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Research article

Novel Berezin number and norm inequalities for operator sums and products

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Abstract: Let $(\mathcal{X}_{\mathscr{F}}, \langle\cdot,\cdot\rangle)$ be a reproducing kernel Hilbert space over a non-empty set $\mathscr{F}$. Let $\widehat{u}_\lambda$ and $\widehat{u}_\mu$ denote the normalized reproducing kernels of $\mathcal{X}_{\mathscr{F}}$. The Berezin number and the Berezin norm of a bounded linear operator $\mathcal{B}$ acting on $\mathcal{X}_{\mathscr{F}}$ are, respectively, defined by

$$\mathbf{ber}(\mathcal{B}) = \sup_{\lambda\in\mathscr{F}} \left|\langle \mathcal{B}\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right| \quad \text{and} \quad \|\mathcal{B}\|_{\mathbf{ber}} = \sup_{\lambda,\mu\in\mathscr{F}} \left|\langle \mathcal{B}\widehat{u}_\lambda, \widehat{u}_\mu\rangle\right|.$$

In this work, we establish new upper bounds for these two quantities. In particular, we derive bounds for their sums and obtain novel estimates for a specific type of product, namely $\mathbf{ber}(C^*\mathcal{B})$, where C^* denotes the adjoint of C. Some of our results also involve another Berezin-type norm that is equivalent to the quantities mentioned above. Several applications and improvements of existing results in the literature are provided.

Keywords: Berezin number; Berezin norm; reproducing kernel Hilbert space; operator inequalities; numerical radius; bounded linear operators
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1. Introduction

Inequalities are fundamental tools in operator theory. They allow us to compare different measures of operators, such as norms or numerical values, and to understand their structure. Classical results on operator inequalities can be found in [1,2], while recent bounds for the numerical radius have been established in [3–5]. More recent studies have focused on partial isometries and projections in semi-Hilbertian spaces [6,7], A-spectral and A-numerical radius inequalities [8–10], and the properties of A-normal operators [11].

Reproducing kernel Hilbert spaces (RKHSs) provide a natural and fruitful setting for studying operators. These spaces, first introduced by Aronszajn [12], have the reproducing property: Evaluating a function at a point can be expressed as an inner product with a special function called the reproducing kernel. This property has made RKHSs important in complex analysis [13], probability theory [14], and applications involving operator inequalities. The theory of Berezin symbols was pioneered in [15,16] and further utilized in operator theory in [17]. Recent applications to inequalities can be found in [18–20].

Studying operator inequalities in RKHSs allows us to explore quantities such as the Berezin number and Berezin norm, which have been the subject of increasing attention in the literature. Initial estimates were provided in [21,22], while further refinements and developments appear in [23–25]. These studies show that establishing sharp bounds and refinements is both mathematically interesting and useful for applications in functional analysis and operator theory.

We begin with some definitions and notations. Let $\mathscr{F}$ be a non-empty set serving as the domain for functions in the RKHS (in applications, $\mathscr{F}$ is often a subset of $\mathbb{R}^n$ or $\mathbb{C}$) and let $\mathcal{FU}(\mathscr{F})$ denote the set of all functions from $\mathscr{F}$ to $\mathbb{C}$. A subset $\mathcal{X}_{\mathscr{F}} \subseteq \mathcal{FU}(\mathscr{F})$ is called a RKHS on $\mathscr{F}$ if it satisfies the following conditions:

(1) $\mathcal{X}_{\mathscr{F}}$ is a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$.
(2) For every $\lambda \in \mathscr{F}$, the evaluation map $E_\lambda : \mathcal{X}_{\mathscr{F}} \to \mathbb{C}$, defined by $E_\lambda(f) = f(\lambda)$ for all $f \in \mathcal{X}_{\mathscr{F}}$, is continuous.

By the Riesz representation theorem, for each $\lambda \in \mathscr{F}$, there exists a unique function $u_\lambda \in \mathcal{X}_{\mathscr{F}}$ such that

$$f(\lambda) = E_\lambda(f) = \langle f, u_\lambda \rangle, \quad \forall f \in \mathcal{X}_{\mathscr{F}}.$$

This function u_λ is referred to as the reproducing kernel at λ, and its normalized form is given by $\widehat{u}_\lambda = \frac{u_\lambda}{\|u_\lambda\|}$. The mapping $k(z, \lambda) = \langle u_\lambda, u_z \rangle$ defines the reproducing kernel function of the (RKHS) $\mathcal{X}_{\mathscr{F}}$.

If $\{e_n\}$ denotes an orthonormal basis for $\mathcal{X}_{\mathscr{F}}$, then the kernel function can be expressed as

$$k(z, \lambda) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(\lambda)}.$$

An example of an RKHS is the Hardy space $H^2(\mathbb{D})$ over the unit disk $\mathbb{D} \subset \mathbb{C}$, which consists of all square-summable holomorphic functions on $\mathbb{D}$. The corresponding reproducing kernel, known as the Szegő kernel, is given by

$$k(z, \lambda) = \frac{1}{1 - z\bar{\lambda}}, \quad \text{for } z, \lambda \in \mathbb{D}.$$

For foundational theory on RKHSs, we refer the reader to [12,14]. Further discussions on Berezin symbols and their boundary values can be found in [26–28].

Let $(\mathcal{X}_{\mathscr{F}}, \langle \cdot, \cdot \rangle)$ be an RKHS over $\mathscr{F}$ with associated norm $\| \cdot \|$. For any $\lambda \in \mathscr{F}$, we define the normalized reproducing kernel $\widehat{u}_\lambda = \frac{u_\lambda}{\|u_\lambda\|}$, where u_λ is the reproducing kernel at λ. The collection $\{\widehat{u}_\lambda : \lambda \in \mathscr{F}\}$ forms a total subset of $\mathcal{X}_{\mathscr{F}}$.

Given an operator $\mathcal{B} \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$, its Berezin symbol $\widetilde{\mathcal{B}}$ is defined on $\mathscr{F}$ by

$$\widetilde{\mathcal{B}}(\lambda) = \langle \mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda} \rangle.$$

This concept was introduced by Berezin in [15, 16] and has since played a crucial role in operator theory.

It is well-known that, in standard RKHSs such as Bergman, Hardy, Fock, and Dirichlet spaces, the operator is uniquely determined by its Berezin transform. Specifically, for $\mathcal{B}, C \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$, we have $\mathcal{B} = C$ if and only if $\widetilde{\mathcal{B}}(\lambda) = \widetilde{C}(\lambda)$ for all $\lambda \in \mathscr{F}$. For further details, see [13, 17].

The Berezin set, Berezin number, and Berezin norm of an operator $\mathcal{B}$ are defined, respectively, as

$$\mathbf{Ber}(\mathcal{B}) := \left\{ \langle \mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda} \rangle : \lambda \in \mathscr{F} \right\}, \quad \mathbf{ber}(\mathcal{B}) := \sup_{\lambda \in \mathscr{F}} \left| \langle \mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda} \rangle \right|,$$

$$\text{and } \|\mathcal{B}\|_{\mathbf{ber}} := \sup \left\{ \left| \langle \mathcal{B}\widehat{u_\lambda}, \widehat{u_\mu} \rangle \right| : \lambda, \mu \in \mathscr{F} \right\}.$$

We also recall the quantity

$$\|\mathcal{B}\|_{\widetilde{\mathbf{ber}}} := \sup_{\lambda \in \mathscr{F}} \|\mathcal{B}\widehat{u_\lambda}\|.$$

Since the set $\{\widehat{u_\lambda} : \lambda \in \mathscr{F}\}$ is complete in $\mathcal{X}_{\mathscr{F}}$, it follows that $\mathbf{ber}(\cdot), \|\cdot\|_{\mathbf{ber}}, \|\cdot\|_{\widetilde{\mathbf{ber}}}$ are norms on $\mathbb{L}(\mathcal{X}_{\mathscr{F}})$, satisfying the inequality

$$\mathbf{ber}(\mathcal{B}) \leq \|\mathcal{B}\|_{\mathbf{ber}} \leq \|\mathcal{B}\|_{\widetilde{\mathbf{ber}}} \leq \|\mathcal{B}\|, \quad \forall\, \mathcal{B} \in \mathbb{L}(\mathcal{X}_{\mathscr{F}}). \tag{1.1}$$

Let us explore how these quantities behave under powers and adjoints. In particular, the property $\mathbf{ber}(\mathcal{B}) = \mathbf{ber}(\mathcal{B}^*)$ and $\|\mathcal{B}\|_{\mathbf{ber}} = \|\mathcal{B}^*\|_{\mathbf{ber}}$ holds for all $\mathcal{B} \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$. However, the identity $\|\mathcal{B}\|_{\widetilde{\mathbf{ber}}} = \|\mathcal{B}^*\|_{\widetilde{\mathbf{ber}}}$ may fail in general, as shown through counterexamples in [29].

In [23, Proposition 2.11], it was shown that if $\mathcal{B}$ is a positive operator on $\mathcal{X}_{\mathscr{F}}$, denoted by $\mathcal{B} \geq 0$, meaning that $\langle \mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda} \rangle \geq 0$ for all $\lambda \in \mathscr{F}$, then

$$\mathbf{ber}(\mathcal{B}) = \|\mathcal{B}\|_{\mathbf{ber}}. \tag{1.2}$$

Furthermore, we note that $\mathbf{Ber}(\mathcal{B}) \subseteq W(\mathcal{B})$, where $W(\mathcal{B})$ denotes the numerical range of $\mathcal{B}$, and this implies $\mathbf{ber}(\mathcal{B}) \leq \omega(\mathcal{B})$, where $\omega(\cdot)$ is the numerical radius. While $\omega(\mathcal{B}^n) \leq \omega^n(\mathcal{B})$ holds for all positive integers n, this inequality does not necessarily extend to $\mathbf{ber}(\mathcal{B}^n)$, even when $\mathcal{B}$ is a positive operator.

