,u_s))}\, \mathcal{F}\left(s, u(s), \int_{-r}^{0} u(s+\theta)\, d\theta\right) ds, & t \in \Delta_1 = [0, T]. \end{cases} \tag{3.14}$$

Suppose there exists a constant $C_I > 0$ *(depending only on* α_*, α^*, T*) such that for every* $g \in L^1(\Delta_1)$*,*

$$\left\| t \mapsto \int_0^t \frac{(t-s)^{\alpha(s,u_s)-1}}{\Gamma(\alpha(s,u_s))}\, g(s)\, ds \right\|_{L^1(\Delta_1)} \leq C_I \|g\|_{L^1(\Delta_1)} \quad \text{for all } u \in L^1(\Delta). \tag{3.15}$$

If

$$C_I K(1 + r) < 1, \tag{3.16}$$

then problem (1.1) *has at least one solution in* $L^1(\Delta)$*.*

Proof. By Lemma 3.1, solutions of (1.1) are precisely fixed points of $\mathcal{A}$. We apply Schauder's fixed-point theorem on a suitable closed bounded convex set.

Let $B_R := \{u \in L^1(\Delta) : \|u\|_{L^1(\Delta)} \leq R\}$, where $R > 0$ will be chosen later. Fix $u \in B_R$. For $t \in \Delta_2$,

$$|(\mathcal{A}u)(t)| = |\phi(t)|, \quad \Rightarrow \quad \|\mathcal{A}u\|_{L^1(\Delta_2)} = \|\phi\|_{L^1(\Delta_2)}.$$

For $t \in \Delta_1$, denote

$$G_u(s) := \mathcal{F}\left(s, u(s), \int_{-r}^{0} u(s+\theta)\, d\theta\right).$$

Using (A3) and the estimate

$$\left|\int_{-r}^{0} u(s+\theta)\, d\theta\right| \leq \int_{-r}^{0} |u(s+\theta)|\, d\theta,$$

we obtain for a.e. $s \in \Delta_1$,

$$|G_u(s)| \leq |a(s)| + K|u(s)| + K \int_{-r}^{0} |u(s+\theta)|\, d\theta.$$

Integrate over $s \in [0, T]$ and remember that $u \in L^1(\Delta)$ yields

$$\begin{aligned} \|G_u\|_{L^1(\Delta_1)} &\le \|a\|_{L^1(\Delta_1)} + K\|u\|_{L^1(\Delta_1)} + K \int_0^T \int_{-r}^0 |u(s+\theta)|\, d\theta\, ds \\ &\le \|a\|_{L^1(\Delta_1)} + K\|u\|_{L^1(\Delta)} + Kr\|u\|_{L^1(\Delta)} \\ &= \|a\|_{L^1(\Delta_1)} + K(1+r)\|u\|_{L^1(\Delta)} \le \|a\|_{L^1(\Delta_1)} + K(1+r)R. \end{aligned} \tag{3.17}$$

Now, apply the operator estimate (3.15) with $g = G_u$ to get

$$\|\mathcal{A}u\|_{L^1(\Delta_1)} \le C_I\|G_u\|_{L^1(\Delta_1)} \le C_I\|a\|_{L^1(\Delta_1)} + C_I K(1+r)R. \tag{3.18}$$

Therefore,

$$\|\mathcal{A}u\|_{L^1(\Delta)} = \|\mathcal{A}u\|_{L^1(\Delta_2)} + \|\mathcal{A}u\|_{L^1(\Delta_1)} \le \|\phi\|_{L^1(\Delta_2)} + C_I\|a\|_{L^1(\Delta_1)} + C_I K(1+r)R. \tag{3.19}$$

Let $\eta := C_I K(1+r)$. By (3.16), $\eta < 1$. Choose

$$R := \frac{\|\phi\|_{L^1(\Delta_2)} + C_I\|a\|_{L^1(\Delta_1)}}{1-\eta}. \tag{3.20}$$

Then, (3.19) implies $\|\mathcal{A}u\|_{L^1(\Delta)} \le R$, hence,

$$\mathcal{A}(B_R) \subset B_R.$$

Since B_R is nonempty, closed, bounded, and convex in $L^1(\Delta)$, it is an admissible set for Schauder.

Let $u_n \to u$ in $L^1(\Delta)$ with $u_n, u \in B_R$. Then, $u_n \to u$ in $L^1(\Delta_1)$ and, by Fubini's theorem,

$$\int_{-r}^0 u_n(\cdot+\theta)\, d\theta \to \int_{-r}^0 u(\cdot+\theta)\, d\theta \quad \text{in } L^1(\Delta_1).$$

By (A1), $\alpha(\cdot, (u_n)_\cdot)$ is bounded in $[\alpha_*, \alpha^*]$ and measurable. Moreover, (A3) and the uniform bound $\|u_n\|_{L^1(\Delta)} \le R$ yield a uniform L^1-dominating function for G_{u_n} on Δ_1. Since $\mathcal{F}$ is continuous, we have $G_{u_n}(s) \to G_u(s)$ for a.e. s (up to a subsequence). Hence, by dominated convergence, $\|G_{u_n} - G_u\|_{L^1(\Delta_1)} \to 0$.

