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Research article

Quasi-periodic solutions of Sawada-Kotera equation

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Abstract: This paper establishes the existence of quasi-periodic solutions to the Sawada-Kotera equation on the torus $\mathbb{T}$. By implementing a Nash-Moser iterative method, we demonstrate that for a prescribed Diophantine frequency vector $\lambda\tilde{\omega}$, such solutions exist for a set of parameters λ of positive Lebesgue measure.

Keywords: Sawada-Kotera equation; quasi-periodic solutions; Nash-Moser iteration; small divisor problem; Hamiltonian PDEs
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1. Introduction and main result

We consider the Sawada-Kotera (SK) equation

$$u_t + u_{xxxxx} + 45u^2u_x + 15(u_xu_{xx} + uu_{xxx}) = 0, \tag{1.1}$$

posed on the torus $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. The unknown $u = u(\omega t, x)$ is assumed to be quasi-periodic in time with frequency vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_\nu) \in \mathbb{R}^\nu$.

The Sawada-Kotera (SK) equation, first derived by Sawada and Kotera [12] as a fifth-order extension of the Korteweg-de Vries (KdV) equation, plays a central role in the study of nonlinear wave phenomena. It serves as a canonical model for various physical systems, such as long-wave propagation in shallow water [14], dynamics of one-dimensional nonlinear lattices [8], chains of coupled oscillators [7], and magneto-acoustic waves in plasmas [10]. Due to its broad applicability and rich mathematical structure, the SK equation remains a fundamental subject in mathematical physics.

Before presenting our main results, we briefly place our work within the existing literature. Two principal methodologies have been developed for constructing quasi-periodic solutions of Hamiltonian partial differential equations. *The Craig–Wayne–Bourgain Approach* combines Lyapunov–Schmidt reduction with a Nash–Moser iteration to handle the small-divisor problem. Foundational contributions are due to Craig and Wayne [4] and Bourgain [3]; recent generalizations can be found in [1, 2].

Alternatively, *Kolmogorov-Arnold-Moser (KAM) Theory* uses Birkhoff normal forms combined with a KAM iterative procedure, as developed in [5, 11] and extended in recent work [6, 9]. Both frameworks are essentially quadratic convergence algorithms generalizing Newton's method. The analysis typically involves studying carefully parameterized families of partial differential equations (PDEs), where parameters are chosen to avoid resonances. Even in this controlled setting, the small-divisor problem presents a fundamental analytical challenge.

The construction of quasi-periodic solutions through a quadratic iteration scheme demands fine control over the linearized operator

$$\mathcal{L} = \mathcal{L}(u)$$

around an approximate solution $u(t, x)$. The nature of this control differs significantly between the two approaches:

(1) In the Nash–Moser framework, it suffices to impose relatively mild *reducibility* conditions. One only needs to establish the existence of a left inverse for $\mathcal{L}$ while maintaining sharp estimates on the loss of regularity (tame estimates). This allows the iteration to proceed without requiring a complete spectral description of the operator [1].

(2) The KAM scheme, in contrast, demands significantly stronger *non-resonance* and *non-degeneracy* conditions. These ensure not only the invertibility of $\mathcal{L}$ but also provide explicit lower bounds on its eigenvalues and their spectral gaps. This stronger structure enables a powerful conclusion: the linearized operator at the final quasi-periodic solution can be diagonalized via an analytic, time-dependent (but periodic in x) coordinate transformation.

In this paper, we adopt the first, more flexible strategy. We employ a Nash-Moser iteration to construct quasi-periodic solutions for the Sawada-Kotera equation (1.1). The central difficulty consists in proving suitable tame estimates for the corresponding linearized problem.

Main result

Applying the scaling transformation $u = \epsilon\tilde{u}$ to (1.1) yields the equivalent equation

$$\tilde{u}_t + \tilde{u}_{xxxxx} + 45\epsilon^2\tilde{u}^2\tilde{u}_x + 15\epsilon(\tilde{u}_x\tilde{u}_{xx} + \tilde{u}\tilde{u}_{xxx}) = 0, \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}. \tag{1.2}$$

For notational simplicity, we drop the tilde and write the unknown again as u. Introducing the phase variable $\boldsymbol{\varphi} = \boldsymbol{\omega} t$ and defining

$$f(u, u_x, u_{xx}, u_{xxx}) := 45\epsilon u^2 u_x + 15(u_x u_{xx} + u u_{xxx}), \tag{1.3}$$

we rewrite (1.2) in the form

$$\boldsymbol{\omega} \cdot \partial_{\boldsymbol{\varphi}} u + u_{xxxxx} + \epsilon f(u, u_x, u_{xx}, u_{xxx}) = 0, \quad x \in \mathbb{T}. \tag{1.4}$$

Observe from (1.3) that the nonlinearity can be expressed as a derivative:

$$f(u, u_x, u_{xx}, u_{xxx}) = \partial_x g(u, u_{xx}), \tag{1.5}$$

where the nonlinear potential g is given by $g(u, u_{xx}) = 15\epsilon u^3 + 15uu_{xx}$.

We now examine Eq (1.4) under the hypotheses that $\epsilon > 0$ is a small parameter and that the temporal dependence is quasi-periodic with a Diophantine frequency vector. Concretely, we set

$$\omega = \lambda\bar{\omega} \in \mathbb{R}^{\nu}, \qquad \lambda \in \Lambda := [\tfrac{1}{2}, \tfrac{3}{2}],$$

and impose on $\bar{\omega}$ the Diophantine condition

$$|\bar{\omega} \cdot \boldsymbol{l}| \geq \frac{3\gamma_0}{|\boldsymbol{l}|^{\tau_0}} \qquad \forall\, \boldsymbol{l} \in \mathbb{Z}^{\nu} \setminus \{0\}, \tag{1.6}$$

with some $\gamma_0 > 0$. For simplicity we keep the exponent $\tau_0 = \nu$ fixed throughout the paper.

Here, the notation $|\cdot|$ denotes the Euclidean norm for vectors and the absolute value for scalars. Specifically:

- For a vector $\boldsymbol{l} \in \mathbb{Z}^{\nu}$ or $\omega \in \mathbb{C}^{\nu}$, $|\boldsymbol{l}| = \sqrt{l_1^2 + \cdots + l_{\nu}^2}$ (and similarly for $|\omega|$).
- For an integer $j \in \mathbb{Z}$, $|j|$ is the usual absolute value.
- For a complex number, $|\cdot|$ denotes its modulus.

We treat the nonlinearity f as a smooth function of its four arguments; with $z = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4$ we simply assume

$$f \in C^{\infty}(\mathbb{R}^4; \mathbb{R}). \tag{1.7}$$

The only free parameter in (1.4) is λ; it rescales the frequency vector and thus corresponds to a time-rescaling.

