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Research article

The edge balance properties of cubic graphs

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Abstract: For a simple, undirected graph $G(V, E)$, let f be an edge labeling f: $E \to \{0, 1\}$ such that, for any vertex v and $i \in \{0, 1\}$, in the edges incident on v, if the number of the edges labeled i is more than the number of edges labeled $1 - i$, then the label of v is defined by i; v is not defined otherwise. In a labeling graph of G, let $i \in \{0, 1\}$, $e_f(i) = |\{e \in E : f(e) = i\}|$ and $v_f(i) = |\{v \in V$: the label of v is $i\}|$. After f runs over all edge labelings satisfying $|e_f(1) - e_f(0)| \leq 1$, the set $\{|v_f(1) - v_f(0)| : |e_f(1) - e_f(0)| \leq 1\}$ is called the edge-balance index set of graph G, denoted by $EBI(G)$. In this paper, the set $\{v_f(1)-v_f(0) : |e_f(1)-e_f(0)| \leq 1\}$ is called the full edge-balance index set of graph G, denoted by $FEBI(G)$. Some results are obtained on $FEBI(G)$, and the relationship between $FEBI(G)$ and $EBI(G)$ is discussed. By finding some closed trails, the $FEBI$ and EBI of some classes of cubic graphs are obtained.

Keywords: cubic graph; edge-balance index set; generalized Petersen graph; $2n$-cycle with warp and weft chords; X-strips graph; Issacs graph; Möbius ladder graph
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1. Introduction

In this paper, all graphs are simple, undirected graphs. In 2009, Kwong and Lee [5] introduced a new concept on the labeling graph.

Definition 1.1. Edge-balance index set of the graph G: A labeling f: $E(G) \to \{0, 1\}$ induces a partial vertex labeling f^+: $V(G) \to \{0, 1\}$ defined by $f^+(v) = 0$ if the edges labeled by i incident to v are more than the number of edges labeled by $1-i$ incident on v, and $f^+(v) = i$ $i \in \{0, 1\}$. $f^+(v)$ is not defined if the edges labeled by 0 are equal to the edges labeled by 1 incident to v. Let $e_f(i) = |\{e \in E(G) : f(e) = i\}|$ and $v_f(i) = |\{v \in V(G) : f^+(v) = i\}|$. The set $\{|v_f(1) - v_f(0)|$: $|e_f(1) - e_f(0)| \leq 1\}$ is said to be the edge-balance index set of G, denoted by $EBI(G)$.

On $EBI(G)$ of the graph G, there are many results; the reader can see [1–3], etc.

Notation. Given an edge labeling f of graph G, if $|e_f(1) - e_f(0)| \leq 1$, then f is an edge-friendly labeling.

In $EBI(G)$ of a graph G, the elements are $|v_f(1) - v_f(0)|$. Now, we consider $v_f(1) - v_f(0)$.

Definition 1.2. Full edge-balance index set of the graph G: For a simple graph $G(V, E)$, let f be an edge labeling, f: $E \to \{0, 1\}$ such that, for any vertex v and $i \in \{0, 1\}$, in the edges incident to v, if the number of the edges labeled i is more than the number of the edges labeled $1 - i$, then the label of v is defined by i; if the number of the edges labeled 1 is equal to the number of the edges labeled 0, then v is not defined. In a labeling graph of G, let $e_f(i) = |\{e \in E : f(e) = i\}|$ and $v_f(i) = |\{v \in V :$ the label of v is $i\}|$. After f runs over all edge labelings that satisfy $|e_f(1) - e_f(0)| \leq 1$, the set $\{v_f(1) - v_f(0) : f$ runs over all edge-friendly labelings$\}$, and is called the full edge-balance index set of the graph G, denoted by $FEBI(G)$.

Given a graph G, we discuss the relationship between $FEBI(G)$ and $EBI(G)$, and obtained some results when G is an odd degree graph.

Now, we present some results to be used in the proofs of this paper.

Definition 1.3. Odd degree graph: A graph is said to be an odd degree graph if the degree of any vertex is odd.

Theorem 1.1. Let graph G be an odd degree graph. For any edge-friendly labeling f of G, the number of vertices with undefined labels is zero.

Proof. Assume f is an edge-friendly labeling of simple graph G. For any vertex v on G, since $d(v)$ is odd, the number of edges labeled with 1 incident to v is not equal to the number of edges labeled with 0 incident to v. Hence, the number of vertices with undefined labels is zero. □

Theorem 1.2. Let f, g be two edge-friendly labelings of the odd degree graph G, then $(v_f(1) - v_f(0)) - (v_g(1) - v_g(0))$ is even.

Proof. Assume f, g are two edge-friendly labelings of the odd degree graph G. By Theorem 1.1, $v_f(1) + v_f(0) = |V(G)|$ and $v_g(1) + v_g(0) = |V(G)|$, so

$(v_f(1) - v_f(0)) - (v_g(1) - v_g(0)) = 2v_f(1) - 2v_g(1) = 2(v_f(1) - v_g(1))$.

The conclusion holds. □

Theorem 1.3. If G is a simple graph, f is an edge-friendly labeling of G, and $v_f(1) - v_f(0) = a$, then $-a \in FEBI(G)$.

Proof. Let f be an edge-friendly labeling of G. Define an edge labeling g: $g(e) = |1 - f(e)|$, $e \in V(G)$, then $e_g(1) = e_f(0)$ and $e_g(0) = e_f(1)$, so g is an edge-friendly labeling. $v_g(1) = v_f(0)$, $v_g(0) = v_f(1)$, $v_g(*) = v_f(*)$ and $v_g(1) - v_g(0) = -(v_f(1) - v_f(0))$, thereby $-a \in FEBI(G)$. □

From Theorem 1.3, for a simple graph G, $FEBI(G)$ is a symmetry set. Given a graph G, by definition of $FEBI(G)$, if $FEBI(G)$ is obtained, then $EBI(G)$ is obtained too. Conversely, by Theorem 1.3, if $EBI(G)$ is obtained, then $FEBI(G)$ is obtained too.

In other sections, we determine the $FEBI$ and EBI of five classes of cubic graphs. The method used is that determining the maximum value and the minimum value of $v_f(k)$ ($k \in \{0, 1\},$) in each graph. Next, we find some closed trails in each graph, and the labeling graph where the maximum value of $v_f(k)$ is determined. Finaliy, we exchange some edge labels to determine the $FEBI$ and EBI of these graphs.

In the following discussions, an edge e is called a k-edge if $f(e) = k$, $k \in \{0, 1\}$, and a vertex v is called a k-vertex if $f^+(v) = k$, $k \in \{0, 1\}$. A vertex v is called a $*$-vertex if $f^+(v)$ is not defined, and the number of the vertices with undefined labels is denoted by $v_f(*)$.

