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Research article

Combinatorial correspondences between colored partitions by Dedekind's level 8 partition identities

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Abstract: The study of Dedekind's η-function and its identities plays a significant role in number theory and combinatorics. In this paper, we study level 8 η-function identities and their applications to colored partitions. Some of these identities arise from algebraic transformations of known mock theta function expansions. By applying these identities, we deduce combinatorial correspondences between specific classes of colored partitions with prescribed color restrictions. Our work extends existing methods and offers a deeper understanding of the combinatorial properties of partitions, contributing to both theoretical advancements and practical applications in partition theory.

Keywords: integer partitions; colored partitions; Dedekind's η-function; Ramanujan's theta functions; q-product identities
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1. Introduction

Integer Partitions: A partition of a positive integer m is a representation of m as a sum of positive integers, where the order of the summands is irrelevant. The summands are called the parts of the partition. For example, the number 5 can be partitioned as $5 = 4 + 1$, where 4 and 1 are the parts of the partition. Other partitions of 5 include:

$$1+1+1+1+1, \quad 2+2+1, \quad 2+1+1+1, \quad 3+1+1, \quad 3+2, \quad \text{and} \quad 5.$$

The number of partitions of an integer depends on its value, as each integer can have multiple distinct partitions. The function that counts the total number of partitions of an integer m is called the partition

function, denoted by $p(m)$.
For large values of m ($m \to \infty$), $p(m)$ can be approximated by the asymptotic formula [10]:

$$p(m) \sim \frac{1}{4\sqrt{3}m} e^{\pi\sqrt{\frac{2m}{3}}}. \tag{1.1}$$

The notation is consistent with the definition of $p(m)$ given earlier.
The generating function for integer partitions is given by

$$\sum_{m=0}^{\infty} p(m) q^m = \frac{1}{(q;q)_\infty}, \tag{1.2}$$

where the q-Pochhammer symbol $(q;q)_\infty$ is defined as

$$(q;q)_\infty = \prod_{k=1}^{\infty} (1 - q^k). \tag{1.3}$$

This infinite product notation is fundamental in the study of q-series and partitions.
For complex numbers a and q with $|q| < 1$, the q-Pochhammer symbol is defined by

$$(a;q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k).$$

More generally, for a finite product,

$$(a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad n \in \mathbb{N}.$$

For $|xy| < 1$, Ramanujan's theta function $f(x,y)$ in [4] is defined as:

$$f(x,y) := \sum_{m=-\infty}^{\infty} x^{m(m+1)/2} y^{m(m-1)/2}.$$

Jacobi's triple-product identity, which was rediscovered by Ramanujan [31], can be stated as:

$$f(x,y) = (-x;xy)_\infty (-y;xy)_\infty (xy;xy)_\infty,$$

or equivalently

$$\begin{aligned} \sum_{m=-\infty}^{\infty} q^{m^2} z^m &= \prod_{m=1}^{\infty} (1 - q^{2m})(1 + zq^{2m-1})(1 + \frac{1}{z} q^{2m-1}) \\ &= (q^2;q^2)_\infty (-zq;q^2)_\infty \left(-\frac{q}{z};q^2\right)_\infty, \quad \text{for } |q| < 1, \quad z \neq 0, \end{aligned}$$

where the notation is consistent with Ramanujan's work on q-series. Three important special cases of $f(x,y)$ are

$$\phi(q) = f(q,q) = \sum_{m=-\infty}^{\infty} q^{m^2} = (-q;q^2)_\infty (q^2;q^2)_\infty, \tag{1.4}$$

$$\psi(q) = f(q, q^3) = \sum_{m=0}^{\infty} q^{m(m+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{1.5}$$

$$f(-q) = f(-q, -q^2) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2} = (q; q)_\infty. \tag{1.6}$$

Ramanujan defined the following auxiliary function:

$$\chi(q) = (-q; q^2)_\infty.$$

If $q = e^{2\pi i\tau}$, then for $\mathrm{Im}(\tau) > 0$, we have

$$f(-q) = q^{-1/24}\eta(\tau),$$

where $\eta(\tau)$ stands for the Dedekind η-function, a fundamental function in number theory and other areas of mathematics.

We restate the definition of the Dedekind η-function here for convenience and to emphasize its role in the identities that follow. The Dedekind η-function, $\eta(\tau)$, is defined for complex numbers τ with a positive imaginary part by the following formula:

$$\eta(\tau) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m), \quad \text{for } \mathrm{Im}(\tau) > 0. \tag{1.7}$$

The study of integer partitions is a fundamental topic in combinatorics that dates back to the 17th century. The French mathematician Marin Mersenne was among the first to explore the representation of numbers as sums of smaller numbers. However, significant advancements in partition theory were made by Leonhard Euler, who introduced several key results, including his famous partition theorem.