In the next elementary example, we show that the Berezin number, the Berezin norm, and the norm $\|\cdot\|_{\widetilde{\mathbf{ber}}}$ do not, in general, satisfy the corresponding power inequalities, even when the operator is positive.

Example 1.1. Consider $\mathbb{C}^2$ as a RKHS and let $\mathcal{B} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{L}(\mathbb{C}^2)$. Then

$$\mathbf{ber}(\mathcal{B}^2) = \mathbf{ber}(\mathcal{B}) = \frac{1}{2} > \frac{1}{4} = \mathbf{ber}^2(\mathcal{B}).$$

Similarly, it can be seen that $\|\mathcal{B}^2\|_{\mathbf{ber}} \leq \|\mathcal{B}\|^2_{\mathbf{ber}}$ and $\|\mathcal{B}^2\|_{\widetilde{\mathbf{ber}}} \leq \|\mathcal{B}\|^2_{\widetilde{\mathbf{ber}}}$ are not verified, even though $\mathcal{B}$ is positive.

The structure of this paper is organized as follows. Section 2 introduces several key and well-known lemmas that serve as the basis for our main results. In Section 3, we present our primary contributions, establishing new upper bounds for the Berezin number ($\mathbf{ber}(\cdot)$) and the Berezin norm ($\|\cdot\|_{\mathbf{ber}}$). In particular, we derive bounds for the sums of these quantities and obtain novel estimates for the product $\mathbf{ber}(C^*\mathcal{B})$, where C^* denotes the adjoint of C. Furthermore, we investigate another Berezin-type norm, $\|\cdot\|_{\widetilde{\mathbf{ber}}}$, showing its equivalence to the main quantities studied. The paper concludes with several applications that yield significant improvements over existing operator inequalities in the literature.

2. Foundational lemmas

In this section, we state several well-known lemmas that will be used later in proving our main results. Throughout this section, $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ denotes a complex Hilbert space, equipped with its standard inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$.

The following result establishes a general, foundational inequality bounding the magnitude of the inner product $|\langle x, y\rangle|$, serving as a refinement of the well-known Cauchy-Schwarz inequality.

Lemma 2.1. *[30] Consider $x, y \in \mathcal{X}$ and $\varepsilon \in [0,1]$. Then, we have*

$$\left|\langle x, y\rangle\right| \leq \sqrt{\varepsilon\|x\|^2\|y\|^2 + (1-\varepsilon)\left|\langle x, y\rangle\right|\|x\|\|y\|} \leq \|x\|\|y\|. \tag{2.1}$$

A refinement of the first inequality in (2.1) appears in [31] as the following lemma, providing a sharper version of the Cauchy-Schwarz inequality.

Lemma 2.2. *Let $x, y \in \mathcal{X}$, and $\varepsilon \in [0,1]$. Then*

$$|\langle x, y\rangle| \leq \sqrt{\Theta_\varepsilon\|x\|^2\|y\|^2 + \Delta_\varepsilon|\langle x, y\rangle|\|x\|\|y\|} \leq \|x\|\|y\|, \tag{2.2}$$

for all $x, y \in \mathcal{X}_{\mathscr{F}}$, where

$$\Theta_\varepsilon = \min\{\varepsilon, 1-\varepsilon\} \quad \text{and} \quad \Delta_\varepsilon = \max\{\varepsilon, 1-\varepsilon\}.$$

The next lemma presents a Buzano-type inequality that plays a crucial role in deriving bounds for the products of inner products.

Lemma 2.3. *[32] Suppose that $x, y, u \in \mathcal{X}$ satisfy $\|u\| = 1$. Then*

$$12\left|\langle x, u\rangle\langle u, y\rangle\right|^2 \leq \|x\|^2\|y\|^2 + \left|\langle x, y\rangle\right|^2 + 2\|x\|\|y\|\left|\langle x, y\rangle\right| + 4\left|\langle x, u\rangle\langle u, y\rangle\right|\left(\|x\|\|y\| + \left|\langle x, y\rangle\right|\right).$$

This lemma is the well-known Hölder-McCarthy inequality, which relates the power of the expectation of a positive operator to the expectation of its power.

Lemma 2.4. *(See [33, Theorem 1.2]) Let $\mathcal{B} \in \mathbb{L}(\mathcal{X})$ be a positive operator $\mathcal{X}$, denoted by $\mathcal{B} \geq 0$, meaning that $\langle\mathcal{B}x, x\rangle \geq 0$ for all $x \in \mathcal{X}$. Then for all $x \in \mathcal{X}$ with $\|x\| = 1$ and for all real numbers $r \geq 1$, the following inequality holds:*

$$\langle\mathcal{B}x, x\rangle^r \leq \langle\mathcal{B}^r x, x\rangle.$$

The following result, often referred to as a Buzano-type inequality, provides an upper bound for the product of two inner products involving a unit vector.

Lemma 2.5. *[34] If $a, b, e \in \mathcal{X}$, and $\|e\| = 1$, then*

$$\left|\langle a, e\rangle\langle e, b\rangle\right| \leq \frac{1}{2}\left(\|a\|\,\|b\| + \left|\langle a, b\rangle\right|\right). \tag{2.3}$$

The following is a generalized Buzano-type inequality (2.3), which serves as a foundation for our subsequent operator inequalities. A version of this result is stated in [30]. Namely, for every $a, b, e \in \mathcal{X}$ with $\|e\| = 1$ and $\beta \in [0, 1]$, we have

$$|\langle a, e\rangle\langle e, b\rangle| \leq \frac{1+\beta}{2}\|a\|\|b\| + \frac{1-\beta}{2}|\langle a, b\rangle|. \tag{2.4}$$

Exploiting the inequality (2.4) together with the convexity of $f(t) = t^s$ ($s \geq 1$), we obtain the following crucial lemma.

Lemma 2.6. *For every $a, b, e \in \mathcal{X}$ with $\|e\| = 1$, $\beta \in [0, 1]$, and $s \geq 1$, we have*

$$|\langle a, e\rangle\langle e, b\rangle|^s \leq \frac{1+\beta}{2}\|a\|^s\|b\|^s + \frac{1-\beta}{2}|\langle a, b\rangle|^s. \tag{2.5}$$

The next lemma is a standard ℓ_p space inequality for norms, derived from the parallelogram law for $p = 2$, extended to $p \geq 2$.

Lemma 2.7. *Let $a, b \in \mathcal{X}$. Then*

$$\|a\|^p + \|b\|^p \leq \frac{1}{2}\left(\|a+b\|^p + \|a-b\|^p\right),$$

for all $p \geq 2$.

The next result is the basic power mean inequality for positive real numbers, essential for separating terms with powers $q \geq 1$.

Lemma 2.8. *Let α_i be positive real numbers ($i = 1, 2, ..., n$). Then for all $q \geq 1$, we have*

$$\left(\sum_{i=1}^{n}\alpha_i\right)^q \leq n^{q-1}\sum_{i=1}^{n}\alpha_i^q.$$

This simple but powerful lemma provides a bound on the real part of an inner product, fundamental for norm calculations.

Lemma 2.9. *Suppose $u, v \in \mathcal{X}$. Then*

$$\Re\left(\langle u, v\rangle\right) \leq \frac{1}{2}\left(\|u\|^2 + \|v\|^2\right),$$

where $\Re(\cdot)$ denotes the real component of complex numbers.

The following inequality gives an alternative bound on the real part of an inner product in terms of the norm of the sum of the vectors.

Lemma 2.10. *Let $u, v \in \mathcal{X}$. Then*

$$\Re\left(\langle u, v\rangle\right) \leq \frac{1}{4}\|u+v\|^2.$$

This lemma contains a fundamental mixed Cauchy-Schwarz inequality for operators, relating the inner product of vectors transformed by an operator to the inner product of vectors transformed by the absolute value of the operator.

Lemma 2.11. *Let $\mathcal{B} \in \mathbb{L}(\mathcal{X})$. Then for all $x, y \in \mathcal{X}$, we have*

$$|\langle \mathcal{B}x, y\rangle| \leq \langle |\mathcal{B}|x, x\rangle^{\frac{1}{2}} \langle |\mathcal{B}^*|y, y\rangle^{\frac{1}{2}}, \tag{2.6}$$

where $|\mathcal{B}| = (\mathcal{B}^*\mathcal{B})^{\frac{1}{2}}$ denotes the absolute value of $\mathcal{B}$.

3. Main results

In this section, we present our main results. Throughout, $(\mathcal{X}_{\mathscr{F}}, \langle\cdot,\cdot\rangle)$ denotes an RKHS defined over a non-empty set $\mathscr{F}$, equipped with its standard inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$.

To establish our first main result, we begin with the following lemma.

Lemma 3.1. *Let $\mathcal{B}, C \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$ be positive operators. Then for all real numbers $r \geq 1$, the following inequality holds:*

$$\left\|\frac{\mathcal{B}+C}{2}\right\|_{ber}^{r} \leq \left\|\frac{\mathcal{B}^r+C^r}{2}\right\|_{ber}.$$

Proof. Let $\widehat{u_\lambda}$ be a normalized reproducing kernel in $\mathcal{X}_{\mathscr{F}}$. Since the function $h(t) = t^r$ is convex on $[0,\infty)$ for $r \geq 1$, we obtain

$$\begin{aligned}
\left(\langle\frac{\mathcal{B}+C}{2}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right)^r &= \left(\frac{\langle\mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle + \langle C\widehat{u_\lambda}, \widehat{u_\lambda}\rangle}{2}\right)^r \\
&\leq \frac{\langle\mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle^r + \langle C\widehat{u_\lambda}, \widehat{u_\lambda}\rangle^r}{2} \\
&\leq \frac{\langle\mathcal{B}^r\widehat{u_\lambda}, \widehat{u_\lambda}\rangle + \langle C^r\widehat{u_\lambda}, \widehat{u_\lambda}\rangle}{2} \quad \text{(by Lemma 2.4)} \\
&= \langle\frac{\mathcal{B}^r+C^r}{2}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle \\
&\leq \mathbf{ber}\left(\frac{\mathcal{B}^r+C^r}{2}\right).
\end{aligned}$$

Since $\frac{\mathcal{B}^r+C^r}{2} \geq 0$, then by applying (1.2), we obtain

$$\left(\langle\frac{\mathcal{B}+C}{2}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right)^r \leq \left\|\frac{\mathcal{B}^r+C^r}{2}\right\|_{\mathbf{ber}}.$$

Taking the supremum over all all $\lambda \in \mathscr{F}$ in the last inequality yields

$$\mathbf{ber}^r\left(\frac{\mathcal{B}+C}{2}\right) \leq \left\|\frac{\mathcal{B}^r+C^r}{2}\right\|_{\mathbf{ber}}.$$

This completes the proof by taking (1.2) into account since $\frac{\mathcal{B}+C}{2} \geq 0$.

