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Research article

Explicit upper bound approximations for the distance between successive zeros of all solutions to first-order differential equations with several monotone delays

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Abstract: This paper examines the distribution of zeros of all solutions to first-order delay differential equations with several increasing delay arguments (DDEs). By advancing and generalizing the existing frameworks of the oscillation theory of delay and neutral differential equations, new explicit upper bounds are established for the separation between successive zeros of solutions. These bounds are constructed by analyzing the properties of positive solutions over a closed interval, providing sufficient conditions to prevent the persistence of such positivity. This approach provides a general analytical framework for identifying conditions under which all nontrivial solutions oscillate. Moreover, we determine the locations of the zeros of solutions to equations with several delays and oscillatory coefficients. Furthermore, the developed approach allows future extensions to functional and difference equations with non-monotone or distributed delays. Several illustrative examples and comparisons with existing criteria are included to demonstrate the accuracy and effectiveness of the proposed results.

Keywords: delay differential equations; oscillation criteria; monotone delays; distribution of zeros; upper bounds; functional and difference equations
Mathematics Subject Classification: 34K11, 34K06, 39A10, 39A21

1. Introduction

Consider the first-order differential equation with several increasing delays

$$w'(t) + \sum_{r=1}^{m} \theta_r(t)\, w(q_r(t)) = 0, \qquad t \geq t_0, \tag{1.1}$$

where $m \in \mathbb{N}$ and $t_0 \in \mathbb{R}$. Each coefficient $\theta_r \in C([t_0,\infty),[0,\infty))$ and each delay $q_r \in C([t_0,\infty),\mathbb{R})$ satisfy that $q_r(t) < t$, is increasing, and $\lim_{t\to\infty} q_r(t) = \infty$ for $r = 1, 2, \ldots, m$.

For convenience, define the maximal inverse-delay operator

$$S(t) := \max_{1\le r\le m} q_r^{-1}(t),$$

where $q_r^{-1}(t)$ denotes the inverse function of $q_r(t)$, $r = 1, 2, \ldots, m$. Moreover,

$$S^a(t) := (S^{a-1} \circ S)(t), \quad a = 2, 3, \ldots,$$

which serves as a fundamental tool for precisely characterizing the distance between zeros for all solutions.

In dynamical models, delay and oscillation scenarios are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems (see, e.g., [22, 24, 25]). The oscillation theory of DDEs has long been a cornerstone of the qualitative analysis of functional differential equations. Such models naturally arise in diverse applications where the present rate of change of a process depends on its past states, including biological systems, control theory, population dynamics, and economics (see, e.g., [1, 13, 16]). The presence of delay arguments has a significant impact on the qualitative behavior of differential systems, leading to oscillatory solutions, stability switches, and profound changes in the distribution of zeros. These phenomena have long motivated analytical investigations aimed at understanding how delay mechanisms govern the dynamical evolution of solutions.

Historically, the principal objective of oscillation theory was to determine whether all nontrivial solutions of a given delay equation oscillate—i.e., possess infinitely many zeros. Foundational contributions by Myshkis [27], Kolmanovskii and Myshkis [23], Halanay [18], and several others have established classical oscillation and stability criteria for first-order delay differential equations with monotone delays and positive coefficients, often through integral inequalities and comparison methods. Subsequent advances by Koplatadze and Chanturija [14], Koplatadze and Kvinikadze [15], and their collaborators extended these results to many types of DDEs, considerably broadening the applicability of the theory (see, e.g., [17, 19, 21]).

More recently, attention has shifted from the mere existence of oscillations to its quantitative structure—in particular the distribution and spacing between zeros of solutions. The distance between consecutive zeros, sometimes called the zero gap, provides deeper insight into the oscillatory nature of delayed equations. Considerable effort has been devoted to studying the zeros of all solutions of first-order DDEs with a single constant delay; see, for example, [9, 11, 28]. Subsequent research has addressed equations with variable delays, particularly those exhibiting monotonic growth, as discussed in [8, 29, 30]. Equations incorporating several delays have also been analyzed in [10, 12, 26], revealing a richer and more complex oscillatory structure. Significant progress has also been made in the study of first-order neutral differential equations; see, for example, [2, 6, 7]. Furthermore, results on the zeros of solutions of second-order delay equations were obtained in [9].