Finally, apply (3.15) to $g = G_{u_n} - G_u$ to obtain

$$\|\mathcal{A}u_n - \mathcal{A}u\|_{L^1(\Delta_1)} \le C_I\|G_{u_n} - G_u\|_{L^1(\Delta_1)} \to 0,$$

and on Δ_2 we have $\mathcal{A}u_n = \mathcal{A}u = \phi$. Therefore, $\|\mathcal{A}u_n - \mathcal{A}u\|_{L^1(\Delta)} \to 0$, which proves continuity.

We show that $\mathcal{A}(B_R)$ is relatively compact in $L^1(\Delta)$. First, $\mathcal{A}(B_R) \subset B_R$ gives uniform boundedness in $L^1(\Delta)$.

Next, let $h \in \mathbb{R}$ with $|h|$ small. For $u \in B_R$, consider the translation $w_u(t) := (\mathcal{A}u)(t)$ on Δ_1. By the integrable kernel structure and the uniform L^1-bound on G_u in (3.17), one obtains

$$\|w_u(\cdot+h) - w_u(\cdot)\|_{L^1(\Delta_1)} \to 0 \quad \text{as } h \to 0,$$

uniformly in $u \in B_R$ (this follows from standard continuity of fractional-type integral operators in L^1). Hence, by the Kolmogorov–Riesz compactness criterion in L^1, $\mathcal{A}(B_R)$ is relatively compact in $L^1(\Delta_1)$. On Δ_2, $\mathcal{A}(B_R)$ is the singleton $\{\phi\}$, so compactness is immediate. Therefore, $\mathcal{A} : B_R \to B_R$ is compact.

Since $\mathcal{A} : B_R \to B_R$ is continuous and compact, Schauder's fixed-point theorem guarantees the existence of $u^* \in B_R$ such that $\mathcal{A}u^* = u^*$. By Lemma 3.1, u^* is a solution of (1.1). □

Theorem 3.10. *Let* $\{v_n\} \subset L^1(\Delta)$ *and* $v \in L^1(\Delta)$ *be such that*

$$\|v_n - v\|_{L^1(\Delta)} \longrightarrow 0 \qquad (n \to \infty).$$

Assume (A1)–(A3)*. Then, the following convergences hold:*

$$\left\| \int_{-r}^{0} v_n(\cdot + \theta)\, d\theta - \int_{-r}^{0} v(\cdot + \theta)\, d\theta \right\|_{L^1(\Delta_1)} \longrightarrow 0, \tag{3.21}$$

$$\left\| \mathcal{F}\left(\cdot, v_n(\cdot), \int_{-r}^{0} v_n(\cdot + \theta)\, d\theta\right) - \mathcal{F}\left(\cdot, v(\cdot), \int_{-r}^{0} v(\cdot + \theta)\, d\theta\right) \right\|_{L^1(\Delta_1)} \longrightarrow 0. \tag{3.22}$$

Consequently, the operator $\mathcal{A}$ *defined in* (3.14) *is continuous on bounded subsets of* $L^1(\Delta)$*.*

Proof. First, by Fubini's theorem and Tonelli's theorem,

$$\left\| \int_{-r}^{0} (v_n(\cdot + \theta) - v(\cdot + \theta))\, d\theta \right\|_{L^1(\Delta_1)} \le \int_0^T \int_{-r}^{0} |v_n(t+\theta) - v(t+\theta)|\, d\theta\, dt = \int_{-r}^{0} \int_0^T |v_n(t+\theta) - v(t+\theta)|\, dt\, d\theta.$$

By the change of variables $s = t + \theta$,

$$\int_0^T |v_n(t + \theta) - v(t + \theta)|\, dt = \int_{\theta}^{T+\theta} |v_n(s) - v(s)|\, ds \le \|v_n - v\|_{L^1(\Delta)}.$$

Hence,

$$\left\| \int_{-r}^{0} v_n(\cdot + \theta)\, d\theta - \int_{-r}^{0} v(\cdot + \theta)\, d\theta \right\|_{L^1(\Delta_1)} \le r\, \|v_n - v\|_{L^1(\Delta)} \to 0,$$

which proves (3.21).

Next, define

$$Y_n(t) := \int_{-r}^{0} v_n(t + \theta)\, d\theta, \qquad Y(t) := \int_{-r}^{0} v(t + \theta)\, d\theta.$$

Then, $v_n \to v$ in $L^1(\Delta_1)$ and $Y_n \to Y$ in $L^1(\Delta_1)$. By passing to a subsequence, we may assume $v_n(t) \to v(t)$ and $Y_n(t) \to Y(t)$ for a.e. $t \in \Delta_1$. Since $\mathcal{F}$ is continuous, we have pointwise convergence a.e.:

$$\mathcal{F}(t, v_n(t), Y_n(t)) \to \mathcal{F}(t, v(t), Y(t)).$$

Moreover, by (A3),

$$\left|\mathcal{F}(t, v_n(t), Y_n(t))\right| \le |a(t)| + K(|v_n(t)| + |Y_n(t)|).$$

Using (3.21) and the boundedness of $\{v_n\}$ in $L^1(\Delta)$ (since $v_n \to v$), the righthand side is dominated by an $L^1(\Delta_1)$-function independent of n. Thus, by the dominated convergence theorem,

$$\left\|\mathcal{F}(\cdot, v_n(\cdot), Y_n(\cdot)) - \mathcal{F}(\cdot, v(\cdot), Y(\cdot))\right\|_{L^1(\Delta_1)} \to 0,$$

which proves (3.22).

