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Research article

On smooth numbers of forms $\left\lfloor \frac{x}{n} \right\rfloor$ and $\left\lfloor \frac{x}{n^c} \right\rfloor$

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Abstract: A positive integer n is called y-smooth, if its largest prime factor $P^+(n)$ is at most y. Smooth numbers are widely recognized as playing a pivotal role in analytic number theory, and the study of their distribution constitutes a core focus in relevant research. On the other hand, Bordellès, Dai, Heyman, Pan, and Shparlinski first studied the primes of the form $\left\lfloor \frac{x}{n} \right\rfloor$, and later, a paper written by Bordellès, Dai, Heyman, and Nikolic concentrated on studying primes of the form $\left\lfloor \frac{x}{n^c} \right\rfloor$. Combining the two lines of research, in this paper we studied the smooth number of the form $\left\lfloor \frac{x}{n} \right\rfloor$ and $\left\lfloor \frac{x}{n^c} \right\rfloor$, where $\lfloor t \rfloor$ denoted the integral part of a real number t. Clearly, studies concerning the distribution of smooth numbers play an important role in number theory, which makes our studies valuable and rather interesting.

Keywords: smooth numbers; exponential sums; floor function; asymptotic formulae; largest prime factor of an integer
Mathematics Subject Classification: 11L07, 11N25

1. Introduction

Recall that for a given positive integer $y \geq 2$, a positive integer n is called y-smooth if its largest prime factor $P^+(n)$ is at most y. For $2 \leq y \leq x$, let

$$S(x, y) := \{n \leq x : P^+(n) \leq y\}, \quad \Psi(x, y) := |S(x, y)|.$$

The size of the parameter y with respect to x is of great importance in the study of smooth numbers. The smaller y is, the sparser the set $S(x, y)$ is, and the more complicated the set $S(x, y)$ becomes. If we write $y = x^{\frac{1}{u}}$, then it is known that $\Psi(x, y) \sim \rho(u)x$ where $\rho(u)$ denotes the Dickman's function defined by $\rho(u) = 1$ for $0 \leq u \leq 1$, and for $u \geqslant 1$ it is defined as the unique continuous solution to the differential-difference equation $u\rho'(u) = -\rho(u-1)$. This asymptotic formula was first published by

Dickman [5] for fixed values of u and as $x \to \infty$. We remark that the studies of smooth numbers play a crucial role in number theory, especially the distribution of smooth numbers; interested readers can see [1] and [14] for examples. Readers can also see the surveys [6] and [10] for an account of classical theory on smooth numbers and their applications.

On the other hand, Bordellès-Dai-Heyman-Pan-Shparlinski [2] established an asymptotic formula of

$$S_f(x) := \sum_{n \le x} f\Big(\Big\lfloor \frac{x}{n} \Big\rfloor\Big), \tag{1.1}$$

under some simple assumptions of f. Subsequently, Wu [13] and Zhai [15] improved their results independently. Later on, Ma and Wu [9] study the case $f(n) = \Lambda(n)$; more precisely, their work focused on studying primes of the form $\left\lfloor \frac{x}{n} \right\rfloor$. Very recently, Bordellès [3] and Liu, Wu, and Yang [8] improved their result subsequently. In another paper [4] written by Bordellès-Dai-Heyman-Nikolic, they concentrated on studying primes of the form $\left\lfloor \frac{x}{n^c} \right\rfloor$. Combining the two lines of research, in this paper we study the smooth numbers of the form $\left\lfloor \frac{x}{n} \right\rfloor$ and $\left\lfloor \frac{x}{n^c} \right\rfloor$. Our main theorems also reflect some properties of smooth numbers, especially the distribution of smooth numbers, so our results are meaningful as well as fascinating. Specifically, we have established the following two theorems.

Theorem 1.1. *Let x and y be real numbers with $\log x < y < x^{1/2}$. Then, we have the following asymptotic formula.*

$$\sum_{\substack{n \le x \\ P^+(\lfloor \frac{x}{n} \rfloor) \le y}} 1 = x + O(x/y + x^{1/2+\varepsilon}).$$

Theorem 1.2. *Let x and y be real numbers with $\log x < y < x$. Let $1 < c < 2$ be a real number, and $\gamma = 1/c$. Then, we have the following asymptotic formula.*

$$\sum_{\substack{n \le x \\ P^+(\lfloor \frac{x}{n^c} \rfloor) \le y}} 1 = \zeta(1+\gamma)\gamma x^{\gamma} + O(x^{\gamma}/y^{\gamma} + x^{\frac{\gamma}{1+\gamma}+\varepsilon}).$$

2. Notations

Throughout this paper, let N be an integer. Furthermore, $c > 1$ is a fixed real number. We define $P^+(n)$ to be the largest prime factor of n and $P^-(n)$ to be the smallest prime factor of n. We shall frequently use ε to mean a small positive number, possibly a different one each time. Given a real number x, we write $e(x) = e^{2\pi i x}$, $\lfloor x \rfloor$ for the greatest integer not exceeding x, and we write $\mathcal{L} = \log H$. We recall that for functions F and real nonnegative G, the notations $F \ll G$ and $F = O(G)$ are equivalent to the statement that the inequality $|F| \le \alpha G$ holds for some constant $\alpha > 0$. We also write $F \sim G$ to indicate that $F \ll G$ and $G \ll F$.

