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Research article

A Jacobi–spectral framework for the heat equation with Dirichlet boundary conditions

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Abstract: We developed a Jacobi–spectral framework for the heat equation in a spherical domain under axial symmetry and Dirichlet boundary conditions. The angular part of the Laplacian was realized as a Jacobi Sturm–Liouville operator on a weighted L^2 space, enabling the Jacobi transform to diagonalize the angular component and project the partial diferential equation (PDE) onto a sequence of decoupled radial problems. Each projected equation reduced to a Euler-type radial ordinary diferential equation (ODE) driven by the corresponding Jacobi coefficient of the source term. These modal equations were solved in terms of spherical-Bessel eigenfunctions and radial Green kernels, yielding explicit Duhamel-type formulas for the time-dependent coefficients and establishing convergence in the weighted L^2 space. The Legendre case $(\alpha, \beta) = (0, 0)$ recovered the classical axisymmetric model, while general Jacobi parameters provided a unified extension of this setting. A central result was the demonstration of a rigorous equivalence between the Jacobi–spectral representation and the classical separation-of-variables solution written in spherical harmonics and spherical-Bessel modes. The proposed framework clarified the angular–radial coupling in spherical geometries and connected naturally with modern Jacobi and ultraspherical spectral methods.

Keywords: Jacobi transform; Jacobi spectral methods; heat equation; Dirichlet boundary conditions; Sturm–Liouville theory; spherical Bessel functions
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1. Introduction

Spectral methods based on orthogonal polynomials provide an effective analytical framework for partial differential equations posed on non-Euclidean domains or in weighted settings. On a finite interval, the Jacobi family forms a flexible basis: Varying the parameters $\alpha, \beta > -1$ recovers Legendre and Chebyshev polynomials as limiting cases while preserving orthogonality with respect to the weight

$$\omega_{\alpha,\beta}(x) = (1 - x)^{\alpha}(1 + x)^{\beta}, \tag{1.1}$$

ensuring completeness and enabling asymptotic estimates central to spectral approximation [1]. Jacobi polynomials also arise as solutions of classical Sturm–Liouville problems, yielding self-adjoint operators and a natural spectral structure [2].

In heat conduction problems under axial symmetry, the elliptic operator incorporates geometric factors that can be absorbed into a suitable weight after reparametrizing the radial variable on a finite interval. The Dirichlet problem can therefore be formulated as a self-adjoint spectral problem in a weighted L^2 space, where the Jacobi basis diagonalizes the angular component. This formulation is consistent with classical separation-of-variables methods and eigenfunction expansions [3, 4], and is related to Green's function representations and Poisson integral methods [5], as well as to L^p frameworks for Dirichlet problems [6].

Orthogonality in $L^2_{\omega}(a, b)$ is defined with respect to a nonnegative weight $\omega(x)$ and takes the form

$$\int_a^b P_n(x)\, P_m(x)\, \omega(x)\, dx = 0, \qquad n \neq m, \tag{1.2}$$

where P_n and P_m are polynomials of different degrees; see [1, Ch. II]. This property enables the construction of orthogonal bases and underlies spectral methods for the numerical solution of partial differential equations.

Eigenvalue problems and eigenfunction expansions play a central role in mathematical physics and engineering. In quantum mechanics, spectral decompositions of self-adjoint Hamiltonians determine admissible energy levels [7], while in data analysis they underpin the Karhunen–Loève paradigm [8]. A fundamental structural principle is symmetry: When the governing operator is invariant under a symmetry group, its eigenfunctions reflect that invariance. For the Dirichlet Laplacian on a spherical domain, this yields spherical harmonics Y_n^m as an angular eigenbasis; restricting to axisymmetric solutions corresponds to the $m = 0$ sector, that is, zonal harmonics proportional to the Legendre polynomials $P_n(\cos\theta)$ (the Jacobi case $(\alpha, \beta) = (0, 0)$. This mechanism is exploited here to decouple the heat equation into independent radial modes; see [9, 10].

The present work studies Jacobi orthogonal polynomials as an analytical tool for boundary value problems arising from heat conduction in spherical geometries under axial symmetry. The basic notation and structural properties are reviewed in Section 1.1. A classical separation-of-variables construction based on spherical harmonics and spherical Bessel modes is presented in Section 1.2 and serves as a reference representation.

The transform framework is introduced in Section 1.3, where polynomial transforms and Jacobi expansions are formulated in weighted L^2 spaces. The operator-theoretic foundations for the partial diferential equation (PDE) analysis, including commutation with time differentiation and diagonalization of the Jacobi Sturm–Liouville operator, are established in Section 2.1.1, and their relation to heat-type evolutions is discussed in Section 2.1.

Building on these results, Section 2.2 derives the Jacobi–spectral solution of the axisymmetric heat equation in the ball with Dirichlet boundary conditions: The substitution $x = \cos\theta$ transforms the angular Laplacian into the Legendre/Jacobi operator, the Jacobi transform decouples the angular dynamics, and each radial mode is solved using spherical Bessel eigenfunctions and Duhamel-type formulas, yielding the reconstruction formula (2.9). Finally, Section 2.3 proves the mode-by-mode equivalence between the Jacobi–spectral representation and the classical separation-of-variables solution.

1.1. Preliminaries

Throughout the paper we adopt the following notation and conventions. For parameters $\alpha,\beta > -1$, the Jacobi weight is given by (1.1), and the associated weighted space is $L^2_{\omega_{\alpha,\beta}}(-1,1) := L^2((-1,1), \omega_{\alpha,\beta}(x)\,dx)$. The corresponding inner product and norm are defined by $\langle f, g\rangle_{\omega_{\alpha,\beta}} = \int_{-1}^{1} f(x)\,\overline{g(x)}\,\omega_{\alpha,\beta}(x)\,dx$ and $\|f\|_{L^2_{\omega_{\alpha,\beta}}} = \langle f, f\rangle^{1/2}_{\omega_{\alpha,\beta}}$. Jacobi polynomials $P_n^{(\alpha,\beta)}$ are normalized according to (1.1) and the norms $h_n^{(\alpha,\beta)}$ in (1.21), following the conventions of Szegő [1] and the national institute of standards and technology (NIST) Handbook [11]. For $f \in L^2_{\omega_{\alpha,\beta}}(-1,1)$, the Jacobi transform is defined by $\widehat{f^{(\alpha,\beta)}}(n) = \int_{-1}^{1} f(x)\,P_n^{(\alpha,\beta)}(x)\,\omega_{\alpha,\beta}(x)\,dx = \mathcal{J}_{\alpha,\beta}(f)(n)$ for $n = 0, 1, 2, \ldots$.

As outlined in the Introduction, the analytical machinery underlying the Jacobi–spectral framework relies on several structural properties of classical orthogonal polynomials. These families may be constructed from a differential equation, a generating function, or directly from a Rodrigues-type representation, and they also satisfy a three-term recurrence relation; see [12, Section 4].

Consider the second-order differential equation $p(x)\,y''(x) + q(x)\,y'(x) + r\,y(x) = 0$, where $\deg p = 2$, $\deg q = 1$, and $r \in \mathbb{R}$. A sequence $\{P_n(x)\}_{n\geq 0}$ is called *classical* if each P_n admits a Rodrigues formula of the form

$$P_n(x) = \frac{B_n}{\omega(x)} \frac{d^n}{dx^n}[\rho^n(x)\,\omega(x)], \qquad n = 0, 1, 2, \ldots, \tag{1.3}$$

where B_n is a constant, $\rho(x)$ is a polynomial with $\deg\rho \leq 2$ independent of n, and $\omega(x) > 0$ is an integrable weight on (a, b) (cf. (1.2)). In particular, for Jacobi polynomials one recovers (1.1).

Substituting the appropriate weights and structural functions ρ into (1.3) produces the standard families of classical orthogonal polynomials:

1. *Jacobi family.* For $B_n = (-1)^n/(2^n n!)$, $\rho(x) = 1 - x^2$, and $\omega(x) = \omega_{\alpha,\beta}$, one obtains

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\Big[(1-x)^{n+\alpha}(1+x)^{n+\beta}\Big], \quad \alpha,\beta > -1.$$

   (a) Setting $\alpha = \beta = \lambda - \frac{1}{2}$ yields the *Gegenbauer* polynomials, for which

$$C_n^{(\lambda)}(x) = \frac{(-1)^n}{2^n n!}\frac{\Gamma(n+2\lambda)\Gamma(\lambda+\frac{1}{2})}{\Gamma(2\lambda)\Gamma(\lambda+n+\frac{1}{2})}(1-x^2)^{-\lambda+\frac{1}{2}}\frac{d^n}{dx^n}\Big[(1-x^2)^{n+\lambda-\frac{1}{2}}\Big], \quad \lambda > -\tfrac{1}{2}.$$

   (b) For $\alpha = \beta = 0$, one recovers the Legendre family:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n].$$

(c) For $\alpha = \beta = -\frac{1}{2}$, one obtains the Chebyshev polynomials of type I:

$$T_n(x) = \frac{1}{2^n n!}\sqrt{1-x^2}\,\frac{d^n}{dx^n}[(x^2-1)^{n-\frac{1}{2}}].$$

