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Research article

An Artinian version of the Artin-Rees lemma and duality

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Abstract: We have presented a practical theorem for Artinian modules in this study, which is called the Artinian version of the Artin-Rees lemma. Using his own theorem [5, Theorem 1], D. Kirby demonstrated an Artinian analogue of the Artin-Rees lemma. Using the duality inferred from [2, Theorem 10.2.12]), we demonstrated how such an Artinian version may be directly and simply deduced from the original result [10, Theorem 8.5]. It can be generalized in a way that does not need the ring used to be local. We should point out that Matsumura [10] was not the first author to publish the Artin-Rees lemma. Note that Lang [7] published this theory earlier, in 1965. Also, Rees [13] published the ideal version of this theory in 1956.

Keywords: Noetherian modules; Artinian modules; duality
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1. Introduction

All rings examined in this work will be non-trivial and commutative with multiplicative identities; these rings will always be indicated by the letter Λ. Λ is referred to as *local* when it has precisely one maximal ideal, let us say $\mathfrak{m}$, and is Noetherian. This is represented by $(\Lambda, \mathfrak{m})$. It is precisely when Λ is Noetherian and has a finite number of maximal ideals that we say it is *semi-local*. We use the same terminology Sharp used in [20].

After giving an application of the Artin-Rees lemma in Section 2, we make progress on Matlis duality in Section 3. One may apply this tool to obtain Noetherian results, without a restriction on the ring used, to get the Artinian case of the original claim. Theorem 2 is our main result, which is the objective in Section 4.

We cannot deal, in some ways, with Artinian modules without using the concept of "secondary representation" of a(n) (Artinian) module. (We should point out that the concept of secondary representation of a module applies more widely than just to the class of Artinian modules. For example, injective modules (and these do not need to be Artinian) over a Noetherian ring also have secondary representation: see Sharp [16].) Sharp applied this theory to the structure of local cohomology [17], and to injective modules [16].

In a particular manner, we consider the theory of secondary representation of a module and the theory of attached prime ideals, which are dual to the theory of primary decomposition of a module and the theory of associated prime ideals. MacDonald [8] worked on this theory, as well. For more information about these theories and the terminology, see Chapter 9 of Sharp [21].

Also Kirby [6] independently discovered the theory of secondary (he called them coprimary) modules. Throughout, we follow MacDonald's terminology [8] and refer the reader to his paper for definitions and related results.

Let us specify E as the direct sum of injective envelopes $\bigoplus_{\mathfrak{m}} E(\Lambda/\mathfrak{m})$, where the sum is taken over all maximal ideals of Λ. The additive, exact, Λ-linear [12, p. 35] functor $\mathrm{Hom}(, E)$ from $C(\Lambda)$, the Λ-module category, to itself will be denoted by D and, under this circumstance, for a Λ-module M, $D(M)$ will stand for $\mathrm{Hom}_{\Lambda}(M, E)$ and $DD(M)$ for $\mathrm{Hom}_{\Lambda}(\mathrm{Hom}_{\Lambda}(M, E), E)$. Brodmann and Sharp presented a different approach to the original work of Matlis [9] (see §2 of Chapter 10 of [2]). Their approach makes use of Melkersson's Theorem [11, Theorem 1.3] and applied it to certain modules. Moreover, they presented the main aspects of Matils duality by [2, Theorem 10.2.12] over a complete local ring. By this work, they made the Matlis theorem more functional, so that it became much easier to obtain Artinian versions of some well-known Noetherian results. By their expression in §2 of Chapter 10 of [2], the Matlis duality theorem allows statements about Noetherian Λ-modules to be translated into "dual" statements about Artinian Λ-modules, and vice versa. In [19], Sharp proved that, for a Noetherian Λ-module M, $\mathrm{Ass}_{\Lambda}(M)$, the set of associated primes of M are equal to the $\mathrm{Att}_{\Lambda}(D(M))$, the set of attached primes of Artinian Λ-module $D(M)$. Also for an Artinian Λ-module A, $\mathrm{Att}_{\Lambda}(A)$ and $\mathrm{Ass}_{\Lambda}(D(A))$ are equal sets. Also, in future works, one can apply Matlis duality theory to the works of [15, 4, 3, 14].

In this the paper, we present the main theorem in Section 4. We found Chapters 7 and 8 of Sharp's book [21] and [2, 19, 20] very useful to deal with Artinian modules. We therefore refer the reader to them for related basic definitions, lemmas, and theorems. Recall that:

Lemma 1. *Let Λ be a semi-local ring and $\mathfrak{m}_1, \dots, \mathfrak{m}_s$, $s \in \mathbb{N}$, all its maximal ideals, and let $E = E(\bigoplus_{i=1}^{s} \Lambda/\mathfrak{m}_i)$ be the injective envelope (see [22, p. 44]) of $\bigoplus_{i=1}^{s} \Lambda/\mathfrak{m}_i$. Then E is an injective cogenerator for Λ. (For the proof, we refer the reader to [22, p. 46]).*

1.1. Notation

Let A be a non-zero Artinian Λ-module with a finite number of maximal ideals $\mathfrak{m}$ of Λ for which $\mathrm{Soc}(A)$, the socle of A, has a submodule isomorphic to $\Lambda/\mathfrak{m}$. Let the distinct maximal ideals be $\mathfrak{m}_1, \dots, \mathfrak{m}_s$. Set $J := \bigcap_{i=1}^{s} \mathfrak{m}_i$ and $\hat{\Lambda}^{(J)} := \varprojlim_{n} \Lambda/J^n$, the J-adic completion of Λ [20].