3. Preliminary lemmas

In this section, we shall cite some lemmas before proving the theorems.

Lemma 3.1. *Put $\psi(x) = x - [x] - 1/2$. Then, there are numbers a_t, b_t such that*

$$\left|\psi(x) - \sum_{0<|t|\le H} a_t e(xt)\right| \le \sum_{|t|\le H} b_t e(xt),$$

where $a_t \ll \frac{1}{|t|}$, $b_t \ll \frac{1}{H}$.

Proof. The proof of this lemma can be found in [11]. □

Lemma 3.2. *Suppose that $2 \le y \le R \le n \le x$, with $n \in S(x, y)$. Then, there is a unique triple (p, u, v) satisfying,*

(i) $n = puv$,
(ii) $p \le y$,
(iii) $R/p < v \le R$ *with* $P^-(v) \ge p$ *and* $P^+(v) \le y$,
(iv) $u \le x/pv$ *with* $P^+(u) \le p$.

Proof. This is Lemma 10.1 in [12]. □

Lemma 3.3. *Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $\delta \in \mathbb{R}$ be some constants. For $X > 0$, $H \geqslant 1$, $M \geqslant 1$, and $N \geqslant 1$, define*

$$S_\delta = S_\delta(H, M, N) := \sum_{h\sim H}\sum_{m\sim M}\sum_{n\sim N} a_{h,m} b_n \mathrm{e}\Big(X\frac{M^\beta N^\gamma}{H^\alpha}\frac{h^\alpha}{m^\beta n^\gamma + \delta}\Big), \tag{3.1}$$

where the $a_{h,m}$ and b_n are complex numbers such that $|a_{h,m}| \le 1$ and $|b_n| \le 1$. Then, for any $\varepsilon > 0$, we have

$$S_\delta \ll ((XHMN)^{1/2} + (HM)^{1/2}N + HMN^{1/2} + X^{-1/2}HMN)X^\varepsilon, \tag{3.2}$$

uniformly for $M \geqslant 1$, $N \geqslant 1$, $H \le M^{\beta-1}N^\gamma$, and $|\delta| \le 1/\varepsilon$, and the implied constant depends on $\alpha, \beta, \gamma, \varepsilon$ only.

Proof. This is formula (3.2) of Proposition 3.1 in [8]. □

Lemma 3.4. *If f is continuously differentiable, f' is monotonic, and $\|f'\| \ge \lambda > 0$ on I, then*

$$\sum_{n\in I} e(f(n)) \ll \lambda^{-1}.$$

Proof. This is Theorem 2.1 in [7]. □

Lemma 3.5. *If Z is large enough, then*

$$\int_{-1/2}^{1/2}\left|\sum_{r\sim I} e(\alpha r)\right| d\alpha \ll \log Z,$$

where I is an interval with $I \subset [Z, 2Z]$.

Proof. We have

$$\int_{-1/2}^{1/2}\left|\sum_{r\sim I} e(\alpha r)\right| d\alpha = \int_{-1/Z}^{1/Z}\left|\sum_{r\sim I} e(\alpha r)\right| d\alpha + \int_{1/Z}^{1/2}\left|\sum_{r\sim I} e(\alpha r)\right| d\alpha + \int_{-1/2}^{-1/Z}\left|\sum_{r\sim I} e(\alpha r)\right| d\alpha$$
$$=: I_1 + I_2 + I_3.$$

By trivial estimation, we have

$$I_1 \le \int_{-1/Z}^{1/Z} Z d\alpha \ll 1. \tag{3.3}$$

By Lemma 3.4, we have

$$I_2 \le \int_{1/Z}^{1/2} \alpha^{-1} d\alpha \ll \log Z. \tag{3.4}$$

Similarly,

$$I_3 \ll \log Z. \tag{3.5}$$

Combining (3.3)–(3.5), we have

$$\int_{-1/2}^{1/2}\left|\sum_{r\sim I} e(\alpha r)\right| d\alpha \ll \log Z.$$

This completes the proof of this lemma. □

4. Proof of Theorem 1.1

Let $N < x$ be a parameter which can be chosen later. First we write

$$S = \sum_{\substack{n\le x\\ P^+(\lfloor \frac{x}{n}\rfloor)\le y}} 1 := S_1(x) + S_2(x) \tag{4.1}$$

where

$$S_1(x) := \sum_{\substack{n\le N\\ P^+(\lfloor \frac{x}{n}\rfloor)\le y}} 1, \qquad S_2(x) := \sum_{\substack{N<n\le x\\ P^+(\lfloor \frac{x}{n}\rfloor)\le y}} 1. \tag{4.2}$$