For sufficiently small $\epsilon > 0$ and for a set of parameters $\lambda \in \Lambda$ of positive Lebesgue measure, we look for $(2\pi)^{\nu+1}$-periodic solutions $u(\boldsymbol{\varphi}, x)$ of (1.4) belonging to the Sobolev space

$$\begin{aligned}\mathcal{H}^s &:= \mathcal{H}^s(\mathbb{T}^{\nu} \times \mathbb{T}; \mathbb{R}) \\ &= \Big\{u = \sum_{(\boldsymbol{l},j)\in\mathbb{Z}^{\nu}\times\mathbb{Z}} u_{\boldsymbol{l},j} e^{\mathrm{i}(\boldsymbol{l}\cdot\boldsymbol{\varphi}+jx)} \,\Big|\, \overline{u}_{\boldsymbol{l},j} = u_{-\boldsymbol{l},-j},\ \|u\|_s < \infty\Big\},\end{aligned} \tag{1.8}$$

whose norm is defined by

$$\|u\|_s^2 := \sum_{(\boldsymbol{l},j)\in\mathbb{Z}^{\nu}\times\mathbb{Z}} \langle \boldsymbol{l}, j\rangle^{2s} |u_{\boldsymbol{l},j}|^2, \qquad \langle \boldsymbol{l}, j\rangle := \max\{1, |\boldsymbol{l}|, |j|\}. \tag{1.9}$$

We fix $s_0 := (\nu + 2)/2$, which exceeds the Sobolev embedding threshold $(\nu + 1)/2$. Hence, for every $s \geq s_0$ the space $\mathcal{H}^s$ is a Banach algebra and embeds continuously into $C(\mathbb{T}^{\nu+1})$.

Our main theorem reads as follows.

Theorem 1.1. *Assume that $\bar{\omega}$ satisfies the Diophantine condition* (1.6) *with $\tau_0 = \nu$. Then there exist constants $s^* := s(\nu) > s_0$ and $\epsilon_0 = \epsilon_0(f, \nu, \gamma_0) > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ one can find a Cantor set $\mathcal{O}_{\epsilon} \subset \Lambda$ with*

$$\big|\mathcal{O}_{\epsilon}\big| \to 1 \qquad \text{as } \epsilon \to 0, \tag{1.10}$$

and for each $\lambda \in \mathcal{O}_{\epsilon}$ equation (1.4) *possesses a solution $u(\epsilon, \lambda) \in \mathcal{H}^{s^*}$ satisfying*

$$\|u(\epsilon, \lambda)\|_{s^*} \to 0 \qquad \text{as } \epsilon \to 0.$$

Remark 1.2 (Regularity index s^*). *The regularity index s^* in Theorem 1.1 is any number larger than $s_0 = (\nu + 2)/2$. The proof requires $s^* > s_0 + \beta$, where $\beta = 8\tau + 28$ and $\tau > \nu + 1/2$ (see (4.3)). Hence one can take, for example,*

$$s^* = \frac{\nu + 2}{2} + 8\nu + 30 + \varepsilon,$$

for an arbitrary $\varepsilon > 0$.

The paper is structured as follows. Section 2 presents the functional framework and the norms employed in the analysis. In Section 3 we address the invertibility of the linearized operator $\mathcal{L}$ and derive crucial estimates for its inverse. The core of the proof, the Nash-Moser iteration, is carried out in Section 4, where we prove Theorem 4.2 (the iterative construction) and then deduce Theorem 1.1. Section 5 contains some concluding remarks and open problems. Finally, an appendix collects a standard result on composition operators needed in the estimates.

2. Functional space and the limitation of the method

2.1. Notation

Let $(E, \|\cdot\|_E)$ be a Banach space and let $\Lambda_0 \subseteq \mathbb{R}$ be an interval. For a function $f : \Lambda_0 \to E$, we define the following norms and semi-norms:

(1) The *supremum norm*:

$$\|f\|_E^{\sup} := \sup_{\lambda \in \Lambda_0} \|f(\lambda)\|_E. \tag{2.1}$$

(2) The *Lipschitz semi-norm*:

$$\|f\|_E^{\mathrm{lip}} := \sup_{\substack{\lambda_1,\lambda_2 \in \Lambda_0 \\ \lambda_1 \neq \lambda_2}} \frac{\|f(\lambda_1) - f(\lambda_2)\|_E}{|\lambda_1 - \lambda_2|}. \tag{2.2}$$

(3) For a parameter $\gamma > 0$, the *weighted Lipschitz norm*:

$$\|f\|_E^{\mathrm{Lip}(\gamma)} := \|f\|_E^{\sup} + \gamma \|f\|_E^{\mathrm{lip}}. \tag{2.3}$$

When the underlying space is $E = \mathcal{H}^s$, we abbreviate

$$\|f\|_s^{\sup} := \|f\|_{\mathcal{H}^s}^{\sup}, \qquad \|f\|_s^{\mathrm{lip}} := \|f\|_{\mathcal{H}^s}^{\mathrm{lip}}, \qquad \|f\|_s^{\mathrm{Lip}(\gamma)} := \|f\|_{\mathcal{H}^s}^{\mathrm{Lip}(\gamma)}.$$

We shall also employ the following comparison notation:

$$a \leq_s b \quad \text{means} \quad a \leq C(s)\, b$$

for some constant $C(s) > 0$ depending on the Sobolev index s. When $s = s_0$ we write simply $a \lessdot b$. More generally, the symbol $\lessdot$ indicates an inequality up to a constant factor that may depend on the nonlinearity f, the number of frequencies ν, the fixed Diophantine vector $\bar{\omega}$, and the Diophantine exponent τ_0 (which appears in condition (1.6)).

2.2. Limitations of the method

The Nash–Moser approach we have employed, while powerful, has several inherent limitations:

(1) **Perturbative nature.** The method requires the nonlinearity to be multiplied by a small parameter ϵ, which is equivalent to studying small-amplitude solutions. Consequently, our results are confined to the regime of weak nonlinearities and do not extend to solutions of arbitrary size.
(2) **Diophantine conditions.** The frequency vector must satisfy the Diophantine condition (1.6). Although the set of Diophantine vectors has full Lebesgue measure, its complement (which includes resonant frequencies) is topologically dense. This restriction is typical in small divisor problems. Extending the results to a broader class of frequencies (for instance, Brjuno vectors) remains an interesting open problem.
(3) **Regularity loss.** The Nash–Moser iteration incurs a fixed loss of regularity, quantified by the constants s and $s + 2\tau + 1$ in (3.8). This loss limits the regularity of the solutions we can obtain and forces the initial Sobolev index s to be taken sufficiently high. In particular, our solutions belong to a Sobolev space $\mathcal{H}^s$ with s larger than a certain threshold s_0.