2. The edge-balance properties of generalized Petersen graphs

In 1969, Watkins [6] defined generalized Petersen graphs and put forward a conjecture: In all generalized Petersen graphs, only the Petersen graph does not have a Tait coloring.

In this section, we study generalized Petersen graph $G(n, m)$ and determine the edge-balance index sets of $G(n, m)$ when $m = 1, 2, 3$. If $m = 1$, then $G(n, 1) \cong C_n \times K_2$, and if $m = 2$, then $G(n, 2)$ is a Petersen graph.

Definition 2.1. Generalized Petersen graph: Let $G = (V, E)$ be a simple, undirected, and connected graph, $n \geq 3$, $1 \leq m \leq n$,

$V(G) = \{u_i | 1 \leq i \leq n\} \cup \{v_i | 1 \leq i \leq n\}$, and

$E(G) = \{u_i u_{i+1} | 1 \leq i \leq n\} \cup \{u_i v_i | 1 \leq i \leq n\} \cup \{v_i v_{i+m} | 1 \leq i \leq n\}$, where all subscripts are taken modulo n, and graph G is called a generalized Petersen graph, which is denoted by $G(n, m)$.

In generalized Petersen graph $G(n, m)$, $|V| = 2n$, and $|E| = 3n$.

For studying the full edge-balance index sets of $G(n, m)$, we now have some preliminary results.

Theorem 2.1. Given generalized Petersen graph $G(n, m)$, if $n \equiv 2\ (mod\ 4)$ and $m = 1$ or 3 is odd, then the length of any closed trail is not odd.

Proof. Because $n \equiv 2\ (mod\ 4)$, $\frac{3n}{2}$ is odd.

If $m = 1$, in $G(n, 1)$, suppose there exists a closed trail. Because $|\{u_i\}| = |\{v_i\}| = n$, there are vertices in $\{u_i | 1 \leq i \leq n\}$ and $\{v_i | 1 \leq i \leq n\}$ on this closed trail. Without loss of generality, let u_1, v_1 be on this closed trail, and v_j be the final vertex of $\{v_i\}$ on this closed trail. Suppose v_i is on this closed trail, then $i < j$. Thus, for $1 < i < j$, the sum of the numbers of (v_i, v_{i+1}) and (u_l, u_{l+1}) on this closed trail is $j - 1$, and v_j is adjacent to u_j. In path $u_j - - - u_1$, the length is $j - 1$ or $n - j + 1$. Because n is even, $j - 1$ and $n - j + 1$ have the same parity, and the number of edges (u_i, v_i) on this closed trail is even. Hence, the length of this closed trail is even.

If $m = 3$, in $G(n, 3)$, suppose there exists a closed trail. Because $|\{u_i\}| = |\{v_i\}| = n$, there are vertices in $\{u_i | 1 \leq i \leq n\}$ and $\{v_i | 1 \leq i \leq n\}$ on this closed trail. Without loss of generality, let u_1, v_1 be on this closed trail.

Let this closed trail be composed of $u_1 v_1$, $v_1 \cdots v_{i_1}$, $v_{i_1} u_{i_1}$, $u_{i_1} \cdots u_{i_2}$, $u_{i_2} v_{i_2}$, $v_{i_2} \cdots v_{i_3}$, $v_{i_3} u_{i_3}$, $u_{i_3} \cdots u_{i_4}$, $u_{i_4} v_{i_4}$, $\ldots$, $v_{i_{j-1}} \cdots v_{i_j}$, $v_{i_j} u_{i_j}$, $u_{i_j} \cdots u_1$.

First, it is easy to know that the number of (u_1, v_1), (v_{i_1}, u_{i_1}), (u_{i_2}, v_{i_2}), $\ldots$, (v_{i_j}, u_{i_j}) is even.

Next, because $m = 3$ is odd, the length of $v_s \cdots v_{i_s}$ has the same parity as the length of $u_s \cdots u_{i_s}$. n is even, so on this closed trail, the number of (v_1, v_{i_j}) has the same parity as the number of (u_1, u_{i_j}). Thus, the length of this closed trail is even. □

Now, we study the edge-balance properties of generalized Petersen graph $G(n, m)$ for $m = 1, 2, 3$. First, we discuss the case of $m = 1$.

Theorem 2.2. For any edge-friendly labeling f of $G(n, 1)$, $k \in \{0, 1\}$,
(1) if $n \equiv 0\ (mod\ 4)$, then $v_f(k) \leq \frac{3n}{2}$;
(2) if n is odd, then $v_f(k) \leq \frac{3n+1}{2}$;
(3) if $n \equiv 2\ (mod\ 4)$, then $v_f(k) \leq \frac{3n-2}{2}$.

Proof. Let f be an edge-friendly labeling of $G(n, 1)$. Because $G(n, 1)$ is a cubic graph, by Theorem 2.1, $v_f(*) = 0$. Because the degree of each vertex is 3, if the label of a vertex is k, then there are at least two k-edges incident to this vertex. Thus, when all k-edges are on some cycles, the maximum value of $v_f(k)$ is obtained.

Case 1. When $n \equiv 0\ (mod\ 4)$, $e_f(k) = \frac{3n}{2}$ is even in $G(n, 1)$, there are even cycles, so $v_f(k) \leq \frac{3n}{2}$.
Case 2. When n is odd, $\frac{3n+1}{2}$ is even in $G(n, 1)$, and there are even cycles, so $v_f(k) \leq \frac{3n+1}{2}$.
Case 3. When $n \equiv 2\ (mod\ 4)$, $\frac{3n}{2}$ is odd, so by Theorem 2.1, $v_f(k) < \frac{3n}{2}$, thus $v_f(k) \leq \frac{3n-2}{2}$. □

In the following discussions, we know that the maximum value of $v_f(k)$ will be obtained.

Lemma 2.3. For generalized Petersen graph $G(n, 1)$, if $n \geq 3$ is odd, then $FEBI(G(n, 1)) = \{0, \pm 2, \ldots, \pm (n-1), \pm(n+1)\}$.

Proof. Because n is odd, if f is an edge-friendly labeling of $G(n, 1)$, by Theorem 2.2, $v_f(1) \leq \frac{3n+1}{2}$. Now, we find a closed trail with the length $\frac{3n+1}{2}$.

Case 1. $n \equiv 1\ (mod\ 4)$. $|E| = 3n$ is odd, so, for any edge-friendly labeling, $\frac{3n+1}{2}$ is even, and $e_f(1) \leq \frac{3n+1}{2}$.