We now in a position to investigate our first main result concerning the Berezin number in $\mathbb{L}(\mathcal{X}_{\mathscr{F}})$, elucidating the bounds on the operator products.

Theorem 3.1. *Let $\mathcal{B}, C \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Then for any $\varepsilon \in [0,1]$, we have the following inequality:*

$$\begin{aligned} \mathbf{ber}^2(C^*\mathcal{B}) &\leq \frac{(1-\varepsilon)}{2}\mathbf{ber}(C^*\mathcal{B})\left\||\mathcal{B}|^2+|C|^2\right\|_{\mathbf{ber}}+\frac{\varepsilon}{2}\left\||\mathcal{B}|^4+|C|^4\right\|_{\mathbf{ber}} \\ &\leq \frac{1}{2}\left\||\mathcal{B}|^4+|C|^4\right\|_{\mathbf{ber}}. \end{aligned}$$

Proof. Fix $\varepsilon \in [0,1]$, and let $\widehat{u_\lambda}$ be a normalized reproducing kernel in $\mathcal{X}_{\mathscr{F}}$. Applying Lemma 2.1 with $x = \mathcal{B}\widehat{u_\lambda}$ and $y = C\widehat{u_\lambda}$, we obtain

$$\begin{aligned} \left|\langle \mathcal{B}\widehat{u_\lambda}, C\widehat{u_\lambda}\rangle\right|^2 &\leq (1-\varepsilon)\|\mathcal{B}\widehat{u_\lambda}\|\|C\widehat{u_\lambda}\|\left|\langle \mathcal{B}\widehat{u_\lambda}, C\widehat{u_\lambda}\rangle\right| + \varepsilon\|\mathcal{B}\widehat{u_\lambda}\|^2\|C\widehat{u_\lambda}\|^2 \\ &= (1-\varepsilon)\sqrt{\langle|\mathcal{B}|^2\widehat{u_\lambda},\widehat{u_\lambda}\rangle}\sqrt{\langle|C|^2\widehat{u_\lambda},\widehat{u_\lambda}\rangle}\left|\langle C^*\mathcal{B}\widehat{u_\lambda},\widehat{u_\lambda}\rangle\right| + \varepsilon\langle|\mathcal{B}|^2\widehat{u_\lambda},\widehat{u_\lambda}\rangle\langle|C|^2\widehat{u_\lambda},\widehat{u_\lambda}\rangle. \end{aligned}$$

Now, applying the arithmetic-geometric mean inequality yields

$$\left|\langle C^*\mathcal{B}\widehat{u_\lambda},\widehat{u_\lambda}\rangle\right|^2 \leq \frac{(1-\varepsilon)}{2}\mathbf{ber}(C^*\mathcal{B})\langle(|\mathcal{B}|^2+|C|^2)\widehat{u_\lambda},\widehat{u_\lambda}\rangle + \frac{\varepsilon}{2}\left(\langle|\mathcal{B}|^2\widehat{u_\lambda},\widehat{u_\lambda}\rangle^2 + \langle|C|^2\widehat{u_\lambda},\widehat{u_\lambda}\rangle^2\right).$$

Since both $|\mathcal{B}|^2$ and $|C|^2$ are positive operators, we can apply Lemma 2.4 to get

$$\begin{aligned} \left|\langle C^*\mathcal{B}\widehat{u_\lambda},\widehat{u_\lambda}\rangle\right|^2 &\leq \frac{(1-\varepsilon)}{2}\mathbf{ber}(C^*\mathcal{B})\langle(|\mathcal{B}|^2+|C|^2)\widehat{u_\lambda},\widehat{u_\lambda}\rangle + \frac{\varepsilon}{2}\langle\left(|\mathcal{B}|^4+|C|^4\right)\widehat{u_\lambda},\widehat{u_\lambda}\rangle \\ &\leq \frac{(1-\varepsilon)}{2}\mathbf{ber}(C^*\mathcal{B})\,\mathbf{ber}(|\mathcal{B}|^2+|C|^2) + \frac{\varepsilon}{2}\mathbf{ber}\left(|\mathcal{B}|^4+|C|^4\right). \end{aligned}$$

Taking the supremum over all $\lambda \in \mathscr{F}$ in the last inequality gives

$$\mathbf{ber}^2(C^*\mathcal{B}) \leq \frac{(1-\varepsilon)}{2}\mathbf{ber}(C^*\mathcal{B})\,\mathbf{ber}(|\mathcal{B}|^2+|C|^2) + \frac{\varepsilon}{2}\mathbf{ber}\left(|\mathcal{B}|^4+|C|^4\right).$$

Since $|\mathcal{B}|^2 + |C|^2 \geq 0$ and $|\mathcal{B}|^4 + |C|^4 \geq 0$, then we apply (1.2) to obtain

$$\mathbf{ber}^2(C^*\mathcal{B}) \leq \frac{(1-\varepsilon)}{2}\mathbf{ber}(C^*\mathcal{B})\left\||\mathcal{B}|^2+|C|^2\right\|_{\mathbf{ber}} + \frac{\varepsilon}{2}\left\||\mathcal{B}|^4+|C|^4\right\|_{\mathbf{ber}}.$$

This proves the first inequality in Theorem 3.1.

Next, since both $|\mathcal{B}|^2$ and $|C|^2$ are positive operators, then by using Lemma 3.1 for $r = 2$, we deduce that

$$\begin{aligned} \mathbf{ber}^2(C^*\mathcal{B}) &\leq \frac{(1-\varepsilon)}{4}\left\||\mathcal{B}|^2+|C|^2\right\|_{\mathbf{ber}}^2 + \frac{\varepsilon}{2}\left\||\mathcal{B}|^4+|C|^4\right\|_{\mathbf{ber}} \\ &= (1-\varepsilon)\left\|\frac{|\mathcal{B}|^2+|C|^2}{2}\right\|_{\mathbf{ber}}^2 + \frac{\varepsilon}{2}\left\||\mathcal{B}|^4+|C|^4\right\|_{\mathbf{ber}} \\ &\leq (1-\varepsilon)\left\|\frac{|\mathcal{B}|^4+|C|^4}{2}\right\|_{\mathbf{ber}} + \frac{\varepsilon}{2}\left\||\mathcal{B}|^4+|C|^4\right\|_{\mathbf{ber}} \\ &= \frac{(1-\varepsilon)}{2}\left\||\mathcal{B}|^4+|C|^4\right\|_{\mathbf{ber}} + \frac{\varepsilon}{2}\left\||\mathcal{B}|^4+|C|^4\right\|_{\mathbf{ber}}. \end{aligned}$$

This establishes the second inequality in Theorem 3.1, completing the proof.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane. The Hardy space $H^2(\mathbb{D})$ consists of analytic functions f on $\mathbb{D}$ such that the coefficients of their power series expansion are square-summable. The reproducing kernel for $H^2(\mathbb{D})$ is the Szegő kernel, defined by

$$k_\lambda(z) = \frac{1}{1 - \bar{\lambda}z}, \quad \lambda, z \in \mathbb{D}.$$

By the reproducing property, the squared norm of the kernel is given by

$$\|k_\lambda\|^2 = \langle k_\lambda, k_\lambda\rangle = k_\lambda(\lambda) = \frac{1}{1 - |\lambda|^2}.$$

Therefore, the normalized reproducing kernel $\widehat{k_\lambda}$ is obtained by

$$\widehat{k_\lambda}(z) = \frac{k_\lambda(z)}{\|k_\lambda\|} = \frac{\sqrt{1 - |\lambda|^2}}{1 - \bar{\lambda}z}.$$

For a function $\phi \in L^\infty(\mathbb{T})$, the Toeplitz operator T_ϕ on $H^2(\mathbb{D})$ is defined by $T_\phi f = P(\phi f)$, where P is the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$. The Berezin symbol of T_ϕ, denoted by $\widetilde{T_\phi}$, is given by

$$\widetilde{T_\phi}(\lambda) = \langle T_\phi \widehat{k_\lambda}, \widehat{k_\lambda}\rangle.$$

This transform corresponds to the Poisson extension of the symbol ϕ. The Berezin number of the Toeplitz operator is then

$$\mathbf{ber}(T_\phi) = \sup_{\lambda\in\mathbb{D}} |\widetilde{T_\phi}(\lambda)|.$$

We can apply Theorem 3.1 to estimate the Berezin number of the product of two Toeplitz operators. The following corollary provides a novel bound for the product operator $T_\psi^* T_\phi$.

Corollary 3.1. *Let $\phi, \psi \in L^\infty(\mathbb{T})$. For any $\varepsilon \in [0, 1]$, we have*

$$\boldsymbol{ber}^2(T_\psi^* T_\phi) \le \frac{1-\varepsilon}{2} \boldsymbol{ber}(T_\psi^* T_\phi) \Big\| |T_\phi|^2 + |T_\psi|^2 \Big\|_{\boldsymbol{ber}} + \frac{\varepsilon}{2} \Big\| |T_\phi|^4 + |T_\psi|^4 \Big\|_{\boldsymbol{ber}}.$$

This result offers an explicit upper bound for the Berezin number of the product $T_\psi^* T_\phi$ in terms of the Berezin norms of the moduli of the individual Toeplitz operators, providing a useful tool for analyzing operator products in function spaces.