3. Invertibility of $\mathcal{L}$

For parameters $\gamma \le \gamma_0/2$ and $\tau \ge \tau_0$, we define the parameter set

$$\Lambda^{2\gamma} := \Big\{\lambda \in \Lambda : \big|\mathrm{i}\lambda\bar{\omega}\cdot \boldsymbol{l} + \mu_j - \mu_k\big| \ge \frac{2\gamma\,|j^5 - k^5|}{\langle \boldsymbol{l}\rangle^{\tau}},\ \forall\, \boldsymbol{l} \in \mathbb{Z}^{\nu},\ j,k \in \mathbb{Z}\Big\}, \tag{3.1}$$

where $\mu_j = \mathrm{i}j^5$. From this definition one immediately obtains the first-order Melnikov (non-resonance) conditions:

$$\big|\mathrm{i}\lambda\bar{\omega}\cdot \boldsymbol{l} + \mu_j\big| \ge \frac{2\gamma\,|j^5|}{\langle \boldsymbol{l}\rangle^{\tau}}, \qquad \forall\, \lambda \in \Lambda^{2\gamma},\ (\boldsymbol{l}, j) \neq (\boldsymbol{0}, 0). \tag{3.2}$$

We next introduce several operators and subspaces that will be used throughout the analysis:

$$\mathcal{L} := \omega\cdot\partial_{\varphi} + \partial_{xxxxx}, \tag{3.3}$$

$$\Pi_C u := \frac{1}{(2\pi)^{\nu+1}} \int_{\mathbb{T}^{\nu+1}} u(\boldsymbol{\varphi}, x)\,\mathrm{d}\boldsymbol{\varphi}\,\mathrm{d}x, \tag{3.4}$$

$$\mathcal{H}^s_0 := \{u \in \mathcal{H}^s(\mathbb{T}^{\nu+1}) : \Pi_C u = 0\}. \tag{3.5}$$

Under the structural condition (1.5), the linearized operator $\mathcal{L}$ maps

$$\mathcal{L} : \mathcal{H}^{s+5} \longrightarrow \mathcal{H}^s_0, \qquad s \ge s_0. \tag{3.6}$$

Lemma 3.1. *For every $\lambda \in \Lambda^{2\gamma}$ and every $g \in \mathcal{H}^s_0$, the equation $\mathcal{L}w = g$ admits a unique zero?mean solution, given by*

$$\mathcal{L}^{-1} g(\boldsymbol{\varphi}, x) = \sum_{(\boldsymbol{l}, j)\neq(\boldsymbol{0},0)} \frac{g_{\boldsymbol{l},j}}{\mathrm{i}\lambda\bar{\omega}\cdot\boldsymbol{l} + \mu_j}\, e^{\mathrm{i}(\boldsymbol{l}\cdot\boldsymbol{\varphi} + jx)}. \tag{3.7}$$

Moreover, if $g = g(\lambda)$ is a Lipschitz family in $\mathcal{H}^s_0$, then

$$\big\|\mathcal{L}^{-1} g\big\|_s^{\mathrm{Lip}(\gamma)} \le C\,\gamma^{-1}\,\|g\|_{s+2\tau+1}^{\mathrm{Lip}(\gamma)}, \tag{3.8}$$

where the constant $C > 0$ depends only on τ and ν.

Proof. For $\lambda \in \Lambda^{2\gamma}$, condition (3.2) guarantees that the formula (3.7) is well defined and satisfies the pointwise estimate

$$\|\mathcal{L}^{-1}(\lambda)g(\lambda)\|_s \lessdot \gamma^{-1}\|g(\lambda)\|_{s+\tau}. \tag{3.9}$$

To establish the Lipschitz bound (3.8), pick $\lambda_1, \lambda_2 \in \Lambda^{2\gamma}$ and decompose the difference as

$$\begin{aligned}\mathcal{L}^{-1}(\lambda_1)g(\lambda_1) - \mathcal{L}^{-1}(\lambda_2)g(\lambda_2)\\ = \mathcal{L}^{-1}(\lambda_1)(g(\lambda_1) - g(\lambda_2))\\ + (\mathcal{L}^{-1}(\lambda_1) - \mathcal{L}^{-1}(\lambda_2))g(\lambda_2).\end{aligned} \tag{3.10}$$

For the first term we combine (3.9) with the definition of the Lipschitz norm:

$$\begin{aligned}\gamma\left\|\mathcal{L}^{-1}(\lambda_1)(g(\lambda_1) - g(\lambda_2))\right\|_s\\ \lessdot \|g(\lambda_1) - g(\lambda_2)\|_{s+\tau}\\ \leq \|g\|^{\mathrm{lip}}_{s+\tau}|\lambda_1 - \lambda_2|\\ \leq \gamma^{-1}\|g\|^{\mathrm{Lip}(\gamma)}_{s+\tau}|\lambda_1 - \lambda_2|.\end{aligned} \tag{3.11}$$

For the second term we introduce $\delta_{lj}(\lambda) := \mathrm{i}\lambda\bar{\omega}\cdot \boldsymbol{l} + \mu_j$ and expand

$$(\mathcal{L}^{-1}(\lambda_1) - \mathcal{L}^{-1}(\lambda_2))g(\lambda_2) = \sum_{(\boldsymbol{l},j)\neq(\boldsymbol{0},0)} \frac{\delta_{lj}(\lambda_2) - \delta_{lj}(\lambda_1)}{\delta_{lj}(\lambda_1)\delta_{lj}(\lambda_2)}\, g_{lj}(\lambda_2)\, e^{\mathrm{i}(\boldsymbol{l}\cdot\boldsymbol{\varphi}+jx)}. \tag{3.12}$$

Applying (3.2), we obtain the following, for every $(\boldsymbol{l}, j) \neq (\boldsymbol{0}, 0)$,

$$\begin{aligned}\gamma\frac{|\delta_{lj}(\lambda_2) - \delta_{lj}(\lambda_1)|}{|\delta_{lj}(\lambda_1)\delta_{lj}(\lambda_2)|} &\lessdot \gamma\frac{|\boldsymbol{l}|\,\langle\boldsymbol{l}\rangle^{2\tau}}{\gamma^2|j|^{10}}\,|\lambda_1 - \lambda_2|\\ &\lessdot \gamma^{-1}\langle\boldsymbol{l}\rangle^{2\tau+1}\,|\lambda_1 - \lambda_2|.\end{aligned} \tag{3.13}$$

From (3.12) and (3.13) it follows that

$$\gamma\left\|(\mathcal{L}^{-1}(\lambda_1) - \mathcal{L}^{-1}(\lambda_2))g(\lambda_2)\right\|_s \lessdot \gamma^{-1}\|g\|^{\mathrm{Lip}(\gamma)}_{s+2\tau+1}\,|\lambda_1 - \lambda_2|. \tag{3.14}$$

Putting together (3.9), (3.11) and (3.14) yields precisely (3.8), which concludes the proof. □

4. The Nash-Moser iteration

This section is devoted to the proof of Theorem 1.1. We begin by introducing a scale of finite-dimensional subspaces consisting of trigonometric polynomials. For each integer $n \geq 0$ define

$$\mathcal{H}_n := \Big\{u \in L^2(\mathbb{T}^{\nu+1}) :\ u(\boldsymbol{\varphi}, x) = \sum_{\langle\boldsymbol{l},j\rangle\leq N_n} u_{\boldsymbol{l},j}\, e^{\mathrm{i}(\boldsymbol{l}\cdot\boldsymbol{\varphi}+jx)}\Big\},$$

where $\langle\boldsymbol{l}, j\rangle := \max\{1, |\boldsymbol{l}|, |j|\}$ as in (1.9), and the truncation parameters $\{N_n\}_{n\geq-1}$ satisfy

$$N_{-1} := 1, \qquad N_n := N_0^{\chi^n}\ (n \geq 0), \quad \chi := \tfrac{3}{2}.$$

Consequently $N_{n+1} = N_n^{\chi}$ for all $n \geq 0$; the initial value N_0 will be chosen sufficiently large later. Denote by

$$\Pi_n := \Pi_{N_n} : L^2(\mathbb{T}^{\nu+1}) \longrightarrow \mathcal{H}_n, \qquad \Pi_n^{\perp} := I - \Pi_n,$$

the orthogonal projections onto $\mathcal{H}_n$ and its orthogonal complement, respectively.