Finding a closed trail $u_1u_2 \cdots u_{\frac{3n+1}{4}} v_{\frac{3n+1}{4}} v_{\frac{3n-3}{4}} \cdots v_1u_1$, its length is $\frac{3n+1}{2}$, and all edges on this closed trail are labeled by 1. The remaining edges are labeled by 0, then $e_f(1) = \frac{3n+1}{2}$, and $e_f(0) = \frac{3n-1}{2}$. By Theorem 1.1, $v_f(1) = \frac{3n+1}{2}$, $v_f(0) = 2n - \frac{3n+1}{2} = \frac{n-1}{2}$, and $v_f(1) - v_f(0) = n + 1$, thus the labeling graph on which $v_f(1) - v_f(0) = n + 1$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n + 1$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}), where $1 \leq i \leq \frac{3n-7}{4}$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{3n-7}{4}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n - 1, n - 3, \ldots, \frac{9-n}{2}$ are obtained.

When $n \geq 9$, $\frac{9-n}{2} \leq 0$, from Theorem 1.3, $FEBI(G(n, 1)) = \{0, \pm 2, \ldots, \pm(n-3), \pm(n-1)\}$.

If $n = 5$ in the labeling graph on which $v_f(1) - v_f(0) = 2$, exchange the labels of $(u_{\frac{3n+1}{4}}, v_{\frac{3n+1}{4}})$ and $(u_{\frac{3n+1}{4}}, u_{\frac{3n+5}{4}})$, then the labeling graph on which $v_f(1) - v_f(0) = 0$ is obtained.

Case 2. $n \equiv 3\ (mod\ 4)$.

Finding a closed trail $u_1u_2 \cdots u_{\frac{n+3}{2}} v_{\frac{n+3}{2}} v_{\frac{n+5}{2}} u_{\frac{n+5}{2}} u_{\frac{n+7}{2}} v_{\frac{n+7}{2}} v_{\frac{n+9}{2}} u_{\frac{n+9}{2}} u_{\frac{n+11}{2}} \cdots u_{n-2}v_{n-2}v_{n-1}u_{n-1}u_nv_nv_1u_1$, its length is $\frac{3n+1}{2}$. All edges on this closed trail are labeled by 1, and the remaining edges are labeled by 0. Then $e_f(1) = \frac{3n+1}{2}$ and $e_f(0) = \frac{3n-1}{2}$. By Theorem 1.1, $v_f(1) = \frac{3n+1}{2}$, $v_f(0) = 2n - \frac{3n+1}{2} = \frac{n-1}{2}$, and $v_f(1) - v_f(0) = n + 1$, thus, the labeling graph on which $v_f(1) - v_f(0) = n + 1$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n + 1$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}), where $1 \le i \le \frac{n-1}{2}$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{n-1}{2}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n-1, n-3, \ldots, 2$ are obtained.

In the labeling graph with $v_f(1) - v_f(0) = 2$, exchange the labels of (v_1, v_2) and (v_n, v_1), and the labeling graph on which $v_f(1) - v_f(0) = 0$ is obtained.

By Theorem 1.3, $FEBI(G(n,1)) = \{0, \pm 2, \ldots, \pm(n-1), \pm(n+1)\}$ for n is odd.

This completes the proof. □

Lemma 2.4. For generalized Petersen graph $G(n, 1)$, if $n > 3$ is even, then $FEBI(G(n,1))$ equals $\{0, \pm 2, \ldots, \pm(n-2), \pm n\}$ when $n \equiv 0 \ (mod\ 4)$ and $\{0, \pm 2, \ldots, \pm(n-4), \pm(n-2)\}$ when $n \equiv 2 \ (mod\ 4)$.

Proof. **Case 1.** $n \equiv 0 \ (mod\ 4)$.

Finding a closed trail $u_1 u_2 \cdots u_{\frac{3n}{4}} v_{\frac{3n}{4}} v_{\frac{3n-4}{4}} \cdots v_1 u_1$, its length is $\frac{3n}{2}$. All edges on this closed trail are labeled by 1, and the remaining edges are labeled by 0. Then $e_f(1) = 2 \times \frac{3n-4}{4} + 2 = \frac{3n}{2} = e_f(0)$, $v_f(1)\frac{3n}{2}$, $v_f(0) = 2n - \frac{3n}{2} = \frac{n}{2}$, and $v_f(1) - v_f(0) = n$, the labeling graph on which $v_f(1) - v_f(0) = n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}), where $1 \le i \le \frac{3n-4}{4}$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{3n-8}{4}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n-2, n-4, \ldots, \frac{8-n}{2}$ are obtained.

When $n \ge 8$, $\frac{8-n}{2} \le 0$, from Theorem 1.3, $FEBI(G(n,1)) = \{0, \pm 2, \ldots, \pm(n-2), \pm n\}$ for $n \equiv 0 \ (mod\ 4)$, and $n \ge 8$.

If $n = 4$ in the labeling graph on which $v_f(1) - v_f(0) = 2$, exchange the labels of $(u_{\frac{3n}{4}}, v_{\frac{3n}{4}})$ and $(u_{\frac{3n}{4}}, u_{\frac{3n+4}{4}})$, then the labeling graph on which $v_f(1) - v_f(0) = 0$ is obtained.

Case 2. $n \equiv 2 \ (mod\ 4)$.

$|E| = 3n$ and $\frac{3n}{2}$ is odd. By Theorem 2.1, for any edge-friendly labeling f, $e_f(1) \le \frac{3n-2}{2}$.

Finding a closed trail $u_1 u_2 \cdots u_{\frac{3n-2}{4}} v_{\frac{3n-2}{4}} v_{\frac{3n-6}{4}} \cdots v_1 u_1$, its length is $\frac{3n-2}{2}$. All edges on this closed trail and the edge (u_n, u_1) are labeled by 1, and the remaining edges are labeled by 0. Then $e_f(1) = 2 \times \frac{3n-6}{4} + 2 + 1 = \frac{3n}{2} = e_f(0)$. By Theorem 1.1, $v_f(1) = \frac{3n-2}{2}$ and $v_f(0) = 2n - \frac{3n-2}{2} = \frac{n+2}{2}$, and the labeling graph on which $v_f(1) - v_f(0) = n-2$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n-2$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}), where $1 \le i \le \frac{3n-6}{4}$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{3n-10}{4}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n-4, n-6, \ldots, \frac{6-n}{2}$ are obtained.

Because $n \ge 0$, $\frac{6-n}{2} \le 0$, from Theorem 1.3, $FEBI(G(n,1)) = \{0, \pm 2, \ldots, \pm(n-4), \pm(n-2)\}$ for $n \equiv 2 \ (mod\ 4)$. □

Overall,

Theorem 2.5. For generalized Petersen graph $G(n, 1)$,

$$FEBI(G(n,1)) = \begin{cases} \{0, \ \pm 2, \ \ldots, \ \pm(n-1), \ \pm(n+1)\} & n \ is \ odd; \\ \{0, \ \pm 2, \ \ldots, \ \pm(n-2), \ \pm n\} & n \equiv 0 \ (mod\ 4); \\ \{0, \ \pm 2, \ \ldots, \ \pm(n-4), \ \pm(n-2)\} & n \equiv 2 \ (mod\ 4). \end{cases}$$

Corollary 2.6. For generalized Petersen graph $G(n,1)$,

$$EBI(G(n,1)) = \begin{cases} \{0, \ 2, \ \ldots, \ n-1, \ n+1\} & n \text{ is odd}; \\ \{0, \ 2, \ \ldots, \ n-2, \ n\} & n \equiv 0 \ (mod\ 4); \\ \{0, \ 2, \ \ldots, \ n-4, \ n-2\} & n \equiv 2 \ (mod\ 4). \end{cases}$$

In the above discussions, $FEBI(G(n,1))$ is obtained. Next, we discuss $FEBI(G(n,2))$.