The main objectives of this study are as follows: (i) to establish explicit sufficient conditions ensuring the oscillation of all solutions of (1.1); and (ii) to establish computable upper bounds for the maximal distance between successive zeros, expressed via the delay arguments and coefficient functions, through a qualitative analysis of positive solutions on closed intervals that prevents the persistence of positivity.

To illustrate the applicability and effectiveness of the proposed results, several numerical examples are presented. These examples verify that our upper bounds yield a more precise approximation than all previously known results. Moreover, by identifying some analytical solutions in each case and comparing them with the theoretical estimates, we demonstrate the high accuracy of our findings.

2. Main results

Let $DS_{m_0}(1.1)$ denote the upper bound of the distance between successive zeros for every solution of Eq (1.1) on $[m_0, \infty)$.

Define the sequence of positive real numbers $\{H_r^a\}_{a\geq 0}$ by setting $H_r^0 = 0$ and $H_r^1 = 1$. For $r = 1, 2, \ldots, m$, define H_r^a by

$$\exp\left(\int_{q_r(t)}^{t} \sum_{r_1=1}^{m} H_{r_1}^{a-1}\, \theta_{r_1}(l)\, dl\right) \geq H_r^a, \qquad a = 2, 3, \ldots, \qquad t \geq S(t_0).$$

Lemma 2.1. *Assume that $r \in \{1, 2, \ldots, m\}$, $a \in \mathbb{N}_0$, and $w(t)$ is a positive solution of Eq (1.1) on $[M_0, M_1]$, $M_0 \geq t_0$, and $M_1 \geq S^{a+1}(M_0)$. Then*

$$\frac{w(q_r(t))}{w(t)} \geq H_r^a \qquad \textit{for } t \in [S^{a+2}(M_0), M_1]. \tag{2.1}$$

Proof. Since $w(t) > 0$ on the interval $[M_0, M_1]$, it follows from Eq (1.1) that

$$w'(t) \leq 0, \qquad t \in [S(M_0), M_1]. \tag{2.2}$$

Consequently, we have

$$\frac{w(q_r(t))}{w(t)} \geq 1 = H_r^0, \qquad t \in [S^2(M_0), M_1], \quad r \in \{1, 2, \ldots, m\}.$$

Dividing Eq (1.1) by $w(t)$ and integrating both sides from $q_r(t)$ to t, we obtain

$$\frac{w(q_r(t))}{w(t)} = \exp\left(\int_{q_r(t)}^{t} \sum_{r_1=1}^{m} \theta_{r_1}(l)\, \frac{w(q_{r_1}(l))}{w(l)}\, dl\right), \qquad t \in [S^1(M_0), M_1]. \tag{2.3}$$

By using (2.2), we have

$$\frac{w(q_r(l))}{w(l)} \geq 1, \qquad l \in [q_r(t), t], \quad t \in [S^3(M_0), M_1], \quad r \in \{1, 2, \ldots, m\}.$$

Substituting into (2.3), we get

$$\frac{w(q_r(t))}{w(t)} \geq \exp\left(\int_{q_r(t)}^{t} \sum_{r_1=1}^{m} \theta_{r_1}(l)\, dl\right) \geq H_r^1, \qquad t \in [S^3(M_0), M_1]. \tag{2.4}$$

Substituting (2.4) into (2.3), we obtain

$$\frac{w(q_r(t))}{w(t)} \geq \exp\left(\int_{q_r(t)}^{t} \sum_{r_1=1}^{m} H_{r_1}^1\, \theta_{r_1}(l)\, dl\right) \geq H_r^2, \qquad t \in [S^4(M_0), M_1]. \tag{2.5}$$

Repeating this procedure successively a times, we obtain (2.1). $\square$

Theorem 2.1. *Assume that there exist $a \in \mathbb{N}_0$ and $m_0 \geq t_0$ such that, for each $k \in \{1, 2, \ldots, m\}$,*

$$\sum_{r_1=1}^{m} \int_{q_{r_1}(t)}^{t} \theta_k(l) \, \exp\left(\int_{q_k(l)}^{q_k(t)} \sum_{r_2=1}^{m} \theta_{r_2}(l_1) \, H^a_{r_2} \, dl_1 \right) dl \geq 1. \tag{2.6}$$

