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Research article

A dynamic inequality with average condition and applications to predator-prey systems on time scales

Yang-Yang Yu[1,*], **Lin-Lin Wang**[2] **and Zhong-Xin Ma**[3]

[1] School of Mathematics and Statistics, Xuzhou University of Technology, 02 Lishui Rd., 221018 Xuzhou, China

[2] School of Mathematics and Statistics Science, Ludong University, 186 Hongqi Middle Rd., 264025 Yantai, China

[3] School of Science, Zhejiang University of Science & Technology, 318 Liuhe Rd., 310023 Hangzhou, China

* **Correspondence:** Email: yyyumath@126.com.

Abstract: In this paper, a dynamic inequality with average condition on time scales is established. Using this inequality, we study the uniform ultimate boundedness of solutions of two predator-prey systems on time scales. Our results extend and generalize some known results about these types of problems. We verify our main results by means of a numerical example.

Keywords: average condition; dynamic inequality; uniform ultimate boundedness; time scale; predator-prey system
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1. Introduction and preliminaries

The theory of dynamic equations on time scales, originally introduced by Hilger in his seminal PhD thesis in 1988 [11], provides a unified framework for the study of continuous and discrete dynamical systems. Since then, this theory has attracted sustained attention and has been extensively developed; see, for instance, [3, 13] and the references therein. By incorporating differential and difference equations into a single setting, time scale theory offers a powerful tool for analyzing hybrid dynamical models arising in applied sciences, recently see [2, 15, 19, 22].

Throughout this paper, we consider a time scale $\mathbb{T}$ satisfying $\sup \mathbb{T} = \infty$ and assume that the jump function $\mu(t) = \sigma(t) - t$ is bounded above by a constant l. Without loss of generality, we further assume that $0 \in \mathbb{T}$.

For bounded rd-continuous functions $g : \mathbb{T} \to \mathbb{R}$ denoted by

$$g^M = \sup \{g(t), t \in \mathbb{T}\}, \, g^L = \inf \{g(t), t \in \mathbb{T}\},$$

we define the notions of lower and upper averages of g by

$$A_L(g) = \lim_{r\to+\infty} \inf_{\substack{t-s\geq r\\ t,s\in\mathbb{T}}} \frac{1}{t-s}\int_s^t g(\tau)\Delta\tau, A_M(g) = \lim_{r\to+\infty} \sup_{\substack{t-s\geq r\\ t,s\in\mathbb{T}}} \frac{1}{t-s}\int_s^t g(\tau)\Delta\tau, \tag{1.1}$$

which play a central role in the subsequent analysis. Obviously, (1.1) is valid. Similar as that in [8], the following remarks are presented:

(1) If g is ω-periodic, $s+\omega \in \mathbb{T}$ holds true for any $s \in \mathbb{T}$ and $\inf \mathbb{T} = -\infty$, then

$$A_L(g) = A_M(g) = A_\omega(g) \text{ (which is defined as } \frac{1}{\omega}\int_0^\omega g(\tau)\Delta\tau).$$

(2) The following inequalities hold true:

$$g^L \leq A_L(g) \leq A_M(g) \leq g^M.$$

(3) For any $\alpha, \beta \in \mathbb{R}$, and a bounded rd-continuous function h, the lower average satisfies

$$A_L(\alpha g + \beta h) = \alpha A_L(g) + \beta A_L(h).$$

Proof. As in [8], we only prove that (1) holds. Setting $t - s = n\omega + \alpha_n$, where $\alpha_n \in [0, \omega]$, we then have

$$\frac{1}{t-s}\int_s^t g(\tau)\Delta\tau = \frac{1}{n\omega+\alpha_n}\int_s^{s+n\omega} g(\tau)\Delta\tau + \frac{1}{n\omega+\alpha_n}\int_{s+n\omega}^t g(\tau)\Delta\tau.$$