Theorem 2.7. For any edge-friendly labeling f of $G(n,2)$ $(n \geq 5)$, $k \in \{0,1\}$, then $v_f(k) \leq \frac{3n}{2}$ for n is even and $v_f(k) \leq \frac{3n+1}{2}$ for n is odd.

Proof. Let f be an edge-friendly labeling of $G(n,2)$. Then $e_f(k) = \frac{3n}{2}$ for n is even or $e_f(k) = \frac{3n+1}{2}$ for n is odd. Because the degree of each vertex is 3, if the label of one vertex is k, then there are at least two k-edges incident on this vertex. Thus, when all k-edges are on some cycles, the maximum value of $v_f(k)$ is obtained. In $G(n,2)$, there exist an even cycle and an odd cycle. Hence,
$v_f(k) \leq \frac{3n}{2}$ for n is even; $v_f(k) \leq \frac{3n+1}{2}$ for n is odd. □

In the following discussions, we know that the maximum value of $v_f(k)$ will be obtained.

Lemma 2.8. For generalized Petersen graph $G(n,2)$, if $n \geq 5$ is odd, then $FEBI(G(n,2)) = \{0, \pm 2, \ldots, \pm\, n+1\}$.

Proof. Finding a closed trail $u_1u_2 \cdots u_nv_nv_{n-2}v_{n-4} \cdots v_1u_1$, its length is $\frac{3n+1}{2}$. All edges on this closed trail are labeled by 1, and the remaining edges are labeled by 0. Then $e_f(1) = n - 1 + 2 + \frac{n-1}{2} = \frac{3n+1}{2}$, $e_f(0) = \frac{3n-1}{2}$, $v_f(1) = \frac{3n+1}{2}$, and $v_f(0) = 2n - \frac{3n+1}{2} = \frac{n-1}{2}$, and the labeling graph on which $v_f(1) - v_f(0) = n+1$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n+1$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}), where $1 \leq i \leq \frac{n+3}{2}$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{n+1}{2}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n-1, n-3, \ldots, 0$ are obtained.

By Theorem 1.3, $FEBI(G(n,1)) = \{0, \pm 2, \ldots, \pm(n-1), \pm(n+1)\}$ for n is odd.

This completes the proof. □

Lemma 2.9. For generalized Petersen graph $G(n,2)$, if $n > 5$ is even, then $FEBI(G(n,2)) = \{0, \pm 2, \ldots, \pm\, n\}$.

Proof. Finding two closed trails $u_1u_2 \cdots u_nu_1$ and $v_1v_3v_5 \cdots v_{n-1}v_1$, its length is $\frac{3n}{2}$. All edges on two closed trails are labeled by 1, and the remaining edges are labeled by 0. Then $e_f(1) = n + \frac{n}{2} = \frac{3n}{2}$, $e_f(0) = \frac{3n}{2}$, $v_f(1) = \frac{3n}{2}$, $v_f(0) = \frac{n}{2}$, and $v_f(1) - v_f(0) = n$, and the labeling graph on which $v_f(1) - v_f(0) = n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}), where $1 \leq i \leq \frac{n+2}{2}$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{n}{2}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n-2, n-4, \ldots, 0$ are obtained.

By Theorem 1.3, $FEBI(G(n,2)) = \{0, \pm 2, \ldots, \pm(n-2), \pm n\}$ for n is even.

This completes the proof. □

Overall,

Theorem 2.10. For generalized Petersen graph $G(n,2)$,

$$FEBI(G(n,2)) = \begin{cases} \{0, \pm2, \ldots, \pm(n-1), \pm(n+1)\} & n \text{ is odd}; \\ \{0, \pm2, \ldots, \pm(n-2), \pm n\} & n \text{ is even}. \end{cases}$$

Corollary 2.11. For generalized Petersen graph $G(n,2)$,

$$EBI(G(n,2)) = \begin{cases} \{0, 2, \ldots, n-1, n+1\} & n \text{ is odd}; \\ \{0, 2, \ldots, n-2, n\} & n \text{ is even}. \end{cases}$$

Finally, we discuss $FEBI(G(n,3))$.

Theorem 2.12. For any edge-friendly labeling f of $G(n,3)$, $k \in \{0,1\}$, then $v_f(k) \leq \frac{3n}{2}$ for n is even and $v_f(k) \leq \frac{3n+1}{2}$ for n is odd.

(1) if $n \equiv 0 \ (mod\ 4)$, then $v_f(k) \leq \frac{3n}{2}$;
(2) if n is odd, then $v_f(k) \leq \frac{3n+1}{2}$;
(3) if $n \equiv 2 \ (mod\ 4)$, then $v_f(k) \leq \frac{3n-2}{2}$.

Proof. Let f be an edge-friendly labeling of $G(n,3)$. Because $G(n,3)$ is a cubic graph, by Theorem 2.1, $v_f(*) = 0$. Because the degree of each vertex is 3, if the label of a vertex is k, then there are at least two k-edges incident on this vertex. Thus, when all k-edges are on some cycles, the maximum value of $v_f(k)$ is obtained.

Case 1. When $n \equiv 0 \ (mod\ 4)$, $e_f(k) = \frac{3n}{2}$ is even in $G(n,3)$, and there are even cycles, so $v_f(k) \leq \frac{3n}{2}$.
Case 2. When n is odd, $e_f(k) = \frac{3n+1}{2}$ is even in $G(n,3)$, and there are even cycles, so $v_f(k) \leq \frac{3n+1}{2}$.
Case 3. When $n \equiv 2 \ (mod\ 4)$, $e_f(k) = \frac{3n}{2}$ is odd, so by Theorem 2.1, $v_f(k) < \frac{3n}{2}$, thus, $v_f(k) \leq \frac{3n-2}{2}$. □

In the following discussions, we know that these maximum value of $v_f(k)$ will be obtained.

Lemma 2.13. For generalized Petersen graph $G(n,3)$, if $n \equiv 0 \ (mod\ 12)$, then $FEBI(G(n,3)) = \{0, \pm2, \ldots, \pm(n-2), \pm n\}$.