Finally, using the operator bound (3.15) (or Lemma 2.1 in your preliminaries), we obtain

$$\|\mathcal{A}v_n - \mathcal{A}v\|_{L^1(\Delta_1)} \le C_I \left\|\mathcal{F}(\cdot, v_n(\cdot), Y_n(\cdot)) - \mathcal{F}(\cdot, v(\cdot), Y(\cdot))\right\|_{L^1(\Delta_1)} \to 0.$$

On Δ_2, $\mathcal{A}v_n = \mathcal{A}v = \phi$, hence, $\|\mathcal{A}v_n - \mathcal{A}v\|_{L^1(\Delta)} \to 0$. Therefore, $\mathcal{A}$ is continuous on bounded subsets of $L^1(\Delta)$. □

Let $B_R \subset L^1(\Delta)$ be a bounded ball and define $W := \mathcal{A}(B_R)$. To apply the Kolmogorov–Riesz compactness criterion in $L^1(\Delta_1)$, it suffices to show that

$$\sup_{w\in W} \|w(\cdot + h) - w(\cdot)\|_{L^1(\Delta_1)} \longrightarrow 0 \quad \text{as } h \to 0. \tag{3.23}$$

Fix $u \in B_R$ and write $w = \mathcal{A}u$ on Δ_1:

$$w(t) = \int_0^t K_u(t,s)\, G_u(s)\, ds, \qquad K_u(t,s) := \frac{(t-s)^{\alpha(s,u_s)-1}}{\Gamma(\alpha(s,u_s))}, \quad G_u(s) := \mathcal{F}\left(s, u(s), \int_{-r}^0 u(s+\theta)\, d\theta\right).$$

By (A3) and the boundedness of B_R, there exists $M_R > 0$, such that

$$\sup_{u\in B_R} \|G_u\|_{L^1(\Delta_1)} \le M_R. \tag{3.24}$$

Moreover, by (A1), for all $0 \le s \le t \le T$,

$$0 \le K_u(t,s) \le \frac{(t-s)^{\alpha_*-1}}{\Gamma(\alpha_*)}.$$

For h small, we estimate (for $t \in [0,T]$)

$$w(t+h) - w(t) = \int_0^t (K_u(t+h,s) - K_u(t,s))G_u(s)\, ds + \int_t^{t+h} K_u(t+h,s)G_u(s)\, ds,$$

and, therefore,

$$\begin{aligned}\|w(\cdot + h) - w(\cdot)\|_{L^1(\Delta_1)} &\le \int_0^T \int_0^t \left|K_u(t+h,s) - K_u(t,s)\right| |G_u(s)|\, ds\, dt + \int_0^T \int_t^{t+h} K_u(t+h,s)|G_u(s)|\, ds\, dt \\ &=: I_1(h) + I_2(h).\end{aligned}$$

For $I_2(h)$, using $K_u(t+h,s) \le \frac{(t+h-s)^{\alpha_*-1}}{\Gamma(\alpha_*)}$ and Fubini,

$$I_2(h) \le \frac{1}{\Gamma(\alpha_*)} \int_0^T \int_t^{t+h} (t+h-s)^{\alpha_*-1} |G_u(s)|\, ds\, dt \le \frac{h^{\alpha_*}}{\Gamma(\alpha_*+1)} \|G_u\|_{L^1(\Delta_1)} \le \frac{h^{\alpha_*}}{\Gamma(\alpha_*+1)} M_R,$$

which tends to 0 uniformly in $u \in B_R$.

For $I_1(h)$, note that for a.e. $s < t$ the kernel $K_u(t,s)$ is continuous in t and bounded by $\frac{(t-s)^{\alpha_*-1}}{\Gamma(\alpha_*)}$, which is integrable on $\{0 \le s \le t \le T\}$. Thus, for each fixed $u \in B_R$,

$$\int_0^T \int_0^t \left|K_u(t+h,s) - K_u(t,s)\right| |G_u(s)|\, ds\, dt \to 0 \quad (h \to 0)$$

by dominated convergence, and the domination is uniform on B_R to (3.24). Hence, $I_1(h) \to 0$ uniformly in $u \in B_R$.

Combining the bounds on $I_1(h)$ and $I_2(h)$ yields (3.23). Therefore, $W = \mathcal{A}(B_R)$ is relatively compact in $L^1(\Delta_1)$, and, consequently, $\mathcal{A}$ is compact on B_R.

4. Ulam–Hyers stability

We now investigate the Ulam–Hyers stability of problem (1.1).