Integer partitions have captivated researchers due to their wide-ranging applications in mathematics, statistics, cryptography, physics, and computer science [5–7, 22, 28, 29].

Colored Partitions: The concept of colored partitions is an extension of integer partitions, first studied by S.S. Huang [11].

Definition 1.1. A colored partition of a non negative integer m with κ colors is a pair (λ, ξ), where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ is a partition of m, i.e., $\lambda_i \in \mathbb{Z}_{>0}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j$, $\sum_{i=1}^{j} \lambda_i = m$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_j)$ is a sequence such that each $\xi_i \in \{1, 2, \ldots, \kappa\}$, assigning one of κ possible colors to the part λ_i.
Two colored partitions (λ, ξ) and (λ', ξ') are considered distinct if either the partitions $\lambda \neq \lambda'$, or the color sequences $\xi \neq \xi'$.

Proposition 1.1. Let $p_\kappa(m)$ denote the number of colored partitions of m with κ colors, where m is a non-negative integer and each part of the partition can appear in any one of the κ available colors. Then the generating function for $p_\kappa(m)$ is given by the following q-series:

$$\sum_{m=0}^{\infty} p_\kappa(m) q^m = \prod_{n=1}^{\infty} \left(\frac{1}{(1 - q^n)^\kappa} \right) \quad (|\, q \,| < 1).$$

This expression follows from the fact that each part size n can appear in κ distinct colors, contributing a geometric series with exponent κ in the product.

For example, let $m = 3$ and $\kappa = 2$. The ordinary partitions of 3 are

$$(3), \quad (2,1), \quad (1,1,1).$$

When a partition contains repeated parts, colorings that differ only by a permutation of identical parts are considered the same.

- (3): The single part can be colored in 2 ways.
- $(2, 1)$: The two distinct parts can be colored independently, giving $2^2 = 4$ colorings.
- $(1, 1, 1)$: The three identical parts can be colored using either one color or both colors, giving 4 distinct colorings.

Hence, the total number of 2-colored partitions of 3 is

$$p_2(3) = 2 + 4 + 4 = 10.$$

For instance, M. Merca [18] introduced colored partitions in the sum-over-partitions framework by allowing each integer n to appear in n colors. Other researchers, including Bandyopadhyay and Baruah [33], extended this idea by allowing n to appear in $n+t$ colors to develop results related to the n-color partition function [1,2,17,23]. An alternative approach involves permitting all parts of a partition to appear in exactly k colors, leading to another variant of colored partitions [3,16,19,20,24,25].

As an example, consider partitioning the number 4 using three colors: {purple, white, blue}. The possible partitions are as follows:

$$\begin{array}{cccc}
1_p+1_p+1_p+1_p, & 1_p+1_p+1_p+1_w, & 1_p+1_p+1_p+1_b, & 1_p+1_p+1_w+1_w,\\
1_p+1_p+1_w+1_b, & 1_p+1_p+1_b+1_b, & 1_p+1_w+1_w+1_w, & 1_p+1_w+1_w+1_b,\\
1_p+1_w+1_b+1_b, & 1_p+1_b+1_b+1_b, & 1_w+1_w+1_w+1_w, & 1_w+1_w+1_w+1_b,\\
1_w+1_w+1_b+1_b, & 1_w+1_b+1_b+1_b, & 1_b+1_b+1_b+1_b, & 1_p+1_p+2_p,\\
1_p+1_p+2_w, & 1_p+1_p+2_b, & 1_p+1_w+2_p, & 1_p+1_w+2_w,\\
1_p+1_w+2_b, & 1_p+1_b+2_p, & 1_p+1_b+2_w, & 1_p+1_b+2_b,\\
1_w+1_w+2_p, & 1_w+1_w+2_w, & 1_w+1_w+2_b, & 1_w+1_b+2_p,\\
1_w+1_b+2_w, & 1_w+1_b+2_b, & 1_b+1_b+2_p, & 1_b+1_b+2_w,\\
1_b+1_b+2_b, & 1_p+3_p, & 1_p+3_w, & 1_p+3_b,\\
1_w+3_p, & 1_w+3_w, & 1_w+3_b, & 1_b+3_p,\\
1_b+3_w, & 1_b+3_b, & 2_p+2_p, & 2_p+2_w,\\
2_p+2_b, & 2_w+2_w, & 2_w+2_b, & 2_b+2_b,\\
4_p, & 4_w, & 4_b, &
\end{array} \tag{1.8}$$

where subscripts p, w, and b denote purple, white, and blue, respectively. That is, parts such as 1_p, 1_w, and 1_b represent the integer 1 assigned the colors purple, white, and blue, respectively. Similarly, 2_p, 2_w, 2_b, 3_p, 3_w, 3_b, and 4_p, 4_w, 4_b represent the integers 2, 3, and 4, each distinguished by the corresponding color.