Proof. $|E| = 3n$, and $\frac{|E|}{2} = \frac{3n}{2}$ is even. Finding $\frac{n}{4}$ closed trails $u_{4i-3}v_{4i}u_{4i}u_{4i-1}u_{4i-2}u_{4i-3}$ for $1 \leq i \leq \frac{n}{3}$, their total length is $\frac{3n}{2}$. Define the labels of the edges on these closed trails as 1. The labels of the remaining edges are 0. Then $e_f(1) = \frac{3n}{2}$, $e_f(0) = 3n - \frac{3n}{2} = \frac{3n}{2}$, $v_f(1) = \frac{3n}{2}$, and $v_f(0) = 2n - \frac{3n}{2} = \frac{n}{2}$. The labeling graph on which $v_f(1) - v_f(0) = n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n$, successively exchange the labels of (u_{4i-3}, u_{4i-2}) and (u_{4i-2}, v_{4i-2}); and (u_{4i-2}, u_{4i-1}) and (u_{4i-1}, v_{4i-1}). Each exchange occurs such that $v_f(1)$ is decreased by 1and $v_f(0)$ is increased by 1, so there are $\frac{n}{2}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n-2, n-4, \ldots, 0$ are obtained.

By Theorem 1.3, $FEBI(G(n,3)) = \{0, \pm2, \ldots, \pm(n-2), \pm n\}$.

This completes the proof. □

Lemma 2.14. For generalized Petersen graph $G(n,3)$, if $n \equiv 1 \ (mod\ 12)$, then $FEBI(G(n,3)) = \{0, \pm2, \ldots, \pm(n-1), \pm(n+1)\}$.

Proof. $|E| = 3n$ is odd, and $\frac{|E|+1}{2} = \frac{3n+1}{2}$ is even. Finding $\frac{n+11}{12}$ closed trails $u_1v_1v_{n-2}v_{n-5}\cdots v_2v_{n-1}v_{n-4}\cdots$

$v_{\frac{n-1}{2}}u_{\frac{n-1}{2}}u_{\frac{n-3}{2}}\cdots u_2u_1$ and $u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}v_{\frac{n+7}{2}}u_{\frac{n+7}{2}}u_{\frac{n+5}{2}}u_{\frac{n+3}{2}}u_{\frac{n+1}{2}}$, $u_{\frac{n+13}{2}}v_{\frac{n+13}{2}}v_{\frac{n+19}{2}}u_{\frac{n+19}{2}}u_{\frac{n+17}{2}}u_{\frac{n+15}{2}}u_{\frac{n+13}{2}}$, ..., $u_{n-6}v_{n-6}v_{n-3}$ $u_{n-3}u_{n-4}u_{n-5}u_{n-6}$, their total length is $6\times\frac{n-1}{12}+\frac{n-3}{2}+2+\frac{n-1}{3}+\frac{n+5}{6}=\frac{3n+1}{2}$. Define the labels of the edges on these closed trails as 1, and the labels of the remaining edges are 0. Then $e_f(1)=\frac{3n+1}{2}$, $e_f(0)=3n-\frac{3n+1}{2}=\frac{3n-1}{2}$, $v_f(1)=\frac{3n+1}{2}$, and $v_f(0)=2n-\frac{3n+1}{2}=\frac{n-1}{2}$. The labeling graph on which $v_f(1)-v_f(0)=n+1$ is obtained.

Starting with the labeling graph on which $v_f(1)-v_f(0)=n+1$, successively exchange the labels of (u_i,u_{i+1}) and (u_{i+1},v_{i+1}), where $1\le i\le\frac{n-3}{2}$; (u_{n-6},u_{n-5}) and $(u_{n-5}v_{n-5})$; (u_{n-5},u_{n-4}) and (u_{n-4},v_{n-4}); and (u_{n-4},u_{n-3}) and (u_{n-3},u_{n-2}). Each exchange occurs such that $v_f(1)$ is decreased by 1and $v_f(0)$ is increased by 1, so there are $\frac{n-5}{2}+3=\frac{n+1}{2}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1)-v_f(0)=n-1, n-3, \ldots, 0$ are obtained.

By Theorem 1.3, this completes the proof. □

Lemma 2.15. For generalized Petersen graph $G(n,3)$, if $n\equiv 2\ (mod\ 12)$, then $FEBI(G(n,3))=\{0, \pm2,\ldots,\pm(n-4),\pm(n-2)\}$.

Proof. $|E|=3n$ is even, and $\frac{3n}{2}$ is odd. By Theorem 2.12, $v_f(1)\frac{3n-2}{2}$. Finding a closed trail $u_1v_1v_{n-2}v_{n-5}\cdots v_3v_nv_{n-3}v_{n-6}\cdots v_2v_{n-1}v_{n-4}\cdots v_{\frac{3n-14}{4}}u_{\frac{3n-14}{4}}u_{\frac{3n-18}{4}}\cdots u_4u_1$, define the labels of the edges on this closed trail and edge u_nu_1 as 1, and the labels of the remaining edges are 0. Then $e_f(1)=2\times(\frac{n-2}{3}+1)+\frac{n+10}{12}+2+\frac{3n-18}{4}+1=\frac{3n}{2}$, $e_f(0)=\frac{3n}{2}$, $v_f(1)=2\times(\frac{n-2}{3}+1)+\frac{n+10}{12}+1+\frac{3n-14}{4}=\frac{3n}{2}-1$, and $v_f(0)=2n-\frac{3n}{2}+1=\frac{n}{2}+1$. The labeling graph on which $v_f(1)-v_f(0)=n-2$ is obtained.

Starting with the labeling graph on which $v_f(1)-v_f(0)=n-2$, successively exchange the labels of (u_i,u_{i+1}) and (u_{i+1},v_{i+1}), where $1\le i\le\frac{n-2}{2}$; and $(u_{\frac{3n-18}{4}},u_{\frac{3n-14}{4}})$ and $(u_{\frac{3n-14}{4}},u_{\frac{3n-10}{4}})$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{n-5}{2}+3=\frac{n+1}{2}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1)-v_f(0)=n-4, n-6, \ldots, 0$ are obtained.

By Theorem 1.3, this completes the proof. □

Lemma 2.16. For generalized Petersen graph $G(n,3)$, if $n\equiv 3\ (mod\ 12)$, then $FEBI(G(n,3))=\{0, \pm2,\ldots,\pm(n-1),\pm(n+1)\}$.