We have trivially

$$S_1(x) \ll N. \tag{4.3}$$

Next we treat $S_2(x)$. Putting $d = \lfloor x/n\rfloor$, then $x/n - 1 < d \le x/n \Leftrightarrow x/(d+1) < n \le x/d$. Thus, we can write

$$\begin{aligned} S_2(x) &= \sum_{\substack{d\le x/N\\ P^+(d)\le y}} \sum_{x/(d+1)<n\le x/d} 1 \\ &= \sum_{\substack{d\le x/N\\ P^+(d)\le y}} \Big(\frac{x}{d} - \psi\Big(\frac{x}{d}\Big) - \frac{x}{d+1} + \psi\Big(\frac{x}{d+1}\Big)\Big) \\ &= x\sum_{\substack{d\le x/N\\ P^+(d)\le y}} \frac{1}{d(d+1)} + O(T_1+T_2), \end{aligned} \tag{4.4}$$

where

$$T_1 = \sum_{\substack{d \le x/N \\ P^+(d) \le y}} \psi\Big(\frac{x}{d}\Big),$$

and

$$T_2 = \sum_{\substack{d \le x/N \\ P^+(d) \le y}} \psi\Big(\frac{x}{d+1}\Big).$$

If $x/N \le y$, we have

$$\begin{aligned}
S_2(x) &= x \sum_{d \le [x/N]} \frac{1}{d(d+1)} + O(T_1 + T_2) \\
&= x - \frac{1}{[x/N] + 1} + O(T_1 + T_2) \\
&= x + O(N/x) + O(T_1 + T_2) \\
&= x + O(1) + O(T_1 + T_2).
\end{aligned}$$

If $y < x/N$, we have

$$\begin{aligned}
S_2(x) = x \sum_{d \le y} \frac{1}{d(d+1)} + x \sum_{\substack{y < d \le x/N \\ P^+(d) \le y}} \frac{1}{d(d+1)} + O(T_1 + T_2) \\
&= x \sum_{d \le [y]} \frac{1}{d(d+1)} + O\left(x \sum_{[y]+1 \le d \le [x/N]} \frac{1}{d(d+1)} \right) + O(T_1 + T_2) \\
&= x \left(1 - \frac{1}{[y] + 1} \right) + O\left(\frac{x}{[y] + 1} \right) + O(T_1 + T_2) \\
&= x + O(x/y) + O(T_1 + T_2).
\end{aligned}$$

So in any cases,

$$S_2(x) = x + O(x/y) + O(T_1 + T_2).$$

For T_2, we have

$$T_2 = \sum_{\substack{d \le x/N \\ P^+(d) \le y}} \psi\left(\frac{x}{d}\right) - \psi(x) + \psi\left(\frac{x}{\lfloor x/N + 1 \rfloor}\right) = T_1 + O(1).$$

So we only need to estimate T_1. By Lemma 3.1 with $H = x^{\varepsilon}$, we have

$$\begin{aligned}
T_1 &= \sum_{\substack{d \le x/N \\ P^+(d) \le y}} \sum_{1 \le |h| \le H} \frac{1}{h} e\left(\frac{hx}{d}\right) + O\left(\frac{\Psi(x/N, y)}{H}\right) \\
&= \sum_{\substack{d \le x/N \\ P^+(d) \le y}} \sum_{1 \le |h| \le H} \frac{1}{h} e\left(\frac{hx}{d}\right) + O\left(\frac{x^{1-\varepsilon}}{N}\right)
\end{aligned}$$

$$= \sum_{\substack{x^{4\varepsilon} \le d \le x/N \\ P^+(d) \le y}} \sum_{1 \le |h| \le H} \frac{1}{h} e\left(\frac{hx}{d}\right) + O\left(\frac{x^{1-\varepsilon}}{N}\right) + O(x^{5\varepsilon}). \tag{4.5}$$

By Lemma 3.2, we have

$$\sum_{\substack{d \sim M \\ P^+(d) \le y}} e\left(\frac{hx}{d}\right) = \sum_{p \le y} \sum_{\substack{M_1/p < u \le M_1 \\ P^-(u) \ge p \\ P^+(u) \le y}} \sum_{\substack{v \sim M/(up) \\ P^+(v) \le p}} e\left(\frac{hx}{puv}\right),$$

where $x^{4\varepsilon} \le M \le x/N$ and $M_1 = x^{1/2} N^{-1/2} y^{-1/2}$.