Notice that $\widetilde{\mathbb{T}} = \{s+\omega \mid s \in \mathbb{T}\} = \mathbb{T}$, and then $\widetilde{\Delta}(\tau) = \Delta\tau$. Assume that $\tau = n\omega + \xi$, and it follows from the property of ω-periodic that

$$\begin{aligned}
\int_s^{s+n\omega} g(\tau)\Delta\tau &= \int_s^0 g(\tau)\Delta\tau + \int_0^{n\omega} g(\tau)\Delta\tau + \int_{n\omega}^{s+n\omega} g(\tau)\Delta\tau \\
&= \int_s^0 g(\tau)\Delta\tau + \int_0^{n\omega} g(\tau)\Delta\tau + \int_0^s g(\xi)\widetilde{\Delta}(\xi) \\
&= \int_0^{n\omega} g(\xi)\Delta\xi = n\int_0^\omega g(\xi)\Delta\xi.
\end{aligned}$$

Otherwise, note that $\alpha_n g^L \leq \int_{s+n\omega}^t g(\tau)\Delta\tau \leq \alpha_n g^M$, it is easy to see $\frac{1}{n\omega+\alpha_n}\int_{s+n\omega}^t g(\tau)\Delta\tau$ converges uniformly to 0.

Thus,

$$A_L(g) = A_M(g) = \lim_{n\to+\infty}\frac{n}{n\omega+\alpha_n}\int_0^\omega g(\tau)\Delta\tau + 0 = \frac{1}{\omega}\int_0^\omega g(\tau)\Delta\tau,$$

which completes the proof. □

One finds that these average quantities extend the classical time-averaged concepts on $\mathbb{R}$ and $\mathbb{Z}$, and the fundamental properties remain valid in the general time scale setting.

The concept of permanence is a cornerstone in population ecology, as it characterizes the long-term coexistence of interacting species. Over the past decades, permanence has been investigated for a wide range of deterministic, stochastic, continuous, and discrete population models, such as in [6, 8, 10, 12, 21]. Du et al. [6] investigated a stochastic SIR model subject to a complex class of random perturbations and established sufficient and nearly necessary conditions for permanence by introducing an appropriate threshold parameter. He et al. [10] discussed the permanence of an impulsive logistic model with time delay, where the analysis relied on the boundedness of solutions to the associated autonomous system. Their approach was mainly based on comparison principles and Lyapunov functional techniques. By using strong endotacticity, Johnston et al. [12] obtained the permanence of a well-studied circadian clock mechanism for the first time. Moreover, in [21], by using the stochastic Lyapunov methods together with comparison theorems and the strong ergodic theorem, Wang and Jin obtained sufficient conditions for the extinction, permanence, and existence of stationary distributions.

Among ecological models, predator-prey systems constitute one of the most fundamental and extensively studied classes; see, for instance, [4, 5, 8, 9, 14, 16–18, 20, 23]. A variety of functional responses and environmental effects have been incorporated to better describe realistic interactions. Ma et al. [16] considered the stability of equilibria and derived criteria for uniform persistence in a predator-prey model. In [23], for the three-dimensional case, Zhao et al. considered a delayed predator-prey system and established fundamental properties of its solutions, including existence, uniqueness, nonnegativity, and boundedness. In addition, Ouyang et al. [17] addressed a stochastic ratio-dependent predator-prey model with Markovian switching and obtained permanence results by analyzing the asymptotic bounds of time-averaged sample paths.

For periodic environments, Cui et al. [4] considered the permanence of a predator-prey system with Beddington-DeAngelis functional response

$$\begin{cases} x'(t) = x(t)\left[a(t) - b(t)x(t) - \frac{c(t)y(t)}{\alpha(t)+\beta(t)x(t)+\gamma(t)y(t)}\right], \\ y'(t) = y(t)\left[-d(t) + \frac{f(t)x(t)}{\alpha(t)+\beta(t)x(t)+\gamma(t)y(t)}\right], \end{cases} \quad t \in \mathbb{R}, \tag{1.2}$$

where $a, b, c, d, f, \alpha, \beta, \gamma \in C(\mathbb{R}, \mathbb{R})$ are periodic functions with common period ω and $b(t) \geq 0$, $c(t), d(t), f(t), \alpha(t), \beta(t), \gamma(t) > 0$. They derived sufficient conditions for permanence of above predator-prey system as follows.