As a further step to the 1.1 notation above, we justify the claim about the Jacobson radical of $\hat{\Lambda}$ in [20, (2.4)] as a lemma below.

Lemma 2. *Let the situation be as in the 1.1 notation and assume that* $\mathrm{Soc}(A) \cong \bigoplus_{i=1}^{s} \Lambda/\mathfrak{m}_i$. *Also set $A_n = (0 :_A J^n)$ and $I_n := (0 :_{\Lambda} A_n) = (0 :_{\Lambda} (0 :_A J^n))$, for each $n \in \mathbb{N}_0$.*

Define $\hat{\Lambda} := \varprojlim_n \Lambda/I_n$. *For each* $n \in \mathbb{N}_0$, *let* $\pi_n : \Lambda \longrightarrow \Lambda/I_n$ *denote the natural map, and let*

$$\hat{I}_n = \{(r_n + I_n)_{n\in\mathbb{N}_0} \in \hat{\Lambda} : r_n + I_n = 0\} = \{(r_n + I_n) \in \hat{\Lambda} : r_n \in I_n\},$$

an ideal of $\hat{\Lambda}$. *Then* $\hat{J} := \hat{I}_1$ *is the Jacobson radical of* $\hat{\Lambda}$.

Proof. If $\hat{r} \in \varprojlim_n \Lambda/I_n$, then $\hat{r} = (r_n + I_n)_{n\in\mathbb{N}_0}$ where, for $m \geq n$, $r_m - r_n \in I_n$. By the definition of $\hat{I}_n$,

$$\hat{I}_1 = \{(r_n + I_n)_{n\in\mathbb{N}_0} \in \hat{\Lambda} : r_1 + I_1 = 0\} = \{(r_n + I_n) \in \hat{\Lambda} : r_1 \in I_1\}.$$

We know that, for $n \in \mathbb{N}$, $J^n \subseteq I_n \subseteq I_1 = J$. Therefore $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ are precisely the maximal ideals $\mathfrak{m}$ of Λ for which $\mathfrak{m} \supseteq I_n$. Thus for all $n \geq 1$, the set $\mathrm{Max}(\Lambda/I_n)$ of all maximal ideals of Λ/I_n is $\{\mathfrak{m}_1/I_n, \ldots, \mathfrak{m}_s/I_n\}$. Consider the natural map $\phi : \Lambda \longrightarrow \hat{\Lambda}$ with $r \longmapsto (r+I_n)_{n\in\mathbb{N}_0}$. Let $\hat{r} = (r_n+I_n)_{n\in\mathbb{N}_0} \in \hat{\Lambda}$ belong to $\mathrm{Jac}(\hat{\Lambda})$, the Jacobson radical of $\hat{\Lambda}$. Therefore, for all $a \in \Lambda$, $(a + I_n)_{n\in\mathbb{N}_0} = \phi(a) \in \hat{\Lambda}$ and, by [21, (3.17)], $1-\hat{r}\phi(a) = (1-r_n a+I_n)_{n\in\mathbb{N}_0}$ is a unit of $\hat{\Lambda}$, and so $1-r_1a+I_1$ is a unit of Λ/I_1, for all choices of $a \in \Lambda$. Thus $(1 + I_1) - (r_1 + I_1)(a + I_1)$ is a unit of Λ/I_1, for all $a + I_1$. Therefore $r_1 + I_1 \in \mathrm{Jac}(\Lambda/I_1) = \bigcap_{i=1}^{s} \mathfrak{m}_i/I_1 = I_1/I_1$ since $J = I_1$, so that $r_1 \in I_1$ and $\mathrm{Jac}(\hat{\Lambda}) \subseteq \hat{I}_1$. (Note that $J = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_s$ and $I_1 = (0 :_\lambda (0 :_A J))$. Since $J(0 :_A J) = 0$, we have $J \subseteq I_1$. On the other hand, $\mathrm{Soc}(A) \subseteq A$ and $\mathrm{Soc}(A) \cong \Lambda/\mathfrak{m}_1 \oplus \Lambda/\mathfrak{m}_2 \oplus \ldots \oplus \Lambda/\mathfrak{m}_s$. Thus J annihilates $\mathrm{Soc}(A)$. Therefore $\mathrm{Soc}(A) \subseteq (0 :_A J)$. Now $I_1(0 :_A J) = 0$ and so $I_1(\mathrm{Soc}(A)) = 0$. Thus $I_1(\Lambda/\mathfrak{m}_1 \oplus \Lambda/\mathfrak{m}_2 \oplus \ldots \oplus \Lambda/\mathfrak{m}_s) = 0$. For each $i = 1, \ldots, s$, $I_1(\Lambda/\mathfrak{m}_i) = 0$ and so $I_1 \subseteq \mathfrak{m}_i$. Therefore $I_1 \subseteq \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_s = J$, that is, $I_1 \subseteq J$. Conversely, let $\hat{r} = (r_n + I_n)_{n\in\mathbb{N}_0} \in \hat{I}_1$. Thus $r_1 \in I_1$. Let $\hat{a} = (a_n + I_n)_{n\in\mathbb{N}_0} \in \hat{\Lambda}$ and consider $1 - \hat{r}\hat{a} = (1 - r_n a_n + I_n)_{n\in\mathbb{N}_0}$. We wish to show that this is a unit of $\hat{\Lambda}$. Take $d_1 = 1$. Since $r_1 \in I_1$, $(d_1 + I_1)(1 - r_1a_1 + I_1) = d_1 + I_1 = 1 + I_1$. Suppose, inductively, that we have constructed $d_1 = 1, d_2, \ldots, d_n \in \Lambda$, $n \geq 1$, such that $d_i - d_{i+1} \in I_i$ for all $i = 1, \ldots, n-1$ and $(d_i + I_i)(1 - r_ia_i + I_i) = 1 + I_i$ for all $i = 1, \ldots, n-1, n$. Now we try to construct a d_{n+1} to extend the sequence. Consider $1 - r_{n+1}a_{n+1} + I_{n+1}$. We know that $1 - r_{n+1}a_{n+1} + I_n = 1 - r_na_n + I_n$. We claim that $1 - r_{n+1}a_{n+1} + I_{n+1}$ is a unit of Λ/I_{n+1}: if not, then $1 - r_{n+1}a_{n+1} + I_{n+1} \in \mathfrak{m}_i/I_{n+1}$ for some $i = 1, \ldots, s$. Therefore $1 - r_{n+1}a_{n+1} \in \mathfrak{m}_i$ and so $1 - r_na_n = 1 - r_{n+1}a_{n+1} - (1 - r_{n+1}a_{n+1} - (1 - r_na_n))$ is in $\mathfrak{m}_i$. Then, $1 - r_na_n + I_n \in \mathfrak{m}_i/I_n$, and so it is not a unit of Λ/I_n. This is a contradiction to the inductive step. Therefore there exist $d_{n+1} \in \Lambda$ such that $(1 - r_{n+1}a_{n+1} + I_{n+1})(d_{n+1} + I_{n+1}) = 1 + I_{n+1}$. Also, if we apply the natural ring homomorphism $\Lambda/I_{n+1} \longrightarrow \Lambda/I_n$ to the last equation, we get that $(1 - r_{n+1}a_{n+1} + I_n)(d_{n+1} + I_n) = 1 + I_n$. But $1 - r_{n+1}a_{n+1} + I_n = 1 - r_na_n + I_n$, since $1 - \hat{r}\hat{a} \in \hat{\Lambda}$. Therefore $(1 - r_na_n + I_n)(d_{n+1} + I_n) = 1 + I_n$, and $d_{n+1} + I_n$ is the inverse of $1 - r_na_n + I_n$ in Λ/I_n. But $d_n + I_n$ is this inverse, so that $d_{n+1} + I_n = d_n + I_n$ and $d_{n+1} - d_n \in I_n$. This completes the inductive step, and so, by induction, we construct a sequence $1 = d_1, d_2, \ldots, d_n, \ldots$, of elements of Λ such that $d_i - d_{i+1} \in I_i$ for all $i \geq 1$, and $(d_i + I_i)(1 - r_ia_i + I_i) = 1 + I_i$ for all $i \geq 1$. Thus $\hat{d} = (d_i + I_i)_{i\in\mathbb{N}_0} \in \hat{\Lambda}$ and $\hat{d}(1 - \hat{r}\hat{a}) = 1_{\hat{\Lambda}}$. This shows that $1 - \hat{r}\hat{a}$ is a unit of $\hat{\Lambda}$ for all $\hat{a} \in \hat{\Lambda}$. Therefore $\hat{r} \in \mathrm{Jac}(\hat{\Lambda})$ and so $\hat{I}_1 = \mathrm{Jac}(\hat{\Lambda})$. □