This formulation allows each part of a partition to be assigned exactly k colors without restrictions, making this type of colored partition sometimes referred to as a uniformly k-colored partition. Results derived from this framework can be found in [12, 15, 26, 27], among others. Some authors have also considered variations where only specific parts appear in k colors, while the remaining parts can take any color freely.

For example, if we restrict the integer 1 to appear in exactly two colors (say, yellow (y) and purple (p)), while other integers (like 2) can only appear in one color (uncolored), the partitions of 2 are:

$$2,\ 1_y + 1_y,\ 1_p + 1_p,\ \text{and}\ 1_y + 1_p,$$

where parts such as 1_{y} and 1_{p} represent the integer 1 assigned the colors yellow and purple, respectively.

Many researchers [13, 14, 21] have employed this approach to connect q-product identities with colored partitions. These novel perspectives provide valuable insights into the combinatorial properties of numbers.

The collection of colored partitions of a non-negative integer m into k distinct colors can be determined using the formula given in [12].

$$p_k(m) = \frac{1}{m!}\begin{vmatrix} k\sigma(1) & k\sigma(2) & k\sigma(3) & \cdots & k\sigma(m) \\ -1 & k\sigma(1) & k\sigma(2) & \cdots & k\sigma(m-1) \\ 0 & -2 & k\sigma(1) & \cdots & k\sigma(m-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0\cdots & -(m-1) & k\sigma(1) \end{vmatrix}, \tag{1.9}$$

where $\sigma(m)$ is the divisor function, which represents the sum of all positive divisors of m:

$$\sigma(m) = \sum_{d|m} d. \tag{1.10}$$

Remark. Setting $k = 1$ in Eq (1.9) yields the total number of partitions of m. For convenience, we use the following notation:

$$(q^{a^{\pm}}; q^m)_\infty = (q^a, q^{m-a}; q^m)_\infty, \quad (a < m), \quad a, m \in \mathbb{N}. \tag{1.11}$$

We define the following shorthand for parts modulo 8:

- Parts of the form $8n \pm 1$ and $8n \pm 3$ (i.e., exponents congruent to 1, 3, 5, 7 mod 8) are denoted by the superscript 'a'.
- Parts of the form $8n \pm 2$ (i.e., exponents congruent to 2, 6 mod 8) are denoted by the superscript 'b'.
- Parts of the form $8n \pm 4$ (i.e., exponents congruent to 4 mod 8) are denoted by the superscript 'c'.

In other words,

$$q^a \equiv q^{1\pm}, q^{3\pm}, \quad q^b \equiv q^{2\pm}, \quad q^c \equiv q^4 \qquad (\text{mod } q^8), \tag{1.12}$$

where

$$(q^{1\pm}; q^8)_\infty \equiv (q^1, q^7; q^8)_\infty, \quad (q^{2\pm}; q^8)_\infty \equiv (q^2, q^6; q^8)_\infty,$$
$$(q^{3\pm}; q^8)_\infty \equiv (q^3, q^5; q^8)_\infty.$$

The identities (1.13), (1.14), and (1.15) form the algebraic foundation for our colored partition results. These identities emerge from the analytic study of mock theta functions, which play a crucial role in the modern theory of modular forms, partitions, and q-series. Mock theta functions, introduced by Ramanujan, possess deep modular properties and encode rich arithmetic information in their Fourier coefficients.

The motivation behind studying such identities stems from the fact that certain coefficients of mock theta functions exhibit elegant patterns and modular transformations when isolated algebraically. Using specific coefficient sequences, such as $b(2m)$ and $b(6m+4)$ from a second-order mock theta function and $v_0(16m+14)$ from an eighth-order one, we obtain closed-form expressions involving products of q-shifted factorials. These expressions, once simplified using Lemma 1.1, give rise to the algebraic identities (1.13), (1.14), and (1.15).