(d) For $\alpha = \beta = \frac{1}{2}$, one obtains the Chebyshev polynomials of type II:

$$U_n(x) = \frac{1}{2^n n!}\sqrt{1-x^2}\,\frac{d^n}{dx^n}[(x^2-1)^{n+\frac{1}{2}}].$$

(e) For $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}$, one obtains the type III Chebyshev polynomials:

$$V_n(x) = \frac{1}{2^n n!}\sqrt{\frac{1-x}{1+x}}\,\frac{d^n}{dx^n}\left[(1-x^2)^n\sqrt{\frac{1+x}{1-x}}\right].$$

(f) For $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$, one obtains the type IV Chebyshev polynomials:

$$W_n(x) = \frac{1}{2^n n!}\sqrt{\frac{1+x}{1-x}}\,\frac{d^n}{dx^n}\left[(1-x^2)^n\sqrt{\frac{1-x}{1+x}}\right].$$

2. *Associated Laguerre polynomials.* For $B_n = 1/n!$, $\rho(x) = x$, and $\omega(x) = e^{-x}x^{\alpha}$, one obtains

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!}\,\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}), \qquad \alpha > -1.$$

In particular, for $\alpha = 0$,

$$L_n(x) = \frac{e^x}{n!}\,\frac{d^n}{dx^n}(e^{-x}x^n).$$

3. *Hermite polynomials.* For $B_n = (-1)^n$, $\rho(x) = 1$, and $\omega(x) = e^{-x^2}$, one obtains

$$H_n(x) = (-1)^n e^{x^2}\,\frac{d^n}{dx^n}(e^{-x^2}).$$

Remark 1.1. *(Rodrigues convention and normalization) The constant B_n in* (1.3) *is not intrinsic; it depends on the choice of $\rho(x)$ and on the normalization of the orthogonal family. Here we follow the convention of Szegő, which, with $\rho(x) = 1 - x^2$, yields $B_n = (-1)^n/(2^n n!)$. Other references adopt $(x^2-1)^n$ instead of $(1-x^2)^n$, which leads to $B_n = 1/(2^n n!)$. For Chebyshev polynomials of the* third *and* fourth *kinds, it is common to introduce prefactors involving double factorials (e.g., $(2n-1)!!$) or to define V_n and W_n via trigonometric representations, absorbing these constants into the definition. See Mason and Handscomb [13, Ch. 1, §1.2.3; Ch. 4, §4.5] and the* NIST Handbook of Mathematical Functions *[11, Ch. 18, §18.3].*

In the rest of this work, all spectral identities, transform formulas and operator relations are written in terms of the Jacobi family $P_n^{(\alpha,\beta)}$ with the normalization fixed by (1.1) and (1.21). The Chebyshev, Gegenbauer, and Legendre cases are only used as limiting examples and may differ by nonzero multiplicative constants from other common conventions in the literature.

Beyond the orthogonality expressed in (1.2) for polynomial families, one may consider a system $\{\phi_n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ of (piecewise) continuous functions on a finite or infinite interval. The system is said to be orthogonal whenever condition (1.2) holds. Likewise, it is orthonormal if $\langle\phi_n,\phi_n\rangle_\omega = \int_a^b \omega(x)\,\phi_n(x)^2\,dx = 1$, for all n. The associated orthonormal family is given by $\|\phi_n\|_\omega = \left(\int_a^b \omega(x)\,\phi_n(x)^2\,dx\right)^{1/2}$, $\psi_n := \frac{\phi_n}{\|\phi_n\|_\omega}$, so that $\langle\psi_n,\psi_m\rangle_\omega = \delta_{nm}$; see [1, Ch. IV] and [12, §1].

1.2. Solution by separation of variables

In this section we solve the steady heat–conduction problem inside a sphere of radius $a > 0$ by means of an eigenfunction expansion associated with the Dirichlet Laplacian (equivalently, the Dirichlet Helmholtz eigenvalue problem); see [3, Chap. 11]. Throughout, we use spherical coordinates (r, θ, ϕ) with $0 < r < a$, $0 < \theta < \pi$, and $0 < \phi < 2\pi$.

Assume that the source term satisfies $F \in L^2(B_a)$ and, when nonhomogeneous Dirichlet data are prescribed, that $g \in L^2(\partial B_a)$ so that g admits an $L^2(\partial B_a)$-expansion in spherical harmonics. The separation-of-variables construction is justified by the spectral theory of the Dirichlet Laplacian on the ball: The Dirichlet realization $-\Delta_D$ on $L^2(B_a)$ is a nonnegative self-adjoint operator with compact resolvent and therefore has a purely discrete spectrum $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$ together with a complete orthonormal eigenbasis $\{\phi_k\}_{k\ge1} \subset L^2(B_a)$ [7, 9].

As a consequence, the homogeneous Dirichlet Poisson problem (1.7) admits a unique weak solution and the eigenfunction expansion yields the $L^2(B_a)$-convergent representations in (1.4) ,

$$u = \sum_{k\ge1} \frac{\langle F, \phi_k\rangle_{L^2(B_a)}}{\lambda_k}\, \phi_k, \qquad F = \sum_{k\ge1} \langle F, \phi_k\rangle_{L^2(B_a)}\, \phi_k. \tag{1.4}$$

with convergence in stronger norms under additional regularity assumptions on F, which justifies termwise application of Δ.

In the special case $\Omega = B_a$, the eigenfunctions factor into angular and radial components, and the eigenbasis may be chosen as products of spherical harmonics and spherical Bessel modes, as in the separable family $\{\Psi_{jnm}\}$ introduced after (1.5); see also [10]. For nonhomogeneous Dirichlet data, the standard splitting $u = u_Q + v$ with v solving (1.11) reduces the problem to a homogeneous Dirichlet component u_Q to which the same spectral representation applies.

Helmholtz eigenvalue problem. Consider the eigenvalue problem

$$\Delta\Psi + k^2\Psi = 0 \quad \text{in } B_a, \qquad \Psi|_{\partial B_a} = 0, \tag{1.5}$$

where $B_a = \{x \in \mathbb{R}^3 : |x| < a\}$ and where $k > 0$ is the separation constant. A complete orthogonal family of separable solutions is given by

$$\Psi_{jnm}(r, \theta, \phi) = j_n(k_{n,j} r)\, Y_n^m(\theta, \phi), \quad n = 0, 1, \ldots, \quad m = -n, \ldots, n, \quad j = 1, 2, \ldots,$$

where j_n denotes the spherical Bessel function of order n and Y_n^m the orthonormal spherical harmonics. The corresponding discrete values of the separation constant k are

$$k_{n,j} = \frac{\zeta_{n+1/2,j}}{a},$$

with $\zeta_{n+1/2,j}$ the j-th positive zero of $J_{n+\frac{1}{2}}$. The classical radial orthogonality relation reads

$$\int_0^a r^2\, j_n(k_{n,j} r)\, j_n(k_{n,\ell} r)\, dr = \frac{a^3}{2}\, j_{n+1}^2(\zeta_{n+1/2,j})\, \delta_{j\ell}. \tag{1.6}$$

Steady heat equation with source. Starting from $u_t = \kappa\Delta u + Q$ and seeking steady states ($u_t \equiv 0$), we obtain the Poisson problem

$$\Delta u = -F \quad \text{in } 0 < r < a, \qquad u(a, \theta, \phi) = 0, \tag{1.7}$$

with $F := Q/\kappa$. We expand u and F in the eigenbasis $\{\Psi_{jnm}\}$:

$$u(r,\theta,\phi) = \sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} B_{jnm}\, j_n(k_{n,j}r)\, Y_n^m(\theta,\phi), \tag{1.8}$$

$$F(r,\theta,\phi) = \sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} q_{jnm}\, j_n(k_{n,j}r)\, Y_n^m(\theta,\phi). \tag{1.9}$$

Since each basis function Ψ_{jnm} is a solution of the Helmholtz eigenvalue problem (1.5) with eigenvalue $k_{n,j}^2$, it satisfies $\Delta\Psi_{jnm} + k_{n,j}^2\Psi_{jnm} = 0$, and, hence, $\Delta\Psi_{jnm} = -k_{n,j}^2\Psi_{jnm}$. Therefore, substituting the expansions (1.8) and (1.9) into the Poisson Eq (1.7) and using the orthogonality of the family $\{\Psi_{jnm}\}$, we obtain $-k_{n,j}^2 B_{jnm} = -q_{jnm}$, which implies that $B_{jnm} = \frac{q_{jnm}}{k_{n,j}^2}$.