Theorem 1.1. *Soppose that* $A_\omega(a(t)) > 0$, $A_\omega(b(t)) > 0$ *and* $A_\omega\left(-d(t) + \frac{f(t)x^*(t)}{\alpha(t)+\beta(t)x^*(t)}\right) > 0$ *hold, where* $x^*(t)$ *is given in* [4]. *Then, system* (*1.2*) *is permanent.*

In the discrete-time case, Fan et al. [8] studied a bounded predator-prey system with Holling type II functional response

$$\begin{cases} N_1(k+1) = N_1(k)\exp\left\{a(k) - b(k)N_1(k) - \frac{c(k)N_2(k)}{N_1(k)+\gamma(k)N_2(k)}\right\}, \\ N_2(k+1) = N_2(k)\exp\left\{-d(k) + \frac{f(k)N_1(k)}{N_1(k)+\gamma(k)N_2(k)}\right\}, \end{cases} \quad k \in \mathbb{Z}, \tag{1.3}$$

where the sequences a, b, c, d, f, γ are bounded and $b(k), c(k), f(k), \gamma(k) > 0$. They obtained average conditions for the permanence of system (1.3) as follows.

Theorem 1.2. *Assume that* $A_L\left(a(k)-\frac{c(k)}{\gamma(k)}\right)>0$, $A_L(f(k)-d(k))>0$, $A_L(d(k))>0$ *hold. Then, the system* (*1.3*) *is permanent.*

These results highlight the effectiveness of average-based techniques in studying long-term population dynamics. Motivated by these studies, it is natural to ask whether such average conditions can be formulated and applied in a unified time scale framework, thereby covering continuous and discrete predator-prey systems simultaneously. In this work, we investigate the permanence of two classes of predator-prey systems defined on an arbitrary time scale $\mathbb{T}$. By employing appropriate logarithmic transformations, the original population models are converted into equivalent dynamic systems amenable to time scale analysis.

For the first system:

$$\begin{cases} x_1^{\Delta}(t)=a(t)-b(t)\exp\{x_1(t)\}-\frac{c(t)\exp\{x_2(t)\}}{\alpha(t)+\beta(t)\exp\{x_1(t)\}+\gamma(t)\exp\{x_2(t)\}}, \\ x_2^{\Delta}(t)=-d(t)+\frac{f(t)\exp\{x_1(t)\}}{\alpha(t)+\beta(t)\exp\{x_1(t)\}+\gamma(t)\exp\{x_2(t)\}}, \end{cases} \tag{1.4}$$

on $\mathbb{T}$ with the initial condition

$$x_1(t_0),x_2(t_0)\in\mathbb{R},\ t_0\in\mathbb{T},$$

where $a,b,c,d,f,\alpha,\beta,\gamma\in C_{rd}(\mathbb{T},\mathbb{R})$ are assumed to be bounded, $b(t),c(t),f(t),\alpha(t),\beta(t),\gamma(t)>0$, and Δ stands for the delta derivative.

Let $x(t)=\exp\{x_1(t)\}$ and $y(t)=\exp\{x_2(t)\}$. Thus, if $\mathbb{T}=\mathbb{R}$, then (1.4) reduces to (1.2).

Throughout this paper, we assume that $b^L,a^M>0$, and all the coefficients of systems are periodic functions in some periodic cases.

For the second system:

$$\begin{cases} x_3^{\Delta}(t)=a(t)-b(t)\exp\{x_3(t)\}-\frac{c(t)\exp\{x_4(t)\}}{\exp\{x_3(t)\}+\gamma(t)\exp\{x_4(t)\}}, \\ x_4^{\Delta}(t)=-d(t)+\frac{f(t)\exp\{x_3(t)\}}{\exp\{x_3(t)\}+\gamma(t)\exp\{x_4(t)\}}, \end{cases} \tag{1.5}$$

on $\mathbb{T}$ with the initial condition

$$x_3(t_0),x_4(t_0)\in\mathbb{R},\ t_0\in\mathbb{T},$$

where all the coefficients are the same as (1.4).