These projectors enjoy the following standard smoothing properties: for every $\alpha \geq 0$ and every $s \geq s_0$,

$$\begin{aligned} \left\|\Pi_n u\right\|_{s+\alpha}^{\mathrm{Lip}(\gamma)} &\leq N_n^{\alpha} \|u\|_s^{\mathrm{Lip}(\gamma)}, && \forall\, u(\lambda) \in \mathcal{H}^s, \\ \left\|\Pi_n^{\perp} u\right\|_{s}^{\mathrm{Lip}(\gamma)} &\leq N_n^{-\alpha} \|u\|_{s+\alpha}^{\mathrm{Lip}(\gamma)}, && \forall\, u(\lambda) \in \mathcal{H}^{s+\alpha}. \end{aligned} \tag{4.1}$$

Both estimates are immediate consequences of the definition of the Fourier truncation and the norm $\|\cdot\|_s^{\mathrm{Lip}(\gamma)}$.

Define the nonlinear operator

$$F(u) := F(\lambda, u) := \lambda\bar{\omega} \cdot \partial_{\varphi} u + u_{xxxxx} + \epsilon f(u, u_x, u_{xx}, u_{xxx}), \tag{4.2}$$

where f is given in (1.3). For the Nash-Moser iteration we shall fix two constants

$$\kappa := 6\tau + 21, \qquad \beta := 8\tau + 28, \tag{4.3}$$

which measure the loss of regularity that occurs in the scheme.

Remark 4.1 (Choice of κ and β)**.** *The exponents κ and β are chosen to balance the various growth rates that appear in the recursive estimates of Lemma 4.4. Specifically, the exponents $2\tau + 4$, $2\tau + 7$, etc., arise from the regularity loss encountered when inverting the linearized operator, together with the smoothing properties of the projection operators. The particular values*

$$\kappa = 6\tau + 21, \qquad \beta = 8\tau + 28$$

ensure that the recurrences (4.13)-(4.14) *close. These values are obtained by solving a simple linear system that matches the exponents of N_{n+1} on both sides of the inequalities.*

Theorem 4.2. *Assume $0 < \gamma \leq \min\{\gamma_0, 1/24\}$ and $\tau > \nu + \frac{1}{2}$. Then one can find constants $\delta > 0$, $C_* > 0$ and an integer $N_0 \in \mathbb{N}$ (which may depend on τ) such that, whenever $\epsilon\gamma^{-1} < \delta$, the following statements hold for every $n \geq 0$.*

($\mathcal{P}1_n$) *There exists a map $u_n : \mathcal{G}_n \subset \Lambda \to \mathcal{H}_n$, $\lambda \mapsto u_n(\lambda)$ satisfying $\|u_n\|_{s_0+3}^{\mathrm{Lip}(\gamma)} \leq 1$. The initial approximation u_0 is the unique solution in $\mathcal{H}_0$ of the projected equation $\Pi_0 F(u_0) = 0$. The nested parameter sets $\mathcal{G}_n \subset \Lambda := [1/2, 3/2]$ are defined recursively by*

$$\mathcal{G}_0 := \Lambda, \qquad \mathcal{G}_{n+1} := \mathcal{G}_n \cap \left\{\lambda \in \Lambda : \left|\mathrm{i}\lambda\bar{\omega} \cdot \boldsymbol{l} + \mu_j - \mu_k\right| \geq \frac{2\gamma|j^5 - k^5|}{\langle \boldsymbol{l} \rangle^{\tau}},\ \forall \boldsymbol{l} \in \mathbb{Z}^{\nu},\ j, k \in \mathbb{Z}\right\}. \tag{4.4}$$

For $n \geq 1$ the successive corrections $h_n := u_n - u_{n-1}$ obey

$$\|h_n\|_{s_0+3}^{\mathrm{Lip}(\gamma)} \leq C_* \epsilon\gamma^{-1} N_n^{-6}. \tag{4.5}$$

($\mathcal{P}2_n$) $\|F(u_n)\|_{s_0}^{\mathrm{Lip}(\gamma)} \leq C_* \epsilon N_n^{-\frac{1}{3}\kappa}$.

($\mathcal{P}3_n$) $\|u_n\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)} \le C_* \,\epsilon\gamma^{-1} N_n^{\kappa}$ *and* $\|F(u_n)\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)} \le C_* \,\epsilon N_n^{\kappa}$.
($\mathcal{P}4_n$) *The Cantor-like sets satisfy the measure estimates*

$$|\mathcal{G}_0 \setminus \mathcal{G}_1| \le C_*\gamma, \qquad |\mathcal{G}_n \setminus \mathcal{G}_{n+1}| = 0 \ \ (n \ge 1). \tag{4.6}$$

All Lipschitz norms are evaluated over the corresponding parameter sets $\mathcal{G}_n$.

Remark 4.3 (Choice of C^* and δ)**.** *The constant C^* depends on the constants appearing in the tame estimates for the nonlinearity (i.e., the bounds on f and its derivatives) and on the geometric progression of the truncation parameters N_n. More precisely, C^* can be taken as*

$$C^* = \max\{C_0, C_1, C_2\},$$

where C_0 comes from Lemma 4.4, C_1 from the base case (implicit function theorem), and C_2 from the measure estimates. All these constants depend polynomially on $\|f\|_{C^r}$ for a sufficiently high r, on τ, and on ν. The smallness parameter δ is then determined by the condition that the series $\sum_k N_k^{-6}$ converges and is bounded by a fixed constant; this leads to $\delta = (2C^*)^{-1}$ after choosing N_0 large enough.

Proof of Theorem 4.2. The proof proceeds by induction on n and is divided into three steps.

Step 1: Base case ($n = 0$). We construct the initial approximation u_0 as the unique solution in $\mathcal{H}_0$ of the projected equation

$$\Pi_0 F(u_0) = 0.$$

Because the linear part of F is invertible on $\mathcal{H}_0$ (the eigenvalues $i(\lambda\bar{\omega} \cdot l + j^5)$ are bounded away from zero for $(l, j) \neq (0, 0)$ thanks to the Diophantine condition on $\bar{\omega}$) and the nonlinearity is of order ϵ, the implicit function theorem yields a solution $u_0 = O(\epsilon)$ satisfying

$$\|u_0\|_{s_0+3}^{\mathrm{Lip}(\gamma)} \le C\epsilon, \qquad \|\Pi_0^{\perp} F(u_0)\|_{s_0}^{\mathrm{Lip}(\gamma)} \le C\epsilon N_0^{-\alpha},$$

for some $\alpha > 0$. Moreover, since $u_0 \in \mathcal{H}_0$, the smoothing properties (4.1) give

$$\|u_0\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)} \le N_0^{\beta} \|u_0\|_{s_0}^{\mathrm{Lip}(\gamma)} \le C\epsilon N_0^{\beta},$$

and similarly for $F(u_0)$. Choosing N_0 sufficiently large and taking $\alpha = \frac{1}{3}\kappa$ (with $\kappa = 6\tau + 21$), we can adjust the constant C_* so that

$$\|u_0\|_{s_0+3}^{\mathrm{Lip}(\gamma)} \le 1, \qquad \|F(u_0)\|_{s_0}^{\mathrm{Lip}(\gamma)} \le C_*\epsilon N_0^{-\frac{1}{3}\kappa},$$

$$\|u_0\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)} \le C_*\epsilon\gamma^{-1} N_0^{\kappa}, \qquad \|F(u_0)\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)} \le C_*\epsilon N_0^{\kappa}.$$

Thus properties $(\mathcal{P}1)_0$, $(\mathcal{P}2)_0$ and $(\mathcal{P}3)_0$ hold.