We now clearly distinguish between the mock theta functions and their coefficient sequences, which play different roles in our analysis. The second-order mock theta function $B(q)$, as given in [9], is defined by

$$B(q) := \sum_{m=0}^{\infty} \frac{q^m(-q;q^2)_m}{(q;q^2)_{m+1}},$$

with the following expansion:

$$B(q) = \sum_{m=0}^{\infty} b(m)q^m,$$

where $b(m)$ denotes the coefficient of q^m in the q-series expansion of the function $B(q)$. In particular, we focus on the coefficients $b(2m)$ and $b(6m+4)$, which through algebraic manipulation of $B(q)$ and application of Lemma 1.1, lead directly to identities (1.13) and (1.14).

Similarly, the eighth-order mock theta function $V_0(q)$, as given in [8], is defined as

$$V_0(q) := -1 + 2\sum_{m=0}^{\infty} q^{m^2} \frac{(-q;q^2)_m}{(q;q^2)_m},$$

with the following expansion:

$$V_0(q) = \sum_{m=0}^{\infty} v_0(m)q^m,$$

where $v_0(m)$ denotes the coefficient q^m in the q-series expansion of the function $V_0(q)$. We focus on the coefficient $v_0(16m+14)$, which through algebraic manipulation of $V_0(q)$ and the application of Lemma 1.1, directly leads to identity (1.15).

These identities are not proven from scratch but are derived as transformations of known mock theta expansions, revealing new modular-type relations. More importantly, they serve as a bridge between analytic q-series theory and combinatorics. In Section 2, we reinterpret these identities in terms of colored partitions, establishing direct combinatorial meaning for these modular structures.

$$f_1^4 f_4^{14} + 4q f_1^4 f_2^4 f_4^2 f_8^8 - f_2^{14} f_8^4 = 0, \tag{1.13}$$

$$16q^2 f_1^8 f_2^8 f_4^4 f_8^{16} + 8q f_1^8 f_2^4 f_4^{16} f_8^8 + f_1^8 f_4^{28} - f_2^{28} f_8^8 = 0, \tag{1.14}$$

$$\begin{aligned}&3f_1^{12} f_4^{42} + 28q f_1^{12} f_2^4 f_4^{30} f_8^8 + 80q^2 f_1^{12} f_2^8 f_4^{18} f_8^{16} + 64q^3 f_1^{12} f_2^{12} f_4^6 f_8^{24} \\ &- 2f_1^4 f_2^{28} f_4^{14} f_8^8 - f_2^{42} f_8^{12} = 0,\end{aligned} \tag{1.15}$$

where

$$f_k = (q^k; q^k)_\infty,$$

for positive integers k. We recall from Eq (1.7) that the Dedekind η-function is related to the infinite product:

$$\eta(\tau) = q^{1/24}(q;q)_\infty, \quad \text{where } q = e^{2\pi i\tau},\ \mathrm{Im}(\tau) > 0.$$

By (1.7), we can express $f_k = (q^k; q^k)_\infty$ in terms of the Dedekind η-function as:

$$f_k = (q^k; q^k)_\infty = q^{-k/24}\,\eta(k\tau).$$

Therefore, we obtain

$$f_k^s = (q^k; q^k)_\infty^s = q^{-sk/24}\eta(k\tau)^s.$$

This form reveals the modular structure of our identities and will be useful when interpreting the expressions as η-products in modular form theory.

Remark 1. Identity (1.13) is not new and appears, for instance, in Michael Somos' list of q-product identities [32]. However, to the best of our knowledge, the identities given in (1.14) and (1.15) do not appear in Somos' list, which has been carefully checked.

The following identities will be used in the subsequent analysis. In particular, they provide the algebraic foundation for transforming the analytic mock theta function expansions into product forms suitable for colored partition interpretations.

Lemma 1.1.

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{1.16}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}. \tag{1.17}$$

The proofs of (1.16) and (1.17) may be found in Xia and Yao [30], offering novel approaches.