Then every solution of Eq (1.1) is oscillatory and possesses at least one zero on each interval $[m_0,\ S^{a+4}(m_0)]$, and

$$DS_{m_0}(1.1) \ \leq \ \sup_{t \geq m_0} (S^{a+4}(t) - t)$$

when $S(t) - t$ is bounded. Moreover, if there exists a constant $\delta > 0$ such that $S(t) - t \leq \delta$, then

$$DS_{m_0}(1.1) \ \leq \ (a+4)\,\delta.$$

Proof. Assume, without loss of generality, that Eq (1.1) admits a solution $w(t)$ satisfying

$$w(t) > 0, \quad t \in [M_0,\ M_1],$$

where $M_0 \geq m_0$ and $M_1 \geq S^{a+4}(M_0)$. Therefore,

$$w'(t) = -\sum_{r=1}^{m} \theta_r(t)\, w(q_r(t)) \leq 0 \qquad \text{for } t \in [S(M_0),\ M_1].$$

From Eq (1.1) it follows for every $k \in \{1, 2, \ldots, m\}$, that

$$w(t) - w(q_k(t)) = \int_{q_k(t)}^{t} w'(l)\, dl = -\sum_{r=1}^{m} \int_{q_k(t)}^{t} \theta_r(l)\, w(q_r(l))\, dl, \qquad t \in [S(M_0),\ M_1]. \tag{2.7}$$

Upon dividing Eq (1.1) by $w(t)$ and integrating over the interval $[q_r(l),\ q_r(t)]$, one obtains

$$w(q_r(l)) = w(q_r(t)) \exp\left(\int_{q_r(l)}^{q_r(t)} \sum_{r_1=1}^{m} \theta_{r_1}(l_1) \frac{w(q_{r_1}(l_1))}{w(l_1)}\, dl_1 \right), \quad q_k(t) \leq l \leq t, \quad t \in [S^2(M_0),\ M_1].$$

From this and (2.7), we have

$$w(t) - w(q_k(t)) + \sum_{r=1}^{m} w(q_r(t)) \int_{q_k(t)}^{t} \theta_r(l) \, \exp\left(\int_{q_r(l)}^{q_r(t)} \sum_{r_1=1}^{m} \theta_{r_1}(l_1) \frac{w(q_{r_1}(l_1))}{w(l_1)}\, dl_1 \right) dl = 0, \qquad t \in [S^2(M_0),\ M_1].$$

This together with (2.1) implies that

$$w(t) - w(q_k(t)) + \sum_{r=1}^{m} w(q_r(t)) \int_{q_k(t)}^{t} \theta_r(l) \, \exp\left(\int_{q_r(l)}^{q_r(t)} \sum_{r_1=1}^{m} \theta_{r_1}(l_1)\, H^a_{r_1}\, dl_1 \right) dl \leq 0, \qquad t \in [S^{a+4}(M_0),\ M_1].$$

Summing the previous expression with respect to $k = 1, 2, \ldots, m$ gives

$$m\, w(t) + \sum_{k=1}^{m} w(q_k(t)) \left(\sum_{r_1=1}^{m} \int_{q_{r_1}(t)}^{t} \theta_k(l) \, \exp\left(\int_{q_k(l)}^{q_k(t)} \sum_{r_2=1}^{m} \theta_{r_2}(l_1)\, H^a_{r_2}\, dl_1 \right) dl - 1 \right) \leq 0 \tag{2.8}$$

for $t \in [S^{a+4}(M_0),\ M_1]$. This contradicts condition (2.6). Therefore, every solution has at least one zero within any interval $[M_0,\ S^{a+4}(M_0)]$. When $S(t) - t$ is bounded, then

$$DS_{m_0}(1.1) \ \leq\ \sup_{t \geq m_0}(S^{a+4}(t) - t).$$

If $S(t) - t \leq \delta$, then by successive application of S, we have

$$\begin{aligned} S^{a+4}(t) - t &= (S^{a+4}(t) - S^{a+3}(t)) + (S^{a+3}(t) - S^{a+2}(t)) + \cdots + (S(t) - t) \\ &= \sum_{k=0}^{a+3} (S(S^k(t)) - S^k(t)) \\ &\leq (a+4)\,\delta, \end{aligned}$$