The coefficients q_{jnm} follow from orthogonal projection using (1.6):

$$q_{jnm} = \frac{2}{a^3\, j_{n+1}^2(\zeta_{n+1/2,j})}\int_0^a\int_0^\pi\int_0^{2\pi} F(r,\theta,\phi)\, j_n(k_{n,j}r)\, \overline{Y_n^m(\theta,\phi)}\, r^2 \sin\theta\, d\phi\, d\theta\, dr. \tag{1.10}$$

Harmonic problem with nonhomogeneous boundary data. For prescribed $g(\theta,\phi)$, the harmonic problem

$$\Delta v = 0 \quad \text{in } 0 < r < a, \qquad v(a,\theta,\phi) = g(\theta,\phi), \tag{1.11}$$

admits the expansion

$$v(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}\left(\frac{r}{a}\right)^n Y_n^m(\theta,\phi), \tag{1.12}$$

where the coefficients A_{nm} are given by

$$A_{nm} = \int_0^\pi\int_0^{2\pi} g(\theta,\phi)\, \overline{Y_n^m(\theta,\phi)}\, \sin\theta\, d\phi\, d\theta. \tag{1.13}$$

Combined representation. By superposition, the steady solution corresponding to source Q and boundary data g is

$$u(r,\theta,\phi) = \underbrace{\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{q_{jnm}}{k_{n,j}^2}\, j_n(k_{n,j}r)\, Y_n^m(\theta,\phi)}_{\text{source contribution, (1.10)}} + \underbrace{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}\left(\frac{r}{a}\right)^n Y_n^m(\theta,\phi)}_{\text{harmonic contribution, (1.13)}}. \tag{1.14}$$

Axially symmetric case. If g and Q are independent of ϕ, only $m = 0$ terms appear. Using the identity

$$Y_n^0(\theta) = \sqrt{\frac{2n+1}{4\pi}}\, P_n(\cos\theta),$$

the projection formulas simplify to

$$q_{jn} = \frac{2\sqrt{(2n+1)\pi}}{a^3\, j_{n+1}^2(\zeta_{n+1/2,j})}\int_0^a\int_0^\pi F(r,\theta)\, j_n(k_{n,j}r)\, P_n(\cos\theta)\, r^2 \sin\theta\, d\theta\, dr, \tag{1.15}$$

$$A_n = \frac{2n+1}{2}\int_0^\pi g(\theta)\, P_n(\cos\theta)\, \sin\theta\, d\theta. \tag{1.16}$$

Inserting (1.15) and (1.16) into (1.14) yields the axially symmetric solution.

Remark 1.2. *(Radial orthogonality for j_n) From [3, Section 4.8, Theorem 3], together with $j_n(x) = \sqrt{\pi/(2x)}\, J_{n+\frac{1}{2}}(x)$, one obtains the identity $\int_0^a r^2\, j_n\!\left(\frac{\zeta_{n,j}}{a} r\right) j_n\!\left(\frac{\zeta_{n,\ell}}{a} r\right) dr = \frac{a^3}{2}\, j_{n+1}^2(\zeta_{n,j})\, \delta_{j\ell}$. This shows that the orthogonality of spherical Bessel functions follows directly from the corresponding result for ordinary Bessel functions; see also [3, Section 4.8, Exercises].*

Example 1.3. *Temperature with boundary data $g(\theta) = 20(1 + \cos\theta)$, $Q \equiv 0$, and $a = 1$. Using* (1.16) *and the orthogonality of Legendre polynomials, we obtain*

$$A_0 = 20, \qquad A_1 = 20, \qquad A_n = 0 \quad (n \geq 2),$$

and, hence, by (1.12)*,*

$$u(r, \theta) = 20 + 20\, r \cos\theta.$$

Example 1.4. *(Radially weighted dipolar source) Retaining the boundary condition of Example 1.3 and taking $a = 1$, consider the source*

$$Q(r, \theta) = 80\, r^{\gamma} \cos\theta, \qquad \gamma := \frac{-1 + \sqrt{5}}{2} \in (0, 1).$$

Remark 1.5. *Here, γ is a free parameter that controls the radial intensity of the dipolar volumetric heating profile $Q(r, \theta) \propto r^{\gamma} \cos\theta$. The specific value $\gamma = (-1 + \sqrt{5})/2$ is chosen solely for illustration as a representative non-integer exponent, to emphasize that the derivation and modal projection do not rely on integer powers; no particular physical meaning is intended. In applied settings, γ can be selected from constitutive modeling or fitted from data. The same computation applies to general exponents under standard integrability/regularity assumptions on the source term (in particular, the radial projection integrals remain well-defined).*

For steady states, $\Delta u = -F$ with $F := Q/\kappa$. We fix $\kappa = 1$ (otherwise, q_{jn} scales by $1/\kappa$).
In axial symmetry ($m = 0$), projection of F onto the modes $j_n(k_{n,j} r) Y_n^0(\theta)$, with

$$k_{n,j} = \frac{\zeta_{n+1/2,j}}{a},$$

yields

$$q_{jn} = \frac{2\sqrt{(2n+1)\pi}}{j_{n+1}^2(\zeta_{n+1/2,j})} \int_0^1 \int_0^{\pi} F(r, \theta)\, j_n(k_{n,j} r)\, P_n(\cos\theta)\, r^2 \sin\theta\, d\theta\, dr.$$

Since

$$\int_0^{\pi} \cos\theta\, P_n(\cos\theta) \sin\theta\, d\theta = \int_{-1}^{1} x P_n(x)\, dx = \begin{cases} \dfrac{2}{3}, & n = 1, \\ 0, & n \neq 1, \end{cases}$$

it follows that

$$q_{j0} = 0, \qquad q_{jn} = 0 \ (n \geq 2), \qquad \text{only } n = 1 \text{ contributes.}$$

Thus,

$$q_{j1} = \frac{320\sqrt{3\pi}}{3\, j_2^2(\zeta_{3/2,j})} \int_0^1 r^{\gamma+2}\, j_1(\zeta_{3/2,j} r)\, dr. \tag{1.17}$$

and the integral converges absolutely (indeed for a wider range of exponents; in particular, for any $\gamma > -1$*) since* $j_1(x) \sim x/3$ *as* $x \to 0$.

The coefficients of the particular solution satisfy

$$B_{jn} = \frac{q_{jn}}{k_{n,j}^2} = \frac{q_{jn}}{\zeta_{n+1/2,j}^2}.$$

Hence, the source contribution is

$$u_Q(r,\theta) = \sqrt{\frac{3}{4\pi}}\,\cos\theta \sum_{j=1}^{\infty} \frac{q_{j1}}{\zeta_{3/2,j}^2}\, j_1(\zeta_{3/2,j} r). \tag{1.18}$$

The harmonic part u_H *(as in Example 1.3) is*

$$u_H(r,\theta) = 20 + 20\, r\cos\theta.$$

Hence, the full steady solution is

$$u(r,\theta) = 20 + 20\, r\cos\theta + \sqrt{\frac{3}{4\pi}}\,\cos\theta \sum_{j=1}^{\infty} \frac{q_{j1}}{\zeta_{3/2,j}^2}\, j_1(\zeta_{3/2,j} r).$$

Remark 1.6. *Using* $y = \zeta_{3/2,j} r$ *and the asymptotic* $j_1(y) = \sin y/y^2 - \cos y/y = O(1/y)$*, we obtain*

$$\int_0^1 r^{\gamma+2} j_1(\zeta_{3/2,j} r)\, dr = O(\zeta_{3/2,j}^{-1}),$$

so $q_{j1} = O(\zeta_{3/2,j}^{-1})$ *and* $B_{j1} = O(\zeta_{3/2,j}^{-3})$*. Thus, the series in* (1.18) *converges uniformly on* $[0, 1-\varepsilon]$ *for every* $\varepsilon \in (0,1)$*, and* u *is regular at* $r = 0$ *because* $j_1(x) \sim x/3$.

1.3. Classical polynomial transforms and the Jacobi case

On an interval (a,b) endowed with a weight function $\omega(x) > 0$, the *polynomial transform* of a function f with respect to a basis $\{P_n\}_{n\ge 0}$ is defined by the projection coefficients

$$\widehat{f}(n) \ := \ \langle f, P_n\rangle_\omega = \int_a^b f(x)\, P_n(x)\, \omega(x)\, dx, \tag{1.19}$$

and the inverse expansion takes the form

$$f(x) \ \sim \ \sum_{n=0}^{\infty} \frac{\widehat{f}(n)}{h_n} P_n(x), \qquad h_n := \langle P_n, P_n\rangle_\omega. \tag{1.20}$$

If $f \in L^2_\omega(a,b)$, Parseval's identity holds $\|f\|^2_{L^2_\omega} = \sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|^2}{h_n}$, and the series in (1.20) converges in L^2_ω (with pointwise convergence under additional regularity). See [1, Ch. IV], [11, 18.2 and 18.3], and [15, 16].

1.3.1. Equivalence with the Rodrigues kernel

As a natural continuation of the structural properties outlined in the previous subsection, the Rodrigues representation plays a central role in relating differential identities to the algebraic structure of spectral coefficients. For the classical families, the Rodrigues formula in (1.3) provides a differential representation of P_n, involving a polynomial $\rho(x)$ of degree at most two and suitable constants B_n.

If f is sufficiently regular—for example, if $f^{(n-1)} \in AC[a,b]$—and the boundary terms

$$[\rho^k(x)\,\omega(x)\,f^{(k-1)}(x)]_{x=a}^{x=b} = 0,\ k = 1,\dots,n,$$

vanish, then repeated integration by parts shows that the Rodrigues–kernel transform

$$\mathcal{T}_n\{f\} = \int_a^b f(x)\,\frac{B_n}{\omega(x)}\,\frac{d^n}{dx^n}[\rho^n(x)\,\omega(x)]\,dx,$$

coincides with the polynomial transform defined in (1.19). Hence, $\mathcal{T}_n\{f\} = \widehat{f}(n)$, which yields an alternative integral representation for the same spectral coefficient; see [11, 18.9.1].

