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Research article

Lie theory and Humbert function Ψ_1

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Abstract: The paper is devoted to bringing hypergeometric functions (HFs) within the purview of Lie theory by constructing a dynamical symmetry algebra of Humbert function Ψ_1. Multiplier representation theory of Lie groups and algebras is then used to obtain a generating function for basic analogue of Humbert function Ψ_1.

Keywords: Lie theory; Lie algebra method; dynamical symmetry algebra; Weisner's group theoretic method; Humbert function Ψ_1; group theoretical analysis
Mathematics Subject Classification: 16W25, 17B40, 22E30, 33C80, 70G65, 76M60

1. Introduction and definitions

Special functions have extensive applications in mathematics. Many of these applications are found in statistics, physics, engineering, the theory of elasticity, quantum theory, and Lie theory. Special functions serve as foundational basis functions within representation spaces for Lie algebras, and many of their properties can be derived through this perspective. Lie theory, in particular, holds a significant place in mathematical physics, serving as a valuable tool for exploring differential equations, special functions, and perturbation theory. Its influence extends beyond elementary particle and nuclear physics to diverse areas such as continuum mechanics, solid-state physics, cosmology, and control theory. As a rapidly evolving field, it offers opportunities to leverage numerous advanced methods from modern mathematics. This approach has been thoroughly explored in the works of

Manocha [1, 2], MoBrlde [3], as well as Miller [4]. The qualitative and geometric foundations of Lie group theory relevant to special functions are outlined in standard references, such as Miller's contributions [5]. Here, we will place particular focus on the Lie group theory presented by Miller, Khan [6], Srinivasa Bhagavan [7], Thakare [8], and Weisner [9], as this aspect of the subject is among the most familiar to physicists. Moreover, it is directly applicable to the study of special functions, particularly in extracting the intricate fine structure of Humbert function Ψ_1.

The confluent hypergeometric function (CHF) ${}_g\mathfrak{F}_g(a_1; b_1; \zeta)$ is defined by (see [10])

$$ {}_g\mathfrak{F}_g(a_1; b_1; \zeta) = \sum_{\kappa=0}^{\infty} \frac{(a_1)_\kappa \zeta^\kappa}{\kappa!(b_1)_\kappa}, \tag{1.1}$$

where a_1, b_1 and ζ represent real or complex numbers, with the stipulation that b_1 does not assume the value of a negative integer and $(a_1)_\kappa$ denotes Pochhammer's symbol

$$ (a_1)_\kappa = \begin{cases} (a_1)(a_1+1)(a_1+2)\ldots(a_1+\kappa-1), & \kappa \in \mathbb{N}; \\ 1, & \kappa = 0. \end{cases} $$

The CHF, denoted as ${}_g\mathfrak{F}_g(a_1; b_1; \zeta)$, adheres to the differential recursion relations

$$ \begin{aligned} D_\zeta \; {}_g\mathfrak{F}_g(a_1; b_1; \zeta) =& \frac{a_1}{b_1} \; {}_g\mathfrak{F}_g(a_1+1; b_1+1; \zeta), \quad D_\zeta = \frac{d}{d\zeta}, \\ (D_\zeta - 1) \; {}_g\mathfrak{F}_g(a_1; b_1; \zeta) =& \frac{a_1 - b_1}{b_1} \; {}_g\mathfrak{F}_g(a_1; b_1+1; \zeta), \end{aligned} \tag{1.2}$$

$$ \begin{aligned} (\Theta + a_1) \; {}_g\mathfrak{F}_g(a_1; b_1; \zeta) =& a_1 \; {}_g\mathfrak{F}_g(a_1+1; b_1; \zeta), \quad \Theta = \zeta \frac{d}{d\zeta}, \\ (\Theta + b_1 - 1) \; {}_g\mathfrak{F}_g(a_1; b_1; \zeta) =& (b_1 - 1) \; {}_g\mathfrak{F}_g(a_1; b_1 - 1; \zeta), \end{aligned} \tag{1.3}$$

and

$$ (\Theta + a_2 - b_1 - \zeta) \; {}_g\mathfrak{F}_g(a_1; b_1; \zeta) = (b_1 - a_1) \; {}_g\mathfrak{F}_g(a_1 - 1; b_1; \zeta). \tag{1.4}$$

The hypergeometric function (HF) ${}_y\mathfrak{F}_g(a_1, a_2; b_1; \zeta)$ is formally defined as follows (see [10]):

$$ {}_y\mathfrak{F}_g(a_1, a_2; b_1; \zeta) = \sum_{\kappa=0}^{\infty} \frac{(a_1)_\kappa (a_2)_\kappa \zeta^\kappa}{\kappa!(b_1)_\kappa}, \tag{1.5}$$

where a_1, a_2, b_1 and ζ are real or complex numbers, and for b_1, it is neither zero nor a negative integer.

The HF ${}_y\mathfrak{F}_g(a_1, a_2; b_1; \zeta)$ satisfies the following differential recursion formulas:

$$ D_x \; {}_y\mathfrak{F}_g(a_1, a_2; b_1; \zeta) = \frac{a_1 a_2}{b_1} \; {}_y\mathfrak{F}_g(a_1+1, a_2+1; b_1+1; \zeta), \tag{1.6}$$

and

$$ \begin{aligned} (\Theta + a_1) \; {}_y\mathfrak{F}_g(a_1, a_2; b_1; \zeta) &= a_1 \; {}_y\mathfrak{F}_g(a_1+1, a_2; b_1; \zeta), \\ (\Theta + a_2) \; {}_y\mathfrak{F}_g(a_1, a_2; b_1; \zeta) &= a_2 \; {}_y\mathfrak{F}_g(a_1, a_2+1; b_1; \zeta), \\ (\Theta + b_1 - 1) \; {}_y\mathfrak{F}_g(a_1, a_2; b_1; \zeta) &= (b_1 - 1) \; {}_y\mathfrak{F}_g(a_1, a_2; b_1 - 1; \zeta), \end{aligned} \tag{1.7}$$

$$[\zeta(1-\zeta)D_\zeta - a_2\zeta + b_1 - a_1]\ {}_y\mathfrak{F}_g(a_1,a_2;b_1;\zeta) = (b_1-a_1)\ {}_y\mathfrak{F}_g(a_1-1,a_2;b_1;\zeta), \tag{1.8}$$
$$[\zeta(1-\zeta)D_\zeta - a_1\zeta + b_1 - a_2]\ {}_y\mathfrak{F}_g(a_1,a_2;b_1;\zeta) = (b_1-a_2)\ {}_y\mathfrak{F}_g(a_1,a_2-1;b_1;\zeta), \tag{1.9}$$
$$[(1-\zeta)D_\zeta + b_1 - a_1 - a_2]\ {}_y\mathfrak{F}_g(a_1,a_2;b_1;\zeta) = (b_1-a_1)(b_1-a_2)\ {}_y\mathfrak{F}_g(a_1,a_2;b_1+1;\zeta), \tag{1.10}$$