Our subsequent result is presented below, with our core method relying on Lemma 2.3.

Theorem 3.2. *Let $\mathcal{B} \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Then*

$$\begin{aligned} \boldsymbol{ber}^4(\mathcal{B}) \le{}& \frac{1}{24} \Big\| |\mathcal{B}|^4 + |\mathcal{B}^*|^4 \Big\| + \frac{1}{12} \boldsymbol{ber}^2(\mathcal{B}^2) + \frac{1}{3} \boldsymbol{ber}^2(\mathcal{B}) \boldsymbol{ber}(\mathcal{B}^2) \\ &+ \frac{1}{12} \Big\| |\mathcal{B}|^2 + |\mathcal{B}^*|^2 \Big\| \Big(\boldsymbol{ber}(\mathcal{B}^2) + 2 \boldsymbol{ber}^2(\mathcal{B}) \Big). \end{aligned}$$

Proof. Let $\widehat{u_\lambda}$ be a normalized reproducing kernel in $\mathcal{X}_{\mathscr{F}}$. By applying Lemma 2.3 with $u = \widehat{u_\lambda}$ and then replacing x and y with $\mathcal{B}\widehat{u_\lambda}$ and $\mathcal{B}^*\widehat{u_\lambda}$, respectively, we have

$$\begin{aligned}
12\left|\langle \mathcal{B}\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right|^4 &\leq \|\mathcal{B}\widehat{u}_\lambda\|^2\|\mathcal{B}^*\widehat{u}_\lambda\|^2 + \left|\langle \mathcal{B}\widehat{u}_\lambda, \mathcal{B}^*\widehat{u}_\lambda\rangle\right|^2 + 2\|\mathcal{B}\widehat{u}_\lambda\|\|\mathcal{B}^*\widehat{u}_\lambda\|\left|\langle \mathcal{B}\widehat{u}_\lambda, \mathcal{B}^*\widehat{u}_\lambda\rangle\right| \\
&\quad + 4\left|\langle \mathcal{B}\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right|^2\left(\|\mathcal{B}\widehat{u}_\lambda\|\|\mathcal{B}^*\widehat{u}_\lambda\| + |\langle \mathcal{B}\widehat{u}_\lambda, \mathcal{B}^*\widehat{u}_\lambda\rangle|\right) \\
&= \langle |\mathcal{B}|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\langle |\mathcal{B}^*|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle + 2\sqrt{\langle |\mathcal{B}|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\langle |\mathcal{B}^*|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle}\left|\langle \mathcal{B}^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right| \\
&\quad + 4\left|\langle \mathcal{B}\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right|^2\left(\sqrt{\langle |\mathcal{B}|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\langle |\mathcal{B}^*|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle} + |\langle \mathcal{B}^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle|\right) + \left|\langle \mathcal{B}^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right|^2.
\end{aligned}$$

Further, by applying the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
12\left|\langle \mathcal{B}\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right|^4 &\leq \frac{1}{2}\left(\langle |\mathcal{B}|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle^2 + \langle |\mathcal{B}^*|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle^2\right) + \left|\langle \mathcal{B}^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right|\left(\langle |\mathcal{B}|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle + \langle |\mathcal{B}^*|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right) \\
&\quad + 2\left|\langle \mathcal{B}\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right|^2\left(\langle |\mathcal{B}|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle + \langle |\mathcal{B}^*|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle + 2|\langle \mathcal{B}^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle|\right) + \left|\langle \mathcal{B}^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right|^2.
\end{aligned}$$

Further, since $|\mathcal{B}|^2$ and $|\mathcal{B}^*|^2$ are positive, then by applying Lemma 2.4, we get

$$\begin{aligned}
12\left|\langle \mathcal{B}\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right|^4 &\leq \frac{1}{2}\left(\langle [|\mathcal{B}|^4 + |\mathcal{B}^*|^4]\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right) + \mathbf{ber}(\mathcal{B}^2)\left(\langle [|\mathcal{B}|^2 + |\mathcal{B}^*|^2]\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right) \\
&\quad + 2\,\mathbf{ber}^2(\mathcal{B})\left(\langle (|\mathcal{B}|^2 + |\mathcal{B}^*|^2)\widehat{u}_\lambda, \widehat{u}_\lambda\rangle + 2\,\mathbf{ber}(\mathcal{B}^2)\right) + \mathbf{ber}^2(\mathcal{B}^2) \\
&\leq \frac{1}{2}\,\mathbf{ber}\left(|\mathcal{B}|^4 + |\mathcal{B}^*|^4\right) + \mathbf{ber}(\mathcal{B}^2)\,\mathbf{ber}\left(|\mathcal{B}|^2 + |\mathcal{B}^*|^2\right) \\
&\quad + 2\,\mathbf{ber}^2(\mathcal{B})\left[\mathbf{ber}\left(|\mathcal{B}|^2 + |\mathcal{B}^*|^2\right) + 2\,\mathbf{ber}(\mathcal{B}^2)\right] + \mathbf{ber}^2(\mathcal{B}^2).
\end{aligned}$$

Since $|\mathcal{B}|^4 + |\mathcal{B}^*|^4 \geq 0$ and $|\mathcal{B}|^2 + |\mathcal{B}^*|^2 \geq 0$, then we apply (1.2) to obtain

$$\begin{aligned}
12\left|\langle \mathcal{B}\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right|^4 &\leq \frac{1}{2}\left\||\mathcal{B}|^4 + |\mathcal{B}^*|^4\right\|_{\mathbf{ber}} + \mathbf{ber}(\mathcal{B}^2)\left\||\mathcal{B}|^2 + |\mathcal{B}^*|^2\right\|_{\mathbf{ber}} \\
&\quad + 2\,\mathbf{ber}^2(\mathcal{B})\left(\left\||\mathcal{B}|^2 + |\mathcal{B}^*|^2\right\|_{\mathbf{ber}} + 2\,\mathbf{ber}(\mathcal{B}^2)\right) + \mathbf{ber}^2(\mathcal{B}^2) \\
&= \frac{1}{2}\left\||\mathcal{B}|^4 + |\mathcal{B}^*|^4\right\|_{\mathbf{ber}} + \mathbf{ber}^2(\mathcal{B}^2) + 4\,\mathbf{ber}^2(\mathcal{B})\,\mathbf{ber}(\mathcal{B}^2) \\
&\quad + \left\||\mathcal{B}|^2 + |\mathcal{B}^*|^2\right\|_{\mathbf{ber}}\left(\mathbf{ber}(\mathcal{B}^2) + 2\,\mathbf{ber}^2(\mathcal{B})\right).
\end{aligned}$$

This gives the following:

$$\begin{aligned}
\left|\langle \mathcal{B}\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right|^4 &\leq \frac{1}{24}\left\||\mathcal{B}|^4 + |\mathcal{B}^*|^4\right\|_{\mathbf{ber}} + \frac{1}{12}\,\mathbf{ber}^2(\mathcal{B}^2) + \frac{1}{3}\,\mathbf{ber}^2(\mathcal{B})\,\mathbf{ber}(\mathcal{B}^2) \\
&\quad + \frac{1}{12}\left\||\mathcal{B}|^2 + |\mathcal{B}^*|^2\right\|_{\mathbf{ber}}\left(\mathbf{ber}(\mathcal{B}^2) + 2\,\mathbf{ber}^2(\mathcal{B})\right).
\end{aligned}$$

This proves Theorem 3.2 by taking the supremum over all $\lambda \in \mathscr{F}$.

The next result establishes a new bound for the Berezin number of an operator sum in $\mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Our proof fundamentally relies on Lemma 2.5. The inspiration for our investigation comes from recent work [9].

Theorem 3.3. *Let $\mathcal{B}, C \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Then*

$$\boldsymbol{ber}(\mathcal{B}+C) \leq \sqrt{\min\left\{\tau(\mathcal{B},C), \tau(C,\mathcal{B})\right\} + \boldsymbol{ber}^2(\mathcal{B}) + \boldsymbol{ber}^2(C)},$$

where

$$\tau(X,Y) := \frac{1}{2}\left\||X|^2 + |Y^*|^2\right\|_{\boldsymbol{ber}} + \boldsymbol{ber}(YX).$$

Proof. Let $\widehat{u_\lambda}$ be a normalized reproducing kernel in $\mathcal{X}_{\mathscr{F}}$. Then

$$\begin{aligned}
\left|\langle (\mathcal{B}+C)\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right|^2 &\leq \left(\left|\langle \mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right| + \left|\langle C\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right|\right)^2 \\
&= \left|\langle \mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right|^2 + \left|\langle C\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right|^2 + 2\left|\langle \mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right|\left|\langle \widehat{u_\lambda}, C^*\widehat{u_\lambda}\rangle\right| \\
&\leq \left|\langle \mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right|^2 + \left|\langle C\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right|^2 + \left\|\mathcal{B}\widehat{u_\lambda}\right\|\left\|C^*\widehat{u_\lambda}\right\| + \left|\langle \mathcal{B}\widehat{u_\lambda}, C^*\widehat{u_\lambda}\rangle\right|,
\end{aligned}$$

where the last inequality follows by applying Lemma 2.5. Furthermore, by using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
\left|\langle (\mathcal{B}+C)\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right|^2 &\leq \left|\langle \mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right|^2 + \left|\langle C\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right|^2 + \frac{1}{2}\left(\left\|\mathcal{B}\widehat{u_\lambda}\right\|^2 + \left\|C^*\widehat{u_\lambda}\right\|^2\right) + \left|\langle C\mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right| \\
&= \left|\langle \mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right|^2 + \left|\langle C\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right|^2 + \frac{1}{2}\langle \left(|\mathcal{B}|^2 + |C^*|^2\right)\widehat{u_\lambda}, \widehat{u_\lambda}\rangle + \left|\langle C\mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right| \\
&\leq \mathbf{ber}^2(\mathcal{B}) + \mathbf{ber}^2(C) + \frac{1}{2}\mathbf{ber}\left(\mathcal{B}|^2 + |C^*|^2\right) + \mathbf{ber}(C\mathcal{B}).
\end{aligned}$$