Step 2: Inductive step.

Let us assume that for a certain $n \ge 0$ the properties $(\mathcal{P}1)_n$, $(\mathcal{P}2)_n$ and $(\mathcal{P}3)_n$ are valid. For every parameter $\lambda \in \mathcal{G}_{n+1}$ (which is defined recursively in (4.4)), the linearized operator

$$\mathcal{L} = F'(\lambda, u_n(\lambda))$$

is invertible on $\mathcal{H}_0^s$ by Lemma 3.1, and we have the estimates for any Lipschitz family $h(\lambda)$ defined on $\mathcal{G}_{n+1}$:

$$\|\mathcal{L}^{-1}h\|_s^{\mathrm{Lip}(\gamma)} \leq_s \gamma^{-1}\|h\|_{s+2\tau+1}^{\mathrm{Lip}(\gamma)}, \tag{4.7}$$

$$\|\mathcal{L}^{-1}h\|_{s_0}^{\mathrm{Lip}(\gamma)} \leq_{s_0} \gamma^{-1}N_{n+1}^{2\tau+1}\|h\|_{s_0}^{\mathrm{Lip}(\gamma)}, \tag{4.8}$$

provided that $h(\lambda) \in \mathcal{H}_{n+1}$ for all $\lambda \in \mathcal{G}_{n+1}$.

We define the correction term by

$$\begin{cases} u_{n+1} := u_n + h_{n+1} \in \mathcal{H}_{n+1}, \\ h_{n+1} := -\Pi_{n+1}\mathcal{L}^{-1}\Pi_{n+1}F(u_n). \end{cases} \tag{4.9}$$

Applying Taylor's expansion to F at u_n yields

$$F(u_{n+1}) = F(u_n) + \mathcal{L}h_{n+1} + \epsilon Q(u_n, h_{n+1}), \tag{4.10}$$

with $Q(u_n, h_{n+1}) = f(u_{n+1}) - f(u_n)$. Inserting (4.9) we obtain

$$\begin{aligned} F(u_{n+1}) = {} & \Pi_{n+1}^{\perp}F(u_n) + \epsilon Q(u_n, h_{n+1}) \\ & + [\mathcal{L}, \Pi_{n+1}^{\perp}]\mathcal{L}^{-1}\Pi_{n+1}F(u_n). \end{aligned} \tag{4.11}$$

Lemma 4.4. *Define*

$$U_n := \|u_n\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)} + \gamma^{-1}\,\|F(u_n)\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)}, \qquad w_n := \gamma^{-1}\|F(u_n)\|_{s_0}^{\mathrm{Lip}(\gamma)}. \tag{4.12}$$

There exists a constant $C_0 = C_0(\nu, \beta) > 0$ such that the following recursive estimates hold:

$$w_{n+1} \leq C_0 N_{n+1}^{-\beta}\, U_n + C_0 N_{n+1}^{2\tau+4}\, w_n, \tag{4.13}$$

$$U_{n+1} \leq C_0 N_{n+1}^{2\tau+7}\,(1 + w_n)\, U_n. \tag{4.14}$$

Proof. By Lemma A.1 in the Appendix, the operator $Q(u_n, \cdot)$ satisfies the following estimates for any Lipschitz family $h(\lambda)$ defined on the parameter set $\mathcal{G}_{n+1}$:

$$\|Q(u_n, h)\|_s^{\mathrm{Lip}(\gamma)} \leq_s \big(\|h\|_{s+3}^{\mathrm{Lip}(\gamma)} + \|u_n\|_{s+3}^{\mathrm{Lip}(\gamma)}\|h\|_{s_0+3}^{\mathrm{Lip}(\gamma)}\big), \tag{4.15}$$

$$\|Q(u_n, h)\|_{s_0}^{\mathrm{Lip}(\gamma)} \leq_{s_0} N_{n+1}^{3}\,\|h\|_{s_0}^{\mathrm{Lip}(\gamma)}, \tag{4.16}$$

provided that $h(\lambda) \in \mathcal{H}_{n+1}$ for all $\lambda \in \mathcal{G}_{n+1}$. Here the constant implicit in the symbol $\leq_s$ depends on the Sobolev index s. Inequality (4.16) follows from (4.15) with $s = s_0$ together with the bound $\|u_n\|_{s_0+3}^{\mathrm{Lip}(\gamma)} \leq 1$, the fact that both u_n and h (when restricted to $\mathcal{G}_{n+1}$) belong to the finite?dimensional space $\mathcal{H}_{n+1}$, and the smoothing estimate (4.1).

From (4.1) and (4.7)–(4.9), we obtain

$$\|h_{n+1}\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)} \leq_{s_0+\beta} \gamma^{-1}N_{n+1}^{2\tau+1}\|F(u_n)\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)}, \tag{4.17}$$

$$\|h_{n+1}\|_{s_0}^{\mathrm{Lip}(\gamma)} \leq_{s_0} \gamma^{-1}N_{n+1}^{2\tau+1}\|F(u_n)\|_{s_0}^{\mathrm{Lip}(\gamma)}. \tag{4.18}$$

Using (4.9) again we also have

$$\|u_{n+1}\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)} \leq_{s_0+\beta} \|u_n\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)} + \gamma^{-1} N_{n+1}^{2\tau+1} \|F(u_n)\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)}. \tag{4.19}$$

Assume now that $\epsilon\gamma^{-1} \leq 1$. From (4.11) together with (4.1), (4.16), and (4.18), we deduce

$$\begin{aligned}\|F(u_{n+1})\|_{s_0}^{\mathrm{Lip}(\gamma)} \leq_{s_0+\beta} & N_{n+1}^{-\beta} \|F(u_n)\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)} \\ & + N_{n+1}^{2\tau+4} \|F(u_n)\|_{s_0}^{\mathrm{Lip}(\gamma)}.\end{aligned} \tag{4.20}$$

Similarly, employing the "highnorm" estimate (4.15) together with (4.1) and (4.17), we obtain

$$\begin{aligned}\|F(u_{n+1})\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)} \leq_{s_0+\beta} & (1 + N_{n+1}^{2\tau+7})\Big(\|F(u_n)\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)} \\ & + \|u_n\|_{s_0+\beta}^{\mathrm{Lip}(\gamma)} \|F(u_n)\|_{s_0}^{\mathrm{Lip}(\gamma)}\Big).\end{aligned} \tag{4.21}$$

Inequalities (4.19)–(4.21) directly imply (4.13) and (4.14). □

By $(\mathcal{P}2)_n$ we have, for $\epsilon\gamma^{-1}$ sufficiently small,

$$w_n \leq C_* \epsilon \gamma^{-1} N_n^{-\frac{1}{3}\kappa} \leq 1. \tag{4.22}$$

Inserting (4.22) and the estimate $U_n \leq C_* \epsilon \gamma^{-1} N_n^{\kappa}$ (which follows from $(\mathcal{P}3)_n$) into (4.14) we obtain, for sufficiently large N_0 and small $\epsilon\gamma^{-1}$,

$$U_{n+1} \leq C_* \epsilon \gamma^{-1} N_{n+1}^{\kappa},$$

provided we choose $\kappa := 6\tau + 21$. This proves the first part of $(\mathcal{P}3)_{n+1}$.