Application of identity (1.13) **to the function** $b(2m)$**.** Multiplying both sides of (1.13) by $\left(\frac{1}{f_1^4 f_2^9 f_8^4}\right)$ and applying identity (1.16) together with the definition and coefficient extraction of the second-order mock theta function $B(q)$ as given in [9, Eq (1.5)], we obtain

$$\begin{aligned}\sum_{m=0}^{\infty} b(2m)q^m &= \frac{f_2^5}{f_1^4}\\ &= f_2^5\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right).\end{aligned}$$

Application of identity (1.14) **to the function** $b(6m+4)$**.** Multiplying both sides of (1.14) by $\left(9\frac{f_3^4 f_6}{f_1^8 f_2^{24} f_8^8}\right)$ and applying identity (1.16) together with the definition and coefficient extraction of the second-order mock theta function $B(q)$ as given in [9, Eq (1.9)], we obtain

$$\sum_{m=0}^{\infty} b(6m+4)q^m = 9\frac{f_2^4 f_3^4 f_6}{f_1^8}$$

$$= 9f_2^4f_3^4f_6\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2.$$

Application of identity (1.15) **to the function** $v_0(16m + 14)$**.** Multiplying both sides of (1.15) by $\left(16\frac{1}{f_1^{13}f_2^{27}f_4^3f_8^{10}}\right)$ and applying identity (1.16) together with the definition and coefficient extraction of the eighth-order mock theta function $V_0(q)$ as given in [8], we derive

$$\begin{aligned}\sum_{m=0}^{\infty} v_0(16m + 14)q^m &= 16\frac{f_2^{15}f_8^2}{f_1^{13}f_4^3} + 32\frac{f_2f_4^{11}}{f_1^9f_8^2}\\ &= 16\frac{f_2^{15}f_8^2}{f_1f_4^3}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^3\\ &\quad + 32\frac{f_2f_4^{11}}{f_1f_8^2}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2.\end{aligned}$$

Before stating the main results, we briefly clarify the notation used for the colored partition functions. Throughout this section, $\mu_1(m)$, $\mu_2(m)$, $\mu_3(m)$, ... denote counting functions that enumerate colored partitions of the integer m under specific congruence and coloring restrictions. The parameters a, b, and c correspond to residue classes modulo 8 as defined in (1.12), while the coefficients indicate the number of available colors assigned to each allowed part. This clarification is intended to guide the reader through the combinatorial interpretation of the identities.

Recall that parts of the form $8n+a$, $8n+b$ and $8n+c$ refer to the residue classes modulo 8 as defined in (1.12).

Remark 2. (Color-indexed q-notation). In the generating functions considered throughout this paper, subscripts attached to q indicate the number of available colors assigned to a part. Specifically, the notation q_i denotes the contribution of a part counted with i distinct colors. For example, q_{14} represents the generating variable corresponding to parts that may appear in exactly 14 different colors. This convention allows us to encode color multiplicities directly within the q-product representations of colored partition generating functions.

Furthermore, in our rewritings of η-products, we find it convenient to use the notation q_0^c to indicate that there are no occurrences of type c parts in the respective infinite products.

2. A set of main results

In this section, we present a set of identities related to level 8 η-functions. These identities reveal important connections to colored partition theory, highlighting their dual significance in both mathematical and combinatorial contexts.

Theorem 2.1. Let $m, n \in \mathbb{N}$. Define $\mu_1(m)$ as the partition function that enumerates the colored partitions of m, where each part is of the form $8n + b$, and each part is colored with ten distinct colors. Similarly, define the following partition functions:

- $\mu_2(m)$ as the partition function that enumerates the colored partitions of m, where each part is of the form $8n + b$, colored with 6 distinct colors, and $8n + c$, colored with 8 distinct colors.

- $\mu_3(m)$ as the partition function that enumerates the colored partitions of m, where each part is of the form $8n + b$ and $8n + c$, each colored with 4 distinct colors.

Then, for $m \geq 1$, the following identity holds:

$$\mu_1(m) + 4\mu_2(m-1) - \mu_3(m) = 0.$$

Proof. By dividing Eq (1.13) by f_1^{18}, we obtain

$$\frac{f_1^4 f_4^{14}}{f_1^{18}} + 4q\frac{f_1^4 f_2^4 f_4^2 f_8^8}{f_1^{18}} - \frac{f_2^{14} f_8^4}{f_1^{18}} = 0. \tag{2.1}$$

First, consider the expression $\frac{f_4^{14}}{f_1^{14}}$. By applying Eqs (1.11) and (1.12), we obtain

$$\frac{f_4^{14}}{f_1^{14}} = \frac{1}{(q_{14}^{1\pm,3\pm}, q_{14}^{2\pm}, q_0^4; q^8)_\infty} = \frac{1}{(q_{14}^a, q_{14}^b, q_0^c; q^8)_\infty}.$$