2. An application of the Artin-Rees lemma

Now we would like to point out how Lemma 4 below has been proven by the Artin-Rees lemma. First;

Lemma 3. *Let the situation be as in the 1.1 notation. Then there exists* $t \in \mathbb{N}$ *and submodules* $A_1, A_2, \ldots, A_t$ *of* A *such that* $0 = \bigcap_{j=1}^{t} A_j$ *and, for each* $j = 1, \ldots, t$, *the Artinian* Λ*-module* A/A_j *satisfies*

$$\mathrm{Soc}(A/A_j) \cong \bigoplus_{i=1}^{s} \Lambda/\mathfrak{m}_i.$$

Proof. We refer the reader to [20, pp. 31–32]. □

Lemma 4. *Let the situation be as in the 1.1 notation and consider [20, Theorem 3.2]. Then* $\Lambda' := \hat{\Lambda}^{(J)}/(0 :_{\hat{\Lambda}^{(J)}} A)$ *is* Λ'*-isomorphic to a submodule, let us call it* V*, of*

$$U := \bigoplus_{i=1}^{t} \Lambda_i$$

where $\Lambda_i := \hat{\Lambda}^{(J)}/(A_i :_{\hat{\Lambda}^{(J)}} A)$ *and* A_i *is defined as in Lemma 3 for* $i = 1, \ldots, t$*, which is* J'*-adically complete, where* J' *is the Jacobson radical of* Λ'*, and* V *is* J'*-adically complete.*