Let $x(t)=\exp\{x_3(t)\}$ and $y(t)=\exp\{x_4(t)\}$. Thus, if $\mathbb{T}=\mathbb{R}$, then (1.5) reduces to the continuous predator-prey system

$$\begin{cases} x'(t)=x(t)\left[a(t)-b(t)x(t)-\frac{c(t)y(t)}{x(t)+\gamma(t)y(t)}\right], \\ y'(t)=y(t)\left[-d(t)+\frac{f(t)x(t)}{x(t)+\gamma(t)y(t)}\right], \end{cases} \quad t\in\mathbb{R}. \tag{1.6}$$

Arditi et al. [1] first proposed a ratio-dependent functional response and obtained the system (1.6) with constant coefficients, where $c(t)$, $f(t)$ and $\gamma(t)$ represent capturing rate, conversion rate, and half capturing saturation constant, respectively. Much attention has been attracted by this predator-prey system or its analogs in [4, 9].

Let $N_1(k)=\exp\{x_3(k)\}$ and $N_2(k)=\exp\{x_4(k)\}$. Thus, if $\mathbb{T}=\mathbb{Z}$, then (1.5) is reformulated as (1.3) (see [8]). In this paper, some approaches are used to discuss the permanence of this system or its other forms. In [7], by using the semi-cycle and related concepts, Fan et al. considered the permanence of system (1.3).

The main contribution of this work is to use average conditions on $\mathbb{T}$ to prove that the solutions $x_i(t)$, $i = 1,2,3,4$ of systems (1.4) and (1.5) are uniformly ultimate bounded, which implies that systems (1.2), (1.3), and (1.6) are permanent.

It is worth mentioning that the objective of this work is not to increase model complexity by introducing time delays or diffusion effects, but rather to develop a unified analytical framework for permanence based on average conditions on arbitrary time scales. Although persistence and stability have been extensively studied for predator-prey systems with delays or diffusion, most existing results are restricted to purely continuous or discrete cases and rely on specific structural assumptions.

In contrast, the present study focuses on the role of time scales in unifying continuous, discrete, and hybrid dynamics within a single theoretical framework. The proposed average condition approach allows us to derive permanence results under relatively weak assumptions and recovers several known results as special cases. From this perspective, the novelty of the paper lies in the methodological unification and the extension of permanence theory to general time scales, rather than in proposing a more complicated biological model. We believe that this framework provides a useful foundation for future investigations of more complex systems, including models with time delays (see Theorem 2.2 later and its applications) or diffusion, on time scales.

The rest of the paper is organized as follows. In Section 2, we establish a dynamic inequality with average condition on time scales. Section 3 is devoted to uniform ultimate boundedness of solutions of systems (1.4) and (1.5). The final section of the paper contains a numerical example to support the results.

2. Dynamic inequality

It is easy to obtain the following lemma.

Lemma 2.1. *Assume that* $a_1, b_1 \in C_{rd}(\mathbb{T}, \mathbb{R})$ *with* $a_1^M, b_1^L > 0$ *and* $x(0) = x_0$, *and further suppose* $x(t)$ *satisfies that*

$$x^{\Delta}(t) \leq a_1(t) - b_1(t) \exp\{x(t)\}, \text{ for } t > 0, \tag{2.1}$$

then

$$\lim_{t\to\infty} \sup x(t) \leq a_1^M l + \ln \frac{a_1^M}{b_1^L}. \tag{2.2}$$

Using the average technique, we establish the following dynamic inequality.

Lemma 2.2. *Assume that* $x(0) = x_0$ *and* $a_2, b_2 \in C_{rd}(\mathbb{T}, \mathbb{R})$ *with*

$$A_L(a_2) > 0, \tag{2.3}$$

and further suppose $x(t)$ *satisfies that*

$$x^{\Delta}(t) \geq a_2(t) - b_2(t) \exp\{x(t)\}, \tag{2.4}$$

and $x(t)$ *is bounded above. Then, there exists a constant* L *such that*

$$\lim_{t\to\infty} \inf x(t) \geq L. \tag{2.5}$$

Proof. By means of mathematical analysis, the rest of the proof can be divided into two steps.