where $S^0(t) = t$. Hence,

$$DS_{m_0}(1.1) \leq (a+4)\,\delta.$$

□

Theorem 2.2. *Let $m_0 \geq t_0$ and $\theta(t)$ be a non-negative continuous function on $[m_0,\ \infty)$ such that $\theta_k(t) \geq \theta(t)$ for every $k \in \{1, 2, \ldots, m\}$, $t \geq m_0$ and*

$$\sum_{r=1}^{m} \int_{q_r(t)}^{t} \theta(l)\, dl \ \geq\ 1, \qquad\qquad t \geq S(m_0). \tag{2.9}$$

Then every solution of Eq (1.1) is oscillatory and possesses at least one zero in each interval $[m_0,\ S^3(m_0)]$. Moreover,

$$DS_{m_0}(1.1) \ \leq\ \sup_{t \geq m_0}(S^3(t) - t)$$

whenever $S(t) - t$ is bounded. In particular, if there exists a constant $\delta > 0$ such that $S(t) - t \leq \delta$, then

$$DS_{m_0}(1.1) \ \leq\ 3\,\delta.$$

Proof. Using the same approach as in the proof of Theorem 2.1, we obtain

$$w(t) - w(q_k(t)) + \sum_{r=1}^{m} \int_{q_k(t)}^{t} \theta_r(l)\, w(q_r(l))\, dl = 0, \qquad t \in [S(M_0),\ M_1], \tag{2.10}$$

where $w(t)$ is a solution of Eq (1.1) such that

$$w(t) > 0, \quad t \in [M_0,\ M_1]$$

and $M_0 \geq m_0$ and $M_1 \geq S^3(M_0)$. Since $w'(t) \leq 0$ for $t \in [S(M_0),\ M_1]$ and $\theta(t) \geq \theta_k(t)$ for $k \in \{1, 2, \ldots, m\}$, it follows that

$$w(t) - w(q_k(t)) + \sum_{r=1}^{m} w(q_r(t)) \int_{q_k(t)}^{t} \theta(l)\, dl \leq 0, \quad t \in [S^3(M_0),\ M_1].$$

Summing this inequality over $k = 1, 2, \ldots, m$ yields

$$m\, w(t) + \sum_{k=1}^{m} w(q_k(t)) \left(\sum_{r=1}^{m} \int_{q_r(t)}^{t} \theta(l)\, dl - 1 \right) \leq 0$$

for $t \in [S^3(M_0),\ M_1]$. From this and (2.9), we obtain

$$0 < m\, w(t) \leq m\, w(t) + \sum_{k=1}^{m} w(q_k(t)) \left(\sum_{r=1}^{m} \int_{q_r(t)}^{t} \theta(l)\, dl - 1 \right) \leq 0.$$

This contradiction completes the proof. □

In the following, we generalize some of our results to differential equations with several delays and oscillatory coefficients.

Theorem 2.3. *Let $\{M_\alpha\}_{\alpha \geq 1}$ be an unbounded sequence of real numbers satisfying $M_{\alpha+1} > S^3(M_\alpha)$, $\alpha \geq 1$. Suppose that $\theta_k(t) \geq 0$ for all $k \in \{1, 2, \ldots, m\}$ and all $t \in [S(M_\alpha),\ S^3(M_\alpha)]$, $\alpha \geq 1$. Assume further that, for each $k \in \{1, 2, \ldots, m\}$, the following condition holds:*

$$\sum_{r=1}^{m} \int_{q_r(S^3(M_\alpha))}^{S^3(M_\alpha)} \theta_k(l)\, dl \ \geq\ 1 \qquad \textit{for all } \alpha \geq 1. \tag{2.11}$$

Then every solution of Eq (1.1) is oscillatory and possesses at least one zero in each interval $[M_\alpha,\ S^3(M_\alpha)]$.