and

$$[\zeta(1-\zeta)D_\zeta - a_2\zeta + b_1 - a_1][\Theta + a_1]\ {}_y\mathfrak{F}_g(a_1,a_2;b_1;\zeta) = a_1(b_1-a_1-1)\ {}_y\mathfrak{F}_g(a_1,a_2;b_1;\zeta), \tag{1.11}$$
$$[\zeta(1-\zeta)D_\zeta - a_1\zeta + b_1 - a_2][\Theta + a_2]\ {}_y\mathfrak{F}_g(a_1,a_2;b_1;\zeta) = a_2(b_1-a_2-1)\ {}_y\mathfrak{F}_g(a_1,a_2;b_1;\zeta), \tag{1.12}$$

and

$$\begin{aligned}&[(1-\zeta)D_\zeta + b_1 - a_1 - a_2][\Theta + b_1 - 1]\ {}_y\mathfrak{F}_g(a_1,a_2;b_1;\zeta)\\ &= (b_1-a_1-1)(b_1-a_2-1)\ {}_y\mathfrak{F}_g(a_1,a_2;b_1;\zeta),\end{aligned} \tag{1.13}$$

and

$$\Big((1-\zeta)D - a_2\Big)\ {}_y\mathfrak{F}_g(a_1,a_2;b_1;\zeta) = \frac{(a_1-b_1)a_1}{b_1}\ {}_y\mathfrak{F}_g(a_1,a_2+1;b_1+1;\zeta), \tag{1.14}$$
$$\Big((1-\zeta)D - a_1\Big)\ {}_y\mathfrak{F}_g(a_1,a_2;b_1;\zeta) = \frac{(a_2-b_1)a_1}{b_1}\ {}_y\mathfrak{F}_g(a_1+1,a_2;b_1+1;\zeta). \tag{1.15}$$

We recall here classical definition of Ψ_1 Humbert function defined as follows (see [11, page 59, Eq (42), page 57, Eqs (28)–(31)], [12, page 225, Eq (24)], and [13, page 384, Eq (5)]):

$$\Psi_1(a_1,a_2;b_1,b_2;x,y) = \sum_{\kappa,n=0}^{\infty} \frac{(a_1)_{\kappa+n}(a_2)_\kappa}{(b_1)_\kappa(b_2)_n}\frac{x^\kappa y^n}{\kappa!n!}; |x| < \infty, |y| < \infty, b_1, b_2 \neq 0,-1,-2,\ldots, \tag{1.16}$$

where a_1, a_2, b_1, b_2, x and y are real or complex numbers. We can write it as

$$\Psi_1(a_1,a_2;b_1,b_2;x,y) = \sum_{\kappa,n=0}^{\infty} \frac{(a_1)_n(a_1+n)_\kappa(a_2)_\kappa}{(b_1)_\kappa(b_2)_n}\frac{x^\kappa y^n}{\kappa!n!} = \sum_{n=0}^{\infty}\frac{(a_1)_n y^n}{(b_2)_n n!}\ {}_y\mathfrak{F}_g(a_1+n,a_2;b_1;x) \tag{1.17}$$

or

$$\Psi_1(a_1,a_2;b_1,b_2;x,y) = \sum_{\kappa=0}^{\infty}\frac{(a_1)_\kappa(a_2)_\kappa x^\kappa}{\kappa!(b_1)_\kappa}\ {}_g\mathfrak{F}_g(a_1+\kappa;b_2;y). \tag{1.18}$$

The extended function is denoted by Ψ_1 Humbert function, and its definition is given by

$$\boldsymbol{\Psi}_{a_1,a_2,b_1,b_2} = \Psi_{a_1,a_2,b_1,b_2}(x,y,s,u,v,t) = \Psi_1(a_1,a_2;b_1,b_2;x,y)s^{a_1}u^{a_2}v^{b_1}t^{b_2}. \tag{1.19}$$

The fundamental components constituting a subspace of analytical functions pertaining to five variables, namely x, y, s, u, v, and t, are associated with the hypergeometric function denoted as Ψ_1, which is defined as Ψ_1.

In [14], Agarwal and Manocha derived a novel set of generating functions related to hypergeometric polynomials. Bhagavan employed a group-theoretic approach to analyze generating functions for hypergeometric polynomials in [15]. Bhagavan and Tadikonda derived the generating functions of Chebyshev polynomials by the Weisner method in [16]. Chakrabarti and Chongdar developed a group-theoretic analysis concerning specific generating functions of hypergeometric polynomials in [17]. In their study presented in [18], Chongdar explored the group-theoretic aspects of specific generating functions. In [19], Das studied differential operators for ${}_{y}\mathfrak{F}_{g}(-n,\beta;\nu;x)$. Lastly, in [20], Das explored various aspects of special functions arising from the framework of continuous transformation groups. In [21], Ghosh derived several generating functions involving hypergeometric polynomials through a Lie-algebraic approach. In [22], Jagtap et al. investigated generating functions of a new class of semi-orthogonal polynomials $X_n(x;a,\alpha)$ using Lie group theory. In [23], Agarwal and Jain introduced the dynamical symmetry algebra of ${}_{y}\mathfrak{F}_{g}$ and Jacobi polynomials. Jain and Agrawal examined Lie theory and the generation functions of certain classical polynomials in [24]. In [25], Karande and Thakare derived results concerning generalized hypergeometric polynomials and orthogonal polynomials associated with Hermite polynomials. Khanna and Bhagavan formulated Weisner's method and Lie group theoretical foundations to derive generating relations for generalized hypergeometric polynomials in [26, 27]. In [28], Khanna et al. established the generating relations for hypergeometric functions through the use of a Lie group theoretical method. In [29], Singh et al. derived simplified expressions for bilinear and bilateral generating functions of hypergeometric polynomials. Srivastava and Dhillon [30] studied Lie operator and Lie group theory of classical orthogonal polynomials and generalized hypergeometric functions. In [31], Tadikonda and Bhagavanirredu considered representation of generating relations and $SL(2,C)$ for generalized hypergeometric functions. In [32], Truesdell investigated unified theory of special functions. Truesdell [33] studied the theorems related to addition and multiplication concerning special functions. In [34–36], Miller integrated special functions into Lie group representation theory to provide a strong framework for their analysis. A partial differential equation (PDE) constructed from an ordinary differential equation (ODE) is key to this strategy. Redefining ODE parameters or indices as extra variables expands the domain and makes the ODE a specific instance of a larger PDE. Once this reinterpretation is complete, they discovered a multiplier representation of a Lie algebra that fits the PDE's symmetries. Generalized Lie derivatives, which are small symmetry generators, do this. These derivatives show how the Lie algebra affects the function space and are essential for understanding the differential equations' algebraic structure. Recently, Wang et al. [37–39] derived a class of extended Lie super algebras and multi-component super integrable hierarchies and their applications.