Next, inserting (4.13) together with (4.22) and the bound $w_n \leq C_* \epsilon \gamma^{-1} N_n^{-\frac{1}{3}\kappa}$ from $(\mathcal{P}2)_n$, we obtain, for sufficiently large N_0 and sufficiently small $\epsilon\gamma^{-1}$,

$$w_{n+1} \leq C_* \epsilon \gamma^{-1} N_{n+1}^{-\frac{1}{3}\kappa},$$

with the choice $\kappa := 6\tau + 21$ and $\beta := 8\tau + 28$. This exactly yields $(\mathcal{P}2)_{n+1}$, because $w_{n+1} = \gamma^{-1} \|F(u_{n+1})\|_{s_0}^{\mathrm{Lip}(\gamma)}$.

From (4.18) and $(\mathcal{P}2)_n$ we also obtain

$$\|h_{n+1}\|_{s_0+3}^{\mathrm{Lip}(\gamma)} \leq C_* \epsilon \gamma^{-1} N_{n+1}^{-6}.$$

Consequently, for $\epsilon\gamma^{-1}$ sufficiently small,

$$\|u_{n+1}\|_{s_0+3}^{\mathrm{Lip}(\gamma)} \leq \|u_0\|_{s_0+3} + \sum_{k=1}^{n+1} \|h_k\|_{s_0+3} \leq \sum_{k=1}^{\infty} C_* \epsilon \gamma^{-1} N_k^{-6} \leq 1.$$

Thus $(\mathcal{P}1)_{n+1}$ holds as well. This completes the inductive step.

Step 3: Measure estimates ($(\mathcal{P}4)_n$). We first estimate $|\mathcal{G}_0 \setminus \mathcal{G}_1|$. Write

$$\mathcal{G}_0 \setminus \mathcal{G}_1 = \bigcup_{\boldsymbol{l} \in \mathbb{Z}^\nu} \bigcup_{j,k \in \mathbb{Z}} R_{\boldsymbol{l},j,k}(0), \tag{4.23}$$

where

$$R_{\boldsymbol{l},j,k}(0) := \left\{\lambda \in \mathcal{G}_0 : \left|\mathrm{i}\lambda\bar{\omega} \cdot \boldsymbol{l} + \mu_j - \mu_k\right| < \frac{2\gamma|j^5 - k^5|}{\langle \boldsymbol{l} \rangle^\tau}\right\}. \tag{4.24}$$

Clearly $R_{\boldsymbol{l},j,k}(0) = \emptyset$ when $j = k$; therefore we may assume $j \neq k$ in what follows.

Lemma 4.5. *If $R_{l,j,k}(0) \neq \emptyset$ then*
$$|j^5 - k^5| \leq 8|\bar{\omega} \cdot \boldsymbol{l}|.$$

Proof. Assume $R_{l,j,k}(0) \neq \emptyset$. Hence we can choose a parameter $\lambda \in \Lambda$ such that

$$\left| \mathrm{i}\lambda\bar{\omega} \cdot \boldsymbol{l} + \mu_j - \mu_k \right| < \frac{2\gamma |j^5 - k^5|}{\langle \boldsymbol{l} \rangle^\tau}.$$

Applying the triangle inequality and using $|\lambda| \leq 3/2$ gives

$$|\mu_j - \mu_k| \leq \left| \mathrm{i}\lambda\bar{\omega} \cdot \boldsymbol{l} + \mu_j - \mu_k \right| + |\lambda\bar{\omega} \cdot \boldsymbol{l}| < \frac{2\gamma |j^5 - k^5|}{\langle \boldsymbol{l} \rangle^\tau} + 2|\bar{\omega} \cdot \boldsymbol{l}|. \tag{4.25}$$

Since $|\mu_j - \mu_k| = |j^5 - k^5|$, we obtain

$$|j^5 - k^5| < \frac{2\gamma |j^5 - k^5|}{\langle \boldsymbol{l} \rangle^\tau} + 2|\bar{\omega} \cdot \boldsymbol{l}|.$$

Rearranging yields
$$\left(1 - \frac{2\gamma}{\langle \boldsymbol{l} \rangle^\tau}\right)|j^5 - k^5| < 2|\bar{\omega} \cdot \boldsymbol{l}|.$$

Because $\gamma \leq 1/24$ and $\langle \boldsymbol{l} \rangle^\tau \geq 1$, we have $1 - \frac{2\gamma}{\langle \boldsymbol{l} \rangle^\tau} \geq \frac{11}{12} > \frac{1}{4}$. Consequently,

$$\frac{1}{4}|j^5 - k^5| < 2|\bar{\omega} \cdot \boldsymbol{l}|, \qquad \text{so} \qquad |j^5 - k^5| < 8|\bar{\omega} \cdot \boldsymbol{l}|.$$

This completes the proof of the lemma. □

Lemma 4.6. *For every $\boldsymbol{l} \in \mathbb{Z}^\nu$ and every $j, k \in \mathbb{Z}$ with $j \neq k$,*

$$\left| R_{l,j,k}(0) \right| \leq \frac{32\gamma}{\langle \boldsymbol{l} \rangle^\tau}.$$

Proof. Define the affine function

$$\phi(\lambda) := \mathrm{i}\lambda\bar{\omega} \cdot \boldsymbol{l} + \mu_j - \mu_k = \mathrm{i}(\lambda\bar{\omega} \cdot \boldsymbol{l} + j^5 - k^5), \qquad \lambda \in \Lambda.$$

Then
$$R_{l,j,k}(0) = \left\{ \lambda \in \Lambda : |\phi(\lambda)| < \frac{2\gamma |j^5 - k^5|}{\langle \boldsymbol{l} \rangle^\tau} \right\}.$$

If $R_{l,j,k}(0) = \emptyset$ the estimate is trivial. Assume therefore that the set is nonempty. For any two points $\lambda_1, \lambda_2 \in \Lambda$ we have

$$|\phi(\lambda_1) - \phi(\lambda_2)| = |\bar{\omega} \cdot \boldsymbol{l}|\, |\lambda_1 - \lambda_2|. \tag{4.26}$$

Because the set is nonempty, Lemma 4.5 applies and yields

$$|\bar{\omega} \cdot \boldsymbol{l}| \geq \frac{1}{8}|j^5 - k^5| > 0. \tag{4.27}$$

Inequalities (4.26) and (4.27) together imply that ϕ is strictly monotone on Λ; hence $R_{l,j,k}(0)$ is an interval. Let λ_0 be the unique point where $\phi(\lambda_0) = 0$ (if it lies in Λ; otherwise the interval is shorter). For any λ in this interval,

$$|\phi(\lambda)| = |\phi(\lambda) - \phi(\lambda_0)| \geq \frac{1}{8}|j^5 - k^5|\,|\lambda - \lambda_0|.$$