Step 2.1. Assume that $x(t)$ is a solution of (2.4) and is bounded above by a constant M; if (2.5) does not hold, then there exist sequences $\{t_n\}$ and $\{s_n\}$ ($t_n, s_n \in \mathbb{T}$) such that

$$\begin{aligned} &0 \le s_n < t_n,\, t_n - s_n \ge n,\, x(s_n) \ge x_0 - n, \\ &x(t) \le x_0 - n \text{ for } s_n < t \le t_n. \end{aligned} \tag{2.6}$$

Proof of the Step 2.1. Notice that

$$a_2(t) - b_2(t)\exp\{x(t)\} \ge a_2^L - b_2^M \exp\{M\} \ge -\gamma, \text{ for any } t \in \mathbb{T},$$

where $\gamma > 0$ is a constant.

If (2.5) does not hold, then $\liminf\limits_{t\to\infty} x(t) = -\infty$. Thus, for any $n > 0$, there exists a sequence $\{t_n | \, t_n > 0\}$ on $\mathbb{T}$ such that

$$x(t_n) < x_0 - n - \gamma n.$$

In addition, there exist s_n ($s_n \in \mathbb{T}$) such that $0 \le s_n < t_n$, $x(s_n) \ge x_0 - n$ and $x(t) \le x_0 - n$ for $s_n < t \le t_n$. From (2.4), we have

$$x_0 - n \le x(s_n) \le x(t_n) + \gamma(t_n - s_n) \le x_0 - n - \gamma n + \gamma(t_n - s_n),$$

which yields $t_n - s_n \ge n$. This completes the proof of Step 2.1. □

Step 2.2. The proof is completed by contradiction.

From (2.4), we have

$$a_2(t) \le x^\Delta(t) + b_2(t)\exp\{x(t)\},$$

thanks to Step 2.1, we obtain that if (2.5) does not hold, then for any n, we have

$$\int_{s_n}^{t_n} a_2(t)\,\Delta t \le \int_{s_n}^{t_n} \left(x^\Delta(t) + b_2(t)\exp\{x(t)\}\right)\Delta t,$$

which implies that

$$\begin{aligned} \frac{1}{t_n - s_n}\int_{s_n}^{t_n} a_2(t)\,\Delta t &\le \frac{1}{t_n - s_n}\int_{s_n}^{t_n} \left(x^\Delta(t) + b_2(t)\exp\{x(t)\}\right)\Delta t \\ &= \frac{1}{t_n - s_n}\int_{s_n}^{t_n} x^\Delta(t)\,\Delta t + \frac{1}{t_n - s_n}\int_{s_n}^{t_n} b_2(t)\exp\{x(t)\}\,\Delta t \\ &\le \frac{x(t_n) - x(s_n)}{t_n - s_n} + \frac{1}{t_n - s_n}\int_{s_n}^{t_n} b_2^M \exp\{x_0 - n\}\Delta t \\ &\le \frac{x_0 - n - (x_0 - n)}{t_n - s_n} + b_2^M \exp\{x_0 - n\}. \end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} \frac{1}{t_n - s_n}\int_{s_n}^{t_n} a_2(t)\,\Delta t \le 0.$$

It is in contradiction to (2.3); the proof is complete. □

From Lemmas 2.1 and 2.2, we have the following theorems.

Theorem 2.1. *Let $x(t)$ be a solution of the following inequality*

$$\begin{cases} a_2(t) - b_2(t)\exp\{x(t)\} \le x^{\Delta}(t) \le a_1(t) - b_1(t)\exp\{x(t)\}, \\ x(0) = x_0,\ t \in \mathbb{T}. \end{cases}$$