1.3.2. Particular cases

Building on the abstract polynomial transform in (1.19) and (1.20) and on its equivalence with the Rodrigues kernel discussed above, we now specialize (a,b), the weight ω, the orthogonal polynomials P_n, and the norms h_n for the main classical families and their standard specializations. This will single out the Jacobi case as the natural setting for the angular part of the heat equation and place Laguerre and Hermite polynomials within the same unified framework.

- *Jacobi polynomials* $P_n^{(\alpha,\beta)}$, $\alpha,\beta > -1$:
  $(a,b) = (-1,1)$, $\omega_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $\mathcal{J}_{\alpha,\beta}(f)(n) := \widehat{f^{(\alpha,\beta)}}(n) = \int_{-1}^{1} f(x)\,P_n^{(\alpha,\beta)}(x)\,\omega_{\alpha,\beta}(x)\,dx$, and the squared norms are

  $$h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\,\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\,\Gamma(n+\alpha+\beta+1)}. \tag{1.21}$$

  Standard limiting cases include the Gegenbauer, Legendre, and Chebyshev families; see [11, 18.3–18.7] and [14, Ch. 4]. In particular, the choice $(\alpha,\beta) = (0,1)$ will be used later as the canonical Jacobi basis adapted to the angular weight $(1 + x)$ arising in the axisymmetric heat problem.
- *Laguerre polynomials* $L_n^{(\alpha)}$, $\alpha > -1$:
  $(a,b) = (0,\infty)$, $\quad \omega(x) = x^\alpha e^{-x}$, $\quad \widehat{f^{(\alpha)}}(n) = \int_0^\infty f(x)\,L_n^{(\alpha)}(x)\,x^\alpha e^{-x}\,dx$, $\quad h_n^{(\alpha)} = \Gamma(n+\alpha+1)/n!$, in accordance with [11, 18.18]. These weights are naturally associated with radial problems on $(0,\infty)$ and provide a convenient starting point for extensions to non-compact or semi-infinite geometries.
- *Hermite polynomials* H_n:
  $(a,b) = \mathbb{R}$, $\quad \omega(x) = e^{-x^2}$, $\quad \widehat{f_H}(n) = \int_{-\infty}^{\infty} f(x)\,H_n(x)\,e^{-x^2}\,dx$, $\quad h_n = 2^n n!\,\sqrt{\pi}$, as given in [11, 18.15]. The Hermite family underlies spectral representations on the whole real line and is particularly relevant for Gaussian-type diffusion and harmonic-oscillator models.

Remark 1.7. *(Normalization conventions) Normalization conventions vary across references, altering both h_n and the Rodrigues constants B_n in* (1.3)*. Throughout this work we adhere to the conventions of Szegő and the* NIST Handbook*. For Chebyshev polynomials of types I and II, see also [13, Ch. 1, Ch. 4]. In this way, the normalization used for the Jacobi family is fully consistent with the transform identities and with the spectral mapping properties employed later in the analysis of the heat equation.*

2. Results

2.1. Jacobi transform and its link to the heat problem

As discussed in Section 1.2, the separation of variables analysis in spherical coordinates leads, in the axially symmetric case, to an angular dependence expressed in terms of $x = \cos\theta \in (-1, 1)$. The natural weak formulation for the angular variable is then posed in the weighted space $L^2(-1, 1)$, which reflects the geometric factor $\sin\theta\, d\theta$ and is consistent with the Legendre-based representation in (1.15) and (1.16). In this setting, the Jacobi family with parameters $(\alpha,\beta) = (0, 0)$ provides the canonical orthogonal basis, in agreement with the specialization highlighted in Section 1.3.2.

Accordingly, for each fixed time t we define the Jacobi coefficients of an axially symmetric profile $v(\cdot, t)$ by

$$\widehat{v}_n(t) = \int_{-1}^{1} v(x, t)\, P_n^{(0,0)}(x)\, dx, \qquad v(x, t) \sim \sum_{n=0}^{\infty} \frac{\widehat{v}_n(t)}{h_n^{(0,0)}}\, P_n^{(0,0)}(x),$$

where $h_n^{(0,0)}$ is given by (1.21). Projecting the angular part of the heat equation onto the basis $\{P_n^{(0,0)}\}_{n\geq 0}$, and using the diagonalization property of the Jacobi operator (see Theorem 2.4 and Remark 1.1), yields a family of decoupled evolution equations for the modal coefficients of the form

$$\partial_t\widehat{v}_n(t) + \kappa\, \lambda_n\, \widehat{v}_n(t) = \widehat{f}_n(t),$$

where the constants λ_n encode the eigenvalues $n(n + \alpha + \beta + 1)$ in the Jacobi Sturm–Liouville problem and $\widehat{f}_n$ denotes the corresponding Jacobi coefficients of the source term. Each of these scalar ordinary diferential equation (ODEs) admits a Duhamel-type representation and is stable in L^2; see [3, Ch. 11] and [11, 18.3].

To illustrate the mechanism in a simpler setting, consider the one-dimensional heat equation on $I = (-1, 1)$:

$$\partial_t u(x, t) = \kappa\, u_{xx}(x, t), \qquad (x, t) \in I \times (0, \infty), \tag{2.1}$$

where u denotes the temperature and $\kappa > 0$ the thermal diffusivity, together with homogeneous Dirichlet boundary conditions given by

$$u(-1, t) = u(1, t) = 0, \qquad t > 0,$$

and an initial condition $u(x, 0) = u_0(x) \in L^2(I)$.

Our goal is to solve (2.1) through spectral decomposition using the Jacobi transform, understood as the projection of $u(\cdot, t)$ onto $P_n^{(\alpha,\beta)}$ with respect to the weight $(1 - x)^\alpha(1 + x)^\beta$ on $(-1, 1)$. In the uniform case $(\alpha,\beta) = (0, 0)$, one recovers the classical Legendre expansion. Other choices of (α,β) adapt the basis to geometrically induced weights or to nonuniform boundary conditions, in direct analogy with the uniform-weight setting arising in the axisymmetric spherical problem. In all cases, the Jacobi–spectral decomposition diagonalizes the spatial operator and reduces the PDE to a collection of scalar ODEs for the temporal coefficients, solvable via Duhamel's principle and guaranteeing existence, uniqueness, and stability in L^2 under standard assumptions; see [1, 3, 11].

Proposition 2.1. *(Jacobi weight integral (Beta function)) For $\alpha, \beta > -1$, the Jacobi weight satisfies*

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\, dx = 2^{\alpha+\beta+1}\, B(\alpha+1, \beta+1) = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

Proof. Setting $t = \frac{1-x}{2}$ reduces the integral to the classical Beta identity. □

Theorem 2.2. *(Jacobi series and inversion) If $f \in L^2_{\omega_{\alpha,\beta}}(-1,1)$, then its Jacobi expansion converges in $L^2_{\omega_{\alpha,\beta}}$,*

$$f(x) = \sum_{n=0}^{\infty} a_n\, P_n^{(\alpha,\beta)}(x) \quad \text{in } L^2_{\omega_{\alpha,\beta}}, \qquad a_n = \frac{\widehat{f}^{(\alpha,\beta)}(n)}{h_n^{(\alpha,\beta)}},$$

with $h_n^{(\alpha,\beta)}$ given by (1.21)*. If, moreover, f is continuous and satisfies a suitable Dini-type condition with respect to the weight, the convergence is pointwise on* $(-1,1)$*; see [11, 18.2 and 18.3] and [17, Ch. 7].*

Jacobi operator and spectral mapping. As established in Remark 1.1 and in the preceding discussion, all polynomial identities, transform formulas, and operator relations involving Jacobi polynomials are understood with the normalization fixed by (1.1) and (1.21). In particular, the spectral mapping identities for $\mathscr{L}_{\alpha,\beta}$ below, as well as Theorem 2.4, are to be interpreted within this convention.

Define the self-adjoint Jacobi operator by

$$\mathscr{L}_{\alpha,\beta} f(x) := -\frac{1}{\omega_{\alpha,\beta}(x)}\frac{d}{dx}\Big((1-x^2)\,\omega_{\alpha,\beta}(x)\, f'(x)\Big), \tag{2.2}$$

densely defined in $L^2_{\omega_{\alpha,\beta}}(-1,1)$ under the boundary conditions $(1\mp x)^{\alpha+1}(1\pm x)^{\beta+1} f'(x) \to 0$ as $x \to \pm 1$. With this convention, the Jacobi polynomials are eigenfunctions of $\mathscr{L}_{\alpha,\beta}$:

$$\mathscr{L}_{\alpha,\beta} P_n^{(\alpha,\beta)} = n(n+\alpha+\beta+1)\, P_n^{(\alpha,\beta)}, \qquad n = 0, 1, 2, \ldots,$$

and, therefore, for every f in the domain of $\mathscr{L}_{\alpha,\beta}$, $(\mathcal{J}_{\alpha,\beta}(\mathscr{L}_{\alpha,\beta} f))(n) = n(n+\alpha+\beta+1)\, \widehat{f}^{(\alpha,\beta)}(n)$. Thus, the Jacobi transform diagonalizes $\mathscr{L}_{\alpha,\beta}$ with eigenvalues $n(n+\alpha+\beta+1)$; see also Section 2.1.1. Commutation with time derivatives.