Next, we divide the ranges of p, v, and u into $\log^3 x$ dyadic intervals to get

$$\begin{aligned}
\left| \sum_{p \le y} \sum_{\substack{M_1/p < u \le M_1 \\ P^-(u) \ge p \\ P^+(u) \le y}} \sum_{\substack{v \sim M/(up) \\ P^+(v) \le p}} e\left(\frac{hx}{puv}\right) \right| &\le \sum_{p \le y} \sum_{\substack{M_1/p < u \le M_1 \\ P^-(u) \ge p \\ P^+(u) \le y}} \left| \sum_{\substack{v \sim M/(up) \\ P^+(v) \le p}} e\left(\frac{hx}{puv}\right) \right| \\
&\le \sum_{p \le y} \sum_{M_1/p < u \le M_1} \left| \sum_{\substack{v \sim M/(up) \\ P^+(v) \le p}} e\left(\frac{hx}{puv}\right) \right| \\
&= \sum_{p \le y} \sum_{M_1/p < u \le M_1} \sum_{\substack{v \sim M/(up) \\ P^+(v) \le p}} b(u,p) e\left(\frac{hx}{puv}\right) \\
&\le \sum_{\substack{2 \le P \le y \\ P = 2^k}} \sum_{\substack{M_1/2P \le L \le M_1 \\ L = 2^r}} \sum_{\substack{M/4PL \le K \le 2M/PL \\ K = 2^j}} |S'(P,K,L)|,
\end{aligned}$$

where

$$S' := S'(P,K,L) = \sum_{v \sim K} \sum_{\substack{p \sim P \\ P^+(v) \le p}} \sum_{\substack{u \sim L \\ uvp \sim M}} b(u,p) e\left(\frac{hx}{puv}\right), \tag{4.6}$$

and $b(u,p) \ll 1$ is a complex number.

Here, we note that

$$\begin{aligned}
S' &= \int_{-1/2}^{1/2} \sum_{v \sim K} \sum_{\substack{p \sim P \\ P^+(v) \le p}} \sum_{u \sim L} e(\alpha up) b(u,p) e\left(\frac{hx}{puv}\right) \sum_{f \sim M/v} e(-\alpha f) d\alpha \\
&= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{v \sim K} \sum_{f \sim M/v} e(-\alpha f) \sum_{p \sim P} \sum_{u \sim L} e(\alpha up - \gamma p) b(u,p) e\left(\frac{hx}{puv}\right) \sum_{\substack{w \sim P \\ w \ge P^+(v)}} e(\gamma w) d\alpha d\gamma \\
&= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{v \sim K} \sum_{f \sim M/v} e(-\alpha f) \sum_{\substack{w \sim P \\ w \ge P^+(v)}} e(\gamma w) \sum_{p \sim P} \sum_{u \sim L} e(\alpha up - \gamma p) b(u,p) e\left(\frac{hx}{puv}\right) d\alpha d\gamma.
\end{aligned}$$

So we get

$$|S'| = \left| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{v\sim K} a(v;\alpha,\gamma) \sum_{u\sim L} \sum_{p\sim P} b(u,p;\alpha,\gamma) e\left(\frac{hx}{puv}\right) d\alpha d\gamma \right|,$$

where $a(v;\alpha,\gamma) = \sum_{f\sim M/v} e(-\alpha f) \sum_{\substack{w\sim P \\ w\geq P^+(v)}} e(\gamma w)$ and $b(u,p;\alpha,\gamma) = e(\alpha up - \gamma p) b(u,p)$.

Let $r = up$. Then, we have

$$S' \ll \left| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{v\sim K} a(v;\alpha,\gamma) \sum_{r\sim PL} c(r;\alpha,\gamma) e\left(\frac{hx}{rv}\right) d\alpha d\gamma \right|,$$

where $c(r) = \tau(r) b(u,p;\alpha,\gamma)$ with $\tau(r) = \sum_{d|r} 1$.