Proof. Consider the natural map

$$\phi_V : V \longrightarrow \varprojlim_n V/J'^n V.$$

This map has kernel $\bigcap_{n=1}^{\infty} J'^n V \subseteq \bigcap_{n=1}^{\infty} J'^n U = 0$ since $\phi_U : U \xrightarrow{nat} \varprojlim_n U/J'^n U$ is injective. Now we show ϕ_V is surjective. Let $(v_n + J'^n V)_{n\in\mathbb{N}_0} \in \varprojlim_n V/J'^n V$. Thus, for all m, n with $n \geq m$, $v_n - v_m$ is in $J'^m V$ and $v_n - v_m \in J'^m U$. Therefore $(v_n + J'^n U)_{n\in\mathbb{N}_0} \in \varprojlim_n U/J'^n U$. Since U is complete, there exists $u \in U$ such that $v_n + J'^n U = u + J'^n U$ for all $n \in \mathbb{N}_0$. By the Artin-Rees lemma, (see [10, Theorem (8.5)]), there exists $c \in \mathbb{N}_0$ such that for all $n \geq c$,

$$J'^n U \cap V = J'^{n-c}(J'^c U \cap V) \subseteq J'^{n-c} V.$$

Now, $u = v_n + (u - v_n) \in V + J'^n U$ for all $n \in \mathbb{N}_0$ and so $u + V \in (V + J'^n U)/V$ for all $n \in \mathbb{N}_0$.

Therefore $u + V \in \bigcap_{n=1}^{\infty} J^n(U/V) = 0$ since $J' = \mathrm{Jac}(\Lambda')$ by (8.9) and (8.10) of [10]. Thus $u \in V$. Also $u - v_n \in (J'^n U) \cap V$ for all $n \in \mathbb{N}_0$. Let $n \in \mathbb{N}_0$. Then $u - v_{n+c} \in (J'^{n+c} U) \cap V \subseteq J'^n V$. Also $v_{n+c} - v_n \in J'^n V$, so that $u - v_n = u - v_{n+c} - (v_{n+c} - v_n) \in J'^n V$. Therefore $u + J'^n V = v_n + J'^n V$ for all $n \in \mathbb{N}_0$. Thus $(v_n + J'^n V)_{n\in\mathbb{N}_0} = (u + J'^n V)_{n\in\mathbb{N}_0} \in \mathrm{Im}(\phi_V)$. Consequently ϕ_V is surjective and so V is complete in the J'-adic topology. □

3. A generalization of Matlis duality

Now we are going to show how [2, Theorem 10.2.12] can be extended to the case where the underlying ring does not need to be local, using Sharp's study [20]. For this, we shall need the following basic facts.

3.1. Notation

Let Λ be a complete semi-local (Noetherian) ring with s distinct maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$. Set

$$J := \bigcap_{i=1}^{s} \mathfrak{m}_i.$$

For $n \in \mathbb{N}$, we have that Λ/J^n has finite length, simply because Λ/J^n is an Artinian ring. Its maximal ideals are precisely $\mathfrak{m}_i/J^n$, $i = 1, \ldots, s$. For $n = 1$,

$$\Lambda/J \cong \bigoplus_{i=1}^{s} \Lambda/\mathfrak{m}_i \quad \text{and} \quad E = E(\Lambda/J) = \bigoplus_{i=1}^{s} E(\Lambda/\mathfrak{m}_i),$$

which is the injective envelope (see [22, p. 44]) of Λ/J as a Λ-module by [22, (2.23)]. Now let us denote $E(\Lambda/\mathfrak{m}_i)$ by E_i ; then $E = \bigoplus_{i=1}^{s} E_i$. For all $n \in \mathbb{N}$, $J^n(\Lambda/J) = 0$ and so $\Lambda/J \subseteq (0 :_E J^n)$, the annihilator of J^n on E. Also a subset of $(0 :_E J^n)$ is a Λ-submodule if and only if it is a Λ/J^n-submodule. Therefore $(0 :_E J^n)$ is an essential extension, as a Λ/J^n-module, of Λ/J. If $(0 :_E J^n)$ is injective as a Λ/J^n-module, and in fact it is, then we have

$$(0 :_E J^n) \cong_{\Lambda/J^n} E_{\Lambda/J^n}((\Lambda/J^n)/(J/J^n))$$

and

$$\Lambda/J \cong_{\Lambda/J^n} ((\Lambda/J^n)/(J/J^n)) \cong \bigoplus_{i=1}^{s} ((\Lambda/J^n)/(\mathfrak{m}_i/J^n))\,.$$

Lemma 5. *Let the situation be as in the 3.1 notation, where Λ is complete, so the natural map $\Lambda \longrightarrow \varprojlim_n \Lambda/J^n$ is an isomorphism. Then, for each $\theta \in \mathrm{Hom}_\Lambda(E, E)$, there is a unique $r_\theta \in \Lambda$ such that $\theta(e) = r_\theta e$ for all $e \in E$.*

Proof. Let

$$\theta_n = \theta\,|_{(0:_E J^n)} : (0 :_E J^n) \longrightarrow (0 :_E J^n)$$

be the restriction of θ to $(0 :_E J^n)$. (Note that $\theta((0 :_E J^n)) \subseteq (0 :_E J^n)$ since for $x \in (0 :_E J^n)$ and $r \in J^n$, $r\theta(x) = \theta(rx) = \theta(0) = 0$). For each $n \in \mathbb{N}$, there is an $r_n \in \Lambda$ such that $\theta_n(e) = r_n e$ for all $e \in (0 :_E J^n)$; this follows from [20, p. 27], since Λ/J^n is Artinian and there is a Λ/J^n-isomorphism

$$(0 :_E J^n) \cong \bigoplus_{i=1}^{s} E_{\Lambda/J^n}((\Lambda/J^n)/(\mathfrak{m}_i/J^n))\,.$$