Proof. $|E|=3n$ is odd, and $\frac{3n+1}{2}$ is odd. Finding a closed trail $u_1v_1v_{n-2}v_{n-5}\cdots v_{\frac{n+13}{4}}u_{\frac{n+13}{4}}u_{\frac{n+9}{4}}\cdots u_2u_1$ and $\frac{n-3}{6}$ closed trails with length 6, let $a=\frac{n+17}{4}$. These closed trails are $u_{a+9i}v_{a+9i}v_{a+3+9i}u_{a+3+9i}u_{a+2+9i}u_{a+1+9i}$ u_{a+9i} and $u_{a+4+9i}v_{a+4+9i}v_{a+7+9i}u_{a+7+9i}u_{a+6+9i}u_{a+5+9i}u_{a+4+9i}$ for $i=0,1,\ldots,\frac{n-3}{12}$. For the edges on these closed trails, define the labels of them as 1, and the labels of the remaining edges are 0. Then $e_f(1)=2+\frac{n+9}{4}+\frac{n-3}{4}+n-3=\frac{3n+1}{2}$, $e_f(0)=3n-\frac{3n+1}{2}=\frac{3n-1}{2}$, $v_f(1)=\frac{3n+1}{2}$, and $v_f(0)=2n-\frac{3n+1}{2}=\frac{n-1}{2}$. The labeling graph on which $v_f(1)-v_f(0)=n+1$ is obtained.

Starting with the labeling graph on which $v_f(1)-v_f(0)=n+1$, successively exchange the labels of (u_i,u_{i+1}) and (u_{i+1},v_{i+1}), where $1\le i\le\frac{n+5}{4}$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{n+5}{4}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1)-v_f(0)=n-1, n-3, \ldots, \frac{n-3}{2}$ are obtained.

Starting with the labeling graph on which $v_f(1)-v_f(0)=\frac{n-3}{2}$, in closed trail $u_{a+9i}v_{a+9i}v_{a+3+9i}u_{a+3+9i}$ $u_{a+2+9i}u_{a+1+9i}u_{a+9i}$, where $i=0,1,\ldots,\frac{n-3}{12}$, successively exchange the labels of (u_{a+9i-1},u_{a+9i}) and (v_{a+9i},u_{a+9i}); and (u_{a+9i},u_{a+9i+1}) and (v_{a+9i+1},u_{a+9i+1}). Each exchange occurs such that $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1, so there are $\frac{n-3}{6}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = \frac{n-7}{2}, \frac{n-11}{2}, \ldots, \frac{n-3}{6}$ are obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = \frac{n-3}{6}$, in closed trail $u_{a+4+9i}v_{a+4+9i}v_{a+7+9i}u_{a+7+9i}u_{a+6+9i}u_{a+5+9i}u_{a+4+9i}$ for $i = 0, 1, \ldots, \frac{n-3}{12}$, successively exchange the labels of (u_{a+9i+3}, u_{a+9i+4}) and (v_{a+9i+4}, u_{a+9i+4}); and (u_{a+9i+5}, u_{a+9i+6}) and (v_{a+9i+6}, u_{a+9i+6}). Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{n-3}{6}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = \frac{n-15}{6}, \frac{n-27}{6}, \ldots, -\frac{n-3}{6}$ are obtained.

By Theorem 1.3, this completes the proof. □

Lemma 2.17. For generalized Petersen graph $G(n, 3)$, if $n \equiv 4$ $(mod\ 12)$, then $FEBI(G(n, 3)) =\{0, \pm 2, \ldots, \pm(n-2), \pm n\}$.

Proof. $|E| = 3n$ is even, and $\frac{3n}{2}$ is even. Finding a closed trail $u_1v_1v_{n-2}v_{n-5}\cdots v_2v_{n-1}v_{n-4}\cdots v_3v_nv_{n-3}v_{n-6}\cdots v_{\frac{3n-8}{4}}u_{\frac{3n-8}{4}}u_{\frac{3n-4}{4}}\cdots u_2u_1$, define the labels of the edges on this closed trail as 1, and the labels of the remaining edges are 0. Then $e_f(1) = 2\times\frac{n-1}{3}+2+\frac{n+20}{12}+\frac{3n-12}{4} = \frac{3n}{2}$, $e_f(0) = \frac{3n}{2}$, $v_f(1) = n-\frac{n-8}{4}+\frac{3n-8}{4} = \frac{3n}{2}$, and $v_f(0) = 2n - \frac{3n}{2} = \frac{n}{2}$. The labeling graph on which $v_f(1) - v_f(0) = n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}), where $1 \le i \le \frac{n}{2}$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{n}{2}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n-2, n-4, \ldots, 0$ are obtained.

By Theorem 1.3, this completes the proof. □

Lemma 2.18. For generalized Petersen graph $G(n, 3)$, if $n \equiv 5$ $(mod\ 12)$, then $FEBI(G(n, 3))= \{0, \pm 2, \ldots, \pm(n-1), \pm(n+1)\}$.

Proof. $|E| = 3n$ is odd, and $\frac{3n+1}{2}$ is even. Finding a closed trail $u_1v_1v_{n-2}v_{n-5}\cdots v_3v_nv_{n-3}\cdots v_2v_{n-1}v_{n-4}\cdots v_{\frac{3n+1}{4}}u_{\frac{3n+1}{4}}u_{\frac{3n+5}{4}}\cdots u_nu_1$ and $\frac{n-5}{12}$ closed trail(s) with length 6 $u_{6i-2}v_{6i-2}v_{6i+2}u_{6i+2}u_{6i+1}u_{6i}u_{6i-1}u_{6i-2}$, where $i = 1, 2, \ldots, \frac{n-5}{12}$, define the labels of the edges on these closed trails as 1, and the labels of the remaining edges are 0. Then $e_f(1) = 2\times\frac{n+1}{3}+2+\frac{n-5}{12}+\frac{n+3}{4}+6\times\frac{n-5}{12} = \frac{3n+1}{2}$, $e_f(0) = \frac{3n-1}{2}$, $v_f(1) = 2\times\frac{n+1}{3}+\frac{n+7}{12}+\frac{n+7}{4}+6\times\frac{n-5}{12} = \frac{3n+1}{2}$, and $v_f(0) = \frac{n-1}{2}$. The labeling graph on which $v_f(1) - v_f(0) = n+1$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n+1$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}), where $i = \frac{3n+1}{4}, \frac{3n+5}{4}, \ldots, n-1$, (u_n, u_1) and (u_1, u_2); (u_{6j-2}, u_{6j-1}) and (v_{6j-1}, u_{6j-1}); (u_{6j-1}, u_{6j}) and (v_{6j}, u_{6j}); and (u_{6j}, u_{6j+1}) and (u_{6j+1}, u_{6j+2}), where $j = 1, 2, \ldots, \frac{n-5}{12}$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{n-1}{2}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n-1, n-3, \ldots, 2$ are obtained.

In the labeling graph with $v_f(1) - v_f(0) = 2$, exchange the labels of (u_1, u_2) and (u_2, u_3). $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1. The labeling graph on which $v_f(1) - v_f(0) = 0$ is obtained.

By Theorem 1.3, this completes the proof. □

Lemma 2.19. For generalized Petersen graph $G(n, 3)$, if $n \equiv 6$ $(mod\ 12)$, then $FEBI(G(n, 3)) = \{0, \pm 2, \ldots, \pm(n-4), \pm(n-2)\}$.