By taking the supremum over all $\lambda \in \mathscr{F}$ in the inequality above, we get

$$\mathbf{ber}^2(\mathcal{B}+C) \leq \mathbf{ber}^2(\mathcal{B}) + \mathbf{ber}^2(C) + \frac{1}{2}\mathbf{ber}\left(\mathcal{B}|^2 + |C^*|^2\right) + \mathbf{ber}(C\mathcal{B}).$$

Applying (1.2) gives

$$\mathbf{ber}^2(\mathcal{B}+C) \leq \mathbf{ber}^2(\mathcal{B}) + \mathbf{ber}^2(C) + \frac{1}{2}\left\||\mathcal{B}|^2 + |C^*|^2\right\|_{\mathbf{ber}} + \mathbf{ber}(C\mathcal{B}). \tag{3.1}$$

Similarly, we can show that

$$\mathbf{ber}^2(\mathcal{B}+C) \leq \mathbf{ber}^2(\mathcal{B}) + \mathbf{ber}^2(C) + \frac{1}{2}\left\||\mathcal{B}^*|^2 + |C|^2\right\|_{\mathbf{ber}} + \mathbf{ber}(\mathcal{B}C). \tag{3.2}$$

From (3.1) and (3.2), we have

$$\begin{aligned}
\mathbf{ber}^2(\mathcal{B}+C) \quad \leq \quad & \min\left\{\frac{1}{2}\left\||\mathcal{B}|^2 + |C^*|^2\right\|_{\mathbf{ber}} + \mathbf{ber}(C\mathcal{B}), \frac{1}{2}\left\||\mathcal{B}^*|^2 + |C|^2\right\|_{\mathbf{ber}} + \mathbf{ber}(\mathcal{B}C)\right\} \\
& + \mathbf{ber}^2(\mathcal{B}) + \mathbf{ber}^2(C).
\end{aligned}$$

This completes the proof.

Corollary 3.2. *For any operator $\mathcal{B} \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$, we have*

$$\boldsymbol{ber}(\mathcal{B}) \leq \sqrt{\frac{1}{4}\left\||\mathcal{B}|^2 + |\mathcal{B}^*|^2\right\|_{\boldsymbol{ber}} + \frac{1}{2}\boldsymbol{ber}(\mathcal{B}^2)}.$$

We now explore a key result on the operator norms in $\mathbb{L}(\mathcal{X}_{\mathscr{F}})$, showing how the norm of a specific expression relates to the differences and sums of operators. Our proof fundamentally relies on Lemma 2.9.

Theorem 3.4. *Let $\mathcal{B}, C \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Then*

$$\left\| \frac{|\mathcal{B}|^2 + |C|^2}{2} \right\|_{ber}^{\frac{p}{2}} \leq \frac{1}{4} \left(\|\mathcal{B} - C\|_{\widetilde{ber}}^{p} + \|\mathcal{B} + C\|_{\widetilde{ber}}^{p} \right),$$

for every $p \geq 2$.

Proof. Let $\widehat{u}_\lambda \in \mathcal{X}_{\mathscr{F}}$ be a normalized reproducing kernel and $p \geq 2$. By using Lemma 2.7, we have

$$\frac{1}{2} \left(\left\|(\mathcal{B} + C)\widehat{u}_\lambda\right\|^p + \left\|(\mathcal{B} - C)\widehat{u}_\lambda\right\|^p \right) \geq \left\|\mathcal{B}\widehat{u}_\lambda\right\|^p + \left\|C\widehat{u}_\lambda\right\|^p . \tag{3.3}$$

Furthermore, by applying the second inequality in Lemma 2.8, we infer that

$$\begin{aligned} \left\|\mathcal{B}\widehat{u}_\lambda\right\|^p + \left\|C\widehat{u}_\lambda\right\|^p &= \left(\left\|\mathcal{B}\widehat{u}_\lambda\right\|^2\right)^{\frac{p}{2}} + \left(\left\|C\widehat{u}_\lambda\right\|^2\right)^{\frac{p}{2}} \\ &\geq 2^{1-\frac{p}{2}} \left(\left\|\mathcal{B}\widehat{u}_\lambda\right\|^2 + \left\|C\widehat{u}_\lambda\right\|^2\right)^{\frac{p}{2}} . \end{aligned}$$

Therefore, we deduce that

$$\left\|\mathcal{B}\widehat{u}_\lambda\right\|^p + \left\|C\widehat{u}_\lambda\right\|^p \geq 2^{1-\frac{p}{2}} \left[\left\langle \left(|\mathcal{B}|^2 + |C|^2\right)\widehat{u}_\lambda, \widehat{u}_\lambda \right\rangle \right]^{\frac{p}{2}} . \tag{3.4}$$

Combining (3.3) with (3.4) gives

$$\frac{1}{4} \left(\left\|(\mathcal{B} + C)\,\widehat{u}_\lambda\right\|^p + \left\|(\mathcal{B} - C)\,\widehat{u}_\lambda\right\|^p \right) \geq \left\langle \frac{|\mathcal{B}|^2 + |C|^2}{2} \widehat{u}_\lambda, \widehat{u}_\lambda \right\rangle^{\frac{p}{2}} .$$

Taking the supremum in the inequality above over all $\lambda \in \mathscr{F}$, we get

$$\frac{1}{4} \left(\|\mathcal{B} + C\|_{\widetilde{\mathbf{ber}}}^{p} + \|\mathcal{B} - C\|_{\widetilde{\mathbf{ber}}}^{p} \right) \geq \mathbf{ber}^{\frac{p}{2}} \left(\frac{|\mathcal{B}|^2 + |C|^2}{2} \right).$$

Since $|\mathcal{B}|^2 + |C|^2$ is a positive operator. Thus we deduce the desired result by applying (1.2). This finishes the proof.

By letting $C = \mathcal{B}^*$ in Theorem 3.4, we obtain the following corollary.

Corollary 3.3. *Let $\mathcal{B} \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$ and let $p \geq 2$. Then*

$$\left\| \frac{|\mathcal{B}|^2 + |\mathcal{B}^*|^2}{2} \right\|_{ber}^{\frac{p}{2}} \leq \frac{1}{4} \left(\|\mathcal{B} + \mathcal{B}^*\|_{\widetilde{ber}}^{p} + \|\mathcal{B} - \mathcal{B}^*\|_{\widetilde{ber}}^{p} \right).$$

We now study a significant result of the Berezin norm in $\mathbb{L}(\mathcal{X}_{\mathscr{F}})$, establishing bounds for the sums of operators. Our proof relies on Lemma 2.9.

Theorem 3.5. *Let $\mathcal{B}, C \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Then*

$$\|\mathcal{B}+C\|_{ber} \leq \sqrt{\left\||\mathcal{B}|^2+|C|^2\right\|_{ber}+\frac{1}{2}\,\|C^*\mathcal{B}+\mathcal{B}^*C\|_{\widetilde{ber}}^2+\frac{1}{2}}.$$

Proof. Let $\widehat{u_\lambda} \in \mathcal{X}_{\mathscr{F}}$ be a normalized reproducing kernel. Then, we have

$$\begin{aligned}
\left\|(\mathcal{B}+C)\,\widehat{u_\lambda}\right\|^2 &= \left\|\mathcal{B}\widehat{u_\lambda}\right\|^2+\left\|C\widehat{u_\lambda}\right\|^2+2\Re\left(\langle\mathcal{B}\widehat{u_\lambda}, C\widehat{u_\lambda}\rangle\right)\\
&= \langle|\mathcal{B}|^2\widehat{u_\lambda}, \widehat{u_\lambda}\rangle+\langle|C|^2\widehat{u_\lambda}, \widehat{u_\lambda}\rangle+\Re\left(\langle C^*\mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right)+\Re\left(\langle\widehat{u_\lambda}, \mathcal{B}^*C\widehat{u_\lambda}\rangle\right)\\
&= \left\langle\left(|\mathcal{B}|^2+|C|^2\right)\widehat{u_\lambda}, \widehat{u_\lambda}\right\rangle+\Re\left(\langle C^*\mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right)+\Re\left(\langle\mathcal{B}^*C\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right)\\
&= \left\langle\left(|\mathcal{B}|^2+|C|^2\right)\widehat{u_\lambda}, \widehat{u_\lambda}\right\rangle+\Re\left(\left\langle\left(C^*\mathcal{B}+\mathcal{B}^*C\right)\widehat{u_\lambda}, \widehat{u_\lambda}\right\rangle\right).
\end{aligned}$$

Moreover, in view of Lemma 2.9, we see that

$$\Re\left(\left\langle\left(C^*\mathcal{B}+\mathcal{B}^*C\right)\widehat{u_\lambda}, \widehat{u_\lambda}\right\rangle\right) \leq \frac{1}{2}\left(\left\|(C^*\mathcal{B}+\mathcal{B}^*C)\,\widehat{u_\lambda}\right\|^2+1\right).$$

This implies that

$$\left\|(\mathcal{B}+C)\,\widehat{u_\lambda}\right\|^2 \leq \left\langle\left(|\mathcal{B}|^2+|C|^2\right)\widehat{u_\lambda}, \widehat{u_\lambda}\right\rangle+\frac{1}{2}\left(\left\|(C^*\mathcal{B}+\mathcal{B}^*C)\,\widehat{u_\lambda}\right\|^2+1\right).$$

Taking the supremum in the above inequality over all $\lambda \in \mathscr{F}$, we get

$$\|(\mathcal{B}+C)\|_{\widetilde{\mathbf{ber}}}^2 \leq \mathbf{ber}\left(|\mathcal{B}|^2+|C|^2\right)+\frac{1}{2}\,\|C^*\mathcal{B}+\mathcal{B}^*C\|_{\widetilde{\mathbf{ber}}}^2+\frac{1}{2}.$$

Since $|\mathcal{B}|^2+|C|^2$ is a positive operator, then by (1.2), we have

$$\|(\mathcal{B}+C)\|_{\widetilde{\mathbf{ber}}}^2 \leq \left\||\mathcal{B}|^2+|C|^2\right\|_{\mathbf{ber}}+\frac{1}{2}\,\|C^*\mathcal{B}+\mathcal{B}^*C\|_{\widetilde{\mathbf{ber}}}^2+\frac{1}{2}.$$

Thus the desired result is established by applying (1.1).