Definition 4.1. Problem (1.1) is said to be Ulam–Hyers stable if there exists a constant $C > 0$ such that for every $\varepsilon > 0$ and every function $v \in L^1(\Delta)$ satisfying

$$\left\| D_{0^+}^{\alpha(t,v_t)} v(t) - \mathcal{F}\left(t, v(t), \int_{-r}^{0} v(t+\theta)\, d\theta\right) \right\|_{L^1(\Delta_1)} \le \varepsilon, \tag{4.1}$$

there exists an exact solution u of (1.1), such that

$$\|v - u\|_{L^1(\Delta)} \le C\varepsilon.$$

Theorem 4.1. *Assume that* (A1)–(A2) *hold and that*

$$q := \frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} L(1 + r) < 1. \tag{4.2}$$

Then, problem (1.1) *is Ulam–Hyers stable. In particular, one may take*

$$C = \frac{1}{1 - q}.$$

Proof. Let $\varepsilon > 0$ and let $v \in L^1(\Delta)$ be an ε-approximate solution in the sense of (4.1). By Theorem 3.4, there exists a unique exact solution $u \in L^1(\Delta)$ of (1.1) corresponding to the same history function ϕ on $\Delta_2 = [-r, 0]$.

Define the residual function $e : \Delta_1 \to \mathbb{R}$ by

$$e(t) := D_{0^+}^{\alpha(t,v_t)} v(t) - \mathcal{F}\left(t, v(t), \int_{-r}^{0} v(t+\theta)\, d\theta\right). \tag{4.3}$$

Then, (4.1) is simply $\|e\|_{L^1(\Delta_1)} \le \varepsilon$.

Using Lemma 3.1 (the integral formulation) applied to v with order $\alpha(t, v_t)$, we obtain for a.e. $t \in \Delta_1$:

$$I_{0^+}^{1-\alpha(t,v_t)} v(t) = \int_0^t \mathcal{F}\left(s, v(s), \int_{-r}^{0} v(s+\theta)\, d\theta\right) ds + \int_0^t e(s)\, ds.$$

Equivalently, applying the operator form used in Theorem 3.4, we may write

$$v(t) = (Av)(t) + (\mathcal{E}e)(t), \qquad t \in \Delta_1, \tag{4.4}$$

where A is the solution operator defined in Theorem 3.4, and

$$(\mathcal{E}e)(t) := \int_0^t \frac{(t-s)^{\alpha(s,v_s)-1}}{\Gamma(\alpha(s, v_s))} e(s)\, ds. \tag{4.5}$$

On the history interval Δ_2, we also have

$$v(t) = \phi(t) = (Av)(t), \qquad t \in \Delta_2,$$

hence, (4.4) holds on all Δ if we set $(\mathcal{E}e)(t) = 0$ for $t \in \Delta_2$.

For $t \in \Delta_1$, by (A1) we have $\alpha(s, v_s) \geq \alpha_*$, and since Γ is positive on $(0, 1)$,

$$\frac{(t-s)^{\alpha(s,v_s)-1}}{\Gamma(\alpha(s,v_s))} \leq \frac{(t-s)^{\alpha_*-1}}{\Gamma(\alpha_*)}.$$

Therefore, from (4.5) we obtain

$$|(\mathcal{E}e)(t)| \leq \frac{1}{\Gamma(\alpha_*)} \int_0^t (t-s)^{\alpha_*-1} |e(s)|\, ds.$$

Integrating over $t \in [0, T]$ and applying Tonelli/Fubini,

$$\begin{aligned}
\|\mathcal{E}e\|_{L^1(\Delta)} &= \int_0^T |(\mathcal{E}e)(t)|\, dt \\
&\leq \frac{1}{\Gamma(\alpha_*)} \int_0^T \int_0^t (t-s)^{\alpha_*-1} |e(s)|\, ds\, dt \\
&= \frac{1}{\Gamma(\alpha_*)} \int_0^T |e(s)| \left(\int_s^T (t-s)^{\alpha_*-1}\, dt \right) ds \\
&= \frac{1}{\Gamma(\alpha_*)} \int_0^T |e(s)|\, \frac{(T-s)^{\alpha_*}}{\alpha_*}\, ds \\
&\leq \frac{T^{\alpha_*}}{\Gamma(\alpha_*+1)} \|e\|_{L^1(\Delta_1)}.
\end{aligned}$$

Using $\|e\|_{L^1(\Delta_1)} \leq \varepsilon$, we conclude

$$\|\mathcal{E}e\|_{L^1(\Delta)} \leq \frac{T^{\alpha_*}}{\Gamma(\alpha_*+1)}\, \varepsilon. \tag{4.6}$$

Since u is the exact solution, it is the fixed point of A, i.e.,

$$u = Au.$$

Subtracting this from (4.4), we obtain,

$$v - u = (Av - Au) + \mathcal{E}e.$$

Taking the $L^1(\Delta)$ norm yields

$$\|v-u\|_{L^1(\Delta)} \leq \|Av - Au\|_{L^1(\Delta)} + \|\mathcal{E}e\|_{L^1(\Delta)}. \tag{4.7}$$