Given $n \in \mathbb{N}$, if $s_n \in \Lambda$ is also such that $\theta(e) = s_n e$ for all $e \in (0 :_E J^n)$, then $(r_n - s_n) \in (0_\Lambda : (0 :_E J^n)) = J^n$. In particular, since $\theta(e) = r_{n+1}e$ for all $e \in (0 : J^{n+1}) \supseteq (0 :_E J^n)$, we have $r_{n+1} - r_n \in J^n$ for all $n \in \mathbb{N}$. Therefore $r_{n+k} - r_n \in J^n$ for all $k, n \in \mathbb{N}$. Thus

$$(\ldots, r_n + J^n, \ldots, r_2 + J^2, r_1 + J) \in \varprojlim_n \Lambda/J^n \overset{\cong}{\longleftarrow} \Lambda,$$

since Λ is complete. Therefore there exists $r \in \Lambda$ such that

$$(\ldots, r_n + J^n, \ldots, r_2 + J^2, r_1 + J) = (\ldots, r + J^n, \ldots, r + J^2, r + J),$$

and therefore $r_n - r \in J^n$ for all $n \in \mathbb{N}$. Thus $\theta_n(e) = r_n e = re$ for all $e \in (0 :_E J^n)$. Thus $\theta(e) = re$ for all $e \in E = \bigcup_{n=1}^{\infty} (0 :_E J^n)$. Finally, if $r' \in \Lambda$ is also such that $\theta(e) = r'e$ for all $e \in E$, then $r - r' \in (0 :_\Lambda E)$, and this is zero by Krull's intersection theorem (see [21, (8.25)]); hence $r' = r$. □

Proposition 1. *Let the situation be as in the 3.1 notation and let A be an Artinian Λ-module. Then A can be embedded in a direct sum of a finite number of copies of E.*

Proof. Recall that A is an essential extension of its socle (see [20, (2.3)]). Let

$$\mathrm{Soc}(A) \cong \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{t_i} \Lambda/\mathfrak{m}_i$$

and $t = \mathrm{Max}\{t_i\}_{i=1}^{s}$. Then $\mathrm{Soc}(A)$ can be embedded in

$$E(\bigoplus_{j=1}^{t} \bigoplus_{i=1}^{s} \Lambda/\mathfrak{m}_i) \cong E^t,$$

the direct sum of t copies of E. By [22, (2.18)], A can be embedded in E^t. □

Remark 1. *Let the situation be as in the 3.1 notation and let A, N be Λ-modules. Then in conjunction with Lemma 5 and Proposition 1, by using the homological methods that Brodmann and Sharp used, we generalize [2, Theorem 10.2.12] to the case in which Λ, the ring modules A and N are based on, need not be local.*

4. The main theorem

Before giving the main result, Theorem 2, we present a theorem as an application of Remark 1.

In [18], Sharp shows that, for an Artinian Λ-module A and an ideal I of Λ, $(\mathrm{Att}((0 :_A I^n)))_{n\in\mathbb{N}}$, the attached primes of $(0 :_A I^n)$ are eventually stationary for integers large enough, n. This done with an Artinian version of the work of [1].

Sharp originally proved the result, but our proof is different and illustrates the technique involving Remark 1. For this purpose, we use the result of the study by Brodmann [1].

Theorem 1. *Let A be an Artinian Λ-module and I be an ideal of Λ. Then the set $(\mathrm{Att}_\Lambda((0 :_A I^n)))_{n\in\mathbb{N}}$ is asymptotically stabile.*

That means that there exists $n_0 \in \mathbb{N}$ such that $\mathrm{Att}_\Lambda((0 :_A I^n)) = \mathrm{Att}_\Lambda((0 :_A I^{n_0}))$ for all $n \geq n_0$.

Proof. By [20, (3.2)], there exists a complete semi-local (Noetherian) ring Λ' such that a subset of A is a Λ-submodule if and only if it is a Λ'-submodule. Let $f : \Lambda \longrightarrow \Lambda'$ be the natural ring homomorphism and let I' denote the extension $f(I)\Lambda'$ of I to Λ'. Now in the 3.1 notation, by using [19, (2.4)], for the submodule $(0 :_A I'^n)$ of the Artinian Λ'-module A, we have

$$D((0 :_A I'^n)) \cong D(A)/I'^n D(A)$$

for all $n \in \mathbb{N}$. Since $D(A)$ is a Noetherian Λ'-module (by Remark 1),

$$\mathrm{Att}_{\Lambda'}((0 :_A I'^n)) = \mathrm{Ass}_{\Lambda'}(D(A)/I'^n D(A))$$

for all $n \in \mathbb{N}$ by [19, (2.7)]. Now by using [1], we have that the set $\mathrm{Att}_{\Lambda'}((0 :_A I'^n))$, for all $n \in \mathbb{N}$, is eventually stationary. But $(0 :_A I'^n) = (0 :_A I^n)$ and also

$$\mathrm{Att}_\Lambda((0 :_A I^n)) = \{f^{-1}(P') : P' \in \mathrm{Att}_{\Lambda'}((0 :_A I^n))\}$$

by [19, (1.12)]. Therefore, the result follows. Note that $\mathrm{Att}_\Lambda((0 :_A I^n))$ might have fewer members than $\mathrm{Att}_{\Lambda'}((0 :_A I^n))$ because it might happen that $f^{-1}(P'_i) = f^{-1}(P'_j)$, say, even through $P'_i \neq P'_j$, i, $j \in \mathbb{N}$. □

Now, we shall establish some of the basic consequences of Matlis duality theory in order to apply them later. For proofs of them, we refer the reader to §5.4 of [22] in connection with Remark 1 and use of the well-known lemma of homological algebra, 5-lemma.