If $a_1, b_1, a_2, b_2 \in C_{rd}(\mathbb{T}, \mathbb{R})$ satisfy $a_1^M > 0$, $b_1^L > 0$, $A_L(a_2) > 0$, then there exist constants L_1 and M_1 such that

$$L_1 \le \lim_{t\to\infty}\inf x(t) \le \lim_{t\to\infty}\sup x(t) \le M_1.$$

By Theorem 2.1, we can easily obtain

Theorem 2.2. *Let $x(t)$ be a solution of the following inequality*

$$\begin{cases} a_2(t) - b_2(t)\exp\{x(t-\eta)\} \le x^{\Delta}(t) \le a_1(t) - b_1(t)\exp\{x(t-\eta)\}, \\ t \in \mathbb{T},\ t-\eta \in \mathbb{T},\ x(t_0) = x_0,\ t_0 \in \{t \in \mathbb{T} | -\eta \le t \le 0\}. \end{cases}$$

If $a_1, b_1, a_2, b_2 \in C_{rd}(\mathbb{T}, \mathbb{R})$ satisfy $a_1^M > 0$, $b_1^L > 0$, $A_L(a_2) > 0$, then there exist constants L_2 and M_2 such that

$$L_2 \le \lim_{t\to\infty}\inf x(t) \le \lim_{t\to\infty}\sup x(t) \le M_2,$$

where η is a positive constant.

Using Theorem 2.2, we could address the permanence of some delay systems.

3. Applications

We first give the definition of uniform ultimate boundedness.

Definition 3.1. Solutions of (1.4) and (1.5) are said to be uniformly ultimate bounded if there exists two constants λ_1 and λ_2 such that for any initial condition $(x_1(0), x_2(0))^{\mathbb{T}} \in \mathbb{R}^2$ or $(x_3(0), x_4(0))^{\mathbb{T}} \in \mathbb{R}^2$,

$$\lambda_1 \le \lim_{t\to\infty}\inf x_i(t) \le \lim_{t\to\infty}\sup x_i(t) \le \lambda_2,\ i = 1, 2, 3, 4.$$

Due to the exponential transformations in Section 1, uniform ultimate boundedness of solutions of systems on $\mathbb{T}$ (e.g., (1.4) and (1.5)) are equivalent to the permanence of systems on $\mathbb{R}$ (e.g., (1.2) and (1.6)) or $\mathbb{Z}$ (e.g., (1.3)).

As applications, we now state the following results.

Theorem 3.1. *Assume that*

(A1)

$$A_L\left(a(t) - \frac{c(t)}{\gamma(t)}\right) > 0;$$

(A2)

$$A_L(f(t) - d(t)) > 0;$$

(A3)

$$A_L(d(t)) > 0.$$

Then, the solutions of system (1.5) *are uniformly ultimate bounded.*

Proof. From (1.5), we have

$$a(t) - b(t)\exp\{x_3(t)\} - \frac{c(t)}{\gamma(t)} \le x_3^{\Delta}(t) \le a(t) - b(t)\exp\{x_3(t)\}.$$

Thanks to Theorem 2.1 and the condition (A1), we can obtain there must exist some constants L_3 and M_3 such that

$$L_3 \le \liminf_{t\to\infty} x_3(t) \le \limsup_{t\to\infty} x_3(t) \le M_3. \tag{3.1}$$

It also follows from (1.5) that

$$\begin{aligned} x_4^{\Delta}(t) &= -d(t) + \frac{f(t)\exp\{x_3(t)\}}{\exp\{x_3(t)\} + \gamma(t)\exp\{x_4(t)\}} \\ &= f(t) - d(t) - \frac{f(t)\gamma(t)\exp\{x_4(t)\}}{\exp\{x_3(t)\} + \gamma(t)\exp\{x_4(t)\}} \\ &\ge f(t) - d(t) - \frac{f(t)\gamma(t)\exp\{x_4(t)\}}{\exp\{L_3\}}. \end{aligned}$$

Thanks to Theorem 2.1 and the condition (A2), we can obtain that there must exists a constant L_4 such that

$$\liminf_{t\to\infty} x_4(t) \ge L_4. \tag{3.2}$$

Let $y(t) = -x_4(t)$. From (1.5), we can obtain

$$\begin{aligned} y^{\Delta}(t) &= d(t) - \frac{f(t)\exp\{x_3(t)\}\exp\{y(t)\}}{\exp\{y(t)\}\exp\{x_3(t)\} + \gamma(t)} \\ &\ge d(t) - \frac{f(t)\exp\{M_3\}\exp\{y(t)\}}{\gamma(t)}. \end{aligned}$$