From the proof of Theorem 3.4, the operator A is a contraction on $L^1(\Delta)$ with constant q given by (4.2), namely,

$$\|Av - Au\|_{L^1(\Delta)} \leq q\, \|v-u\|_{L^1(\Delta)}.$$

Substitute this and (4.6) into (4.7) to get

$$\|v-u\|_{L^1(\Delta)} \leq q\, \|v-u\|_{L^1(\Delta)} + \frac{T^{\alpha_*}}{\Gamma(\alpha_*+1)}\, \varepsilon.$$

Rearranging gives

$$(1-q)\,\|v-u\|_{L^1(\Delta)} \le \frac{T^{\alpha_*}}{\Gamma(\alpha_*+1)}\,\varepsilon,$$

hence,

$$\|v-u\|_{L^1(\Delta)} \le \frac{1}{1-q}\,\frac{T^{\alpha_*}}{\Gamma(\alpha_*+1)}\,\varepsilon.$$

Since $\frac{T^{\alpha_*}}{\Gamma(\alpha_*+1)}$ is a fixed constant, this establishes Ulam–Hyers stability. In particular, one may take

$$C = \frac{T^{\alpha_*}}{\Gamma(\alpha_*+1)} \cdot \frac{1}{1-q}.$$

This completes the proof. □

5. Examples

In this section, we provide an illustrative example showing that the hypotheses of Theorems 3.4 and 3.9 can be satisfied for the proposed problem (1.1).

Example 5.1. Let $T = 2$ and $r = 1$. Consider the finite delayed variable-order fractional problem

$$\begin{cases} D_{0^+}^{\alpha(t,u_t)} u(t) = \mathcal{F}\left(t, u(t), \int_{-r}^{0} u(t+\theta)\, d\theta\right), & t \in [0,2], \\ u(t) = \phi(t), & t \in [-1,0], \end{cases} \tag{5.1}$$

where the variable order is defined by

$$\alpha(t,\psi) = \frac{1}{4} + \frac{t}{8} + \frac{1}{20}\,\frac{1}{1+\|\psi\|^2_{L^1([-1,0])}}, \qquad (t,\psi) \in [0,2] \times L^1([-1,0]), \tag{5.2}$$

and the nonlinear term is given by

$$\mathcal{F}(t,x,y) = \frac{e^{-t}}{6}\,\frac{x}{1+|x|} + \frac{e^{-t}}{6}\,\frac{y}{1+|y|}, \qquad (t,x,y) \in [0,2] \times \mathbb{R}^2. \tag{5.3}$$

The history function ϕ is assumed to belong to $L^1([-1,0])$.

Proof. We verify assumptions (A1)–(A3).

From (5.2) we have $\alpha(t,\psi)$ continuous in both variables. Moreover, since $t \in [0,2]$ and $\frac{1}{1+\|\psi\|^2} \in (0,1]$, it follows that

$$\alpha(t,\psi) \ge \frac{1}{4} + 0 + 0 = \frac{1}{4} =: \alpha_*,$$

and

$$\alpha(t,\psi) \le \frac{1}{4} + \frac{2}{8} + \frac{1}{20} = \frac{1}{4} + \frac{1}{4} + \frac{1}{20} = \frac{11}{20} =: \alpha^*.$$

Hence, $0 < \alpha_* \le \alpha(t,\psi) \le \alpha^* < 1$ for all (t,ψ), and (A1) holds.

For any $z \in \mathbb{R}$, the function $g(z) := \frac{z}{1+|z|}$ satisfies

$$|g(z_1) - g(z_2)| \le |z_1 - z_2|, \qquad z_1, z_2 \in \mathbb{R},$$

since g is globally Lipschitz with constant 1. Using (5.3), we obtain, for any $t \in [0, 2]$ and $(x_i, y_i) \in \mathbb{R}^2$,

$$\begin{aligned}|\mathcal{F}(t, x_1, y_1) - \mathcal{F}(t, x_2, y_2)| &\leq \frac{e^{-t}}{6}|g(x_1) - g(x_2)| + \frac{e^{-t}}{6}|g(y_1) - g(y_2)| \\ &\leq \frac{1}{6}(|x_1 - x_2| + |y_1 - y_2|).\end{aligned}$$

Thus, (A2) holds with Lipschitz constant $L = \frac{1}{6}$. Again, using $|g(z)| \leq 1$, we get

$$|\mathcal{F}(t, x, y)| \leq \frac{e^{-t}}{6}|g(x)| + \frac{e^{-t}}{6}|g(y)| \leq \frac{1}{3}.$$

Hence (A3) holds with $a(t) \equiv \frac{1}{3} \in L^1([0, 2])$ and $K = 0$ (or alternatively with any $K > 0$ and $a(t) \equiv \frac{1}{3}$).

Therefore, all hypotheses of Theorem 3.4 are satisfied. In particular, since $L = \frac{1}{6}$ and $r = 1$, the contraction condition

$$\frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} L(1 + r) < 1$$

is fulfilled for $T = 2$ and $\alpha_* = \frac{1}{4}$. Consequently, problem (5.1) admits a unique solution in $L^1(\Delta)$.