After this point, we assume that Λ is a complete, semi-local (Noetherian) ring unless otherwise stated.

4.1. Definition and notation

Let A be a Λ-module and B a submodule of A. Let $D(A)$ be the Matlis dual of A, and let K be a submodule of $D(A)$. Then we define a submodule B^{λ} of $D(A)$ by

$$B^{\lambda} := \{f \in D(A) : f(B) = 0\}$$

and a submodule K^{μ} of A as

$$K^{\mu} := \{a \in A : f(a) = 0 \text{ for all } f \in K\}.$$

Now, let B_1, B_2 be submodules of A and K_1, K_2 submodules of $D(A)$. If $B_1 \subseteq B_2$ and $K_1 \subseteq K_2$, then $B_2^{\lambda} \subseteq B_1^{\lambda}$ and $K_2^{\mu} \subseteq K_1^{\mu}$, so that the correspondence $B \longrightarrow B^{\lambda}$ from the submodules of A to the submodules of $D(A)$, and the correspondence $K \longrightarrow K^{\mu}$ from the submodules of $D(A)$ to the submodules of A, by reverse inclusion. Also note that $B \subseteq B^{\lambda\mu}$ and $K \subseteq K^{\mu\lambda}$ by their definition. Using the preceding facts will give us $B^{\lambda\mu\lambda} = B^{\lambda}$ and $K^{\mu\lambda\mu} = K^{\mu}$.

Proposition 2. *Let the situation be as in the 4.1 definition and notation. Then $K = K^{\mu\lambda}$.*

Proof. Since K is a submodule of $D(A)$, then the sequence

$$0 \longrightarrow K \longrightarrow D(A) \longrightarrow D(A)/K \longrightarrow 0$$

is exact. This gives rise to the exact sequence

$$0 \longrightarrow D(D(A)/K) \longrightarrow DD(A) \longrightarrow D(K) \longrightarrow 0.$$

But $DD(A) \cong A$ and so we can define a natural Λ-homomorphism $\phi_K : A \longrightarrow D(K)$ with $a \longmapsto (f \mapsto f(a))$ for all $f \in K$. ϕ_K has kernel

$$\begin{aligned} \operatorname{Ker}(\phi_K) &= \{a \in A : f(a) = 0 \text{ for all } f \in K\} \\ &= K^{\mu}. \end{aligned}$$

Therefore the sequence

$$0 \longrightarrow K^{\mu} \longrightarrow A \longrightarrow D(K) \longrightarrow 0$$

and also the sequence

$$0 \longrightarrow K^{\mu\lambda\mu} \longrightarrow A \longrightarrow D(K^{\mu\lambda}) \longrightarrow 0$$

are exact. The natural injection $j : K \longrightarrow K^{\mu\lambda}$ gives the natural inclusion map

$$D(j) : D(K^{\mu\lambda}) \longrightarrow D(K).$$

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{\mu\lambda\mu} & \longrightarrow & A & \longrightarrow & D(K^{\mu\lambda}) & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow{\scriptstyle D(j)} & & \\ 0 & \longrightarrow & K^{\mu} & \longrightarrow & A & \longrightarrow & D(K) & \longrightarrow & 0\,. \end{array}$$

By the 5-lemma, $D(j)$ is an isomorphism. Since

$$0 \longrightarrow K \longrightarrow K^{\mu\lambda} \longrightarrow K^{\mu\lambda}/K \longrightarrow 0$$

and therefore

$$0 \longrightarrow D(K^{\mu\lambda}/K) \longrightarrow D(K^{\mu\lambda}) \longrightarrow D(K) \longrightarrow 0$$

are exact, and since $D(K^{\mu\lambda}) \longrightarrow D(K)$ is an isomorphism, $D(K^{\mu\lambda}/K)$ is zero, and so $DD(K^{\mu\lambda}/K) = K^{\mu\lambda}/K$ is zero as well. Thus $K^{\mu\lambda} = K$. □

Remark 2. *Let the situation be as in the 4.1 definition and notation, and let I be an ideal of Λ. Then*

(i) $(B_1 + B_2)^\lambda = B_1^\lambda \cap B_2^\lambda$;
(ii) $(K_1 + K_2)^\mu = K_1^\mu \cap K_2^\mu$;
(iii) $(IB)^\lambda = (B^\lambda :_{D(A)} I)$;
(iv) $(IK)^\mu = (K^\mu :_A I)$;
(v) $(B_1 \cap B_2)^\lambda = B_1^\lambda + B_2^\lambda$;
(vi) $(K_1 \cap K_2)^\mu = K_1^\mu + K_2^\mu$;
(vii) $(B :_A I)^\lambda = IB^\lambda$;
(viii) $(K :_{D(A)} I)^\mu = IK^\mu$.

Remark 3. *Let the situation be as in the 4.1 definition and notation and consider the exact sequence*

$$0 \longrightarrow B \overset{i}{\longrightarrow} A \overset{\pi}{\longrightarrow} A/B \longrightarrow 0 \qquad (I).$$

By duality theory, we have that the induced sequence

$$0 \longrightarrow D(A/B) \overset{D(\pi)}{\longrightarrow} D(A) \overset{D(i)}{\longrightarrow} D(B) \longrightarrow 0 \qquad (II)$$

is exact. Therefore we can identify $D(A/B)$ as a submodule of $D(A)$. Then

(i) $(D(A))^\mu = 0$;
(ii) $(D(A/B))^\mu = B$.