Similarly, we can express the other terms as

$$\frac{f_2^4 f_4^2 f_8^8}{f_1^{14}} = \frac{1}{(q_{14}^a, q_{10}^b, q_8^c; q^8)_\infty},$$
$$\frac{f_2^{14} f_8^4}{f_1^{18}} = \frac{1}{(q_{18}^a, q_4^b, q_4^c; q^8)_\infty}.$$

Substituting these into Eq (2.1), we derive

$$\frac{1}{(q_{14}^a, q_{14}^b, q_0^c; q^8)_\infty} + 4\frac{q}{(q_{14}^a, q_{10}^b, q_8^c; q^8)_\infty} - \frac{1}{(q_{18}^a, q_4^b, q_4^c; q^8)_\infty} = 0. \tag{2.2}$$

Multiplying both sides by $(q_{14}^a, q_4^b, q_0^c; q^8)_\infty$, we obtain

$$\frac{1}{(q_0^a, q_{10}^b, q_0^c; q^8)_\infty} + 4\frac{q}{(q_0^a, q_6^b, q_8^c; q^8)_\infty} - \frac{1}{(q_4^a, q_0^b, q_4^c; q^8)_\infty} = 0. \tag{2.3}$$

We observe that Eq (2.3) generates the functions $\mu_1(m)$, $\mu_2(m)$, and $\mu_3(m)$. Consequently, we obtain

$$\sum_{m=0}^{\infty} \mu_1(m) q^m + 4\sum_{m=0}^{\infty} \mu_2(m) q^{m+1} - \sum_{m=0}^{\infty} \mu_3(m) q^m = 0. \tag{2.4}$$

By setting $\mu_1(0) = \mu_2(0) = \mu_3(0) = 1$, we obtain the required result by extracting the coefficients of q^m.

Example 2.1. Theorem 2.1 states that for $m \geq 1$,

$$\mu_1(m) + 4\mu_2(m-1) - \mu_3(m) = 0.$$

For $m = 1$, Table 1 shows the following values:

$$\mu_1(1) = 0, \quad \mu_2(0) = 1, \quad \mu_3(1) = 4.$$

Substituting these into the identity, we get

$$0 + 4(1) - 4 = 0,$$

which verifies Theorem 2.1 for $m = 1$.

Table 1. Verification of Theorem 2.1 for $m = 1$.

Function	**Partitions**
$\mu_1(1) = 0$:	No partitions
$\mu_2(0) = 1$:	1 partition
$\mu_3(1) = 4$:	$1_r, 1_g, 1_w, 1_y$

Theorem 2.2. Let $m, n \in \mathbb{N}$. Define $\mu_1(m)$ as the partition function that enumerates the colored partitions of m, where each part is of the form $8n+b$ and $8n+c$, colored with 12 and 16 distinct colors, respectively. Similarly, define the following partition functions:

- $\mu_2(m)$ as the partition function that enumerates the colored partitions of m, where each part is of the form $8n + b$ and $8n + c$, colored with 16 and 8 distinct colors, respectively.
- $\mu_3(m)$ as the partition function that enumerates the colored partitions of m, where each part is of the form $8n + b$, colored with 20 distinct colors.
- $\mu_4(m)$ as the partition function that enumerates the colored partitions of m, where each part is of the form $8n + a$ and $8n + c$, each colored with 8 distinct colors.

Then, for $m \geq 2$, the following identity holds:

$$16\mu_1(m-2) + 8\mu_2(m-1) + \mu_3(m) - \mu_4(m) = 0.$$

Proof. Dividing identity (1.14) by f_1^{36}, we rewrite each term using the product decompositions (1.11) and (1.12), which separate factors according to residue classes modulo 8. The resulting expressions are simplified by collecting exponents of q^a, q^b, and q^c, following the same procedure as in the proof of Theorem 2.1. The algebraic reductions rely on Lemma 1.1, and therefore intermediate steps are omitted for brevity.

$$\begin{aligned}&16\frac{q^2}{(q^a_{28}, q^b_{20}, q^c_{16}; q^8)_\infty} + 8\frac{q}{(q^a_{28}, q^b_{24}, q^c_{8}; q^8)_\infty} + \frac{1}{(q^a_{28}, q^b_{28}, q^c_{0}; q^8)_\infty}\\ &- \frac{1}{(q^a_{36}, q^b_{8}, q^c_{8}; q^8)_\infty} = 0.\end{aligned}$$