The extended Humbert function Ψ_1 is a key element of modern computational mathematics, with applications spanning fluid dynamics, statistical distributions, and quantum mechanics. While previous studies (e.g., Manocha [2], Miller [5, 36], Shehata and Kumar [40]) have established its utility in solving complex PDEs, the underlying algebraic structures that govern its behavior remain an area of active investigation. To address this, the current study aim is to introduce a group-theoretic framework built on Lie algebra representations for the generalized extended Humbert function Ψ_1. This Lie-theoretic approach provides for the development of a wide class of generating relations that were previously inaccessible through standard analytical methods. By revealing the function's inherent algebraic symmetries, we have developed new E-operators and reduction formulas that significantly enhance the analytical toolkit for Ψ_1. These associations and discoveries between the Humbert function and representation theory suggest that symmetry-based approaches can provide deeper insights and new analytical tools into special function theory as applications. In the following section, we define the generalized extended Humbert function and detail the Lie-theoretic techniques used to obtain these novel identities.

2. Derivation of main results

Here, the subsequent details will be significant in backing up our primary research outcomes. The E-operators are

$$E_{a_1} = s\Big(x\frac{\partial}{\partial x} + s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}\Big), \tag{2.1}$$

$$E_{a_2} = u\Big(u\frac{\partial}{\partial u} + x\frac{\partial}{\partial x}\Big), \quad E_{a_2'} = \frac{1}{u}\Big(x(1-x)\frac{\partial}{\partial x} - xs\frac{\partial}{\partial s} - xy\frac{\partial}{\partial y} + v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u}\Big), \tag{2.2}$$

$$E_{b_1'} = \frac{1}{v}\Big(v\frac{\partial}{\partial v} + x\frac{\partial}{\partial x} - 1\Big), \tag{2.3}$$

$$E_{b_2'} = \frac{1}{t}\Big(t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} - 1\Big), \tag{2.4}$$

$$E_{a_1,b_1} = sv\Big((1-x)\frac{\partial}{\partial x} - s\frac{\partial}{\partial s} - y\frac{\partial}{\partial y}\Big), \tag{2.5}$$

$$E_{a_1,b_2} = st\frac{\partial}{\partial y}, and \tag{2.6}$$

$$E_{a_1,a_2,b_1} = suv\frac{\partial}{\partial x}. \tag{2.7}$$

We use the relations $(a_1+n)(a_1)_n = a_1(a_1+1)_n$, and

$$x\frac{\partial}{\partial x}x^\kappa = \kappa\, x^\kappa,\ \ y\frac{\partial}{\partial y}y^n = ny^n,\ \ s\frac{\partial}{\partial s}s^{a_1} = a_1 s^{a_1},\ \ u\frac{\partial}{\partial u}u^{a_2} = a_2u^{a_2},\ \ v\frac{\partial}{\partial v}v^{b_1} = b_1v^{b_1},\ \ t\frac{\partial}{\partial t}t^{b_2} = b_2t^{b_2}.$$

The actions of these E-operators on $\mathbf{\Psi}_{a_1,a_2,b_1,b_2}$ are given by

$$\begin{aligned}
E_{a_1}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= E_{a_1}\sum_{n=0}^{\infty}\frac{(a_1)_n y^n}{n!(b_2)_n}\,{}_y\mathfrak{F}_g(a_1+n,a_2;b_1;x)s^{a_1}u^{a_2}v^{b_1}t^{b_2}\\
&= s\Big(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\frac{(a_1)_n}{(b_2)_n}\frac{y^n}{n!}\,{}_y\mathfrak{F}_g(a_1+n,a_2;b_1;x) + s\frac{\partial}{\partial s}\sum_{n=0}^{\infty}\frac{(a_1)_n}{(b_2)_n}\frac{y^n}{n!}\,{}_y\mathfrak{F}_g(a_1+n,a_2;b_1;x)\\
&\quad + y\frac{\partial}{\partial y}\sum_{n=0}^{\infty}\frac{(a_1)_n}{(b_2)_n}\frac{y^n}{n!}\,{}_y\mathfrak{F}_g(a_1+n,a_2;b_1;x)\Big)s^{a_1}u^{a_2}v^{b_1}t^{b_2}\\
&= s\Big(\sum_{n=0}^{\infty}\frac{(a_1)_n y^n}{n!(b_2)_n}(a_1+n)\,{}_y\mathfrak{F}_g(a_1+n+1,a_2;b_1;x) + \sum_{n=0}^{\infty}\frac{(a_1)_n y^n}{n!(b_2)_n}(a_1+n)\,{}_y\mathfrak{F}_g(a_1+n,a_2;b_1;x)\\
&\quad - \sum_{n=0}^{\infty}\frac{(a_1+n)(a_1)_n y^n}{n!(b_2)_n}\,{}_y\mathfrak{F}_g(a_1+n,a_2;b_1;x)\Big)s^{a_1}u^{a_2}v^{b_1}t^{b_2}\\
&= \sum_{n=0}^{\infty}\frac{a_1(a_1+1)_n y^n}{n!(b_2)_n}\,{}_y\mathfrak{F}_g(a_1+n+1,a_2;b_1;x)s^{a_1+1}u^{a_2}v^{b_1}t^{b_2}\\
&= a_1\mathbf{\Psi}_{a_1+1,a_2,b_1,b_2}.
\end{aligned}$$

Hence, we obtain

$$E_{a_1}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} = a_1\mathbf{\Psi}_{a_1+1,a_2,b_1,b_2}. \tag{2.8}$$

Similarly, we can find actions of other E-operators on

$$\begin{aligned}
E_{a_2}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= a_2\mathbf{\Psi}_{a_1,a_2+1,b_1,b_2}, &(2.9)\\
E_{a_2'}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= (b_1-a_2)\mathbf{\Psi}_{a_1,a_2-1,b_1,b_2}, &(2.10)\\
E_{b_1'}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= (b_1-1)\mathbf{\Psi}_{a_1,a_2,b_1-1,b_2}, &(2.11)\\
E_{b_2'}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= (b_2-1)\mathbf{\Psi}_{a_1,a_2,b_1,b_2-1}, &(2.12)\\
E_{a_1,b_1}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= \frac{a_1(a_2-b_1)}{b_1}\mathbf{\Psi}_{a_1+1,a_2,b_1+1,b_2}, &(2.13)\\
E_{a_1,b_2}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= \frac{a_1}{b_2}\mathbf{\Psi}_{a_1+1,a_2,b_1,b_2+1}, \textit{and} &(2.14)\\
E_{a_1,a_2,b_1}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= \frac{a_1a_2}{b_1}\mathbf{\Psi}_{a_1+1,a_2+1,b_1+1,b_2}. &(2.15)
\end{aligned}$$

These E-operators, along with the four maintenance operators I_{a_1}, I_{a_2}, I_{b_1}, and I_{b_2} and identity operator I, form a basis. Here, we get

$$I_{a_1} = s\frac{\partial}{\partial s},\ I_{a_2} = u\frac{\partial}{\partial u},\ I_{b_1} = v\frac{\partial}{\partial v},\ I_{b_2} = t\frac{\partial}{\partial t},\ I = 1, \tag{2.16}$$

and

$$\begin{aligned}
I_{a_1}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= a_1\mathbf{\Psi}_{a_1,a_2,b_1,b_2}, \quad I_{a_2}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} = a_2\mathbf{\Psi}_{a_1,a_2,b_1,b_2},\\
I_{b_1}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= b_1\mathbf{\Psi}_{a_1,a_2,b_1,b_2}, \quad I_{b_2}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} = b_2\mathbf{\Psi}_{a_1,a_2,b_1,b_2}.
\end{aligned} \tag{2.17}$$

Here, a group-theoretic foundation is presented to deduce reduction formulas for Humbert function Ψ_1.