To illustrate the necessity of the linear growth term in assumption (A3), consider the modified nonlinear function

$$\widetilde{F}(t, x, y) = \frac{e^{-t}}{6}\frac{x}{1 + |x|} + \frac{e^{-t}}{6}\frac{y}{1 + |y|} + \frac{1}{5}(x + y).$$

Clearly, $\widetilde{F}$ is continuous in all variables. The first two terms are globally bounded as in Example 5.1, while the additional linear term $\frac{1}{5}(x + y)$ introduces genuine linear growth.

We estimate

$$|\widetilde{F}(t, x, y)| \leq \frac{1}{3} + \frac{1}{5}(|x| + |y|).$$

Hence, assumption (A3) holds with

$$a(t) \equiv \frac{1}{3} \in L^1([0, T]), \qquad K = \frac{1}{5} > 0.$$

In this case, the growth term proportional to $|x| + |y|$ is essential and cannot be absorbed into $a(t)$. Therefore, condition (A3) with $K > 0$ plays a nontrivial role in verifying the hypotheses of Theorems 3.5–3.9. □

Example 5.2. Consider Example 5.1 with $T = 2$ and $r = 1$, and

$$\alpha(t, \psi) = \frac{1}{4} + \frac{t}{8} + \frac{1}{20}\frac{1}{1 + \|\psi\|^2_{L^1([-1,0])}}, \qquad F(t, x, y) = \frac{e^{-t}}{6}\frac{x}{1 + |x|} + \frac{e^{-t}}{6}\frac{y}{1 + |y|}.$$

As shown in Example 5.1, we have $\alpha_* = 1/4$ and the Lipschitz constant in (A2) is $L = 1/6$.

Define

$$q := \frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} L(1 + r).$$

With $T = 2$, $r = 1$, $\alpha_* = 1/4$, and $L = 1/6$, we obtain

$$q = \frac{2^{1/4}}{\Gamma(5/4)} \cdot \frac{1}{6} \cdot 2 = \frac{2^{1/4}}{3\,\Gamma(5/4)} \approx 0.4373 < 1.$$

Hence, the hypotheses of Theorem 4.1 hold.

Therefore, for every $\varepsilon > 0$ and every ε-approximate solution v (in the sense of Definition 4.1), there exists the unique exact solution u such that

$$\|v - u\|_{L^1(\Delta)} \leq C\,\varepsilon, \qquad C = \frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} \cdot \frac{1}{1 - q}.$$

Numerically,

$$C = \frac{2^{1/4}}{\Gamma(5/4)} \cdot \frac{1}{1 - 0.4373} \approx 2.3318.$$

For instance, if $\varepsilon = 10^{-2}$, then every ε-approximate solution satisfies the explicit stability estimate as shown in Figure 1.

$$\|v - u\|_{L^1(\Delta)} \leq 2.3318 \times 10^{-2}.$$

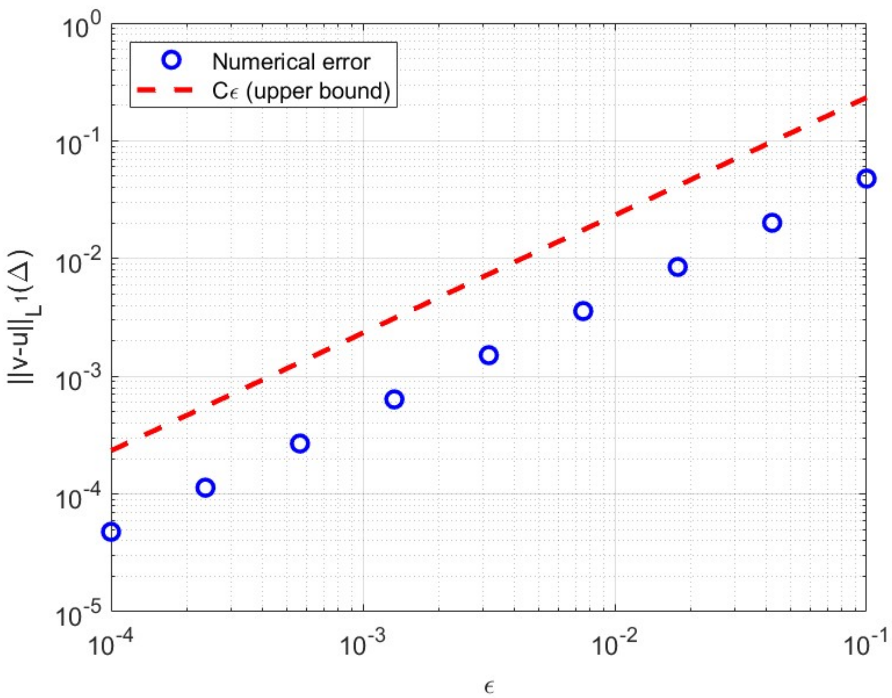

Figure 1. Ulam–Hyers stability for Example 5.1. The numerical error $\|v - u\|_{L^1(\Delta)}$ grows linearly with the perturbation size ε, in agreement with the theoretical bound $\|v-u\|_{L^1(\Delta)} \leq C\varepsilon$.