By taking $C=\mathcal{B}^*$ in Theorem 3.5, we obtain the following Berezin norm inequality for the real part of $\mathcal{B} \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$, given by

$$\Re e(\mathcal{B})=\frac{\mathcal{B}+\mathcal{B}^*}{2}.$$

Corollary 3.4. *Let $\mathcal{B} \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Then*

$$\left\|\Re e(\mathcal{B})\right\|_{ber} \leq \frac{1}{2}\sqrt{\left\|\,|\mathcal{B}|^2+|\mathcal{B}^*|^2\right\|_{ber}+\tfrac{1}{2}\left\|\mathcal{B}^{*2}+\mathcal{B}^2\right\|_{\widetilde{ber}}^2+\tfrac{1}{2}}.$$

Next, we establish a general inequality that provides an upper bound for the Berezin norm of a finite sum of operators in terms of their individual Berezin norms and Berezin numbers of pairwise products.

Theorem 3.6. *Let $\mathcal{B}_1, ..., \mathcal{B}_n \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Then*

$$\left\|\sum_{k=1}^{n}\mathcal{B}_k\right\|_{ber}^2 \leq \left\|\sum_{k=1}^{n}|\mathcal{B}_k|^2\right\|_{ber}+\sum_{1\leq k\neq j\leq n}\boldsymbol{ber}\left(\mathcal{B}_j^*\mathcal{B}_k\right).$$

Proof. Let $\widehat{u}_\lambda \in \mathcal{X}_{\mathscr{F}}$ be a normalized reproducing kernel. Then

$$\begin{aligned}\left\|\sum_{k=1}^{n}\mathcal{B}_k\widehat{u}_\lambda\right\|^2 &= \left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle\mathcal{B}_k\widehat{u}_\lambda,\mathcal{B}_j\widehat{u}_\lambda\rangle\right| \\ &= \left|\sum_{k=1}^{n}\langle\mathcal{B}_k\widehat{u}_\lambda,\mathcal{B}_k\widehat{u}_\lambda\rangle+\sum_{1\le k\neq j\le n}\langle\mathcal{B}_k\widehat{u}_\lambda,\mathcal{B}_j\widehat{u}_\lambda\rangle\right| \\ &\le \left|\sum_{k=1}^{n}\langle|\mathcal{B}_k|^2\widehat{u}_\lambda,\widehat{u}_\lambda\rangle\right|+\left|\sum_{1\le k\neq j\le n}\langle\mathcal{B}_j^*\mathcal{B}_k\widehat{u}_\lambda,\widehat{u}_\lambda\rangle\right|.\end{aligned}$$

This implies that

$$\begin{aligned}\left\|\sum_{k=1}^{n}\mathcal{B}_k\widehat{u}_\lambda\right\|^2 &\le \left|\langle\sum_{k=1}^{n}|\mathcal{B}_k|^2\widehat{u}_\lambda,\widehat{u}_\lambda\rangle\right|+\sum_{1\le k\neq j\le n}\left|\langle\mathcal{B}_j^*\mathcal{B}_k\widehat{u}_\lambda,\widehat{u}_\lambda\rangle\right| \\ &\le \mathbf{ber}\left(\sum_{k=1}^{n}|\mathcal{B}_k|^2\right)+\sum_{1\le k\neq j\le n}\mathbf{ber}\left(\mathcal{B}_j^*\mathcal{B}_k\right).\end{aligned}$$

Taking the supremum in the inequality above over all $\lambda \in \mathscr{F}$, we obtain

$$\left\|\sum_{k=1}^{n}\mathcal{B}_k\right\|_{\widetilde{\mathbf{ber}}}^2 \le \mathbf{ber}\left(\sum_{k=1}^{n}|\mathcal{B}_k|^2\right)+\sum_{1\le k\neq j\le n}\mathbf{ber}\left(\mathcal{B}_j^*\mathcal{B}_k\right).$$

Since $\sum_{k=1}^{n}|\mathcal{B}_k|^2 \geq 0$, by (1.2), we get

$$\left\|\sum_{k=1}^{n}\mathcal{B}_k\right\|_{\widetilde{\mathbf{ber}}}^2 \le \left\|\sum_{k=1}^{n}|\mathcal{B}_k|^2\right\|_{\mathbf{ber}}+\sum_{1\le k\neq j\le n}\mathbf{ber}\left(\mathcal{B}_j^*\mathcal{B}_k\right).$$

Therefore, the desired result follows by taking (1.1) into account.

By taking $n = 2$ and setting $\mathcal{B}_1 = \mathcal{B}$, $\mathcal{B}_2 = C$ in Theorem 3.6, we obtain the following bound for the Berezin norm.

Corollary 3.5. *Let $\mathcal{B}, C \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Then*

$$\left\|\mathcal{B}+C\right\|_{\boldsymbol{ber}} \le \sqrt{\left\||\mathcal{B}|^2+|C|^2\right\|_{\boldsymbol{ber}}+\boldsymbol{ber}(C^*\mathcal{B})+\boldsymbol{ber}(\mathcal{B}^*C)}.$$

The following result presents an n-tuple extension of Theorem 3.5, providing a general Berezin norm inequality for finite families of operators.

Theorem 3.7. *Let $\mathcal{B}_1, ..., \mathcal{B}_n \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Then*

$$\left\|\sum_{k=1}^{n}\mathcal{B}_k\right\|_{\boldsymbol{ber}} \le \sqrt{\left\|\sum_{k=1}^{n}|\mathcal{B}_k|^2\right\|_{\boldsymbol{ber}}+\frac{1}{2}\left\|\sum_{1\le k\neq j\le n}\mathcal{B}_j^*\mathcal{B}_k\right\|_{\widetilde{\boldsymbol{ber}}}^2+\frac{1}{2}}.$$

Proof. Let $\widehat{u}_\lambda$ be a normalized reproducing kernel in $\mathcal{X}_{\mathscr{F}}$. One observes that

$$\begin{aligned}\left\|\sum_{k=1}^{n}\mathcal{B}_k\widehat{u}_\lambda\right\|^2 &= \Re\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\langle\mathcal{B}_k\widehat{u}_\lambda,\mathcal{B}_j\widehat{u}_\lambda\rangle\right)\\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\Re\left(\langle\mathcal{B}_k\widehat{u}_\lambda,\mathcal{B}_j\widehat{u}_\lambda\rangle\right)\\ &= \sum_{k=1}^{n}\left\|\mathcal{B}_k\widehat{u}_\lambda\right\|^2+\sum_{1\le k\ne j\le n}\Re\left(\langle\mathcal{B}_k\widehat{u}_\lambda,\mathcal{B}_j\widehat{u}_\lambda\rangle\right)\\ &= \langle\sum_{k=1}^{n}|\mathcal{B}_k|^2\widehat{u}_\lambda,\widehat{u}_\lambda\rangle+\Re\left(\sum_{1\le k\ne j\le n}\langle\mathcal{B}_j^*\mathcal{B}_k\widehat{u}_\lambda,\widehat{u}_\lambda\rangle\right)\\ &\le \mathbf{ber}\left(\sum_{k=1}^{n}|\mathcal{B}_k|^2\right)+\Re\left(\sum_{1\le k\ne j\le n}\langle\mathcal{B}_j^*\mathcal{B}_k\widehat{u}_\lambda,\widehat{u}_\lambda\rangle\right).\end{aligned}$$

By applying Lemma 2.9, we obtain

$$\left\|\sum_{k=1}^{n}\mathcal{B}_k\widehat{u}_\lambda\right\|^2\le\mathbf{ber}\left(\sum_{k=1}^{n}|\mathcal{B}_k|^2\right)+\frac{1}{2}\left(\left\|\sum_{1\le k\ne j\le n}\mathcal{B}_j^*\mathcal{B}_k\widehat{u}_\lambda\right\|^2+1\right).$$

Taking the supremum in the inequality above over all $\lambda\in\mathscr{F}$ yields

$$\left\|\sum_{k=1}^{n}\mathcal{B}_k\right\|_{\widetilde{\mathbf{ber}}}^2\le\left\|\sum_{k=1}^{n}|\mathcal{B}_k|^2\right\|_{\mathbf{ber}}+\frac{1}{2}\left\|\sum_{1\le k\ne j\le n}\mathcal{B}_j^*\mathcal{B}_k\right\|_{\widetilde{\mathbf{ber}}}^2+\frac{1}{2}.$$

This, in turn, implies the desired result by applying (1.1).

The next result derives a novel bound for the Berezin norm of a sum of the operators in $\mathbb{L}(\mathcal{X}_{\mathscr{F}})$, with our primary tool being Lemma 2.10.