It follows readily, thanks to Theorem 2.1 and the condition (A3), that there must exists a constant M_4 such that

$$\limsup_{t\to\infty} x_4(t) \le M_4. \tag{3.3}$$

By (3.1), (3.2), (3.3), we complete the proof. □

According to Theorem 3.1, it immediately follows

Lemma 3.1. *Assume that* (A1) *holds. Then, for the first equation of* (1.4)*, there exist some constants* L_1 *and* M_1 *such that*

$$L_1 \le \liminf_{t\to\infty} x_1(t) \le \limsup_{t\to\infty} x_1(t) \le M_1.$$

Theorem 3.2. *Assume* (A1) *and* (A3) *hold and further suppose that*

(A4)

$$A_L\left(-d(t) + \frac{f(t)\exp\{L_1\}}{\alpha(t) + \beta(t)\exp\{L_1\}}\right) > 0,$$

where L_1 *is given by Lemma 3.1. Then, the solutions of system* (1.4) *are uniformly ultimate bounded.*

Proof. Noticing Lemma 3.1, we only prove that there exist some constants L_2 and M_2 such that

$$L_2 \leq \lim_{t\to\infty}\inf x_2(t) \leq \lim_{t\to\infty}\sup x_2(t) \leq M_2.$$

From (1.4), we have

$$\begin{aligned} x_2^{\Delta}(t) &= -d(t) + \frac{f(t)}{\beta(t)} - \frac{f(t)\alpha(t) + f(t)\gamma(t)\exp\{x_2(t)\}}{\beta(t)(\alpha(t) + \beta(t)\exp\{x_1(t)\} + \gamma(t)\exp\{x_2(t)\})} \\ &\geq -d(t) + \frac{f(t)}{\beta(t)} - \frac{f(t)\alpha(t) + f(t)\gamma(t)\exp\{x_2(t)\}}{\beta(t)(\alpha(t) + \beta(t)\exp\{x_1(t)\})} \\ &= -d(t) + \frac{f(t)\exp\{x_1(t)\}}{\alpha(t) + \beta(t)\exp\{x_1(t)\}} - \frac{f(t)\gamma(t)\exp\{x_2(t)\}}{\beta(t)(\alpha(t) + \beta(t)\exp\{x_1(t)\})} \\ &\geq -d(t) + \frac{f(t)\exp\{L_1\}}{\alpha(t) + \beta(t)\exp\{L_1\}} - \frac{f(t)\gamma(t)\exp\{x_2(t)\}}{\beta(t)(\alpha(t) + \beta(t)\exp\{L_1\})}. \end{aligned}$$

Thanks to Theorem 2.1 and the condition (A4), it follows readily that there must exists a constant L_2 such that

$$\lim_{t\to\infty}\inf x_2(t) \geq L_2. \tag{3.4}$$

Moreover, by an argument similar to that in Theorem 3.1, we can obtain there exists a constant M_2 such that

$$\lim_{t\to\infty}\sup x_2(t) \leq M_2. \tag{3.5}$$

From (3.4) and (3.5), we complete the proof. □

Remark 3.1. In periodic cases, the general average condition is the same as ω-periodic average condition. Thus, Theorem 3.2 includes Theorems 1.1, 1.2, and 3.1. Therefore, our results extend and generalize the results presented in [4, 8].

4. Numerical example

In this section, we present a numerical example to illustrate and support the theoretical results established in Theorem 3.1. The verification of Theorem 3.2 can be carried out in a similar manner and is therefore omitted.