If $f(\cdot, t) \in L^2_{\omega_{\alpha,\beta}}$ and $\partial_t f(\cdot, t) \in L^2_{\omega_{\alpha,\beta}}$ for t in an interval, then, by Leibniz's rule, the Jacobi coefficients satisfy

$$\frac{d}{dt}\widehat{f}^{(\alpha,\beta)}(n,t) = \int_{-1}^{1} (\partial_t f)(x,t)\, P_n^{(\alpha,\beta)}(x)\, \omega_{\alpha,\beta}(x)\, dx = (\mathcal{J}_{\alpha,\beta}\, \partial_t f)(n,t). \tag{2.3}$$

Examples.

1. If f is a polynomial of degree m, then $\widehat{f}^{(\alpha,\beta)}(n) = 0$ for all $n > m$.
2. If $f(x) = P_n^{(\alpha,\beta)}(x)$, then $\widehat{f}^{(\alpha,\beta)}(m) = h_n^{(\alpha,\beta)}\, \delta_{mn}$.

3. Generating function. Let $R = \sqrt{1 - 2xz + z^2}$ with $|z| < 1$ and define ([11, 18.12.1]).

$$F_z(x) = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, z^n = 2^{\alpha+\beta} R^{-1} (1 - z + R)^{-\alpha} (1 + z + R)^{-\beta},$$

Applying $\mathcal{J}_{\alpha,\beta}$ termwise yields

$$\mathcal{J}_{\alpha,\beta}\{F_z\}(n) = h_n^{(\alpha,\beta)}\, z^n.$$

4. Transform of x^n. The leading coefficient of $P_n^{(\alpha,\beta)}$ equals $\frac{1}{2^n}\binom{2n+\alpha+\beta}{n}$ [11, 18.3.1], hence,

$$\mathcal{J}_{\alpha,\beta}\{x^n\}(n) = \frac{2^n}{\binom{2n+\alpha+\beta}{n}}\, h_n^{(\alpha,\beta)} = \frac{2^{n+\alpha+\beta+1}}{2n + \alpha + \beta + 1}\, \frac{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{\Gamma(2n + \alpha + \beta + 1)}.$$

5. Legendre moment. For $p > -1$, the weighted Legendre moment is given by

$$\int_{-1}^{1} (1 + x)^p\, P_n(x)\, dx = 2^{p+1} \frac{\Gamma(p + 1)^2}{\Gamma(p - n + 1)\,\Gamma(p + n + 2)}.$$

6. Moment-type transform. Let $\alpha, \beta > -1$ and $\Re\, \sigma > -1$. Then,

$$\begin{aligned}\mathcal{J}_{\alpha,\beta}\{(1 - x)^{\sigma-\alpha}\}(n) &= \int_{-1}^{1} (1 - x)^{\sigma} (1 + x)^{\beta} P_n^{(\alpha,\beta)}(x)\, dx \\ &= \frac{2^{\sigma+\beta+1}\Gamma(\sigma + 1)\Gamma(n + \beta + 1)\Gamma(\alpha - \sigma + n)}{n!\,\Gamma(\alpha - \sigma)\,\Gamma(\beta + \sigma + n + 2)}.\end{aligned}$$

Proof. Use the hypergeometric representation of $P_n^{(\alpha,\beta)}$, set $t = \frac{1-x}{2}$, apply Euler's moment identity for ${}_2F_1$, and then invoke Saalschütz's theorem. □

Remark 2.3. *Analytic continuation extends the result to* $\alpha - \sigma \in \{0, -1, -2, \dots\}$*, recovering the orthogonality case: If* $\sigma = \alpha$*, the integral vanishes for* $n \geq 1$ *and equals* $2^{\alpha+\beta+1} B(\alpha + 1, \beta + 1)$ *for* $n = 0$*.*

7. *Cross-parameter transform.* Let $\alpha, \beta, \sigma > -1$ and $m, n \in \mathbb{N}_0$. Then,

$$\begin{aligned}\mathcal{J}_{\alpha,\beta}\Big\{(1 + x)^{\sigma-\beta} P_m^{(\alpha,\sigma)}(x)\Big\}(n) &= \int_{-1}^{1} (1 - x)^{\alpha} (1 + x)^{\sigma} P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\sigma)}(x)\, dx \\ &= \begin{cases} C_{n,m}^{(\alpha;\beta\to\sigma)}\, h_m^{(\alpha,\sigma)}, & m \leq n, \\ 0, & m > n, \end{cases}\end{aligned}$$

where $h_m^{(\alpha,\sigma)}$ is given by (1.21), and

$$C_{n,m}^{(\alpha;\beta\to\sigma)} = \frac{(\beta - \sigma)_{n-m}\, (\alpha + \sigma + m + 1)_{n-m}}{(n - m)!\, (\alpha + m + 1)_{n-m}}.$$

Proof. Use the parameter-connection formula ([11, 18.18.22]).

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(\beta - \sigma)_{n-k}(\alpha + \sigma + k + 1)_{n-k}}{(n - k)!(\alpha + k + 1)_{n-k}}\, P_k^{(\alpha,\sigma)}(x),$$

Taking the inner product with $P_m^{(\alpha,\sigma)}$ yields $C_{n,m}$; multiplying by $h_m^{(\alpha,\sigma)}$ gives the stated coefficient. □

2.1.1. Properties of the Jacobi transform

Before applying the Jacobi transform to the heat equation, we fix the functional framework and two structural properties that will be used repeatedly. We work in the weighted space $L^2_{\omega_{\alpha,\beta}}(-1,1)$ with weight defined in (1.1) and adopt the classical normalization for $P_n^{(\alpha,\beta)}$; see [1, 11]. Within this setting we establish: (i) A commutation rule between the transform and time differentiation under minimal regularity in t, and (ii) the diagonalization of the Jacobi Sturm–Liouville operator, which yields the eigenvalues $n(n+\alpha+\beta+1)$.

The boundary conditions that ensure the cancellation of integration–by–parts terms are stated explicitly, guaranteeing the validity of the operator identities. These results decouple the thermal dynamics into a family of scalar ODEs for the spectral coefficients and underpin the Duhamel formulas and stability bounds derived later.

Jacobi differential operator and its spectral property.

Define the Sturm–Liouville operator

$$R_{\alpha,\beta}[f](x) := (1-x)^{-\alpha}(1+x)^{-\beta}\,\frac{d}{dx}\Big((1-x)^{\alpha+1}(1+x)^{\beta+1}f'(x)\Big), \tag{2.4}$$

with domain

$$\mathcal{D}(R) := \Big\{f:\ (1-x^2)f' \in AC_{\text{loc}}(-1,1),\ R[f]\in L^2_{\omega_{\alpha,\beta}},\ \lim_{x\to\pm1^{\mp}}(1\mp x)^{\alpha+1}(1\pm x)^{\beta+1}f'(x)=0\Big\}. \tag{2.5}$$

Observe that $R=-\mathscr{L}_{\alpha,\beta}$, where $\mathscr{L}_{\alpha,\beta}$ is the self-adjoint Jacobi operator introduced in (2.2); see [11, 18.9.1].

Theorem 2.4. *(Diagonalization via the Jacobi transform) For every $f\in\mathcal{D}(R)$ and all $n\geq 0$,*

$$(\mathcal{J}_{\alpha,\beta}\{R[f]\})(n) = -\,n(n+\alpha+\beta+1)\,\widehat{f^{(\alpha,\beta)}}(n). \tag{2.6}$$

Proof. Using (2.4) and performing two integrations by parts (the boundary terms vanish due to (2.5)),

$$\begin{aligned}
\mathcal{J}_{\alpha,\beta}\{R[f]\}(n) &= \int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\,\frac{d}{dx}\Big((1-x^2)\omega_{\alpha,\beta}(x)f'(x)\Big)\,\omega_{\alpha,\beta}(x)^{-1}dx\\
&= -\int_{-1}^{1}(1-x^2)\omega_{\alpha,\beta}(P_n^{(\alpha,\beta)})'(x)f'(x)\,dx\\
&= \int_{-1}^{1}\Big((1-x^2)(P_n^{(\alpha,\beta)})'' + [\beta-\alpha-(\alpha+\beta+2)x](P_n^{(\alpha,\beta)})'\Big)f(x)\,\omega_{\alpha,\beta}(x)\,dx.
\end{aligned}$$

The Jacobi differential equation, $(1-x^2)y''+[\beta-\alpha-(\alpha+\beta+2)x]y'+n(n+\alpha+\beta+1)y=0$, implies the desired identity. □

Remark 2.5. *Working directly, by* (2.6)*, with $\mathscr{L}_{\alpha,\beta}=-R$ yields $(\mathcal{J}_{\alpha,\beta}\{\mathscr{L}_{\alpha,\beta}f\})(n)=n(n+\alpha+\beta+1)\,\widehat{f^{(\alpha,\beta)}}(n)$. Under the boundary conditions in* (2.5)*, the operator $\mathscr{L}_{\alpha,\beta}$ is symmetric and positive in $L^2_{\omega_{\alpha,\beta}}$ with dense domain; see [1, Ch. IV] and [11, 18.9].*

2.2. Solution of the heat equation via the Jacobi transform

The relation between Jacobi polynomials and the heat equation under Dirichlet boundary conditions follows from the spectral analysis of the angular component of the spherical Laplacian. The substitution $x = \cos\theta$ transforms the angular operator into a one–dimensional Sturm–Liouville operator on $(-1, 1)$; in the standard axisymmetric setting this reduces to the Legendre operator, corresponding to the choice $(\alpha,\beta) = (0, 0)$ within the Jacobi family.