Multiplying both sides by the common factor $(q^a_{28}, q^b_{8}, q^c_{0}; q^8)_\infty$, we obtain

$$\begin{aligned}&16\frac{q^2}{(q^a_{0}, q^b_{12}, q^c_{16}; q^8)_\infty} + 8\frac{q}{(q^a_{0}, q^b_{16}, q^c_{8}; q^8)_\infty} + \frac{1}{(q^a_{0}, q^b_{20}, q^c_{0}; q^8)_\infty}\\ &- \frac{1}{(q^a_{8}, q^b_{0}, q^c_{8}; q^8)_\infty} = 0.\end{aligned} \tag{2.5}$$

Equation (2.5) provides the generating functions for $\mu_1(m)$, $\mu_2(m)$, $\mu_3(m)$, and $\mu_4(m)$, respectively. Consequently, the identity can be rewritten as

$$16\sum_{m=0}^{\infty}\mu_1(m)q^{m+2}+8\sum_{m=0}^{\infty}\mu_2(m)q^{m+1}+\sum_{m=0}^{\infty}\mu_3(m)q^{m}-\sum_{m=0}^{\infty}\mu_4(m)q^{m}=0.$$

By setting $\mu_1(0)=\mu_2(0)=\mu_3(0)=\mu_4(0)=1$, we obtain the desired result by extracting the coefficients of q^m.

Example 2.2. Theorem 2.2 states that for $m \geq 2$,

$$16\mu_1(m-2)+8\mu_2(m-1)+\mu_3(m)-\mu_4(m)=0.$$

For $m=2$, Table 2 gives

$$\mu_1(0)=1, \quad \mu_2(1)=0, \quad \mu_3(2)=20, \quad \mu_4(2)=36.$$

Substituting into the identity, we have

$$16(1)+8(0)+20-36=0,$$

thereby verifying Theorem 2.2 for $m=2$.

Table 2. Verification of the theorem for $m=2$.

Function	Partitions
$\mu_1(0)=1$:	
$\mu_2(1)=0$:	
$\mu_3(2)=20$:	$2_r, 2_g, 2_w, 2_y, 2_b, 2_p, 2_m, 2_v, 2_o, 2_t, 2_i, 2_{rt}, 2_{ry}, 2_{po}, 2_{vg}, 2_{ty}, 2_{pv}, 2_{lb}, 2_{lg}, 2_{gb}$
$\mu_4(2)=36$:	$1_r+1_r, 1_g+1_g, 1_w+1_w, 1_y+1_y, 1_b+1_b, 1_p+1_p, 1_o+1_o, 1_v+1_v, 1_r+1_g, 1_r+1_w,$ $1_r+1_y, 1_r+1_b, 1_r+1_p, 1_r+1_o, 1_r+1_v, 1_g+1_w, 1_g+1_y, 1_g+1_b, 1_g+1_p, 1_g+1_o,$ $1_g+1_v, 1_w+1_y, 1_w+1_b, 1_w+1_p, 1_w+1_o, 1_w+1_v, 1_y+1_b, 1_y+1_p, 1_y+1_o, 1_y+1_v,$ $1_b+1_p, 1_b+1_o, 1_b+1_v, 1_p+1_o, 1_p+1_v, 1_o+1_v$

Here, subscripts such as "r," "g," "w," "y," "p," "o," "m," "v," "t," "i," and "b" denote different colors assigned to the corresponding parts.

Theorem 2.3. Let $m,n \in \mathbb{N}$. Define $\mu_1(m)$ as the partition function that enumerates the colored partitions of m, where each part is of the form $8n+b$, and each part is colored with 30 distinct colors. Similarly, define the following partition functions:

- $\mu_2(m)$ as the partition function that enumerates the colored partitions of m, where each part is of the form $8n+b$ and $8n+c$, colored with 26 and 8 distinct colors, respectively.
- $\mu_3(m)$ as the partition function that enumerates the colored partitions of m, where each part is of the form $8n+b$ and $8n+c$, colored with 22 and 16 distinct colors, respectively.
- $\mu_4(m)$ as the partition function that enumerates the colored partitions of m, where each part is of the form $8n+b$ and $8n+c$, colored with 18 and 24 distinct colors, respectively.
- $\mu_5(m)$ as the partition function that enumerates the colored partitions of m, where each part is of the form $8n+a$, $8n+b$, and $8n+c$, colored with 8, 10, and 8 distinct colors, respectively.

- $\mu_6(m)$ as the partition function that enumerates the colored partitions of m, where each part is of the form $8n + a$ and $8n + c$, each colored with 12 distinct colors.