If $(\alpha,\gamma) \in [-1/2,1/2] \times [-1/2,1/2]$, $\max_{v\sim K} |a(v;\alpha,\gamma)| \neq 0$, and $\max_{r\sim PL} |c(r;\alpha,\gamma)| \neq 0$, let

$$T_0(\alpha,\gamma) = \sum_{v\sim K} \frac{a(v;\alpha,\gamma)}{\max_{v\sim K} |a(v;\alpha,\gamma)|} \sum_{r\sim PL} \frac{c(r;\alpha,\gamma)}{\max_{r\sim PL} |c(r;\alpha,\gamma)|} e\left(\frac{hx}{rv}\right).$$

So we have

$$\begin{aligned} S' &\ll \left| \int\int_{(\alpha,\gamma)\in\Omega} \max_{v\sim K} |a(v;\alpha,\gamma)| \max_{r\sim PL} |c(r;\alpha,\gamma)| |T_0(\alpha,\gamma)| d\alpha d\gamma \right| \\ &= \int\int_{(\alpha,\gamma)\in\Omega} \max_{v\sim K} |a(v;\alpha,\gamma)| \max_{r\sim PL} |c(r;\alpha,\gamma)| f(h) T_0(\alpha,\gamma) d\alpha d\gamma, \end{aligned}$$

where $|f(h)| = 1$.

By Lemmas 3.3 and 3.5 with some fixed v, we have

$$\begin{aligned} T_1 &\ll H^{-1} x^{\varepsilon} \max_{x^{4\varepsilon}\leq M\leq x/N} \sum_{1\leq|h|\leq H} S' + O\left(\frac{x^{1-\varepsilon}}{N}\right) \\ &\ll H^{-1} x^{\varepsilon} \max_{1\leq M\leq x/N} (x^{1/2}H + (HK)^{1/2}PL + HK(PL)^{1/2} + x^{-1/2}M^{3/2}H^{1/2}) \\ &\quad \times \left| \int\int_{(\alpha,\gamma)\in\Omega} \max_{v\sim K} |a(v;\alpha,\gamma)| \max_{r\sim PL} |c(r;\alpha,\gamma)| d\alpha d\gamma \right| + \frac{x^{1-\varepsilon}}{N} \\ &\ll H^{-1} x^{\varepsilon} \max_{1\leq M\leq x/N} (x^{1/2}H + (HK)^{1/2}PL + HK(PL)^{1/2} + x^{-1/2}M^{3/2}H^{1/2}) \\ &\quad \times \left| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \max_{v\sim K} |a(v;\alpha,\gamma)| \max_{r\sim PL} |c(r;\alpha,\gamma)| d\alpha d\gamma \right| + \frac{x^{1-\varepsilon}}{N} \\ &\ll H^{-1} x^{\varepsilon} \max_{1\leq M\leq x/N} (x^{1/2}H + (HK)^{1/2}PL + HK(PL)^{1/2} + x^{-1/2}M^{3/2}H^{1/2}) \\ &\quad \times | \int_{-1/2}^{1/2} \left| \sum_{f\sim M/v} e(-\alpha f) \right| d\alpha \int_{-1/2}^{1/2} \left| \sum_{\substack{w\sim P \\ w\geq P^+(v)}} e(\gamma w) \right| d\gamma + \frac{x^{1-\varepsilon}}{N} \\ &\ll x^{1/2+\varepsilon} + x^{3/4+\varepsilon} y^{1/4} N^{-3/4} + x N^{-3/2} + \frac{x^{1-\varepsilon}}{N}. \end{aligned}$$

So we derive

$$T_1 \ll x^{1/2+\varepsilon} + x^{3/4+\varepsilon}y^{1/4}N^{-3/4} + xN^{-3/2} + \frac{x^{1-\varepsilon}}{N}. \tag{4.7}$$

So we have

$$T_2 \ll x^{1/2+\varepsilon} + x^{3/4+\varepsilon}y^{1/4}N^{-3/4} + xN^{-3/2} + \frac{x^{1-\varepsilon}}{N}. \tag{4.8}$$

Combining (4.1)–(4.4) and (4.7)–(4.8), we have

$$S = x + O(x/y + x^{1/2+\varepsilon} + x^{3/4+\varepsilon}y^{1/4}N^{-3/4} + xN^{-3/2} + \frac{x^{1-\varepsilon}}{N} + N). \tag{4.9}$$

Choosing $N = x^{1/2+\varepsilon}$ and noting that $y \leq x^{1/2}$, we derive

$$S = x + O(x/y + x^{1/2+\varepsilon}). \tag{4.10}$$

This completes the proof of this theorem.