First, we assign the operator

$$E_{a_1} = s\left(x\frac{\partial}{\partial x} + s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}\right)$$

with action

$$E_{a_1}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} = a_1\mathbf{\Psi}_{a_1+1,a_2,b_1,b_2}.$$

To find $\exp(\xi E_{a_1})$, we utilize the standard Lie theoretic approach, which involves solving the differential equations to compute in the following results:

$$\begin{aligned}
\frac{ds(\xi)}{d\xi} &= s^2(\xi) \quad \text{when} \quad s(0) = s \Rightarrow -\frac{1}{s(\xi)} = \xi + \mathfrak{c} \Rightarrow \mathfrak{c} = -\frac{1}{s},\ \therefore s(\xi) = \frac{s}{1-\xi s}, &(2.18)\\
\frac{dx(\xi)}{d\xi} &= x(\xi)s(\xi) \quad \text{with} \quad x(0) = x \Rightarrow x(\xi) = \frac{\mathfrak{c}}{1-\xi s} \Rightarrow \mathfrak{c} = x,\ \therefore x(\xi) = \frac{x}{1-\xi s}, &(2.19)
\end{aligned}$$

and

$$\frac{dy(\xi)}{d\xi} = y(\xi)s(\xi) \ \text{with} \ y(0) = y \Rightarrow y(\xi) = \frac{\mathfrak{c}}{1-\xi s} \Rightarrow \mathfrak{c} = y,\ \therefore y(\xi) = \frac{y}{1-\xi s}. \tag{2.20}$$

Thus, we get

$$\begin{aligned}\exp(\xi\ E_{a_1})\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= \Psi_1\Big(a_1,a_2,b_1,b_2;\frac{x}{1-\xi s},\frac{y}{1-\xi s}\Big)\Big(\frac{s}{1-\xi s}\Big)^{a_1}u^{a_2}v^{b_1}t^{b_2}\\ &= \Psi_1\Big(a_1,a_2,b_1,b_2;\frac{x}{1-\xi s},\frac{y}{1-\xi s}\Big)s^{a_1}(1-\xi s)^{-a_1}u^{a_2}v^{b_1}t^{b_2}.\end{aligned} \tag{2.21}$$

On the other hand, through direct expansion, it provides

$$\begin{aligned}\exp(\xi\ E_{a_1})\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= \sum_{\nu=0}^{\infty}\frac{\xi^\nu}{\nu\ !}\Big(E_{a_1}\Big)^\nu\mathbf{\Psi}_{a_1,a_2,b_1,b_2} = \sum_{\nu=0}^{\infty}\frac{\xi^\nu(a_1)_\nu}{\nu\ !}\mathbf{\Psi}_{a_1+\nu,a_2;b_1,b_2}\\ &= \sum_{\nu=0}^{\infty}\frac{\xi^\nu(a_1)_\nu}{\nu\ !}\Psi_1(a_1+\nu,a_2;b_1,b_2;x,y)s^{a_1+\nu}u^{a_2}v^{b_1}t^{b_2}.\end{aligned} \tag{2.22}$$

By using (2.21) and (2.22), we obtain the formula

$$(1-\xi s)^{-a_1}\Psi_1\Big(a_1,a_2,b_1,b_2;\frac{x}{1-\xi s},\frac{y}{1-\xi s}\Big) = \sum_{\nu=0}^{\infty}\frac{\xi^\nu(a_1)_\nu}{\nu\ !}\Psi_1(a_1+\nu,a_2;b_1,b_2;x,y)s^\nu. \tag{2.23}$$

Equating the two values of $\exp(\xi\ E_{a_1})\mathbf{\Psi}_{a_1,a_2,b_1,b_2}$, we have the expansion

$$\begin{aligned}(1-\xi s)^{-a_1}\Psi_1\Big(a_1,a_2,b_1,b_2;\frac{x}{1-\xi s},\frac{y}{1-\xi s}\Big) &= \sum_{\nu=0}^{\infty}\frac{\xi^\nu(a_1)_\nu}{\nu\ !}\Psi_1(a_1+\nu,a_2;b_1,b_2;x,y)s^\nu\\ &= \sum_{\nu,\kappa,n=0}^{\infty}\frac{\xi^\nu(a_1)_\nu}{\nu\ !}\frac{(a_1+\nu)_{\kappa+n}(a_2)_\kappa x^\kappa y^n}{\kappa!n!(b_1)_\kappa(b_2)_n}s^\nu\\ &= \sum_{\nu,\kappa,n=0}^{\infty}\frac{(a_1)_{\kappa+n+\nu}(a_2)_\kappa}{\kappa!n!\nu\ !(b_1)_\kappa(b_2)_n}x^\kappa y^n(s\xi)^\nu.\end{aligned} \tag{2.24}$$

Next, we proceed with the operator

$$E_{a_2} = u\Big(u\frac{\partial}{\partial\ u}+x\frac{\partial}{\partial\ x}\Big)$$

with action

$$E_{a_2}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} = a_2\mathbf{\Psi}_{a_1,a_2+1,b_1,b_2}.$$

To $\exp(\xi E_{a_2})$, we utilize the conventional methodology from Lie theory, which can be established by addressing the relevant differential equations:

$$\frac{du(\xi)}{d\xi} = u^2(\xi),\ \ \text{when}\ \ u(0)=u \Rightarrow -\frac{1}{u(\xi)} = \xi+\mathfrak{c} \Rightarrow \mathfrak{c} = -\frac{1}{u},\ \therefore\ u(\xi) = \frac{u}{1-\xi u}, \tag{2.25}$$

and

$$\frac{dx(\xi)}{d\xi} = x(\xi)u(\xi),\ \ \text{when}\ \ x(0)=x \Rightarrow x(\xi) = \frac{\mathfrak{c}}{1-\xi u} \Rightarrow \mathfrak{c} = x,\ \therefore\ x(\xi) = \frac{x}{1-\xi u}. \tag{2.26}$$

So, this formula reduces to the expansion

$$\exp(\xi\ E_{a_2})\mathbf{\Psi}_{a_1,a_2,b_1,b_2} = \Psi_1\Big(a_1, a_2+\kappa; b_1, b_2; \frac{x}{1-\xi u}, y\Big)s^{a_1}\Big(\frac{u}{1-\xi u}\Big)^{a_2} v^{b_1} t^{b_2}. \tag{2.27}$$