Consequently, if λ satisfies $|\phi(\lambda)| < \frac{2\gamma|j^5-k^5|}{\langle l\rangle^\tau}$, then necessarily

$$|\lambda - \lambda_0| < \frac{2\gamma|j^5 - k^5|}{\langle \boldsymbol{l}\rangle^\tau} \cdot \frac{8}{|j^5 - k^5|} = \frac{16\gamma}{\langle \boldsymbol{l}\rangle^\tau}.$$

Thus the total length of the interval $R_{l,j,k}(0)$ does not exceed twice this distance, i.e.,

$$\left|R_{l,j,k}(0)\right| \leq 2 \cdot \frac{16\gamma}{\langle \boldsymbol{l}\rangle^\tau} = \frac{32\gamma}{\langle \boldsymbol{l}\rangle^\tau}.$$

This completes the proof. □

Since for $j \neq k$

$$|j^5 - k^5| = |j - k|\left|j^4 + j^3k + j^2k^2 + jk^3 + k^4\right| \geq \frac{1}{2}(j^2 + k^2)^2.$$

From Lemma 4.5, it follows that whenever $R_{l,j,k}(0) \neq \varnothing$,

$$\frac{1}{2}(j^2 + k^2)^2 \leq |j^5 - k^5| \leq 8|\bar{\omega} \cdot \boldsymbol{l}| \leq 8|\bar{\omega}|\,|\boldsymbol{l}|.$$

Thus, for each fixed $\boldsymbol{l}$ the pairs (j, k) contributing to the union $\bigcup_{j,k} R_{l,j,k}(0)$ are restricted to the set

$$j^2 + k^2 \leq 2\sqrt{8|\bar{\omega}|\,|\boldsymbol{l}|}.$$

Consequently the number of such pairs is bounded by $C|\boldsymbol{l}|^{1/2}$ (with a constant C depending on $|\bar{\omega}|$). Applying Lemma 4.6, we then obtain

$$\begin{aligned}\left|\mathcal{G}_0 \setminus \mathcal{G}_1\right| &\leq \sum_{\boldsymbol{l}\in\mathbb{Z}^\nu} \sum_{\substack{j,k\in\mathbb{Z} \\ j\neq k \\ j^2+k^2\leq 2\sqrt{8|\bar{\omega}|\,|\boldsymbol{l}|}}} \left|R_{l,j,k}(0)\right| \\ &\lessdot \sum_{\boldsymbol{l}\in\mathbb{Z}^\nu} |\boldsymbol{l}|^{1/2}\,\frac{\gamma}{\langle \boldsymbol{l}\rangle^\tau} \\ &\lessdot \gamma \sum_{\boldsymbol{l}\in\mathbb{Z}^\nu} \langle \boldsymbol{l}\rangle^{-\tau+1/2}.\end{aligned}$$

The final series converges exactly when $\tau > \nu + \frac{1}{2}$; in that case it is bounded by a constant depending solely on ν and τ. Hence

$$\left|\mathcal{G}_0 \setminus \mathcal{G}_1\right| \leq C\gamma,$$

which yields the first estimate in $(\mathcal{P}4)_n$.

For all $n \geq 1$ the sets $\mathcal{G}_{n+1}$ are constructed by imposing exactly the same resonance conditions as those used for $\mathcal{G}_1$; therefore $\mathcal{G}_n \setminus \mathcal{G}_{n+1} = \varnothing$ holds for every $n \geq 1$. This gives the second assertion in $(\mathcal{P}4)_n$ and finishes the proof of Theorem 4.2. □

4.1. Proof of Theorem 1.1

Proof. Choose $\gamma = \epsilon^a$ with a fixed exponent $a \in (0,1)$. Then $\epsilon\gamma^{-1} = \epsilon^{1-a}$; taking ϵ small enough we can satisfy the condition $\epsilon\gamma^{-1} < \delta$ required in Theorem 4.2. Hence the iterative construction of Theorem 4.2 produces a sequence $\{u_n\}$ defined on the nested Cantor sets $\mathcal{G}_n$ and converging in the norm $\|\cdot\|_{s_0+3}^{\mathrm{Lip}(\gamma)}$ to a limit function $u_\infty : \mathcal{G}_\infty \to \mathcal{H}^{s_0+3}$, where $\mathcal{G}_\infty := \bigcap_{n\geq 0} \mathcal{G}_n$.

From $(\mathcal{P}2)_n$ we have $\|F(u_n)\|_{s_0}^{\mathrm{Lip}(\gamma)} \to 0$; the continuity of F therefore implies $F(\lambda, u_\infty(\lambda)) = 0$ for every $\lambda \in \mathcal{G}_\infty$. Thus $u_\infty(\lambda)$ solves Eq (1.4) with $\omega = \lambda\bar{\omega}$.

The bound $(\mathcal{P}1)_n$ gives $\sup_{\lambda\in\mathcal{G}_\infty} \|u_\infty(\lambda)\|_{s_0+3} \leq C\epsilon\gamma^{-1} = C\epsilon^{1-a}$; in particular $\|u_\infty(\lambda)\|_{s_0+3} \to 0$ as $\epsilon \to 0$.

Consequently, the Lebesgue measure of the excluded parameters is estimated by

$$\begin{aligned} \left|\Lambda \setminus \mathcal{G}_\infty\right| &= \sum_{n\geq 0} \left|\mathcal{G}_n \setminus \mathcal{G}_{n+1}\right| \\ &\leq \left|\mathcal{G}_0 \setminus \mathcal{G}_1\right| \leq C\gamma = C\epsilon^{1-a}, \end{aligned}$$

which vanishes as $\epsilon \to 0$. Hence (1.10) holds, and the proof of Theorem 1.1 is complete. □

5. Conclusions and future work

In this paper, we have proved the existence of small-amplitude quasi-periodic solutions for the Sawada-Kotera equation (1.1) on the torus $\mathbb{T}$. We employed a Nash-Moser iteration scheme that relies on tame estimates for the linearized operator, thereby circumventing the stringent reducibility conditions typical in KAM theory. Our main theorem (Theorem 1.1) demonstrates that, for a set of parameters of asymptotically full Lebesgue measure, Eq (1.4) possesses a solution that tends to zero in an appropriate Sobolev norm as the nonlinearity parameter ϵ approaches zero.

Several natural extensions of our work present themselves for future investigation:

(1) **Other integrable equations.** The Sawada-Kotera equation belongs to a family of fifth-order integrable equations. It would be interesting to apply the same Nash-Moser technique to related models, such as the Kaup-Kupershmidt equation or the Lax fifth-order KdV equation.
(2) **Large-amplitude solutions.** Our result is perturbative, requiring ϵ to be small. Constructing solutions for large values of ϵ (i.e., beyond the small-nonlinearity regime) is a challenging problem that may require a global Nash-Moser theorem or a completely different approach.
(3) **Weaker frequency conditions.** The Diophantine condition (1.6) excludes a dense set of frequencies. Relaxing this condition-for instance to Brjuno-type conditions or to measure-theoretic hypotheses with slower decay would broaden the applicability of the result.
(4) **Higher-dimensional settings.** Extending the analysis to the Sawada-Kotera equation posed on higher-dimensional tori (with respect to the space variable x) and seeking quasi-periodic solutions in both time and space is a natural generalization.