5. Proof of Theorem 1.2

Let N be a parameter which can be chosen later. First, we write

$$R = \sum_{\substack{n \leq x \\ P^+(\lfloor \frac{x}{n^c} \rfloor) \leq y}} 1 := R_1(x) + R_2(x) \tag{5.1}$$

where

$$R_1(x) := \sum_{\substack{n \leq N \\ P^+(\lfloor \frac{x}{n^c} \rfloor) \leq y}} 1, \qquad R_2(x) := \sum_{\substack{N < n \leq x \\ P^+(\lfloor \frac{x}{n^c} \rfloor) \leq y}} 1. \tag{5.2}$$

Obviously, we have

$$R_1(x) \ll N. \tag{5.3}$$

Now we treat R_2. Putting $d = \lfloor x/n^c \rfloor$, then $x/n^c - 1 < d \leq x/n^c \Leftrightarrow (x/(d+1))^\gamma < n \leq (x/d)^\gamma$. Thus, we can write

$$\begin{aligned} R_2(x) &= \sum_{\substack{d \leq x/N^c \\ P^+(d) \leq y}} \sum_{(x/(d+1))^\gamma < n \leq (x/d)^\gamma} 1 \\ &= \sum_{\substack{d \leq x/N^c \\ P^+(d) \leq y}} \left(\left(\frac{x}{d}\right)^\gamma - \psi\left(\left(\frac{x}{d}\right)^\gamma\right) - (\frac{x}{d+1})^\gamma + \psi\left(\left(\frac{x}{d+1}\right)^\gamma\right) \right) \\ &= x^\gamma \sum_{\substack{d \leq x/N^c \\ P^+(d) \leq y}} \left(\frac{1}{d^\gamma} - \frac{1}{(d+1)^\gamma} \right) + O(E) \\ &= \gamma x^\gamma \sum_{d \leq y} \frac{1}{d^{\gamma+1}} + \gamma x^\gamma \sum_{\substack{y < d \leq x/N^c \\ P^+(d) \leq y}} \frac{1}{d^{\gamma+1}} + O\left(\sum_{d \leq x/N^c} \frac{1}{d^{\gamma+2}} \right) + O(E) \\ &= \zeta(1+\gamma)\gamma x^\gamma + O(x^\gamma/y^\gamma) + O(E), \end{aligned} \tag{5.4}$$

where

$$E = \sum_{\substack{d \le x/N^c \\ P^+(d) \le y}} \Big(\psi\Big(\Big(\frac{x}{d}\Big)^\gamma\Big) + \psi\Big(\Big(\frac{x}{d+1}\Big)^\gamma\Big)\Big).$$

By elementary method, we have

$$\begin{aligned} E &= \sum_{\substack{d \le x/N^c \\ P^+(d) \le y}} \psi\Big(\Big(\frac{x}{d}\Big)^\gamma\Big) + \sum_{\substack{d \le x/N^c \\ P^+(d) \le y}} \psi\Big(\Big(\frac{x}{d+1}\Big)^\gamma\Big) \\ &\ll \sum_{\substack{d \le x/N^c \\ P^+(d) \le y}} \psi\Big(\Big(\frac{x}{d}\Big)^\gamma\Big) + \sum_{\substack{1 \le d \le x/N^c \\ P^+(d) \le y}} \psi\Big(\Big(\frac{x}{d}\Big)^\gamma\Big) - \psi(x^\gamma) + \psi\Big(\Big(\frac{x}{(x/N^c+1)}\Big)\Big) \\ &\ll \sum_{\substack{1 \le d \le x/N^c \\ P^+(d) \le y}} \psi\Big(\Big(\frac{x}{d}\Big)^\gamma\Big) + O(1). \end{aligned} \tag{5.5}$$

By Lemma 3.1 with $H = x^\varepsilon$, we have

$$\begin{aligned} E &\ll \sum_{\substack{d \le x/N^c \\ P^+(d) \le y}} \sum_{1 \le |h| \le H} \frac{1}{h} e\left(\frac{hx^\gamma}{d^\gamma}\right) + O\left(\frac{\Psi(x/N^c, y)}{H}\right) \\ &= \sum_{\substack{d \le x/N^c \\ P^+(d) \le y}} \sum_{1 \le |h| \le H} \frac{1}{h} e\left(\frac{hx^\gamma}{d^\gamma}\right) + O\left(\frac{x}{N^c H}\right) \\ &= \sum_{1 \le |h| \le H} \frac{1}{h} \sum_{\substack{d \le x/N^c \\ P^+(d) \le y}} e\left(\frac{hx^\gamma}{d^\gamma}\right) + O\left(\frac{x^{1-\varepsilon}}{N^c}\right) \\ &= \sum_{1 \le |h| \le H} \frac{1}{h} \sum_{\substack{x^{4\varepsilon} \le d \le x/N^c \\ P^+(d) \le y}} e\left(\frac{hx^\gamma}{d^\gamma}\right) + O\left(\frac{x^{1-\varepsilon}}{N^c}\right) + O(x^{5\varepsilon}). \end{aligned} \tag{5.6}$$

By Lemma 3.2, we have

$$\sum_{\substack{d \sim M \\ P^+(d) \le y}} e\left(\frac{hx^\gamma}{d^\gamma}\right) = \sum_{p \le y} \sum_{\substack{M_1/p < u \le M_1 \\ P^-(u) \ge p \\ P^+(u) \le y}} \sum_{\substack{v \sim M/(up) \\ P^+(v) \le p}} e\left(\frac{hx^\gamma}{p^\gamma u^\gamma v^\gamma}\right),$$

where $x^{4\varepsilon} \le M \le x/N^c$ and $M_1 = M^{1/2}$.