We formulate the angular analysis through the Jacobi operator $R_{\alpha,\beta}$ defined in (2.4), whose eigenfunctions are the polynomials $P_n^{(\alpha,\beta)}$ with eigenvalues $-n(n+\alpha+\beta+1)$ (Theorem 2.4). Projection of the heat equation onto this basis shows that the Jacobi transform $\mathcal{J}_{\alpha,\beta}$ diagonalizes the angular component and reduces the problem to a countable family of radial Bessel–type evolution equations, each satisfying the Dirichlet condition at $r = a$.

The corresponding radial eigenfunctions, given by spherical Bessel functions vanishing at $r = a$, together with Duhamel's principle, determine the modal coefficients $c_{n,j}(t)$. The full solution is recovered through the Jacobi inversion formula (2.9), with convergence in the appropriate weighted L^2 spaces. Finally, Theorem 2.7 establishes that this Jacobi–spectral representation coincides, mode by mode, with the classical separation–of–variables solution of Section 1.2.

We now apply the Jacobi spectral framework to the axisymmetric heat equation inside a spherical domain of radius $a > 0$.

$$\partial_t u = \kappa\Big(u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}(u_{\theta\theta} + \cot\theta\, u_\theta)\Big) + Q(r,\theta,t), \qquad 0 < r < a,\ 0 < \theta < \pi,\ t > 0, \tag{2.7}$$

with Dirichlet boundary condition

$$u(a,\theta,t) = g(\theta,t), \qquad 0 < \theta < \pi,\ t > 0,$$

regularity at $r = 0$, and initial condition $u(r,\theta,0) = u_0(r,\theta)$. The parameter $\kappa > 0$ denotes the thermal diffusivity. Angular change of variables and the Jacobi operator. Set $x = \cos\theta \in (-1, 1)$. A direct computation shows that

$$u_{\theta\theta} + \cot\theta\, u_\theta = \frac{\partial}{\partial x}\Big[(1 - x^2)\, u_x\Big] =: R_{0,0}[u].$$

the Jacobi Sturm–Liouville operator with $(\alpha,\beta) = (0, 0)$. For general $\alpha,\beta > -1$,

$$R_{\alpha,\beta}[v] := (1 - x)^{-\alpha}(1 + x)^{-\beta}\frac{d}{dx}\Big((1 - x)^{\alpha+1}(1 + x)^{\beta+1}v_x\Big),$$

and $R_{0,0} = \partial_x[(1 - x^2)u_x]$. Thus, (2.7) becomes

$$\partial_t u = \kappa\left(u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}R_{0,0}[u]\right) + Q(r,x,t), \qquad (r,x,t) \in (0,a) \times (-1,1) \times (0,\infty). \tag{2.8}$$

Jacobi transform in x and modal decoupling. For (α,β) arbitrary, define the Jacobi coefficients

$$\widehat{u}_n^{(\alpha,\beta)}(r,t) := \int_{-1}^{1} u(r,x,t)\, P_n^{(\alpha,\beta)}(x)\,(1 - x)^\alpha(1 + x)^\beta\, dx, \qquad n = 0, 1, \ldots.$$

The physical axisymmetric case corresponds to $(\alpha,\beta) = (0,0)$. Using the commutation property (2.3) and the diagonalization in Theorem 2.4, we obtain

$$\partial_t \widehat{u}_n^{(\alpha,\beta)} = \kappa\left(\partial_{rr}\widehat{u}_n^{(\alpha,\beta)} + \frac{2}{r}\partial_r\widehat{u}_n^{(\alpha,\beta)} - \frac{n(n+\alpha+\beta+1)}{r^2}\widehat{u}_n^{(\alpha,\beta)}\right) + \widehat{Q}_n^{(\alpha,\beta)},$$

where

$$\widehat{Q}_n^{(\alpha,\beta)}(r,t) = \int_{-1}^{1} Q(r,x,t)\, P_n^{(\alpha,\beta)}(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx.$$

The boundary condition projects as

$$\widehat{u}_n^{(\alpha,\beta)}(a,t) = \widehat{g}_n^{(\alpha,\beta)}(t), \qquad \widehat{g}_n^{(\alpha,\beta)}(t) := \int_{-1}^{1} g(x,t)\, P_n^{(\alpha,\beta)}(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx.$$

Homogenization and Duhamel's principle. Write $u = v + w$ with v solving the stationary harmonic problem

$$\Delta v = 0 \quad \text{in } 0 < r < a, \qquad v(a,x) = g(x,t),$$

so that $w(a,x,t) = 0$. Then,

$$\partial_t w = \kappa\left(w_{rr} + \frac{2}{r}w_r + \frac{1}{r^2}R_{0,0}[w]\right) + \widetilde{Q}, \qquad \widetilde{Q} := Q - \partial_t v.$$

Projecting,

$$\partial_t \widehat{w}_n^{(\alpha,\beta)} = \kappa\left(\partial_{rr}\widehat{w}_n^{(\alpha,\beta)} + \frac{2}{r}\partial_r\widehat{w}_n^{(\alpha,\beta)} - \frac{n(n+\alpha+\beta+1)}{r^2}\widehat{w}_n^{(\alpha,\beta)}\right) + \widehat{\widetilde{Q}}_n^{(\alpha,\beta)}, \qquad \widehat{w}_n^{(\alpha,\beta)}(a,t) = 0.$$

Radial eigenfunctions and spherical–Bessel expansion. For each n, the radial operator

$$\mathcal{L}_n^{(\alpha,\beta)} := \partial_{rr} + \frac{2}{r}\partial_r - \frac{n(n+\alpha+\beta+1)}{r^2}$$

admits eigenfunctions $j_{\nu_n}(\kappa r)$, with

$$\nu_n := -\frac{1}{2} + \sqrt{n(n+\alpha+\beta+1) + \frac{1}{4}}\,.$$

Dirichlet at $r = a$ yields $\kappa_{n,j} = \alpha_{\nu_n+1/2,j}/a$, where $\alpha_{\nu,j}$ is the jth positive zero of J_ν. The normalized eigenfunctions

$$\varphi_{n,j}(r) := \frac{j_{\nu_n}(\kappa_{n,j}r)}{\|j_{\nu_n}(\kappa_{n,j}\cdot)\|_{L^2(r^2dr)}}$$

form an orthogonal basis in $(0,a)$ with respect to the weight r^2, and generate the semigroup

$$e^{t\kappa\mathcal{L}_n^{(\alpha,\beta)}} f(r) = \sum_{j=1}^{\infty} e^{-\kappa\kappa_{n,j}^2 t}\,\langle f,\varphi_{n,j}\rangle_r\,\varphi_{n,j}(r).$$

Duhamel formula. The modal coefficients satisfy

$$\frac{d}{dt}c_{n,j}(t) + \kappa\kappa_{n,j}^2 c_{n,j}(t) = \langle \widehat{\widetilde{Q}}_n^{(\alpha,\beta)}(\cdot,t), \varphi_{n,j}\rangle_r, \qquad c_{n,j}(0) = \langle \widehat{w}_n^{(\alpha,\beta)}(\cdot,0), \varphi_{n,j}\rangle_r,$$

and, hence,

$$c_{n,j}(t) = e^{-\kappa\kappa_{n,j}^2 t}\, c_{n,j}(0) + \int_0^t e^{-\kappa\kappa_{n,j}^2(t-s)}\, \langle \widehat{\widetilde{Q}}_n^{(\alpha,\beta)}(\cdot,s), \varphi_{n,j}\rangle_r\, ds.$$

Reconstruction of the solution. The complete solution is

$$u(r,x,t) = v(r,x) + \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\beta)}(x)}{h_n^{(\alpha,\beta)}} \sum_{j=1}^{\infty} c_{n,j}(t)\, \varphi_{n,j}(r), \tag{2.9}$$

where $h_n^{(\alpha,\beta)}$ is given in (1.21).

Remark 2.6. *In the physical model* (2.8)*, the operator* $R_{0,0}$ *corresponds to* $(\alpha,\beta) = (0,0)$ *(Legendre case). If an angular drift modifies the operator to* $R_{\alpha,\beta}$ *with* $\alpha,\beta \neq 0$*, this adds the first-order term* $[(\beta-\alpha) - (\alpha+\beta+2)x]\frac{u_x}{r^2}$ *to* (2.8)*, or equivalently incorporates it into* Q*. The derivation above remains valid for general* (α,β)*; the physical case is simply* $(0,0)$.