In conclusion, the Nash-Moser iteration provides a flexible framework for constructing quasi-periodic solutions of nonlinear PDEs with small divisors. Although the method is inherently perturbative and requires a certain amount of regularity, it yields measure-theoretic results for the parameters and can be applied to equations that do not possess a fully reducible linear part. We hope that this work will stimulate further research on quasi-periodic solutions for integrable equations and beyond.

Author contributions

Wenlei Li: Writing – review, Writing – editing, Validation, Supervision, Formal analysis, Conceptualization; Juanying Huo: Writing – original draft, Writing – editing, Resources, Methodology, Investigation, Formal analysis. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

Appendix

For a function $u \in C^s(\mathbb{T}^d)$ we denote

$$|u|_{s,\infty} := \sum_{|\beta| \leq s} \|D^\beta u\|_{L^\infty}.$$

Let $B_1 := \{y \in \mathbb{R}^m : |y| < 1\}$ and consider a function $f : \mathbb{T}^d \times B_1 \to \mathbb{C}$. For an integer $p \geq 0$ we define the composition operator

$$\tilde{f}(u)(x) := f(x, u(x), Du(x), \ldots, D^p u(x)), \tag{A.1}$$

where $D^k u(x)$ stands for the collection of all partial derivatives $\partial_x^\alpha u(x)$ with $|\alpha| = k$ (the dimension m of the y-variables depends on p and d).

Lemma A.1. *Let $f \in C^r(\mathbb{T}^d \times B_1)$.*

(i) If $u \in \mathcal{H}^{r+p}$ and $|u|_{p,\infty} < 1$, then $\tilde{f}(u)$ is well defined and

$$\|\tilde{f}(u)\|_r \leq C\,\|f\|_{C^r}(1 + \|u\|_{r+p}),$$

with the constant $C = C(r, p, d)$.
Assume furthermore that $f \in C^{r+2}$ and that u, h satisfy $|u|_{p,\infty}, |h|_{p,\infty} < \frac{1}{2}$. Then

$$\|\tilde{f}(u+h) - \tilde{f}(u)\|_r \leq C\,\|f\|_{C^{r+1}}(\|h\|_{r+p} + |h|_{p,\infty}\|u\|_{r+p}),$$

$$\|\tilde{f}(u+h) - \tilde{f}(u) - \tilde{f}'(u)[h]\|_r \leq C\,\|f\|_{C^{r+2}}\,|h|_{p,\infty}(\|h\|_{r+p} + |h|_{p,\infty}\|u\|_{r+p}).$$

(ii) The same estimates hold with the Sobolev norm $\|\cdot\|_r$ replaced by the Hölder norm $|\cdot|_{r,\infty}$.

Proof. The estimates are standard in the theory of nonlinear composition operators in Sobolev and Hölder spaces. We give a sketch of the main ideas and refer to [13, Chapter 13] or [2, Lemma B.2] for complete details.

Part (i). Let $U(x) = (x, u(x), Du(x), \dots, D^p u(x))$. For a multi-index α with $|\alpha| \le r$ we apply the Faà di Bruno formula to $D^\alpha(f(U(x)))$. The result is a finite sum of terms of the form

$$(\partial^\beta f)(U(x)) \prod_{j=1}^{k} D^{\gamma_j} U_{\iota_j}(x), \qquad 1 \le |\beta| \le |\alpha|,$$

where the γ_j are multi-indices with $\sum_{j=1}^k \gamma_j = \alpha$ and each U_{ι_j} is one of the components of U (either a coordinate of x or a derivative of u of order $\le p$). Consequently each factor $D^{\gamma_j} U_{\iota_j}(x)$ is a derivative of u of order at most $p + |\gamma_j| \le p + r$.

Because $|u|_{p,\infty} < 1$, all derivatives of u of order $\le p$ are uniformly bounded by 1. For the remaining factors we use the Sobolev embedding $\mathcal{H}^{r+p} \hookrightarrow C^{p+r-d/2-\varepsilon}(\mathbb{T}^d)$ (valid for $r + p > d/2$), which yields

$$\|D^\gamma u\|_{L^\infty} \le C\|u\|_{r+p}, \qquad |\gamma| \le p + r.$$

To estimate the L^2-norm of a product $\prod_{j=1}^k v_j$ we repeatedly apply the inequality

$$\|vw\|_{L^2} \le \|v\|_{L^\infty}\|w\|_{L^2}.$$

Taking into account that at most one factor in the product contains a derivative of order $> p$, we obtain

$$\Big\|\prod_{j=1}^{k} D^{\gamma_j} U_{\iota_j}\Big\|_{L^2} \le C\|u\|_{r+p}.$$

Since $|(\partial^\beta f)(U(x))| \le \|f\|_{C^r}$ and the number of terms in the sum depends only on r, p, d, we finally get

$$\|D^\alpha \tilde{f}(u)\|_{L^2} \le C\|f\|_{C^r}\|u\|_{r+p}, \qquad |\alpha| \le r,$$

and therefore $\|\tilde{f}(u)\|_r \le C\|f\|_{C^r}(\|u\|_{r+p} + 1)$.

The estimates for the differences follow from Taylor expansions. Write

$$\tilde{f}(u+h) - \tilde{f}(u) = \int_0^1 \tilde{f}'(u+th)[h]\, dt,$$

and

$$\tilde{f}(u+h) - \tilde{f}(u) - \tilde{f}'(u)[h] = \int_0^1 (1-t)\tilde{f}''(u+th)[h,h]\, dt,$$

where the Frchet derivatives are given by

$$\tilde{f}'(u)[h] = \sum_{|\alpha|\le p} \frac{\partial f}{\partial y_\alpha}(U(x))\, D^\alpha h(x),$$

and $\tilde{f}''(u)[h, h]$ involves second derivatives of f and products of two derivatives of h up to order p. Using the same product estimates as above, together with the assumptions $|u|_{p,\infty}, |h|_{p,\infty} < \frac{1}{2}$ (which guarantee that the arguments stay inside B_1), one obtains the bounds

$$\|\tilde{f}(u + h) - \tilde{f}(u)\|_r \leq C\|f\|_{C^{r+1}}(\|h\|_{r+p} + |h|_{p,\infty}\|u\|_{r+p}),$$
$$\|\tilde{f}(u + h) - \tilde{f}(u) - \tilde{f}'(u)[h]\|_r \leq C\|f\|_{C^{r+2}}|h|_{p,\infty}(\|h\|_{r+p} + |h|_{p,\infty}\|u\|_{r+p}).$$

Part (ii). The proofs for the Hörder norms $|\cdot|_{r,\infty}$ are identical, with the L^2 norm replaced by the L^∞ norm and using the obvious estimate $\|vw\|_{L^\infty} \leq \|v\|_{L^\infty}\|w\|_{L^\infty}$. All constants depend only on r, p, d because the condition $|u|_{p,\infty} < 1$ provides a uniform bound for the lower Hörder derivatives. □

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