Proof. Assume, for the sake of contradiction, that Eq (1.1) has a solution $w(t)$ such that

$$w(t) > 0, \qquad t \in [M_\alpha,\ M^*],$$

where $M^* \geq S^3(M_\alpha)$. Integrating Eq (1.1) from $q_k(t)$ to t, for each $k \in \{1, 2, \ldots, m\}$, we have

$$w(t) - w(q_k(t)) + \sum_{r=1}^{m} \int_{q_k(t)}^{t} \theta_r(l)\, w(q_r(l))\, dl = 0, \quad t \in [S(M_\alpha),\ M^*]. \tag{2.12}$$

From Eq (1.1) it follows for $r \in \{1, 2, \ldots, m\}$, that

$$w'(q_r(l)) = -\sum_{r_1=1}^{m} \theta_{r_1}(q_r(l))\, w(q_{r_1}(q_r(l))) \leq 0, \qquad l \in [q_k(t),\ t], \quad t \in [S^3(M_\alpha),\ M^*].$$

Combining this with (2.12), we obtain

$$w(t) - w(q_k(t)) + \sum_{r=1}^{m} w(q_r(t)) \int_{q_k(t)}^{t} \theta_r(l)\, dl \leq 0, \qquad t \in [S^3(M_\alpha),\ M^*].$$

By summing the above inequality over $k = 1, 2, \ldots, m$, we obtain

$$m\, w(t) + \sum_{k=1}^{m} w(q_k(t)) \left[\sum_{r_1=1}^{m} \int_{q_{r_1}(t)}^{t} \theta_k(l)\, dl - 1 \right] \leq 0, \qquad t \in [S^3(M_\alpha),\ M^*].$$

This together with (2.11) implies that

$$0 < m\, w(S^3(M_\alpha)) + \sum_{k=1}^{m} w(q_k(S^3(M_\alpha))) \left[\sum_{r_1=1}^{m} \int_{q_{r_1}(S^3(M_\alpha))}^{S^3(M_\alpha)} \theta_k(l)\, dl - 1 \right] \leq 0.$$

This contradiction completes the proof. □

Theorem 2.4. *Let $\{M_\alpha\}_{\alpha\geq 1}$ be an unbounded sequence of real numbers satisfying $M_{\alpha+1} > S^3(M_\alpha)$, and let $\theta(t)$ be a continuous function on $[m_0, \infty)$ such that $\theta_k(t) \geq \theta(t) \geq 0$ for every $k \in \{1, 2, \ldots, m\}$ and all $t \in [S(M_\alpha), S^3(M_\alpha)]$, where $\alpha \geq 1$. Assume further that*

$$\sum_{r=1}^{m} \int_{q_r(S^3(M_\alpha))}^{S^3(M_\alpha)} \theta(l)\, dl \;\geq\; 1 \qquad \text{for all } \alpha \geq 1. \tag{2.13}$$

Then every solution of Eq (1.1) *is oscillatory and possesses at least one zero in each interval* $[M_\alpha,\, S^3(M_\alpha)]$.

Proof. Suppose, to the contrary, that Eq (1.1) has a solution $w(t)$ such that $w(t) > 0$, $t \in [M_\alpha,\, M^*]$, where $M^* \geq S^3(M_\alpha)$. Proceeding analogously to the proof of Theorem 2.3, we arrive at

$$w(t) - w(q_k(t)) + \sum_{r=1}^{m} w(q_r(t)) \int_{q_k(t)}^{t} \theta_r(l)\, dl \leq 0, \qquad t \in [S^3(M_\alpha),\, M^*].$$

In view of $\theta_k(t) \geq \theta(t) \geq 0$, for all $k = 1, 2, \ldots, m$ and all $t \in [S(M_\alpha),\, S^3(M_\alpha)]$, it follows that

$$w(t) - w(q_k(t)) + \sum_{r=1}^{m} w(q_r(t)) \int_{q_k(t)}^{t} \theta(l)\, dl \leq 0, \qquad t \in [S^3(M_\alpha),\, M^*].$$

Summing the last inequality over all $k = 1, \ldots, m$, we obtain

$$m\, w(t) + \left[\sum_{r=1}^{m} \int_{q_r(t)}^{t} \theta(l)\, dl - 1 \right] \sum_{k=1}^{m} w(q_k(t)) \leq 0, \qquad t \in [S^3(M_\alpha),\, M^*].$$