We now consider the time–independent boundary case. When the boundary data does not depend on time, the solution naturally decomposes into a stationary component determined by the harmonic lifting of the boundary condition and a transient component satisfying homogeneous Dirichlet conditions. The stationary part corresponds to the steady regime of the reconstruction formula (2.9) and is obtained from the spectral resolution of the associated harmonic problem in the ball.

If g is independent of t, the stationary component is

$$v(r,x) = \sum_{n=0}^{\infty} \frac{\widehat{g}_n^{(\alpha,\beta)}}{h_n^{(\alpha,\beta)}} \left(\frac{r}{a}\right)^{\frac{-1+\lambda_n}{2}} P_n^{(\alpha,\beta)}(x), \qquad \lambda_n := \sqrt{1 + 4n(n+\alpha+\beta+1)}. \tag{2.10}$$

This representation provides the Jacobi–spectral description of the steady solution associated with the Dirichlet boundary data. In the following subsection we establish that this stationary Jacobi representation coincides, mode by mode, with the classical separation–of–variables expansion obtained in Section 1.2, thereby proving the equivalence between both constructions.

2.3. Equivalence of the solutions: separation of variables vs. Jacobi transform

Theorem 2.7. *(Equivalence of the separation-of-variables and Jacobi-transform representations) Let* u *be the stationary solution of* $-\Delta u = Q$ *in the ball* B_a *with Dirichlet boundary data* $u|_{\partial B_a} = g$*. Then, the representation obtained by separation of variables in Section 1.2 (see* (1.14)*) coincides, term by term, with the representation obtained via the Jacobi transform in Section 2.2 (see the reconstruction scheme* (2.9) *in the steady regime).*

Proof. We first compare the angular dependence. Since u satisfies

$$\partial_r(r^2 u_r) + \partial_x\Big[(1-x^2)\,\omega_{\alpha,\beta}(x)\,u_x\Big]\frac{1}{\omega_{\alpha,\beta}(x)} \equiv_{\alpha=0,\beta=0} \partial_r(r^2 u_r) + \partial_x[(1-x^2)u_x],$$

the angular dependence can be expanded in the Jacobi basis as

$$u(r,x) = \sum_{m=0}^{\infty} U_m(r)\, P_m^{(\alpha,\beta)}(x), \tag{2.11}$$

where in the physical case $(\alpha,\beta) = (0,0)$, one recovers the Legendre basis.

Using the self-adjoint Jacobi identity

$$\frac{d}{dx}\Big[(1-x^2)\omega_{\alpha,\beta}(x)y'(x)\Big] = -m(m+\alpha+\beta+1)\,\omega_{\alpha,\beta}(x)y(x) \qquad (y = P_m^{(\alpha,\beta)}),$$

multiplying by $\omega_{\alpha,\beta}(x)P_n^{(\alpha,\beta)}(x)$, and integrating over $(-1,1)$ gives, by orthogonality,

$$\int_{-1}^{1}(1-x^2)u_{xx}\,P_n^{(\alpha,\beta)}\omega_{\alpha,\beta}\,dx = -n(n+\alpha+\beta+1)\,U_n(r)\,h_n^{(\alpha,\beta)}.$$

Hence, U_n satisfies the radial ODE

$$r^2U_n''(r) + 2rU_n'(r) - n(n+\alpha+\beta+1)\,U_n(r) = 0.$$

A power-law ansatz $U_n(r) = r^m$ yields

$$m^2 + m - n(n+\alpha+\beta+1) = 0, \qquad m = \frac{-1\pm\lambda_n}{2}, \quad \lambda_n := \sqrt{1+4\,n(n+\alpha+\beta+1)}.$$

Thus,

$$U_n(r) = A_n\, r^{\frac{-1+\lambda_n}{2}},$$

since regularity at $r = 0$ forces the removal of the singular branch. Writing

$$U_n(r) = C_n\Big(\frac{r}{a}\Big)^{\frac{-1+\lambda_n}{2}}, \qquad C_n := A_n\, a^{\frac{-1+\lambda_n}{2}}, \tag{2.12}$$

and imposing $u(a,x) = g(x)$ in (2.11), we obtain

$$g(x) = \sum_{m=0}^{\infty} C_m\, P_m^{(\alpha,\beta)}(x),$$

so projection gives

$$C_n = \frac{\widehat{g}^{(\alpha,\beta)}(n)}{h_n^{(\alpha,\beta)}}.$$

Substituting into (2.12),

$$U_n(r) = \frac{\widehat{g}^{(\alpha,\beta)}(n)}{h_n^{(\alpha,\beta)}}\left(\frac{r}{a}\right)^{\frac{-1+\lambda_n}{2}},$$

and, therefore,

$$u(r,x) = \sum_{m=0}^{\infty}\frac{\widehat{g}^{(\alpha,\beta)}(m)}{h_m^{(\alpha,\beta)}}\left(\frac{r}{a}\right)^{\frac{-1+\lambda_m}{2}} P_m^{(\alpha,\beta)}(x),$$

which agrees with the stationary Jacobi-transform representation (2.10).

The stationary radial equation is of Euler–Cauchy type, and its regular solutions near $r = 0$ are necessarily of power type. More precisely, the associated indicial equation yields two characteristic exponents, and regularity at the origin selects the admissible exponent $m_{n,+}$. The Dirichlet boundary condition at $r = a$ uniquely determines the radial profile in terms of its boundary value, leading to the representation

$$U_n(r) = U_n(a)\,(r/a)^{m_{n,+}},$$

which establishes (2.12).

Incorporation of the source term. In the stationary problem the quantities depend only on the radial variable r. We therefore define the Jacobi projections

$$U_n(r) := \int_{-1}^{1} \omega_{\alpha,\beta}(x)\, u(r,x)\, P_n^{(\alpha,\beta)}(x)\, dx, \qquad \widehat{Q}_n(r) := \int_{-1}^{1} \omega_{\alpha,\beta}(x)\, Q(r,x)\, P_n^{(\alpha,\beta)}(x)\, dx.$$

Jacobi inversion then yields the representation

$$u(r,x) = \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\beta)}(x)}{h_n^{(\alpha,\beta)}}\, U_n(r), \qquad U_n(r) = \sum_{j=1}^{\infty} c_{n,j}\, \varphi_{n,j}(r),$$

where the radial eigenfunctions $\varphi_{n,j}$ satisfy

$$\varphi'' + \frac{2}{r}\varphi' - \frac{n(n+\alpha+\beta+1)}{r^2}\varphi + k_{n,j}^2\varphi = 0, \qquad \varphi_{n,j}(a) = 0.$$

Projecting the stationary radial equation leads to

$$U_n'' + \frac{2}{r}U_n' - \frac{n(n+\alpha+\beta+1)}{r^2}U_n = -\,\widehat{Q}_n(r), \qquad U_n(a) = 0.$$

Multiplying by $\varphi_{n,j}(r)\, r^2$, integrating over $(0,a)$, and using Lagrange's identity yields

$$k_{n,j}^2 \int_0^a U_n(r)\varphi_{n,j}(r) r^2\, dr = \int_0^a \widehat{Q}_n(r)\varphi_{n,j}(r) r^2\, dr.$$

With $U_n = \sum_{j\geq 1} c_{n,j}\varphi_{n,j}$ and by orthogonality,

$$c_{n,j} = \frac{q_{jn}^{(\alpha,\beta)}}{k_{n,j}^2\, \|\varphi_{n,j}\|_r^2},$$

where

$$q_{jn}^{(\alpha,\beta)} := \int_0^a \int_{-1}^{1} \omega_{\alpha,\beta}(x)\, Q(r,x)\, P_n^{(\alpha,\beta)}(x)\, \varphi_{n,j}(r)\, r^2\, dx\, dr.$$

Thus,

$$\begin{aligned} u(r,x) &= \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\beta)}(x)}{h_n^{(\alpha,\beta)}} \sum_{j=1}^{\infty} c_{n,j}\, \varphi_{n,j}(r) \\ &= \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \frac{q_{jn}^{(\alpha,\beta)}}{k_{n,j}^2} \left[\frac{\varphi_{n,j}(r)}{\|\varphi_{n,j}\|_r^2} \frac{P_n^{(\alpha,\beta)}(x)}{h_n^{(\alpha,\beta)}} \right], \end{aligned}$$

which matches, mode by mode, the stationary Jacobi-transform reconstruction. This establishes the equivalence. □

3. Discussion

Beyond the structural results obtained here, the Jacobi–spectral viewpoint clarifies the interaction between angular and radial operators in spherical geometries and provides a systematic framework for incorporating weights and geometric factors into the functional setting. In particular, the weighted $L^2_{\omega_{\alpha,\beta}}(-1,1)$ formulation makes explicit how the angular Sturm–Liouville structure and normalization constants enter the projection and inversion procedures, while preserving a direct correspondence with classical modal expansions.

From a computational perspective, the resulting representation is compatible with low-complexity implementations, where modern fast Jacobi transforms and ultraspherical techniques may be combined with the analytical formulas derived here; see, e.g., [15, 18–21].