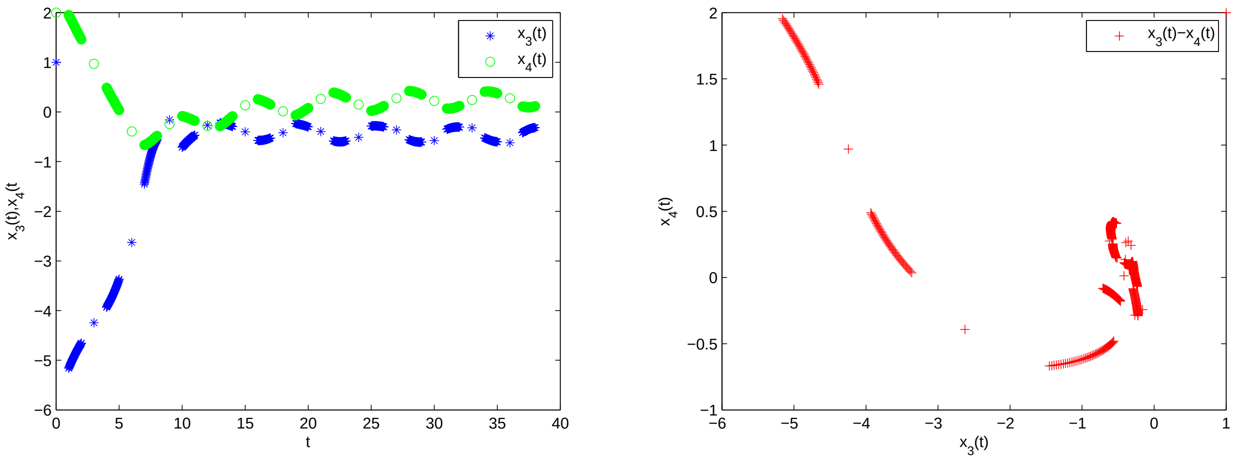
We consider system (1.5) on the time scale

$$\mathbb{T} = \{3k\} \cup [3k+1, 3k+2], k = 0, 1, \ldots,$$

which combines discrete sampling points with continuous intervals. This choice of $\mathbb{T}$ reflects the coexistence of continuous population evolution and discrete events, and thus demonstrates the applicability of our results to nontrivial time scales beyond purely continuous or discrete cases.

The coefficients of system (1.5) are chosen as $a(t) = 2.5 + 0.2\cos(t), b(t) = 3, c(t) = 0.4, d(t) = 0.5, f(t) = 0.7 + 0.2\sin(t)$ and $\gamma(t) = 0.2$, which are bounded and rd-continuous on $\mathbb{T}$. Moreover, the intrinsic growth rate $a(t)$ and the conversion rate $f(t)$ are taken to be time-varying periodic functions, representing seasonal environmental fluctuations, while the remaining parameters are constants for simplicity. A direct calculation shows that the corresponding lower average conditions required in Theorem 3.1 are satisfied.

Using Matlab, we numerically simulate system (1.5) under the above parameter settings. The time evolution of the state variables $x_3(t)$, $x_4(t)$ is shown in Figure 1. From the numerical results, it can be clearly observed that both $x_3(t)$, $x_4(t)$ remain bounded for sufficiently large (t) and eventually enter a compact invariant region independent of the initial values. This behavior confirms the uniform ultimate boundedness of solutions predicted by Theorem 3.1.



(a) (b)

Figure 1. System (1.5) under initial condition $x_3(0) = 1, x_4(0) = 2$. $\exp\{x_3(t)\}$ presents the population rate of prey, $\exp\{x_4(t)\}$ presents the population rate of predator. The predator and prey could coexist in the long run.

In addition, the phase portrait displayed in Figure 1 illustrates the long-term interaction between the predator and prey populations. The trajectories remain confined to a bounded region, further indicating the permanence of the system. These numerical observations are fully consistent with the theoretical analysis and demonstrate the effectiveness of the average condition approach on time scales.

5. Conclusions

Using the average condition, we establish a dynamic inequality within the framework of time scales. Based on this inequality, the uniform ultimate boundedness of solutions to predator-prey systems is investigated. Our results extend and generalize several existing conclusions for related models. Moreover, the proposed approach provides a comparatively straightforward and effective technique for addressing such problems. Finally, a numerical example is presented to illustrate and validate the theoretical findings.

Author contributions

Yang-Yang Yu: Wrote and proofread the paper; Lin-Lin Wang and Zhong-Xin Ma: Mainly responsible for the language correction and technical verification of this paper. All authors read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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