This contradicts condition (2.13). Consequently, Eq (1.1) has no solution that remains positive on the interval $[M_\alpha,\, S^3(M_\alpha)]$, which completes the proof. □

3. Illustrative numerical examples

To illustrate the accuracy and effectiveness of our results, we now present two representative examples. The first example shows that the upper bounds for the spacing of consecutive zeros of Eq (1.1), established here, are not obtainable from any existing criteria in the literature. This claim will be verified through a detailed comparison with all existing criteria. The second example addresses an equation with several delays with oscillatory coefficients. To the best of the author's knowledge, the spacing between successive zeros for this class of equations has not yet been investigated in the existing literature.

Example 3.1. Consider the first-order differential equation with several delays

$$w'(t) + \theta_1(t)\, w(q_1(t)) + \theta_2(t)\, w(q_2(t)) = 0, \qquad t \geq 0, \tag{3.1}$$

where $\theta_1(t) = \theta_2(t) = \mu > 0$, $q_1(t) = t - \frac{3}{2}$, and $q_2(t) = t - 2$. Obviously, $\theta(t) = \theta_1(t) = \theta_2(t) = \mu$, and

$$S(t) := \max_{1 \leq r \leq 2} q_r^{-1}(t) = q_2^{-1}(t) = t + 2 = t + \delta.$$

Moreover,

$$\sum_{r=1}^{2} \int_{q_r(t)}^{t} \theta(l)\, dl = \frac{7\mu}{2}.$$

Hence, condition (2.9) is satisfied for $\mu \geq \frac{2}{7}$, and therefore

$$DS_{m_0}(3.1) \leq 3\delta = 6.$$

To verify the accuracy of our results, we consider the analytical solution of Eq (3.1) with the initial function $\phi(t) = t$ on $[-2, 0]$; see Figure 1. The computed estimate in condition (2.9) deviates from the optimal threshold of the upper bound by 1.93, which is less than the delay parameter $\delta = 2$.

In Table 1, we present the upper bounds of the distances between consecutive zeros for all previously known criteria, along with their deviations from the optimal upper bound. As clearly illustrated in the table, our result yields a sharper approximation and exhibits a smaller deviation from the optimal upper bound. In fact, condition (2.9) yields the estimate 6 for $\mu \geq \frac{2}{7}$, whereas existing estimates fail to give this distance for the same range of μ. By contrast, [5, Theorem 2.1] and [4, Theorem 2] achieve this estimate only when $\mu \geq \frac{1}{2}$.