Proof. $|E| = 3n$ is even, and $\frac{3n}{2}$ is odd. By Theorem 2.12, $v_f(1) \le \frac{3n-2}{2}$. Finding a closed trail with length $\frac{n}{3}v_3v_6v_9\cdots v_nv_3$ and $\frac{n-6}{12}$ closed trail(s) with length 6 $u_{\frac{n-2}{2}+6i}u_{\frac{n}{2}+6i}u_{\frac{n+2}{2}+6i}u_{\frac{n+4}{2}+6i}v_{\frac{n+4}{2}+6i}v_{\frac{n-2}{2}+6i}u_{\frac{n-2}{2}+6i}$, where $i = 1, 2, \ldots, \frac{n-6}{12}$, as well as a closed trail $u_1v_1v_{n-2}v_{n-5}\cdots v_{\frac{n+2}{2}}u_{\frac{n+2}{2}}u_{\frac{n}{2}}\cdots u_2u_1$, define the labels of the edges on these closed trails and (u_n, u_1) as 1, the labels of the remaining edges are 0. Then $e_f(1) = \frac{n}{3} + 6 \times \frac{n-6}{12} + 2 + \frac{n}{6} + \frac{n}{2} + 1 = \frac{3n}{2}$, $e_f(0) = \frac{3n}{2}$, $v_f(1) = \frac{n}{3} + \frac{n-6}{2} + \frac{n}{6} + 1 + \frac{n}{2} + 1 = \frac{3n-2}{2}$, $v_f(0) = 2n - \frac{3n-2}{2} = \frac{n+2}{2}$. The labeling graph on which $v_f(1) - v_f(0) = n - 2$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n - 2$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}), where $1 \leq i \leq \frac{n-2}{2}$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{n}{2}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n - 4, n - 6, \ldots, 0$ are obtained.

By Theorem 1.3, this completes the proof. □

Lemma 2.20. For generalized Petersen graph $G(n, 3)$, if $n \equiv 7$ $(mod\ 12)$, then $FEBI(G(n, 3)) = \{0, \pm 2, \ldots, \pm(n-1), \pm(n+1)\}$.

Proof. $|E| = 3n$ is odd, and $\frac{3n+1}{2}$ is even. Finding a closed trail $u_1 v_1 v_{n-2} v_{n-5} \cdots v_2 v_{n-1} v_{n-4} \cdots v_3 v_n v_{n-3} \cdots v_{\frac{3n-5}{4}} u_{\frac{3n-5}{4}} u_{\frac{3n-9}{4}} \cdots u_2 u_1$, define the labels of the edges on this closed trail as 1, the labels of the remaining edges are 0. Then $e_f(1) = 2 \times \frac{n-1}{3} + \frac{n+17}{12} + 2 + \frac{3n-9}{4} = \frac{3n+1}{2}$, $e_f(0) = 3n - e_f(1) = \frac{3n-1}{2}$, $v_f(1) = \frac{3n+1}{2}$, and $v_f(0) = \frac{n-1}{2}$. The labeling graph on which $v_f(1) - v_f(0) = n + 1$ is obtained.

If $n = 7$, Starting with the labeling graph on which $v_f(1) - v_f(0) = 8$, successively exchange the labels of (u_n, u_1) and (u_1, v_1); (u_1, u_2) and (u_2, v_2); (u_2, u_3) and (u_3, v_3) ; and (u_3, u_4) and (u_4, u_5). Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are 4 exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = 6, 4, 2, 0$ are obtained.

If $n > 7$, starting with the labeling graph on which $v_f(1) - v_f(0) = n + 1$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}) for $1 \leq i \leq \frac{n-2}{2}$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{n+1}{2}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n - 1, n - 3, \ldots, 0$ are obtained.

By Theorem 1.3, this completes the proof. □

Lemma 2.21. For generalized Petersen graph $G(n, 3)$, if $n \equiv 8$ $(mod\ 12)$, then $FEBI(G(n, 3)) = \{0, \pm 2, \ldots, \pm(n-2), \pm n\}$.

Proof. $|E| = 3n$ is even, and $\frac{3n}{2}$ is even. Finding $\frac{n}{4}$ closed trails with length 6 $u_{4i-3} v_{4i-3} v_{4i} u_{4i} u_{4i-1} u_{4i-2} u_{4i-3}$ for $1 \leq i \leq \frac{n}{4}$, define the labels of the edges on these closed trails as 1, the labels of the remaining edges are 0. Then $e_f(1) = 6 \times \frac{n}{4} = \frac{3n}{2}$, $e_f(0) = \frac{3n}{2}$, $v_f(1) = \frac{3n}{2}$, and $v_f(0) = \frac{n}{2}$, the labeling graph on which $v_f(1) - v_f(0) = n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n$, successively exchange the labels of (u_{4i-3}, u_{4i-2}) and (u_{4i-2}, v_{4i-2}); and (u_{4i-2}, u_{4i-1}) and (u_{4i-1}, v_{4i-1}) for $1 \leq i \leq \frac{n}{4}$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{n}{2}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n - 2, n - 4, \ldots, 0$ are obtained.

By Theorem 1.3, this completes the proof. □

Lemma 2.22. For generalized Petersen graph $G(n, 3)$, if $n \equiv 9$ $(mod\ 12)$, then $FEBI(G(n, 3)) = \{0, \pm 2, \ldots, \pm(n-1), \pm(n+1)\}$.

Proof. $|E| = 3n$ is odd, and $\frac{3n+1}{2}$ is even.

Case 1. $n = 9$.

Finding two closed trails $u_1 v_1 v_7 v_4 u_4 u_3 u_2 u_1$ and $u_9 v_9 v_3 v_6 u_6 u_7 u_8 u_9$, define the labels of the edges on these closed trails as 1, the labels of the remaining edges are 0. Then $e_f(1) = 7 + 7 = 14$, $e_f(0) = 13$, $v_f(1) = 14$, and $v_f(0) = 4$. The labeling graph on which $v(1) - v(0) = 10$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = 10$, successively exchange the labels of (u_1, u_2) and (u_2, v_2); (u_2, u_3) and (u_3, v_3); (u_3, u_4) and (u_4, u_5); (u_6, u_7) and (u_7, v_7); and (u_7, u_8) and (u_8, v_8). Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are 5 exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = 8, 6, 4, 2, 0$ are obtained.