To support the theoretical stability result in Example 5.1, we briefly describe the numerical procedure used to generate Figure 1. The variable-order fractional problem (5.1) was first reformulated in its equivalent integral form (3.1). The interval $[0, T]$ with $T = 2$ was discretized using a uniform grid

$$t_n = nh, \qquad n = 0, 1, \ldots, N, \qquad h = \frac{T}{N},$$

with $N = 200$ in the simulations. The variable-order Riemann–Liouville fractional integral was approximated using a fractional trapezoidal-type quadrature rule:

$$I_{0+}^{\alpha(t_n,u_{t_n})}u(t_n) \approx \sum_{k=0}^{n-1} \frac{(t_n - t_k)^{\alpha(t_k,u_{t_k})-1}}{\Gamma(\alpha(t_k, u_{t_k}))} u(t_k)h.$$

The integral memory term

$$\int_{-r}^{0} u(t_n + \theta)\, d\theta$$

was computed using a standard rectangle rule on the delayed grid.

To construct an ε-approximate solution v, we perturbed the exact numerical solution u by adding a small disturbance of magnitude ε, i.e.,

$$v(t_n) = u(t_n) + \varepsilon\xi_n,$$

where $\{\xi_n\}$ is a bounded random sequence with $|\xi_n| \leq 1$. The $L^1(\Delta)$-error was then computed as

$$\|v - u\|_{L^1(\Delta)} \approx h \sum_{n=0}^{N} |v(t_n) - u(t_n)|.$$

The resulting numerical errors for different values of ε exhibit linear growth, in agreement with the theoretical estimate

$$\|v - u\|_{L^1(\Delta)} \leq C\varepsilon,$$

where C is given explicitly in Theorem 4.1.

To further illustrate the applicability of the theoretical results, we investigate the influence of the delay parameter r and the structure of the variable-order function $\alpha(t, \psi)$. Keeping all parameters as in Example 5.1, we vary the delay length r and compute the contraction coefficient

$$q(r) = \frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} L(1 + r).$$

As expected, increasing r enlarges the effective memory contribution and increases the stability constant. Numerical simulations confirm that larger delay values lead to stronger cumulative memory effects and slower decay of perturbations. We compare the history-dependent order (5.3) with a purely time-dependent alternative

$$\tilde{\alpha}(t) = \frac{1}{4} + \frac{t}{8}.$$

The comparison shows that the history-dependent term enhances adaptive behavior: when the L^1-norm of the history increases, the fractional order decreases slightly, strengthening memory persistence. This demonstrates the practical role of the history-dependent mechanism in shaping system dynamics.

5.1. Discussion on the influence of parameters and order function

It is instructive to briefly analyze how variations in the model parameters and in the order function $\alpha(t, u_t)$ influence the qualitative behavior of the system. From Theorems 3.4 and 4.1, the key quantity governing uniqueness and Ulam–Hyers stability is the contraction constant

$$q = \frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} L(1 + r), \tag{5.4}$$

where $\alpha_* = \inf_{(t,\psi)} \alpha(t,\psi)$. The delay parameter appears linearly in the factor $(1 + r)$. Thus, increasing the memory window enlarges the contraction constant q, making the stability condition $q < 1$ more restrictive. Consequently, longer cumulative memory effects require weaker nonlinear feedback (smaller Lipschitz constant L) to preserve uniqueness and stability. The term $\frac{T^{\alpha_*}}{\Gamma(\alpha_*+1)}$ is increasing with respect to α_* on $(0, 1)$. Hence, larger fractional orders enlarge the contraction constant q. In contrast, smaller fractional orders strengthen the smoothing effect of the fractional integral operator and improve stability. This observation reflects the regularizing role of stronger memory effects in fractional systems.

Changes in the functional dependence of α on the history u_t directly affect the lower bound α_*. For instance, if $\alpha(t, u_t)$ decreases when the history norm $\|u_t\|_{L^1([-r,0])}$ increases, then stronger historical influence may reduce α_* and thereby improve stability. Conversely, if α approaches 1, the model becomes closer to a classical differential equation and the contraction condition becomes more restrictive. The Lipschitz constant L scales the stability threshold directly. In fact, the contraction condition $q < 1$ can be rewritten as

$$L < \frac{\Gamma(\alpha_* + 1)}{T^{\alpha_*}(1 + r)}, \tag{5.5}$$

which provides an explicit quantitative criterion for admissible nonlinear response intensity. Smaller values of L enlarge the stability region, while stronger nonlinear amplification may violate the uniqueness condition.

These observations show that the theoretical results obtained in this work provide explicit quantitative relationships describing how the delay length, memory order, and nonlinear strength jointly determine existence, uniqueness, and robustness of solutions.

5.2. *Comparative numerical analysis*

To further investigate the influence of model parameters on the system behavior, we perform additional numerical experiments focusing on the delay length r and on different forms of the variable-order function $\alpha(t, \psi)$.

Keeping all other parameters fixed as in Example 5.1, we vary the delay length and consider $r \in \{0.5, 1, 1.5\}$. Recall that the contraction coefficient

$$q(r) = \frac{T^{\alpha_*}}{\Gamma(\alpha_* + 1)} L(1 + r)$$

depends linearly on $(1 + r)$. Thus, increasing the delay enlarges the effective memory contribution and leads to larger stability constants.