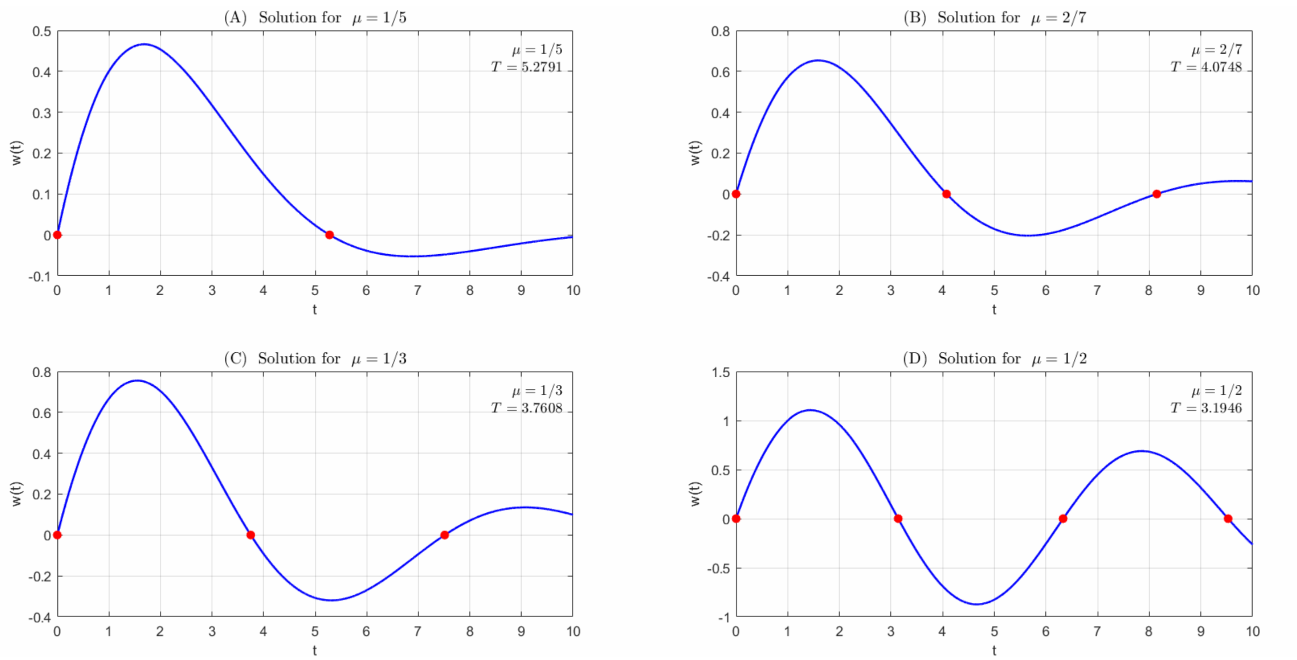

Figure 1. Numerical solutions of Eq (3.1) with the initial function $\phi(t) = t$ on $[-2, 0]$. Panels (A)–(D) correspond to $\mu = 1/5$, $\mu = 2/7$, $\mu = 1/3$, and $\mu = 1/5$, respectively. Red dots indicate the zeros of the solution, and T denotes the maximal distance between two consecutive zeros.

Table 1. Comparison of the upper bounds for the distance between consecutive zeros.

Condition	Values of μ	$DS_{m_0}(3.1) \leq$	Deviation from the optimal upper bound
Condition (2.9)	$\mu \geq \frac{2}{7}$	6	1.93
[5, Theorem 2.1], [4, Theorem 2], and [12, Corollary 2.24]	$\mu \geq \frac{1}{3}$	7	3.22
[5, Theorem 2.1] and [4, Theorem 2]	$\mu \geq \frac{1}{2}$	6	2.57
[5, Theorem 2.2]	$\mu \geq \frac{1}{5}$	8	2.7

Example 3.2. Consider the first-order differential equation with several delays and oscillatory coefficients

$$w'(t) + \theta_1(t)\, w(q_1(t)) + \theta_2(t)\, w(q_2(t)) = 0, \qquad t \geq 0, \tag{3.2}$$

where $q_1(t) = t - 1$, $q_2(t) = t - 2$, and $\theta_1(t) = \theta_2(t) = \mu > 0$ on $[M_\alpha, M_\alpha + 4]$, $\alpha \in \mathbb{N}$, such that $M_{\alpha+1} > M_\alpha + 6$.

Clearly, $\theta(t) = \theta_1(t) = \theta_2(t) = \mu$, and

$$S(t) := \max_{1 \leq r \leq 2} q_r^{-1}(t) = q_2^{-1}(t) = t + 2 = t + \delta.$$

Hence,

$$\sum_{r=1}^{2} \int_{q_r\left(S^3(M_\alpha)\right)}^{S^3(M_\alpha)} \theta(l)\, dl = 3\mu \geq 1 \qquad \text{for } \mu \geq \tfrac{1}{3}.$$

Therefore, condition (2.13) is satisfied, and every solution of Eq (3.2) is oscillatory and possesses at least one zero in each interval $[M_\alpha,\ M_\alpha + 6]$ for all $\alpha \in \mathbb{N}$.

4. Conclusions

In this work, we have established new oscillation criteria for first-order DDEs with several delays. The results generalize several known findings and provide explicit estimates for the upper bounds of the spacing between consecutive zeros. The illustrative examples confirm the effectiveness of our results and show sharper bounds with smaller deviations from the optimal upper bounds. Furthermore, the applicability of the criteria to equations with sign-changing coefficients demonstrates the broader scope of the analysis. The techniques developed here can be extended to other functional differential equations, including neutral and higher-order equations, as well as systems with distributed or state-dependent delays.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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