Conversely, through direct expansions, it produces

$$\begin{aligned}\exp(\xi\ E_{a_2})\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= \sum_{\nu=0}^{\infty}\frac{\xi^{\nu}(a_2)_{\nu}}{\nu\ !}\mathbf{\Psi}_{a_1,a_2+\nu,b_1,b_2}\\ &= \sum_{\nu=0}^{\infty}\frac{\xi^{\nu}(a_2)_{\nu}}{\nu\ !}\Psi_1(a_1, a_2+\nu; b_1, b_2; x, y)s^{a_1}u^{a_2+\nu}v^{b_1}t^{b_2}\\ &= \sum_{\nu=0}^{\infty}\frac{\xi^{\nu}(a_2)_{\nu}}{\nu\ !}\Psi_1(a_1, a_2+\nu; b_1, b_2; x, y)s^{a_1}u^{a_2+\nu}v^{b_1}t^{b_2}.\end{aligned} \tag{2.28}$$

Equating two values of $\exp(\xi\ E_{a_2})\mathbf{\Psi}_{a_1,a_2,b_1,b_2}$, we reach the identity

$$\Psi_1\Big(a_1, a_2+\nu; b_1, b_2; \frac{x}{1-\xi u}, y\Big)(1-\xi u)^{-a_2} = \sum_{\nu=0}^{\infty}\frac{(a_2)_{\nu}}{\nu\ !}\Psi_1(a_1, a_2+\nu; b_1, b_2; x, y)(\xi u)^{\nu}. \tag{2.29}$$

In the case $z = u\xi$ in (2.29), it simplifies to

$$\Psi_1\Big(a_1, a_2+\nu; b_1, b_2; \frac{x}{1-z}, y\Big)(1-z)^{-a_2} = \sum_{\nu=0}^{\infty}\frac{(a_2)_{\nu}}{\nu\ !}\Psi_1(a_1, a_2+\nu; b_1, b_2; x, y)z^{\nu}. \tag{2.30}$$

Finally, we use the operator

$$E_{a_2'} = \frac{1}{u}\Big(x(1-x)\frac{\partial}{\partial\ x} - u\frac{\partial}{\partial\ u} - xs\frac{\partial}{\partial\ s} + v\frac{\partial}{\partial\ v}\Big)$$

with action

$$E_{a_2'}\mathbf{\Psi}_{a_1,a_2,b_1,b_2} = (b_1-a_2)\mathbf{\Psi}_{a_1,a_2-1,b_1,b_2}.$$

Computing the operation of a one-parameter subgroup using the conventional multiplier representation approach for the Humbert function Ψ_1 involves solving the associated differential equations:

$$\frac{du(\xi)}{d\xi} = -1 \ \text{ when }\ u(0) = u \Rightarrow u(\xi) = -\xi + \mathfrak{c} \Rightarrow \mathfrak{c} = u,\ \therefore\ u(\xi) = u - \xi, \tag{2.31}$$

$$\frac{dv(\xi)}{d\xi} = \frac{v(\xi)}{u(\xi)},\ \text{ when }\ v(0) = v \Rightarrow v(\xi) = \frac{\mathfrak{c}}{u-\xi} \Rightarrow \mathfrak{c} = uv,\ \therefore\ v(\xi) = \frac{uv}{u-\xi}, \tag{2.32}$$

and

$$\frac{dx(\xi)}{d\xi} = \frac{x(\xi)(1-x(\xi))}{u(\xi)},\ \text{ when }\ x(0) = x \Rightarrow \frac{x(\xi)}{1-x(\xi)} = \frac{\mathfrak{c}}{u-\xi} \Rightarrow \mathfrak{c} = \frac{xu}{1-x},\ \therefore\ x(\xi) = \frac{xu}{u-\xi(1-x)}, \tag{2.33}$$

$$\frac{ds(\xi)}{d\xi}=-\frac{s(\xi)x(\xi)}{u(\xi)},\quad \text{when}\quad s(0)=s\Rightarrow s(\xi)=\frac{\mathfrak{c}(u-\xi)}{(u-\xi(1-x))}\Rightarrow \mathfrak{c}=s,\ \therefore\ s(\xi)=\frac{s(u-\xi)}{u-\xi(1-x)}. \tag{2.34}$$

If $\exp(\xi\ E_{a_2'})$, the formula reduces to

$$\exp(\xi\ E_{a_2'})\mathbf{\Psi}_{a_1,a_2,b_1,b_2}=\Psi_1\Big(a_1,a_2;b_1,b_2;\frac{xu}{u-\xi(1-x)},y\Big)\Big(\frac{s(u-\xi)}{(u-\xi(1-x))}\Big)^{a_1}(u-\xi)^{a_2}\Big(\frac{uv}{u-\xi}\Big)^{b_1}t^{b_2} \tag{2.35}$$

and

$$\begin{aligned}\exp(\xi\ E_{a_2'})\mathbf{\Psi}_{a_1,a_2,b_1,b_2}&=\sum_{\nu=0}^{\infty}\frac{\xi^{\nu}(b_1-a_2)_{\nu}}{\nu\ !}\mathbf{\Psi}_{a_1,a_2-\nu,b_1,b_2}\\&=\sum_{\nu=0}^{\infty}\frac{\xi^{\nu}(b_1-a_2)_{\nu}}{\nu\ !}\Psi_1(a_1,a_2-\nu;b_1,b_2;x,y)s^{a_1}u^{a_2-\nu}v^{b_1}t^{b_2}\\&=\sum_{\nu=0}^{\infty}\frac{\xi^{\nu}(b_1-a_2)_{\nu}}{\nu\ !}\Psi_1(a_1,a_2-\nu;b_1,b_2;x,y)s^{a_1}u^{a_2-\nu}v^{b_1}t^{b_2}.\end{aligned} \tag{2.36}$$

Thus, simplifying, we obtain the generating function

$$\begin{aligned}&\Psi_1\Big(a_1,a_2;b_1,b_2;\frac{xu}{u-\xi(1-x)},y\Big)\Big(u-\xi(1-x)\Big)^{-a_1}(u-\xi)^{a_1+a_2-b_1}u^{b_1-a_2}\\&=\sum_{\nu=0}^{\infty}\frac{\xi^{\nu}(b_1-a_2)_{\nu}}{\nu\ !}\Psi_1(a_1,a_2-\nu;b_1,b_2;x,y)u^{-\nu}.\end{aligned} \tag{2.37}$$

Next, the operator is utilized

$$E_{b_1'}=\frac{1}{v}\Big(v\frac{\partial}{\partial v}+x\frac{\partial}{\partial x}-1\Big)$$

with action

$$E_{b_1'}\Psi_{a_1,a_2,b_1,b_2}(x,y,s,u,v,t)=(b_1-1)\Psi_{a_1,a_2,b_1-1,b_2}(x,y,s,u,v,t).$$

To find $\exp(\xi E_{b_1'})$, we apply the standard Lie theoretic approach, which involves solving the associated differential equations

$$\frac{dv(\xi)}{d\xi}=1\quad\text{when}\quad v(0)=v\Rightarrow v(\xi)=\xi+\mathfrak{c}\Rightarrow\mathfrak{c}=v,\ \therefore\ v(\xi)=v+\xi \tag{2.38}$$

and

$$\frac{dx(\xi)}{d\xi}=\frac{x(\xi)}{v(\xi)}\quad\text{when}\quad x(0)=x\Rightarrow x(\xi)=\mathfrak{c}(v+\xi)\Rightarrow\mathfrak{c}=\frac{x}{v},\ \therefore\ x(\xi)=\frac{x(v+\xi)}{v}. \tag{2.39}$$