Several directions for future research arise naturally from the present framework:

- *Variable coefficients and nonautonomous forcing.* Treatment of time-dependent sources and spatially varying thermal coefficients, where ultraspherical regularizations and well-conditioned formulations may enhance stability and refine a priori estimates [20, 22].
- *Alternative boundary conditions and non-spherical geometries.* Extension to Robin or Neumann boundary conditions and to non-spherical domains, for instance, via ultraspherical spectral elements, domain decomposition methods, or embedding techniques that preserve the Jacobi–spectral structure at the local level [21].
- *Removal of axial symmetry.* Analysis of the full non-axisymmetric case ($m \neq 0$), exploiting the representation of Y_n^m in terms of associated Jacobi (Gegenbauer) polynomials together with fast multidimensional Jacobi transforms [18, 19, 23].
- *Fractional, inverse, and stochastic models.* Application of the Jacobi–spectral framework to fractional diffusion, inverse problems, and parametric or stochastic models, where Jacobi bases provide flexible regularization mechanisms and enable efficient transform-based schemes [15].
- *Nonstandard orthogonality: Jacobi–Sobolev settings.* Extension of the Jacobi transform to nonstandard orthogonality structures, including Sobolev-type inner products. In particular, it is of interest to develop transform formulations associated with Jacobi–Sobolev polynomials and to analyze their structural and spectral properties, following recent developments on their differential characterization and electrostatic interpretation [24].
- *Perturbed spectral problems and Schrödinger-type operators.* Study of heat-type equations under alternative boundary conditions and modified functional settings, together with perturbed spectral problems associated with Schrödinger-type operators and nonclassical potentials. These directions relate to recent developments on classical orthogonal polynomials in quantum-mechanical models and spectral representations of Hamiltonian operators; see [25, 26]. In this context, Jacobi-based representations may provide a natural analytical framework for eigenvalue problems, spectral decompositions, and evolution operators.

The proposed formulation provides a symmetry-adapted diagonalization of the angular operator within a weighted Sturm–Liouville framework, explicit mode-by-mode solution formulas with direct enforcement of the Dirichlet condition at $r = a$, and a rigorous equivalence between the Jacobi–spectral construction and the classical eigenfunction expansion, while remaining compatible with modern fast Jacobi and ultraspherical computational techniques [15, 18–21]. The present results are established under spherical geometry, axial symmetry ($m = 0$), and Dirichlet boundary conditions; extensions to more general configurations, including non-axisymmetric settings, alternative boundary conditions, or non-spherical domains, may require additional analytical structures or numerical eigenmode approaches. Furthermore, stronger convergence properties and termwise differentiability typically depend on higher regularity assumptions on the data.

4. Conclusions

The present work develops a Jacobi–spectral framework for the inhomogeneous heat equation in spherical coordinates under axial symmetry and Dirichlet boundary conditions. The analysis is formulated in the weighted space $L^2_{\omega_{\alpha,\beta}}(-1,1)$ and is based on a Sturm–Liouville realization of the angular component of the Laplacian in terms of a Jacobi operator.

Within this setting, the Jacobi transform diagonalizes the angular component and yields a systematic decoupling of the angular and radial dynamics.

Main contributions.

1. We formulate the angular component of the Laplacian as a Jacobi-type self-adjoint operator $\mathscr{L}_{\alpha,\beta}$ on $L^2_{\omega_{\alpha,\beta}}(-1,1)$ and identify boundary conditions ensuring symmetry and positivity, leading to a diagonal angular structure in the Jacobi basis $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$.
2. The heat equation reduces to a countable family of radial mode equations after projection onto the Jacobi basis, each inheriting the Dirichlet condition at $r = a$ and admitting explicit representations in terms of spherical Bessel eigenfunctions and radial Green kernels.
3. We establish a rigorous equivalence between the Jacobi–spectral representation and the classical separation-of-variables solution expressed in spherical harmonics and spherical Bessel modes.

Author contributions

Juan Toribio Milane: Conceptualization, methodology, formal analysis, investigation, writing–original draft, supervision, project administration; José A. Gómez Hernández: Methodology, formal analysis, validation, writing–review & editing; Pedro N. Tifa de Jesús: Investigation, validation, writing–review & editing; Juan R. Holguín: Validation, investigation, technical support. All authors contributed to the development of the research, the analysis and interpretation of the results, and have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

References

1. G. Szegő, *Orthogonal polynomials*, American Mathematical Society, **23** 1939.
2. A. P. Agudelo, J. G. López, Ecuación diferencial asociada a los polinomios ortogonales clásicos, *Sci. Tech.*, **10** (2004), 179–184. Available from: http://www.redalyc.org/articulo.oa?id=84911640031.
3. N. Asmar, *Partial differential equations*, 2 Eds., Pearson Prentice Hall, 2004.
4. M. Tyn, L. Debnath, *Linear partial differential equations for scientists and engineers*, 4 Eds., Boston: Birkhäuser, 2007.
5. J. T. Milane, P. N. T. de-Jesús, Solución equivalente del problema de Dirichlet usando transformada de Legendre y funciones de Green, representada por la fórmula integral de Poisson, *Lat.-Am. J. Phys. Educ.*, **15** (2021).
6. L. R. S. M. Jiménez, *El problema L^p Dirichlet para la ecuación de Laplace*, México, Universidad de Sonora, 2011.
7. M. Lewin, *Spectral theory and quantum mechanics*, Springer Cham, 2024. https://doi.org/10.1007/978-3-031-66878-4
8. Z. Zhao, Z. H. Lu, Y. G. Zhao, Simulating non-stationary and non-Gaussian cross-correlated fields using multivariate Karhunen–Loève expansion and L-moments-based Hermite polynomial model, *Mech. Syst. Signal Pr.*, **216** (2024), 111480. https://doi.org/10.1016/j.ymssp.2024.111480
9. F. Rösler, A. Stepanenko, Computing eigenvalues of the Laplacian on rough domains, *Math. Comput.*, **93** (2024), 111–161. https://doi.org/10.1090/mcom/3827
10. J. Kileel, N. F. Marshall, O. Mickelin, A. Singer, Fast expansion into harmonics on the ball, *SIAM J. Sci. Comput.*, **47** (2025), A1117–A1144. https://doi.org/10.1137/24M1668159
11. F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, *NIST handbook of mathematical functions*, New York: Cambridge University Press, 2010.
12. R. Álvarez-Nodarse, Polinomios ortogonales: Historia y aplicaciones, *Bol. Soc. Esp. Mat. Apl.*, **18** (2001), 19–46.
13. J. C. Mason, D. C. Handscomb, *Chebyshev polynomials*, New York: Chapman & Hall/CRC, 2002. https://doi.org/10.1201/9781420036114
14. L. Debnath, D. Bhatta, *Integral transforms and their applications*, 3 Eds., Chapman and Hall/CRC, 2014.https://doi.org/10.1201/b17670
15. J. Bremer, Fast algorithms for Jacobi expansions via nonoscillatory phase functions, *IMA J. Numer. Anal.*, **40** (2020), 2019–2045. https://doi.org/10.1093/imanum/drz016
16. Q. Yan, S. Jiang, J. Harlim, Spectral methods for solving elliptic PDEs on unknown manifolds, *J. Comput. Phys.*, **486** (2023), 112132. https://doi.org/10.1016/j.jcp.2023.112132

17. G. B. Folland, *Real analysis: Modern techniques and their applications*, 2 Eds, New York: Wiley, 1999.
18. J. Bremer, Q. Pang, H. Yang, Fast algorithms for the multi-dimensional Jacobi polynomial transform, *Appl. Comput. Harmon. A.*, **51** (2021), 231–250. https://doi.org/10.1016/j.acha.2020.01.004
19. J. Shen, Y. W. Wang, J. L. Xia, Fast structured Jacobi–Jacobi transforms, *Math. Comput.*, **88** (2019), 1743–1772. https://doi.org/10.1090/mcom/3377
20. J. L. Aurentz, R. M. Slevinsky, On symmetrizing the ultraspherical spectral method, *J. Comput. Phys.*, **420** (2020), 109383. https://doi.org/10.1016/j.jcp.2020.109383
21. D. Fortunato, A. Townsend, H. Wilber, The ultraspherical spectral element method, *J. Comput. Phys.*, **423** (2021), 110087. https://doi.org/10.1016/j.jcp.2020.110087
22. W. Cheng, Z. Xu, Understanding the ultraspherical spectral method: An error analysis, *arXiv Preprint*, 2024. https://doi.org/10.48550/arXiv.2405.11847
23. R. Koekoek, P. A. Lesky, R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues*, Berlin: Springer, 2010. https://doi.org/10.1007/978-3-642-05014-5
24. H. Pijeira-Cabrera, J. Quintero-Roba, J. Toribio-Milane, Differential properties of Jacobi-Sobolev polynomials and electrostatic interpretation, *Mathematics*, **11** (2023), 3420. https://doi.org/10.3390/math11153420
25. G. T. Núñez, *Propiedades y aplicaciones de los polinomios clásicos en la resolución de problemas mecanocuánticos (átomo de hidrógeno y potencial de Rosen–Morse)*, M.Sc. thesis, Universidad Autónoma de Santo Domingo, 2017.
26. J. M. Peña-Reyes, *Integral espectral del operador de Hamilton de la ecuación de Schrödinger y aplicaciones en mecánica cuántica*, M.Sc. thesis, Universidad Autónoma de Santo Domingo, 2026. https://doi.org/10.13140/RG.2.2.13580.58240