The numerical simulations confirm this analytical observation. In particular, the accumulated solution norm

$$\|u\|_{L^1(\Delta)} \approx h \sum_{n=0}^{N} |u(t_n)|$$

increases moderately as r becomes larger, indicating that stronger cumulative memory effects amplify the system response and enhance persistence.

Next, we compare the original history-dependent order function defined in (5.2) with two alternative forms:

$$\alpha_1(t,\psi) = \frac{1}{4} + \frac{t}{8}, \qquad \alpha_2(t,\psi) = \frac{1}{4} + \frac{t}{8} + \frac{\mu}{1 + \|\psi\|_{L^1([-1,0])}},$$

where $\mu > 0$ controls the strength of the history dependence. When $\mu = 0$, the system reduces to a purely time-dependent order, corresponding to a fixed memory evolution determined solely by time. For $\mu > 0$, the fractional order decreases as the history norm increases, which corresponds to stronger adaptive memory effects.

The numerical results (see Figure 2) show that increasing μ enhances the sensitivity of the solution to past states, producing slower decay and stronger memory persistence compared to the purely time-dependent case. This comparison confirms that the history-dependent component of the order plays a significant role in shaping the qualitative dynamics of the system.

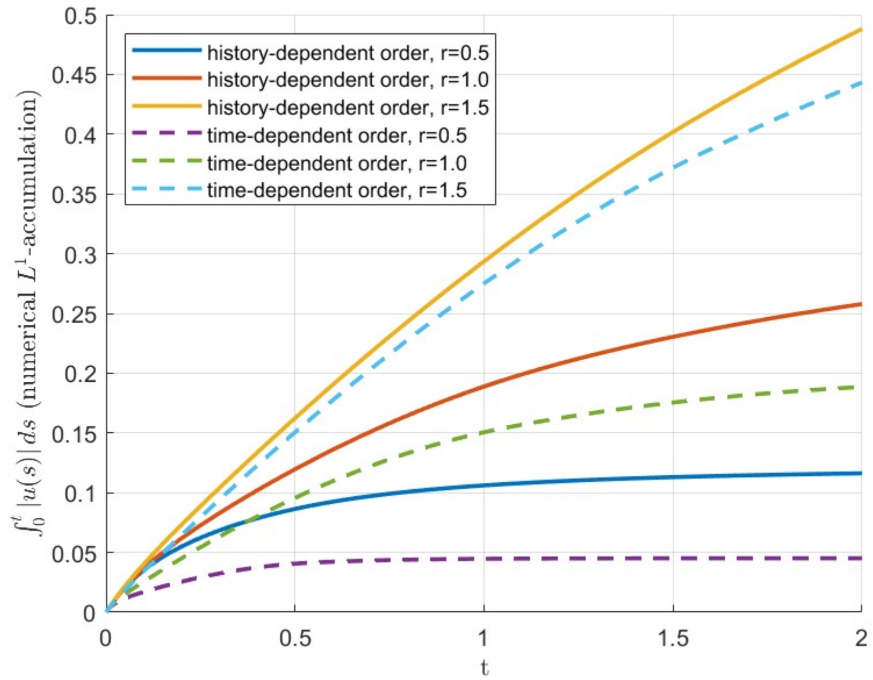

Figure 2. Accumulated L^1-norm $\int_0^t |u(s)|\, ds$ for different delay values r and two variable-order functions. Solid curves denote the history-dependent order, while dashed curves correspond to the purely time-dependent order.

6. Conclusions

In this work, we studied a class of finite-delay fractional differential systems with a nonlinear variable order depending on the past evolution of the state. The proposed model combines two mechanisms: a history-dependent fractional order and an integral memory term in the nonlinear source. This formulation allows cumulative hereditary effects to be incorporated in a flexible and mathematically consistent manner. We reformulated the problem as an equivalent integral equation. Using fixed-point techniques in the Banach space $L^1(\Delta)$, we established existence and uniqueness results under mild assumptions. Both the Banach contraction principle and the Schauder fixed-point theorem were applied. In addition, Ulam–Hyers stability was proved, providing quantitative robustness of solutions with respect to perturbations. An illustrative example was presented to verify the theoretical assumptions and to demonstrate how parameter variations influence stability. The numerical experiment confirms the linear stability behavior predicted by the theory. Future research may address asymptotic behavior and attractivity of solutions. Extensions to infinite-delay problems, impulsive systems, or stochastic effects are also promising directions. The development of efficient numerical schemes and applications to real-world problems in engineering, economics, and biomedical sciences remain important topics for further investigation.

Author contributions

Ibraheem M. Alsulami: Methodology, investigation, developed the software, formal analysis, and performed the review and editing of the manuscript. Ramsha Shafqat: Conceptualized the study, conducted the investigation, curated the resources, developed the software, supervised the research prepared the original draft of the manuscript, performed the review and editing of the manuscript. Both authors contributed to the investigation and resources, read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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