So, the operator becomes

$$\exp(\xi\ E_{b_1'})\mathbf{\Psi}_{a_1,a_2,b_1,b_2}=\Psi_1(a_1,a_2;b_1,b_2;\frac{x(v+\xi)}{v},y)s^{a_1}u^{a_2}(v+\xi)^{b_1}t^{b_2}. \tag{2.40}$$

Conversely, through direct expansion, it results in

$$\begin{aligned}\exp(\xi\, E_{b_1'})\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= \sum_{\nu=0}^{\infty} \frac{\xi^{\nu}(b_1-\nu)_{\nu}}{\nu\,!}\mathbf{\Psi}_{a_1,a_2,b_1-\nu,b_2}\\ &= \sum_{\nu=0}^{\infty} \frac{\xi^{\nu}(b_1-\nu)_{\nu}}{\nu\,!}\Psi_1(a_1,a_2;b_1-\nu,b_2;x,y)s^{a_1}u^{a_2}v^{b_1-\nu}t^{b_2} \qquad (2.41)\\ &= \sum_{\nu=0}^{\infty} \frac{\xi^{\nu}(b_1-\nu)_{\nu}}{\nu\,!}\Psi_1(a_1,a_2;b_1-\nu,b_2;x,y)s^{a_1}u^{a_2}v^{b_1-\nu}t^{b_2},\end{aligned}$$

which gives the generating relation

$$\Psi_1\Big(a_1,a_2;b_1,b_2;\frac{x(v+\xi)}{v},y\Big)\Big(1+\frac{\xi}{v}\Big)^{b_1} = \sum_{\nu=0}^{\infty} \frac{(b_1-\nu)_{\nu}}{\nu\,!}\Psi_1(a_1,a_2;b_1-\nu,b_2;x,y)\Big(\frac{\xi}{v}\Big)^{\nu}. \qquad (2.42)$$

Here, we use the operator

$$E_{b_2'} = \frac{1}{t}\Big(t\frac{\partial}{\partial\, t} + y\frac{\partial}{\partial\, y} - 1\Big)$$

with action

$$E_{b_2'}\mathbf{\Psi}_{a_1,a_2;b_1,b_2} = (b_2-1)\mathbf{\Psi}_{a_1,a_2,b_1,b_2-1}.$$

To find $\exp(\xi E_{b_2'})$, we use the standard Lie theoretic technique

$$\frac{dt(\xi)}{d\xi} = 1 \;\text{ when }\; t(0) = t \Rightarrow t(\xi) = \xi + \mathfrak{c} \Rightarrow \mathfrak{c} = t,\; \therefore\; t(\xi) = t + \xi \qquad (2.43)$$

and

$$\frac{dy(\xi)}{d\xi} = \frac{y(\xi)}{t(\xi)} \;\text{ with }\; y(0) = y \Rightarrow y(\xi) = \mathfrak{c}(t+\xi) \Rightarrow \mathfrak{c} = \frac{y}{t},\; \therefore\; y(\xi) = \frac{y(t+\xi)}{t}. \qquad (2.44)$$

So, we get

$$\exp(\xi\, E_{b_2'})\mathbf{\Psi}_{a_1,a_2,b_1,b_2} = \Psi_1(a_1,a_2;b_1,b_2;x,\frac{y(t+\xi)}{t})s^{a_1}u^{a_2}v^{b_1}(t+\xi)^{b_2}. \qquad (2.45)$$

Conversely, through direct expansion, it produces

$$\begin{aligned}\exp(\xi\, E_{b_2'})\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= \sum_{\nu=0}^{\infty} \frac{\xi^{\nu}(b_2-\nu)_{\nu}}{\nu\,!}\mathbf{\Psi}_{a_1,a_2,b_1,b_2-\nu}\\ &= \sum_{\nu=0}^{\infty} \frac{\xi^{\nu}(b_2-\nu)_{\nu}}{\nu\,!}\Psi_1(a_1,a_2;b_1,b_2-\nu;x,y)s^{a_1}u^{a_2}v^{b_1}t^{b_2-\nu} \qquad (2.46)\\ &= \sum_{\nu=0}^{\infty} \frac{\xi^{\nu}(b_2-\nu)_{\nu}}{\nu\,!}\Psi_1(a_1,a_2;b_1,b_2-\nu;x,y)s^{a_1}u^{a_2}v^{b_1}t^{b_2-\nu},\end{aligned}$$

which gives the generating relation

$$\Psi_1\Big(a_1,a_2;b_1,b_2;x,\frac{y(t+\xi)}{t}\Big)\Big(1+\frac{\xi}{t}\Big)^{b_2}=\sum_{\nu=0}^{\infty}\frac{(b_2-\nu)_\nu}{\nu\,!}\Psi_1(a_1,a_2;b_1,b_2-\nu;x,y)\Big(\frac{\xi}{t}\Big)^{\nu}. \tag{2.47}$$

Here, we use the operator

$$E_{a_1,b_1}=sv\Big((1-x)\frac{\partial}{\partial x}-s\frac{\partial}{\partial s}-y\frac{\partial}{\partial y}\Big)$$

with action

$$E_{a_1,b_1}\mathbf{\Psi}_{a_1,a_2,b_1,b_2}=\frac{a_1(a_2-b_1)}{b_1}\mathbf{\Psi}_{a_1+1,a_2,b_1+1,b_2}.$$

To find $\exp(\xi E_{a_1,b_1})$, we employ the standard Lie theoretic approach, which involves solving differential equations to compute the results

$$\frac{ds(\xi)}{d\xi}=s^2(\xi)v\ \text{ with }\ s(0)=s\Rightarrow-\frac{1}{s(\xi)}=v\xi+\mathfrak{c}\Rightarrow\mathfrak{c}=-\frac{1}{s},\ \therefore\ s(\xi)=\frac{s}{1-\xi vs} \tag{2.48}$$

and

$$\frac{dx(\xi)}{d\xi}=(x(\xi)-1)s(\xi)v\text{ with }x(0)=x\Rightarrow x(\xi)=\frac{\mathfrak{c}}{1-\xi vs}+1\Rightarrow\mathfrak{c}=x-1,\ \therefore x(\xi)=\frac{x-\xi vs}{1-\xi vs}. \tag{2.49}$$

So, we get

$$\exp(\xi\ E_{a_1,b_1})\mathbf{\Psi}_{a_1,a_2,b_1,b_2}=\Psi_1\Big(a_1,a_2;b_1,b_2;\frac{x-\xi vs}{1-\xi vs},y\Big)\Big(\frac{s}{1-\xi vs}\Big)^{a_1}u^{a_2}v^{b_1}t^{b_2}. \tag{2.50}$$