Theorem 3.8. *Let $\mathcal{B}_1,...,\mathcal{B}_n\in\mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Then*

$$\left\|\sum_{k=1}^{n}\mathcal{B}_k\right\|_{ber}\le\sqrt{\left\|\sum_{k=1}^{n}|\mathcal{B}_k|^2\right\|_{ber}+\frac{1}{4}\sum_{k,j=1}^{n}\left\|\mathcal{B}_k+\mathcal{B}_j\right\|_{ber}^2-\sum_{k=1}^{n}\|\mathcal{B}_k\|_{ber}^2}.$$

Proof. Let $\widehat{u}_\lambda$ be a normalized reproducing kernel in $\mathcal{X}_{\mathscr{F}}$. By applying Lemma 2.10, we obtain

$$\Re\left(\langle\mathcal{B}_k\widehat{u}_\lambda,\mathcal{B}_j\widehat{u}_\lambda\rangle\right)\le\frac{1}{4}\left\|\mathcal{B}_k\widehat{u}_\lambda+\mathcal{B}_j\widehat{u}_\lambda\right\|^2.$$

Therefore, we deduce that

$$\sum_{1\le k\ne j\le n}\Re\left(\langle\mathcal{B}_k\widehat{u}_\lambda,\mathcal{B}_j\widehat{u}_\lambda\rangle\right)\le\frac{1}{4}\sum_{1\le k\ne j\le n}\left\|\mathcal{B}_k\widehat{u}_\lambda+\mathcal{B}_j\widehat{u}_\lambda\right\|^2.$$

On the other hand, we have

$$\left\|\sum_{k=1}^{n}\mathcal{B}_k\widehat{u}_\lambda\right\|^2 = \langle \sum_{k=1}^{n}|\mathcal{B}_k|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle + \Re\left(\sum_{1\le k\ne j\le n}\langle \mathcal{B}_j^*\mathcal{B}_k\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right).$$

This implies that

$$\begin{aligned}\left\|\sum_{k=1}^{n}\mathcal{B}_k\widehat{u}_\lambda\right\|^2 &= \langle \sum_{k=1}^{n}|\mathcal{B}_k|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle + \Re\left(\sum_{1\le k\ne j\le n}\langle \mathcal{B}_j^*\mathcal{B}_k\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\right)\\ &\le \langle \sum_{k=1}^{n}|\mathcal{B}_k|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle + \frac{1}{4}\sum_{1\le k\ne j\le n}\left\|\mathcal{B}_k\widehat{u}_\lambda + \mathcal{B}_j\widehat{u}_\lambda\right\|^2\\ &\le \mathbf{ber}\left(\sum_{k=1}^{n}|\mathcal{B}_k|^2\right) + \frac{1}{4}\sum_{1\le k\ne j\le n}\left\|\mathcal{B}_k\widehat{u}_\lambda + \mathcal{B}_j\widehat{u}_\lambda\right\|^2.\end{aligned}$$

Taking the supremum in the inequality above over all $\lambda \in \mathscr{F}$, and taking into account that

$$\mathbf{ber}\left(\sum_{k=1}^{n}|\mathcal{B}_k|^2\right) = \left\|\sum_{k=1}^{n}|\mathcal{B}_k|^2\right\|_{\mathbf{ber}},$$

we get

$$\begin{aligned}\left\|\sum_{k=1}^{n}\mathcal{B}_k\right\|^2_{\widetilde{\mathbf{ber}}} &\le \left\|\sum_{k=1}^{n}|\mathcal{B}_k|^2\right\|_{\mathbf{ber}} + \frac{1}{4}\sum_{1\le k\ne j\le n}\left\|\mathcal{B}_k + \mathcal{B}_j\right\|^2_{\widetilde{\mathbf{ber}}}\\ &= \left\|\sum_{k=1}^{n}|\mathcal{B}_k|^2\right\|_{\mathbf{ber}} + \frac{1}{4}\sum_{k,j=1}^{n}\left\|\mathcal{B}_k + \mathcal{B}_j\right\|^2_{\widetilde{\mathbf{ber}}} - \sum_{k=1}^{n}\|\mathcal{B}_k\|^2_{\widetilde{\mathbf{ber}}}.\end{aligned}$$

Therefore, we obtain the desired result by taking (1.1) into account.

By taking $n = 2$ in Theorem 3.8 and setting $\mathcal{B}_1 = \mathcal{B}$, $\mathcal{B}_2 = C$ in Theorem 3.8, we obtain the following bound for the Berezin norm:

Corollary 3.6. *Let $\mathcal{B}, C \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Then*

$$\left\|\mathcal{B} + C\right\|_{ber} \le \sqrt{\left\||\mathcal{B}|^2 + |C|^2\right\|_{ber} + \frac{1}{2}\|\mathcal{B} + C\|^2_{\widetilde{ber}}}.$$

The following example demonstrates that the two corollaries derived from Theorems 3.6 and 3.8 are not comparable.

Example 3.1. Consider $\mathbb{C}^2$ with the standard orthonormal basis $\{e_1, e_2\}$.

Case 1. Let

$$\mathcal{B} = \begin{pmatrix}0 & 3\\ 0 & 0\end{pmatrix}, \quad C = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

In this case,

$$\|\mathcal{B}+C\|_{\mathbf{ber}} = \sup_{i,j\in\{1,2\}} |\langle(\mathcal{B}+C)e_i, e_j\rangle| = 3.$$

For Corollary 3.5, we have

$$\sqrt{\| |\mathcal{B}|^2 + |C|^2\|_{\mathbf{ber}} + \mathbf{ber}(C^*\mathcal{B}) + \mathbf{ber}(\mathcal{B}^*C)} = \sqrt{15} \approx 3.873,$$

while Corollary 3.6 gives

$$\sqrt{\| |\mathcal{B}|^2 + |C|^2\|_{\mathbf{ber}} + \tfrac{1}{2}\|\mathcal{B}+C\|^2_{\mathbf{ber}}} = \sqrt{13.5} \approx 3.674.$$

Hence, the second bound is tighter.

Case 2. Let

$$\mathcal{B} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

Therefore, $\|\mathcal{B}+C\|_{\mathbf{ber}} = 3$ and $\mathcal{B}^*C = C^*\mathcal{B} = \mathbf{0}$. One may see that the bound in Corollary 3.5 is $\sqrt{9} = 3$, whereas that in Corollary 3.6 is $\sqrt{13.5} \approx 3.674$. Hence, the first bound is tighter.

Our next main goal is to establish a new bound for the Berezin number of operators, which enables us to refine several well-known results from the literature. Our primary tool in this derivation is Lemma 2.11.

Theorem 3.9. *Let $C \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$ and $\varepsilon \in [0,1]$. Then, we have*

$$\boldsymbol{ber}(C) \le \sqrt{\frac{\Theta_\varepsilon}{2}\left\||C|^2 + |C^*|^2\right\|_{\boldsymbol{ber}} + \frac{\Delta_\varepsilon}{2}\,\boldsymbol{ber}(C)\left\||C| + |C^*|\right\|_{\boldsymbol{ber}}}. \tag{3.5}$$

Here,

$$\Theta_\varepsilon = \min\{\varepsilon, 1-\varepsilon\} \quad \textit{and} \quad \Delta_\varepsilon = \max\{\varepsilon, 1-\varepsilon\}.$$

Proof. Applying inequality (2.6), we deduce

$$\begin{aligned} |\langle C\widehat{u}_\lambda, \widehat{u}_\lambda\rangle| &\le \Theta_\varepsilon|\langle C\widehat{u}_\lambda, \widehat{u}_\lambda\rangle| + (1-\Theta_\varepsilon)|\langle C\widehat{u}_\lambda, \widehat{u}_\lambda\rangle| \\ &\le \Theta_\varepsilon\sqrt{\langle |C|\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\langle |C^*|\widehat{u}_\lambda, \widehat{u}_\lambda\rangle} + \Delta_\varepsilon|\langle C\widehat{u}_\lambda, \widehat{u}_\lambda\rangle|, \end{aligned}$$

for any normalized reproducing kernel $\widehat{u}_\lambda$ and $\varepsilon \in [0,1]$. Multiplying the relation above by $\sqrt{\langle |C|\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\,\langle |C^*|\widehat{u}_\lambda, \widehat{u}_\lambda\rangle}$ and noting that

$$|\langle C\widehat{u}_\lambda, \widehat{u}_\lambda\rangle|^2 \le |\langle C\widehat{u}_\lambda, \widehat{u}_\lambda\rangle|\sqrt{\langle |C|\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\,\langle |C^*|\widehat{u}_\lambda, \widehat{u}_\lambda\rangle},$$

we derive the following inequality:

$$|\langle C\widehat{u}_\lambda, \widehat{u}_\lambda\rangle|^2 \le \Theta_\varepsilon\langle |C|\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\langle |C^*|\widehat{u}_\lambda, \widehat{u}_\lambda\rangle + \Delta_\varepsilon|\langle C\widehat{u}_\lambda, \widehat{u}_\lambda\rangle|\sqrt{\langle |C|\widehat{u}_\lambda, \widehat{u}_\lambda\rangle\langle |C^*|\widehat{u}_\lambda, \widehat{u}_\lambda\rangle}.$$

Since the operators $|C|$ and $|C^*|$ are positive, we apply Lemma 2.4 to obtain

$$\langle |C|\widehat{u}_\lambda, \widehat{u}_\lambda\rangle^2 \le \langle |C|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle \quad \text{and} \quad \langle |C^*|\widehat{u}_\lambda, \widehat{u}_\lambda\rangle^2 \le \langle |C^*|^2\widehat{u}_\lambda, \widehat{u}_\lambda\rangle.$$

Therefore, by using the arithmetic-geometric mean inequality, we infer that

$$\begin{aligned}
|\langle C\widehat{u_\lambda}, \widehat{u_\lambda}\rangle|^2 &\le \frac{\Theta_\varepsilon}{2}\left(\langle |C|\widehat{u_\lambda}, \widehat{u_\lambda}\rangle^2 + \langle |C^*|\widehat{u_\lambda}, \widehat{u_\lambda}\rangle^2\right) + \frac{\Delta_\varepsilon}{2}|\langle C\widehat{u_\lambda}, \widehat{u_\lambda}\rangle|\langle (|C| + |C^*|)\widehat{u_\lambda}, \widehat{u_\lambda}\rangle \\
&\le \frac{\Theta_\varepsilon}{2}\langle (|C|^2 + |C^*|^2)\widehat{u_\lambda}, \widehat{u_\lambda}\rangle + \frac{\Delta_\varepsilon}{2}|\langle C\widehat{u_\lambda}, \widehat{u_\lambda}\rangle|\langle (|C| + |C^*|)\widehat{u_\lambda}, \widehat{u_\lambda}\rangle \\
&\le \frac{\Theta_\varepsilon}{2}\,\mathbf{ber}\left(|C|^2 + |C^*|^2\right) + \frac{\Delta_\varepsilon}{2}\,\mathbf{ber}(C)\,\mathbf{ber}\left(|C| + |C^*|\right).
\end{aligned}$$

Taking the supremum over all $\lambda \in \mathscr{F}$ in the last bound yields

$$\mathbf{ber}^2(C) \le \frac{\Theta_\varepsilon}{2}\,\mathbf{ber}\left(|C|^2 + |C^*|^2\right) + \frac{\Delta_\varepsilon}{2}\,\mathbf{ber}(C)\,\mathbf{ber}\left(|C| + |C^*|\right).$$

Finally, by taking the positivity of the operators $|C|^2 + |C^*|^2$ and $|C| + |C^*|$ into account, and by applying (1.2), we obtain the desired result. Hence, the proof is complete.