Then, for $m \geq 3$, the following identity holds:

$$3\mu_1(m) + 28\mu_2(m-1) + 80\mu_3(m-2) + 64\mu_4(m-3) - 2\mu_5(m) - \mu_6(m) = 0.$$

Proof. The result follows by dividing Eq (1.15) by f_1^{54}. We obtain the following expression:

$$3\frac{f_1^{12}f_4^{42}}{f_1^{54}} + 28q\frac{f_1^{12}f_2^{4}f_4^{30}f_8^{8}}{f_1^{54}} + 80q^2\frac{f_1^{12}f_2^{8}f_4^{18}f_8^{16}}{f_1^{54}} + 64q^3\frac{f_1^{12}f_2^{12}f_4^{6}f_8^{24}}{f_1^{54}} - 2\frac{f_1^{4}f_2^{28}f_4^{14}f_8^{8}}{f_1^{54}} - \frac{f_2^{42}f_8^{12}}{f_1^{54}} = 0. \tag{2.6}$$

By applying Eqs (1.11) and (1.12) into Eq (2.6), we obtain

$$3\frac{1}{(q^a_{42}, q^b_{42}, q^c_{0}; q^8)_\infty} + 28\frac{q}{(q^a_{42}, q^b_{38}, q^c_{8}; q^8)_\infty} + 80\frac{q^2}{(q^a_{42}, q^b_{34}, q^c_{16}; q^8)_\infty} + 64\frac{q^3}{(q^a_{42}, q^b_{30}, q^c_{24}; q^8)_\infty} - 2\frac{1}{(q^a_{50}, q^b_{22}, q^c_{8}; q^8)_\infty} - \frac{1}{(q^a_{54}, q^b_{12}, q^c_{12}; q^8)_\infty} = 0.$$

Multiplying both sides by the common factor $(q^a_{42}, q^b_{12}, q^c_{0}; q^8)_\infty$, we obtain

$$3\frac{1}{(q^a_{0}, q^b_{30}, q^c_{0}; q^8)_\infty} + 28\frac{q}{(q^a_{0}, q^b_{26}, q^c_{8}; q^8)_\infty} + 80\frac{q^2}{(q^a_{0}, q^b_{22}, q^c_{16}; q^8)_\infty} + 64\frac{q^3}{(q^a_{0}, q^b_{18}, q^c_{24}; q^8)_\infty} - 2\frac{1}{(q^a_{8}, q^b_{10}, q^c_{8}; q^8)_\infty} - \frac{1}{(q^a_{12}, q^b_{0}, q^c_{12}; q^8)_\infty} = 0. \tag{2.7}$$

Equation (2.7) provides the generating functions for $\mu_1(m)$, $\mu_2(m)$, $\mu_3(m)$, $\mu_4(m)$, $\mu_5(m)$, and $\mu_6(m)$, respectively. Consequently, the identity can be rewritten as

$$3\sum_{m=0}^{\infty}\mu_1(m)q^m + 28\sum_{m=0}^{\infty}\mu_2(m)q^{m+1} + 80\sum_{m=0}^{\infty}\mu_3(m)q^{m+2} + 64\sum_{m=0}^{\infty}\mu_4(m)q^{m+3} - 2\sum_{m=0}^{\infty}\mu_5(m)q^m - \sum_{m=0}^{\infty}\mu_6(m)q^m = 0.$$

By setting the initial conditions as

$$\mu_1(0) = \mu_2(0) = \mu_3(0) = \mu_4(0) = \mu_5(0) = \mu_6(0) = 1,$$

we obtain the desired result by extracting the coefficients of q^m.

3. Conclusions

In this paper, we have explored Dedekind's η-function at level 8 and established its fundamental connections to colored partitions. Our research has led to the development of several significant results, including new identities for the η-function, which deepen our understanding of partition functions subject to color restrictions. These findings contribute to the broader theoretical framework of partition theory and make valuable advancements in the study of modular forms. Furthermore, the mathematical structures introduced in this work offer solutions to key problems in combinatorial theory. Future research will aim to develop efficient algorithms for computing partition numbers, utilizing the newly derived identities and investigating their computational implications. This study not only enhances our understanding of Dedekind's η-function but also introduces novel analytical techniques that will further propel the study of colored partitions in modern mathematical research.

Author contributions

A. Ali: Conceptualization, methodology, formal analysis, writing-original draft; A. Fatima: Conceptualization, methodology, data curation, investigation, writing-review & editing; F. Mofarreh: Funding acquisition; W. Albalawi: Resources, writing-review & editing; A. Alshehri: Validation; Muhammad Hanif: Supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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