Conversely, through direct expansion, we obtain

$$\begin{aligned}\exp(\xi\ E_{a_1,b_1})\mathbf{\Psi}_{a_1,a_2,b_1,b_2}&=\sum_{\nu=0}^{\infty}\frac{\xi^\nu(a_1)_\nu(a_2-b_1)_\nu}{\nu\,!(b_1)_\nu}\mathbf{\Psi}_{a_1+\nu,a_2,b_1+\nu,b_2}\\&=\sum_{\nu=0}^{\infty}\frac{(a_1)_\nu(a_2-b_1)_\nu}{\nu\,!(b_1)_\nu}\Psi_1(a_1+\nu,a_2;b_1+\nu,b_2;x,y)\Big(\xi\ s\ v\Big)^{\nu}s^{a_1}u^{a_2}v^{b_1}t^{b_2}\\&=\sum_{\nu=0}^{\infty}\frac{(a_1)_\nu(a_2-b_1)_\nu}{\nu\,!(b_1)_\nu}\sum_{\kappa,n=0}^{\infty}\frac{(a_1+\nu)_{\kappa+n}(a_2)_\kappa x^\kappa y^n}{\kappa!n!(b_1+\nu)_\kappa(b_2)_n}\Big(\xi\ s\ v\Big)^{\nu}s^{a_1}u^{a_2}v^{b_1}t^{b_2}.\end{aligned} \tag{2.51}$$

Equating the two values of $\exp(\xi\ E_{a_1,b_1})$, we get

$$\begin{aligned}&\Psi_1(a_1,a_2;b_1,b_2;\frac{x-\xi sv}{1-\xi sv},y)\Big(\frac{s}{1-\xi sv}\Big)^{a_1}u^{a_2}v^{b_1}t^{b_2}\\&=\sum_{\nu,\kappa,n=0}^{\infty}\frac{(a_1)_\nu(a_2-b_1)_\nu(a_1+\nu)_{\kappa+n}(a_2)_\kappa x^\kappa y^n}{\kappa!n!\nu\,!(b_1)_\nu(b_1+\nu)_\kappa(b_2)_n}\Big(\xi\ s\ v\Big)^{\nu}s^{a_1}u^{a_2}v^{b_1}t^{b_2}\\&=\sum_{\nu,\kappa,n=0}^{\infty}\frac{(a_2-b_1)_\nu(a_1)_{\nu+\kappa+n}(a_2)_\kappa x^\kappa y^n}{\kappa!n!\nu\,!(b_1)_{\nu+\kappa}(b_2)_n}\Big(\xi\ s\ v\Big)^{\nu}s^{a_1}u^{a_2}v^{b_1}t^{b_2},\end{aligned} \tag{2.52}$$

which finally gives

$$(1-\xi\ sv)^{-a_1}\Psi_1\Big(a_1,a_2;b_1,b_2;\frac{x-\xi\ sv}{1-\xi\ sv},y\Big)=\sum_{\nu=0}^{\infty}\frac{\xi^\nu}{\nu\ !}\Big(E_{a_1,a_2}\Big)^\nu\Psi_{a_1,a_2,b_1,b_2}(x,y,s,u,v,t)$$
$$=\sum_{\nu,\kappa,n=0}^{\infty}\frac{(a_2-b_1)_\nu(a_1)_{\nu+\kappa+n}(a_2)_\kappa x^\kappa y^n}{\kappa!n!\nu\ !(b_1)_{\nu+\kappa}(b_2)_n}\Big(\xi\ s\ v\Big)^\nu. \tag{2.53}$$

Setting $\xi\ sv\to\eta$,

$$\Psi_1\Big(a_1,a_2,a_2-b_1;b_1,b_2;x,y,\eta\Big)=(1-\eta)^{-a_1}\Psi_1\Big(a_1,a_2;b_1,b_2;\frac{x-\eta}{1-\eta},y\Big)$$
$$=\sum_{\nu,\kappa,n=0}^{\infty}\frac{(a_1)_{\nu+\kappa+n}(a_2)_\kappa(a_2-b_1)_\nu x^\kappa y^n}{\kappa!n!\nu\ !(b_1)_{\nu+\kappa}(b_2)_n}\eta^\nu, \tag{2.54}$$

we arrive at the reduction formula

$$\Psi_1\Big(a_1,a_2,a_2-b_1;b_1,b_2;x,y,\eta\Big)=(1-\eta)^{-a_1}\Psi_1\Big(a_1,a_2;b_1,b_2;\frac{x-\eta}{1-\eta},y\Big), \tag{2.55}$$

which gives the generating relation

$$(1-\eta)^{-a_1}\Psi_1\Big(a_1,a_2;b_1,b_2;\frac{x-\eta}{1-\eta},y\Big)=\sum_{\nu=0}^{\infty}\frac{(a_1)_\nu(a_2-b_1)_\nu}{\nu\ !(b_1)_\nu}\Psi_1(a_1+\nu,a_2;b_1+\nu,b_2;x,y)\eta^\nu. \tag{2.56}$$

Now, we use the operator

$$E_{a_1,b_2}=st\frac{\partial}{\partial\ y}$$

with action

$$E_{a_1,b_2}\mathbf{\Psi}_{a_1,a_2,b_1,b_2}=\frac{a_1}{b_2}\mathbf{\Psi}_{a_1+1,a_2,b_1,b_2+1}.$$

To find $\exp(\xi E_{a_1,b_2})$, we employ the standard Lie theoretic approach, which involves solving the associated differential equations

$$\frac{dy(\xi)}{d\xi}=st\ \text{ with }\ y(0)=y\Rightarrow y(\xi)=st\xi+\mathfrak{c}\Rightarrow\mathfrak{c}=y,\ \therefore\ y(\xi)=y+st\xi. \tag{2.57}$$

So, we get

$$\exp(\xi\ E_{a_1,b_2})\mathbf{\Psi}_{a_1,a_2,b_1,b_2}=\Psi_1(a_1,a_2;b_1,b_2;x,y+st\xi)s^{a_1}u^{a_2}v^{b_1}t^{b_2}. \tag{2.58}$$

Conversely, through direct expansion, one can arrive at

$$\exp(\xi\ E_{a_1,b_2})\mathbf{\Psi}_{a_1,a_2,b_1,b_2}=\sum_{\nu=0}^{\infty}\frac{\xi^\nu(a_1)_\nu}{\nu\ !(b_2)_\nu}\mathbf{\Psi}_{a_1+\nu,a_2,b_1,b_2+\nu}$$
$$=\sum_{\nu=0}^{\infty}\frac{\xi^\nu(a_1)_\nu}{\nu\ !(b_2)_\nu}\Psi_1(a_1+\nu,a_2;b_1,b_2+\nu;x,y)s^{a_1+\nu}u^{a_2}v^{b_1}t^{b_2+\nu}, \tag{2.59}$$

which gives the generating relation

$$\Psi_1(a_1, a_2; b_1, b_2; x, y + st\xi) = \sum_{\nu=0}^{\infty} \frac{(a_1)_\nu}{\nu\,!(b_2)_\nu} \Psi_1(a_1 + \nu, a_2 + \nu; b_1, b_2; x, y)(\xi st)^\nu. \tag{2.60}$$