Next, we divide the ranges of p, v, and u into $\log^3 x$ dyadic intervals to get

$$\left| \sum_{p \le y} \sum_{\substack{M_1/p < u \le M_1 \\ P^-(u) \ge p \\ P^+(u) \le y}} \sum_{\substack{v \sim M/(up) \\ P^+(v) \le p}} e\left(\frac{hx^\gamma}{p^\gamma u^\gamma v^\gamma}\right) \right| \le \sum_{p \le y} \sum_{\substack{M_1/p < u \le M_1 \\ P^-(u) \ge p \\ P^+(u) \le y}} \left| \sum_{\substack{v \sim M/(up) \\ P^+(v) \le p}} e\left(\frac{hx^\gamma}{p^\gamma u^\gamma v^\gamma}\right) \right|$$

$$\leq \sum_{p\leq y}\sum_{M_1/p<u\leq M_1}\left|\sum_{\substack{v\sim M/(up)\\ P^+(v)\leq p}} e\left(\frac{hx^\gamma}{p^\gamma u^\gamma v^\gamma}\right)\right|$$
$$= \sum_{p\leq y}\sum_{M_1/p<u\leq M_1}\sum_{\substack{v\sim M/(up)\\ P^+(v)\leq p}} b(u,p)e\left(\frac{hx^\gamma}{p^\gamma u^\gamma v^\gamma}\right)$$
$$\leq \sum_{\substack{2\leq P\leq y\\ P=2^k}}\sum_{\substack{M_1/2P\leq L\leq M_1\\ L=2^r}}\sum_{\substack{M/4PL\leq K\leq 2M/PL\\ K=2^j}} |S'(P,K,L)|,$$

where

$$S' := S'(P,K,L) = \sum_{v\sim K}\sum_{\substack{p\sim P\\ P^+(v)\leq p}}\sum_{\substack{u\sim L\\ uvp\sim M}} b(u,p)e\left(\frac{hx^\gamma}{p^\gamma u^\gamma v^\gamma}\right), \tag{5.7}$$

and $b(u,p) \ll 1$ is a complex number.

Here, we note that

$$S' = \int_{-1/2}^{1/2}\sum_{v\sim K}\sum_{\substack{p\sim P\\ P^+(v)\leq p}}\sum_{u\sim L} e(\alpha up)b(u,p)e\left(\frac{hx^\gamma}{p^\gamma u^\gamma v^\gamma}\right)\sum_{f\sim M/v} e(-\alpha f)d\alpha$$
$$= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\sum_{v\sim K}\sum_{f\sim M/v} e(-\alpha f)\sum_{p\sim P}\sum_{u\sim L} e(\alpha up-\gamma p)b(u,p)e\left(\frac{hx^\gamma}{p^\gamma u^\gamma v^\gamma}\right)\sum_{\substack{w\sim P\\ w\geq P^+(v)}} e(\gamma w)d\alpha d\gamma$$
$$= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\sum_{v\sim K}\sum_{f\sim M/v} e(-\alpha f)\sum_{\substack{w\sim P\\ w\geq P^+(v)}} e(\gamma w)\sum_{p\sim P}\sum_{u\sim L} e(\alpha up-\gamma p)b(u,p)e\left(\frac{hx^\gamma}{p^\gamma u^\gamma v^\gamma}\right)d\alpha d\gamma.$$

So we get

$$|S'| = \left|\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\sum_{v\sim K} a(v;\alpha,\gamma)\sum_{u\sim L}\sum_{p\sim P} b(u,p;\alpha,\gamma)e\left(\frac{hx^\gamma}{p^\gamma u^\gamma v^\gamma}\right)d\alpha d\gamma\right|,$$

where $a(v;\alpha,\gamma) = \sum_{f\sim M/v} e(-\alpha f)\sum_{\substack{w\sim P\\ w\geq P^+(v)}} e(\gamma w)$ and $b(u,p;\alpha,\gamma) = e(\alpha up - \gamma p)b(u,p)$.

Let $r = up$. Then, we have

$$S' \ll \left|\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\sum_{v\sim K} a(v;\alpha,\gamma)\sum_{r\sim PL} c(r;\alpha,\gamma)e\left(\frac{hx^\gamma}{r^\gamma v^\gamma}\right)d\alpha d\gamma\right|,$$

where $c(r) = \tau(r)b(u,p;\alpha,\gamma)$.