Proof. Since E is an injective cogenerator for Λ, then:
(i) We have that $(D(A))^\mu = \{a \in A : f(a) = 0 \text{ for all } f \in D(A)\} = 0$.
(ii) For an element $f \in D(A/B)$, $D(\pi)(f) = f o \pi$, and then

$$\begin{aligned} (D(A/B))^\mu &= \{a \in A : (f o \pi)(a) = 0 \text{ for all } f \in D(A/B)\} \\ &= \{a \in A : f(a + B) = 0 \text{ for all } f \in D(A/B)\} \\ &= B. \end{aligned}$$

Note that, by the exactness of (II), $D(A/B) = B^\lambda$. □

Remark 4. *Let Λ, Γ be arbitrary (commutative, non-trivial) rings and $f : \Lambda \longrightarrow \Gamma$ a ring homomorphism. Let M be an Γ-module and I an ideal of Λ. Assume that a subset of M is a Λ-submodule if and only if it is a Γ-submodule. Let I^e denote the extension $f(I)\Gamma$ of I to Γ. Then for a submodule N of M, $(N :_M I^e) = (N :_M I)$ and, for all $n \in \mathbb{N}$, $(N :_M I^{e^n}) = (N :_M I^n)$.*

Proof. Consider the Γ-submodule $(N :_M I^e)$ of M. We have

$$\begin{aligned}(N :_M I^e) &= \{x \in M : sx \in N \text{ for all } s \in I^e\}\\ &= \{x \in M : f(r)x \in N \text{ for all } r \in I\}\\ &= \{x \in M : rx \in N \text{ for all } r \in I\}\\ &= (N :_M I).\end{aligned}$$

Now, since $I^{e^n} = I^{n^e}$ by [21, (2.43)(ii)], then

$$(N :_M I^n) = (N :_M I^{n^e}) = (N :_M I^{e^n})$$

by the first part. □

Now we give the Artinian version of the Artin-Rees lemma as a theorem and prove it by using the Matlis duality we generalized in the previous section.

Theorem 2 (Artin-Rees lemma for Artinian modules)**.** *Let A be an Artinian Λ-module, and let B be a submodule of A. Then for an ideal I of Λ, there exists a non-negative integer c such that*

$$B + (0 :_A I^n) = ((B + (0 :_A I^c)) :_A I^{n-c})$$

for all $n \geq c$, $n \in \mathbb{N}$.

Proof. We prove this theorem in two steps. First, we consider the special case in which we assume Λ is a complete semi-local (Noetherian) ring. Consider the exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0,$$

which gives rise to the exact sequence

$$0 \longrightarrow D(A/B) \longrightarrow D(A) \longrightarrow D(B) \longrightarrow 0.$$

We can regard $D(A/B)$ as a submodule of $D(A)$. By Remark 1, $D(A)$ is Noetherian, and so too is $D(A/B)$. Let us temporarily denote $D(A)$ by N and $\operatorname{Im}(D(A/B) \longrightarrow D(A))$ by M. Then for an ideal J of Λ, there exists a $d \in \mathbb{N}$ such that

$$J^n N \cap M = J^{n-d}(J^d N \cap M)$$

for all $n \geq d$ by [10, (8.5)]. Then by the 4.1 definition and notation,

$$(J^n N \cap M)^\mu = (J^{n-d}(J^d N \cap M))^\mu.$$

Now consider the left-hand side:

$$
\begin{aligned}
(J^n N \cap M)^\mu &= (J^n N)^\mu + M^\mu \quad \text{by Remark 2 (vi)} \\
&= (N^\mu :_A J^n) + M^\mu \quad \text{by Remark 2 (iv).}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(J^{n-d}(J^d N \cap M))^\mu &= \left((J^d N \cap M)^\mu :_A J^{n-d}\right) \\
&= \left(((J^d N)^\mu + M^\mu) :_A J^{n-d}\right) \\
&= \left((M^\mu + (N^\mu :_A J^d)) :_A J^{n-d}\right)
\end{aligned}
$$

by Remark 2 (iv), (vi). Therefore we have

$$
M^\mu + (N^\mu :_A J^n) = \left((M^\mu + (N^\mu :_A J^d)) :_A J^{n-d}\right).
$$

But $M^\mu = (B^\lambda)^\mu = B$ by Remark 3 (ii) and $N^\mu = (D(A))^\mu = 0$ by Remark 3 (i). Thus

$$
B + (0 :_A J^n) = \left((B + (0 :_A J^d)) :_A J^{n-d}\right).
$$

The proof can now be finished by assuming that the ring Λ is arbitrary (non-trivial). By [20, (3.2)], a complete semi-local ring Λ' exists such that, naturally, the module A is a faithful Artinian Λ'-module; additionally, a subset of A is a Λ-module if and only if it is a Λ'-submodule. This is because A is Artinian. Let $f : \Lambda \longrightarrow \Lambda'$ be the natural ring homomorphism (see §3 of [20]) and let I' denote the extension $f(I)\Lambda'$ of I to Λ'. By the special case, already proved, there exists $c \in \mathbb{N}$ such that

$$
B + (0 :_A I'^n) = ((B + (0 :_A I'^c)) :_A I'^{n-c})
$$

for all $n \geq c$. But since, for any submodule X of A, $(X :_A I^i) = (X :_A I'^i)$ for all $i \in \mathbb{N}$ by Remark 4, we have

$$
B + (0 :_A I^n) = ((B + (0 :_A I^c)) :_A I^{n-c})
$$

for all $n \geq c$. □

5. Conclusions

In this paper, we established an Artinian version of the classical Artin–Rees lemma by means of a generalized Matlis duality over complete semi-local Noetherian rings. Our approach provides a direct dual interpretation of the classical Noetherian result and avoids unnecessary locality assumptions.