By equating two values of $\exp(\xi\ E_{a_1,b_2})$, we get

$$\begin{aligned}\exp(\xi\ E_{a_1,b_2})\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= \sum_{\nu=0}^{\infty} \frac{\xi^\nu (a_1)_\nu}{\nu\,!(b_2)_\nu} \mathbf{\Psi}_{a_1+\nu,a_2+\nu,b_1+\nu,b_2} \\ &= \sum_{\nu=0}^{\infty} \frac{\xi^\nu (a_1)_\nu}{\nu\,!(b_2)_\nu} \Psi_1(a_1 + \nu, a_2; b_1, b_2 + \nu; x, y) s^{a_1+\nu} u^{a_2} v^{b_1} t^{b_2+\nu} \\ &= \sum_{\nu,\kappa,n=0}^{\infty} \frac{\xi^\nu (a_1)_\nu}{\nu\,!(b_2)_\nu} \frac{(a_1 + \nu)_{\kappa+n} (a_2)_\kappa x^\kappa y^n}{\kappa! n! (b_1)_\kappa (b_2 + \nu)_n} s^{a_1+\nu} u^{a_2} v^{b_1} t^{b_2+\nu} \\ &= \sum_{\nu,\kappa,n=0}^{\infty} \frac{\xi^\nu (a_1)_{\kappa+n+\nu} (a_2)_\kappa x^\kappa y^n}{\kappa! n! \nu\,!(b_1)_\kappa (b_2)_{n+\nu}} s^{a_1+\nu} u^{a_2} v^{b_1} t^{b_2+\nu}.\end{aligned} \tag{2.61}$$

So, we can write the generating relation

$$\Psi_1(a_1, a_2; b_1, b_2; x, y + st\xi) = \sum_{\nu,\kappa,n=0}^{\infty} \frac{(a_1)_{\kappa+n+\nu} (a_2)_\kappa}{(b_1)_\kappa (b_2 + \nu)_n} \frac{x^\kappa y^n (\xi st)^\nu}{\kappa! n! \nu\,!}. \tag{2.62}$$

Now, we use the operator

$$E_{a_1,a_2,b_1} = suv \frac{\partial}{\partial\, x}$$

with action

$$E_{a_1,a_2,b_1} \mathbf{\Psi}_{a_1,a_2,b_1,b_2} = \frac{a_1 a_2}{b_1} \mathbf{\Psi}_{a_1+1,a_2+1,b_1+1,b_2}.$$

To find $\exp(\xi E_{a_1,a_2,b_1})$, we employ the standard Lie theoretic approach, which involves solving the corresponding differential equations

$$\frac{dx(\xi)}{d\xi} = suv \ \text{ with } \ x(0) = x \Rightarrow x(\xi) = suv\xi + \mathfrak{c} \Rightarrow \mathfrak{c} = x, \ \therefore \ x(\xi) = x + suv\xi. \tag{2.63}$$

So, we have

$$\exp(\xi\ E_{a_1,a_2,b_1})\mathbf{\Psi}_{a_1,a_2,b_1,b_2} = \Psi_1(a_1, a_2; b_1, b_2; x + suv\xi, y) s^{a_1} u^{a_2} v^{b_1} t^{b_2}. \tag{2.64}$$

On the other hand, direct expansion allows us to achieve

$$\begin{aligned}\exp(\xi\ E_{a_1,a_2,b_1})\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= \sum_{\nu=0}^{\infty} \frac{\xi^\nu (a_1)_\nu (a_2)_\nu}{\nu\,!(b_1)_\nu} \mathbf{\Psi}_{a_1+\nu,a_2+\nu,b_1+\nu} \\ &= \sum_{\nu=0}^{\infty} \frac{\xi^\nu (a_1)_\nu (a_2)_\nu}{\nu\,!(b_1)_\nu} \Psi_1(a_1 + \nu, a_2 + \nu; b_1 + \nu, b_2; x, y) s^{a_1+\nu} u^{a_2+\nu} v^{b_1+\nu} t^{b_2},\end{aligned} \tag{2.65}$$

which gives the generating relation

$$\Psi_1(a_1, a_2; b_1, b_2; x + suv\xi, y) = \sum_{\nu=0}^{\infty} \frac{(a_1)_\nu (a_2)_\nu}{\nu\,!(b_1)_\nu} \Psi_1(a_1 + \nu, a_2 + \nu; b_1 + \nu, b_2; x, y)(\xi suv)^\nu. \tag{2.66}$$

By using (1.16) and (2.66), we get

$$\begin{aligned} \exp(\xi\, E_{a_1,a_2,b_1})\mathbf{\Psi}_{a_1,a_2,b_1,b_2} &= \sum_{\nu,\kappa,n=0}^{\infty} \frac{(a_1)_\nu (a_2)_\nu}{\nu\,!(b_1)_\nu} \frac{(a_1 + \nu)_{\kappa+n} (a_2 + \nu)_\kappa x^\kappa y^n}{\kappa! n! (b_1 + \nu)_\kappa (b_2)_n} s^{a_1+\nu} u^{a_2+\nu} v^{b_1+\nu} t^{b_2} \\ &= \sum_{\nu,\kappa,n=0}^{\infty} \frac{(a_1)_{\kappa+n+\nu} (a_2)_{\kappa+\nu} x^\kappa y^n}{\kappa! n! \nu\,!(b_1)_{\kappa+\nu} (b_2)_n} s^{a_1+\nu} u^{a_2+\nu} v^{b_1+\nu} t^{b_2}. \end{aligned} \tag{2.67}$$

So, we can write the generating relation

$$\Psi_1(a_1, a_2; b_1, b_2; x + suv\xi, y) = \sum_{\nu,\kappa,n=0}^{\infty} \frac{(a_1)_{\kappa+n+\nu} (a_2)_{\kappa+\nu}}{(b_1)_{\kappa+\nu} (b_2)_n} (\xi suv)^\nu \frac{x^\kappa y^n}{\kappa! n! \nu\,!}. \tag{2.68}$$

3. Conclusions and concluding observations

This study introduces a group-theoretic framework based on Lie algebra representations to develop a wide foundation and establish a broad class of generating relations for the Humbert function Ψ_1. The results have been applied to formulate E-operators and derive reduction formulas, thereby enhancing analytical methods for investigating Ψ_1. This approach uncovers the function's underlying algebraic symmetries and paves the way for diverse applications in mathematical physics, statistics, and engineering. The associations between the Humbert function and the representation theory of Lie groups and algebras point to suggest interesting future study and exploration in special function theory and symmetry-based mathematical approaches.

Author contributions

Ayman Shehata: conceptualization, methodology, formal analysis, validation, formal analysis, investigation, supervision, writing–original draft preparation, writing–review and editing; Nada Mostafa: formal analysis, validation, visualization, writing–review and editing; Mohamed M. Awad: validation, investigation, writing–review and editing, project administration, funding acquisition. All authors read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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