If $(\alpha,\gamma)\in\Omega$, $\max_{v\sim K}|a(v;\alpha,\gamma)| \neq 0$ and $\max_{r\sim PL}|c(r;\alpha,\gamma)| \neq 0$, let

$$T_*(\alpha,\gamma) = \sum_{v\sim K}\frac{a(v;\alpha,\gamma)}{\max_{v\sim K}|a(v;\alpha,\gamma)|}\sum_{r\sim PL}\frac{c(r;\alpha,\gamma)}{\max_{r\sim PL}|c(r;\alpha,\gamma)|}e\left(\frac{hx^\gamma}{r^\gamma v^\gamma}\right).$$

So we have

$$
\begin{aligned}
S' &\ll \left| \int\int_{(\alpha,\gamma)\in\Omega} \max_{v\sim K} |a(v;\alpha,\gamma)| \max_{r\sim PL} |c(r;\alpha,\gamma)||T_*(\alpha,\gamma)| d\alpha d\gamma \right| \\
&= \int\int_{(\alpha,\gamma)\in\Omega} \max_{v\sim K} |a(v;\alpha,\gamma)| \max_{r\sim PL} |c(r;\alpha,\gamma)| f(h) T_*(\alpha,\gamma) d\alpha d\gamma,
\end{aligned}
$$

where $|f(h)| = 1$.

By Lemmas 3.3 and 3.5 with some fixed v, we have

$$
\begin{aligned}
E \ll& H^{-1} x^{\varepsilon} \max_{x^{4\varepsilon}\le M\le x/N} \sum_{1\le|h|\le H} S' + x/N^c \\
\ll& H^{-1} x^{\varepsilon} (x^{1/2} H N^{1/2-c/2} + (Hx)^{3/4} N^{-3c/4} + x N^{-c-1/2} H^{1/2}) \\
&\times \left| \int_{-1/2}^{1/2} \sum_{f\sim M/v} e(-\alpha f) d\alpha \int_{-1/2}^{1/2} \sum_{\substack{w\sim P \\ w\ge P^+(v)}} e(\gamma w) d\gamma \right| + x/N^c \\
\ll& x^{1+\varepsilon}/N^c.
\end{aligned} \tag{5.8}
$$

Collecting (5.1)–(5.4) and (5.8), we get

$$
R = \zeta(1+\gamma)\gamma x^{\gamma} + O(x^{\gamma}/y^{\gamma} + N + x^{1+\varepsilon}/N^c). \tag{5.9}
$$

Choosing $N = x^{\frac{1}{1+c}}$, we get

$$
\begin{aligned}
R &= \zeta(1+\gamma)\gamma x^{\gamma} + O(x^{\gamma}/y^{\gamma} + x^{\frac{1}{1+c}+\varepsilon}) \\
&= \zeta(1+\gamma)\gamma x^{\gamma} + O(x^{\gamma}/y^{\gamma} + x^{\frac{\gamma}{1+\gamma}+\varepsilon}).
\end{aligned} \tag{5.10}
$$

This completes the proof of this theorem.

6. Discussion and conclusions

Let $\lfloor t \rfloor$ be the integral part of a real number t. $P^+(n)$ denotes the largest prime factor of an integer n. In this paper, we prove that for any $\varepsilon > 0$, the asymptotic formulae

$$
\sum_{\substack{n\le x \\ P^+(\lfloor \frac{x}{n} \rfloor)\le y}} 1 = x + O(x/y + x^{1/2+\varepsilon})
$$

and

$$
\sum_{\substack{n\le x \\ P^+(\lfloor \frac{x}{n^c} \rfloor)\le y}} 1 = \zeta(1+\gamma)\gamma x^{\gamma} + O(x^{\gamma}/y^{\gamma} + x^{\frac{\gamma}{1+\gamma}+\varepsilon})
$$

hold. Clearly, studies concerning the distribution of smooth numbers play an important role in number theory, which makes our studies worthwhile and engaging.

We give an open problem related to this article. Let c_i $(i = 1, 2, \cdots, k)$ be non-integers, then whether we can give an asymptotic formula for the following sum

$$\sum_{\substack{n\le x\\ P^+(\lfloor \frac{x}{n^{c_1}+n^{c_2}+\cdots+n^{c_k}}\rfloor)\le y}} 1.$$

According to the method we used in the current paper, we should estimate exponential sum with non-integer polynomial, which is the difficulty of this problem. This is why this problem seems gripping and consequential.

Use of Generative-AI tools declaration

The author declare he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author states no conflicts of interest.

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