By extending the duality framework and exploiting the correspondence between associated and attached prime ideals, we obtained a transparent proof of the Artin–Rees-type stabilization for Artinian modules. As an application, we also recovered the asymptotic stability of the sets $\mathrm{Att}_\Lambda((0 :_A I^n))$.

The methods developed here demonstrate that several classical Noetherian results admit natural Artinian counterparts through duality, and they may serve as a foundation for further investigations in the structure theory of Artinian modules.

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Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest. No new data were created or analyzed in this study. Data sharing is not applicable to this article.

References

1. M. Brodmann, Asymptotic stability of Ass$\left(\frac{M}{I^nM}\right)$, *Proc. Amer. Math. Soc.*, **74** (1979), 16–18. https://doi.org/10.2307/2042097
2. M. P. Brodmann, R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications*, 2nd ed., Cambridge University Press, 2013. https://doi.org/10.1017/CBO9781139044059
3. H. Chakraborty, M. K. Patel, Iso M-Artinian (Noetherian) modules and rings, *Algebra Appl.,* Springer Proceedings in Mathematics Statistics, Singapore, 2025, 251–261. https://doi.org/10.1007/978-981-97-6798-4_22
4. Y. El Haddaoui, H. Kim, N. Mahdou, On $\aleph_0$-Noetherian modules and $\aleph_0$-coherent modules, *Commun. Korean Math. Soc.*, **40** (2025), 571–585. https://doi.org/10.4134/CKMS.c240240
5. D. Kirby, Artinian modules and Hilbert polynomials, *Quart. J. Math. Oxford*, **24** (1973), 47–57. https://doi.org/10.1093/qmath/24.1.47
6. D. Kirby, Coprimary decomposition of Artinian modules, *J. London Math. Soc.*, **6** (1973), 571–576. https://doi.org/10.1112/jlms/s2-6.3.571
7. S. Lang, *Algebra.* Addison-Wesley Publishing Co., 1965.
8. I. G. MacDonald, Secondary representation of modules over a commutative ring, *Symposia Math.*, **11** (1973), 23–43.
9. E. Matlis, Injective modules over Noetherian rings, *Pacific J. Math.*, **8** (1958), 511–528. https://doi.org/10.2140/pjm.1958.8.511
10. H. Matsumura, *Commutative Ring Theory,* Cambridge University Press, 1986. https://doi.org/10.1017/CBO9781139171762
11. L. Melkersson, On asymptotic stability for sets of prime ideals connected with the powers of an ideal, *Math. Proc. Cambridge Philos. Soc.*, **107** (1990), 267–271. https://doi.org/10.1017/S0305004100068535

12. D. G. Northcott, *An Introduction to Homological Algebra* Cambridge University Press, 2010. https://doi.org/10.1017/CBO9780511565915

13. D. Rees, Two classical theorems of ideal theory, *Math. Proc. Cambridge Philos. Soc.*, **52** (1956), 155–157. https://doi.org/10.1017/S0305004100031091

14. T. Singh, A. U. Ansari, S. D. Kumar, Existence and uniqueness of S-primary decomposition in S-Noetherian modules, *Comm. Algebra*, **52** (2024), 4515–4524. https://doi.org/10.1080/00927872.2024.2350598

15. R. K. Singh, M. K. Patel, H. Chakraborty, A note on essential Noetherian modules and rings, *Proc. Jangjeon Math. Soc.*, **28** (2025), 733–741. http://dx.doi.org/10.17777/pjms2025.28.4.025

16. R. Y. Sharp, Secondary representation for injective modules over commutative Noetherian rings, *Proc. Edinburgh Math. Soc.*, **20** (1976), 143–151. https://doi.org/10.1017/S0013091500010658

17. R. Y. Sharp, On the attached prime ideals of certain Artinian local cohomology modules, *Proc. Edinburgh Math. Soc.*, **24** (1981), 9–14. https://doi.org/10.1017/S0013091500003941

18. R. Y. Sharp, Asymptotic behaviour of certain sets of attached prime ideals, *J. London Math. Soc.*, **34** (1986), 212–218. https://doi.org/10.1112/jlms/s2-34.2.212

19. R. Y. Sharp, A method for the study of Artinian modules, with an application to asymptotic behaviour, In *Commutative Algebra*, Mathematical Sciences Research Institute Publications no 15, Springer-Verlag, (1989), 443–465. https://doi.org/10.1007/978-1-4612-3660-3_25

20. R. Y. Sharp, Artinian modules over commutative rings, *Math. Proc. Camb. Phil. Soc.*, **111** (1992), 25–33. https://doi.org/10.1017/S0305004100075125

21. R. Y. Sharp, *Steps in Commutative Algebra*, 2nd ed., Cambridge University Press, 2010. https://doi.org/10.1017/CBO9780511623684

22. D. W. Sharpe, P. Vámos, *Injective Modules*, Cambridge University Press, 1972. https://doi.org/10.1002/zamm.19750550208

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