As an application, we obtain a refinement of a result proved in [29]. To achieve this, we first need to establish the following lemma.

Lemma 3.2. *Let $\mathcal{B} \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Then the Berezin number of $\mathcal{B}$ satisfies the inequality*

$$\boldsymbol{ber}(\mathcal{B}) \le \frac{1}{2}\,\||\mathcal{B}| + |\mathcal{B}^*|\|_{\boldsymbol{ber}}\,.$$

Proof. Let $\widehat{u_\lambda}$ be a normalized reproducing kernel in $\mathcal{X}_{\mathscr{F}}$. By Lemma 2.11 and the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
\left|\langle \mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right| &\le \langle |\mathcal{B}|\widehat{u_\lambda}, \widehat{u_\lambda}\rangle^{1/2}\langle |\mathcal{B}^*|\widehat{u_\lambda}, \widehat{u_\lambda}\rangle^{1/2} \\
&\le \frac{1}{2}\left(\langle |\mathcal{B}|\widehat{u_\lambda}, \widehat{u_\lambda}\rangle + \langle |\mathcal{B}^*|\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\right) \\
&= \frac{1}{2}\langle (|\mathcal{B}| + |\mathcal{B}^*|)\widehat{u_\lambda}, \widehat{u_\lambda}\rangle \\
&\le \frac{1}{2}\,\mathbf{ber}\left(|\mathcal{B}| + |\mathcal{B}^*|\right).
\end{aligned}$$

Taking the supremum over all $\lambda \in \mathscr{F}$ yields

$$\mathbf{ber}(\mathcal{B}) \le \frac{1}{2}\,\mathbf{ber}\left(|\mathcal{B}| + |\mathcal{B}^*|\right).$$

Since $|\mathcal{B}| + |\mathcal{B}^*|$ is a positive operator, then an application of (1.2) finishes the proof.

We are now in a position to prove the following corollary.

Corollary 3.7. *Let $C \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$. Then*

$$\begin{aligned}
\boldsymbol{ber}^2(C) &\le \frac{1}{4}\left(\left\||C|^2 + |C^*|^2\right\|_{\boldsymbol{ber}} + \boldsymbol{ber}(C)\left\||C| + |C^*|\right\|_{\boldsymbol{ber}}\right) \\
&\le \frac{1}{2}\left\||C|^2 + |C^*|^2\right\|_{\boldsymbol{ber}}.
\end{aligned}$$

Proof. By taking $\varepsilon = \frac{1}{2}$ in (3.5) and by applying Lemma 3.2, we get

$$\begin{aligned}\mathbf{ber}^2(C) &\leq \frac{1}{4}\left(\left\||C|^2 + |C^*|^2\right\|_{\mathbf{ber}} + \mathbf{ber}(C)\left\||C| + |C^*|\right\|_{\mathbf{ber}}\right)\\ &\leq \frac{1}{4}\left(\left\||C|^2 + |C^*|^2\right\|_{\mathbf{ber}} + \frac{1}{2}\left\||C| + |C^*|\right\|^2_{\mathbf{ber}}\right).\end{aligned}$$

Further, by Lemma 3.1 for $r = 2$, we obtain

$$\begin{aligned}\mathbf{ber}^2(C) &\leq \frac{1}{4}\left(\left\||C|^2 + |C^*|^2\right\|_{\mathbf{ber}} + \left\||C|^2 + |C^*|^2\right\|_{\mathbf{ber}}\right)\\ &= \frac{1}{2}\left\||C|^2 + |C^*|^2\right\|_{\mathbf{ber}}.\end{aligned}$$

Consequently, we deduce the desired result.

The following theorem provides a generalized Berezin numerical radius inequality, offering a refined upper bound that depends on a variable parameter and the powers of the operator.

Theorem 3.10. *For $\mathcal{B} \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$, $\beta \in [0, 1]$, and $s \geq 1$, we have*

$$\boldsymbol{ber}^{2s}(\mathcal{B}) \leq \frac{1+\beta}{4}\left\||\mathcal{B}|^{2s} + |\mathcal{B}^*|^{2s}\right\|_{\boldsymbol{ber}} + \frac{1-\beta}{2}\boldsymbol{ber}^s(\mathcal{B}^2). \tag{3.6}$$

Proof. Let $\widehat{u_\lambda}$ be a normalized reproducing kernel in $\mathcal{X}_{\mathscr{F}}$. By using Lemma 2.6, we obtain

$$\begin{aligned}|\langle\mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle|^{2s} &= |\langle\mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle\langle\widehat{u_\lambda}, \mathcal{B}^*\widehat{u_\lambda}\rangle|^s\\ &\leq \frac{1+\beta}{2}\|\mathcal{B}\widehat{u_\lambda}\|^s\|\mathcal{B}^*\widehat{u_\lambda}\|^s + \frac{1-\beta}{2}|\langle\mathcal{B}\widehat{u_\lambda}, \mathcal{B}^*\widehat{u_\lambda}\rangle|^s.\end{aligned}$$

By using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}|\langle\mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle|^{2s} &\leq \frac{1+\beta}{4}\left(\|\mathcal{B}\widehat{u_\lambda}\|^{2s} + \|\mathcal{B}^*\widehat{u_\lambda}\|^{2s}\right) + \frac{1-\beta}{2}|\langle\mathcal{B}^2\widehat{u_\lambda}, \widehat{u_\lambda}\rangle|^s\\ &\leq \frac{1+\beta}{4}\langle((|\mathcal{B}|^2)^s + (|\mathcal{B}^*|^2)^s)\widehat{u_\lambda}, \widehat{u_\lambda}\rangle + \frac{1-\beta}{2}\mathbf{ber}^s(\mathcal{B}^2)\\ &\leq \frac{1+\beta}{4}\mathbf{ber}\left(|\mathcal{B}|^{2s} + |\mathcal{B}^*|^{2s}\right) + \frac{1-\beta}{2}\mathbf{ber}^s(\mathcal{B}^2).\end{aligned}$$

Since $|\mathcal{B}|^{2s} + |\mathcal{B}^*|^{2s} \geq 0$, so by applying (1.2), we deduce that

$$|\langle\mathcal{B}\widehat{u_\lambda}, \widehat{u_\lambda}\rangle|^{2s} \leq \frac{1+\beta}{4}\left\||\mathcal{B}|^{2s} + |\mathcal{B}^*|^{2s}\right\|_{\mathbf{ber}} + \frac{1-\beta}{2}\mathbf{ber}^s(\mathcal{B}^2).$$

Taking the supremum over all $\lambda \in \mathscr{F}$ in the last inequality yields (3.6).

Remark 3.1. *By letting $s = 1$ and $\beta = 0$ in Theorem 3.10, we get the result established in Corollary 3.2.*

We conclude this paper with an improvement of Theorem 3.1. The proof is omitted here and left to the interested reader. The proof mainly uses (2.2).

Theorem 3.11. *Let $\mathcal{B}, C \in \mathbb{L}(\mathcal{X}_{\mathscr{F}})$ and $\varepsilon \in [0, 1]$. We then have*

$$\begin{aligned}\boldsymbol{ber}^2(C^*\mathcal{B}) &\leq \frac{\Theta_\varepsilon}{2}\left\||\mathcal{B}|^4 + |C|^4\right\|_{\boldsymbol{ber}} + \frac{\Delta_\varepsilon}{2}\boldsymbol{ber}(C^*\mathcal{B})\left\||\mathcal{B}|^2 + |C|^2\right\|_{\boldsymbol{ber}}\\ &\leq \frac{1}{2}\left\||\mathcal{B}|^4 + |C|^4\right\|_{\boldsymbol{ber}},\end{aligned}$$

where $\Theta_\varepsilon = \min\{\varepsilon, 1-\varepsilon\}$ and $\Delta_\varepsilon = \max\{\varepsilon, 1-\varepsilon\}$.

4. Conclusions

In this paper, we investigated the Berezin number and the Berezin norm of bounded linear operators acting on reproducing kernel Hilbert spaces. We established novel upper bounds for these quantities, with a particular focus on the sums and products of operators, yielding new estimates for forms such as $\mathbf{ber}(C^*\mathcal{B})$. Furthermore, we explored the relationship between these measures and another equivalent Berezin-type norm, $\|\cdot\|_{\widetilde{\mathbf{ber}}}$. By applying these foundational inequalities, we successfully deduced several applications that refine and improve upon existing operator inequalities in the current literature.

The results presented in this study not only deepen the understanding of operator inequalities within the framework of reproducing kernel Hilbert spaces but also open avenues for broader generalizations. In particular, this paper may serve as a natural starting point for future research directions, such as investigating the joint Berezin number related to a p-tuple of operators, where $p \geq 1$.

Author contributions

Feryal Aladsani, Asmahan Alajyan, Salma Aljawi and Kais Feki: Conceptualization, Visualization, Funding, Writing–review & editing, Formal analysis, Project administration, Validation, Investigation; Feryal Aladsani: Resources. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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