Case 2. $n > 9$. Finding three closed trails $v_3v_6 \cdots v_nv_3$, $u_1v_1v_{n-2}v_{n-5} \cdots v_{\frac{3n-23}{4}}u_{\frac{3n-23}{4}}u_{\frac{3n-27}{4}} \cdots u_2u_1$, and $u_{\frac{3n-7}{4}}v_{\frac{3n-7}{4}}v_{\frac{3n+5}{4}} \cdots v_{n-1}u_{n-1}u_{n-2} \cdots u_{\frac{3n-3}{4}}u_{\frac{3n-7}{4}}$, define the labels of the edges on these closed trails as 1, and the labels of the remaining edges are 0. Then $e_f(1) = \frac{n}{3} + \frac{3n-23}{4} - 1 + \frac{n-(\frac{3n-23}{4}-1)}{3} + 2 + n - 1 - (\frac{3n-23}{4} + 4) + 2 + \frac{n-\frac{3n-23}{4}-5}{3} = \frac{3n+1}{2}$, $e_f(0) = \frac{3n-1}{2}$, $v_f(1) = \frac{3n+1}{2}$, and $v_f(0) = \frac{n-1}{2}$. The labeling graph on which $v_f(1) - v_f(0) = n + 1$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n + 1$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}) for $1 \le i \le \frac{n+1}{2}$ $(n \ne 21)$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are $\frac{n+1}{2}$ exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n - 1, n - 3, \ldots, 0$ are obtained.

If $n = 21$, starting with the labeling graph on which $v_f(1) - v_f(0) = 22$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}) for $1 \le i \le 8$; and (u_{13+j}, u_{14+j}) and (u_{14+j}, v_{14+j}) for $1 \le j \le 3$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are 11 exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = 20$, 18, ..., 0 are obtained.

By Theorem 1.3, this completes the proof. □

Lemma 2.23. For generalized Petersen graph $G(n, 3)$, if $n \equiv 10 \ (mod\ 12)$, then $FEBI(G(n, 3)) = \{0, \pm 2, \ldots, \pm(n-4), \pm(n-2)\}$.

Proof. $|E| = 3n$ is even, and $\frac{3n}{2}$ is odd. By Theorem 2.12, $v_f(1) \le \frac{3n-2}{2}$. Finding a closed trail with length $\frac{3n-2}{2}$ $u_1v_1v_{n-2}v_{n-5} \cdots v_2v_{n-1}v_{n-4} \cdots v_3v_nv_{n-3} \cdots v_{\frac{3n-14}{4}}u_{\frac{3n-14}{4}}u_{\frac{3n-18}{4}} \cdots u_2u_1$, define the labels of the edges on this closed trail and (u_{n-1}, u_n) as 1, and the labels of the remaining edges are 0. Then $e_f(0) = 2 \times \frac{n-1}{3} + 1 + \frac{n+14}{12} + \frac{3n-14}{4} - 1 + 2 + 1 = \frac{3n}{2}$, $e_f(0) = \frac{3n}{2}$, $v_f(1) = \frac{3n-2}{2}$, and $v_f(0) = \frac{n+2}{2}$. The labeling graph on which $v_f(1) - v_f(0) = n - 2$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n - 2$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}) for $1 \le i \le \frac{n-2}{2}$ $(n \ne 10)$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are 5 exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n - 4, n - 6, \ldots, 0$ are obtained.

When $n = 10$, starting with the labeling graph on which $v_f(1) - v_f(0) = 8$, successively exchange the labels of (u_1, u_2) and (u_2, v_2); (u_2, u_3) and (u_3, v_3); (u_3, u_4) and (u_4, u_5); and (u_4, u_5) and (u_5, u_6). Then the labeling graphs on which $v_f(1) - v_f(0) = 6$, 4, 2, 0 are obtained, respectively.

By Theorem 1.3, this completes the proof. □

Lemma 2.24. For generalized Petersen graph $G(n, 3)$, if $n \equiv 11 \ (mod\ 12)$, then $FEBI(G(n, 3)) = \{0, \pm 2, \ldots, \pm(n-1), \pm(n+1)\}$.

Proof. $|E| = 3n$ is odd, and $\frac{3n+1}{2}$ is odd. Finding a closed trail with length $\frac{3n+1}{2}$ $u_1v_1v_{n-2}v_{n-5} \cdots v_3v_nv_{n-3}$

$\cdots v_2 v_{n-1} v_{n-4} \cdots v_{\frac{3n-5}{4}} u_{\frac{3n-5}{4}} u_{\frac{3n-9}{4}} \cdots u_2 u_1$, define the labels of the edges on this closed trail as 1, and the labels of the remaining edges are 0. Then $e_f(0) = 2 \times \frac{n+1}{3} + \frac{n+1}{12} + \frac{3n-5}{4} - 1 + 2 = \frac{3n+1}{2}$, $e_f(0) = \frac{3n-1}{2}$, $v_f(1) = \frac{3n+1}{2}$, and $v_f(0) = \frac{n-1}{2}$. The labeling graph on which $v(1) - v(0) = n + 1$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n + 1$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}) for $1 \leq i \leq \frac{n+1}{2}$ $(n \neq 11)$. Each exchange occurs such that $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so there are 5 exchanges. After all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n - 1, n - 3, \ldots, 0$ are obtained.

When $n = 11$, starting with the labeling graph on which $v_f(1) - v_f(0) = 12$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, v_{i+1}) for $1 \leq i \leq 5$. Then the labeling graphs on which $v_f(1) - v_f(0) = 10$, $8, \ldots, 2$ are obtained, respectively.

In the labeling graph with $v_f(1) - v_f(0) = 2$, exchange the labels of (u_6, u_7) and (u_7, u_8). Then the labeling graph on which $v_f(1) - v_f(0) = 0$ is obtained.

By Theorem 1.3, this completes the proof. □

To provide readers with a clearer understanding of the correctness of the conclusions, we have included two illustrations of $G(8, 3)$ and $G(11, 3)$ and explained the process of exchanging edge labels to obtain the labeling graphs.

Example 1. For $G(8, 3)$, the process of the labeling graphs on which $v_f(1) - v_f(0) = 8, 6, \ldots, 0$ are obtained is shown in Figure 1.

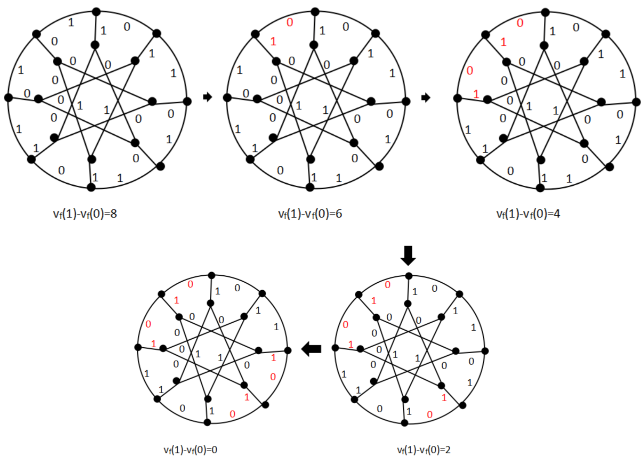

Figure 1. The process of the labeling graphs obtained on $G(8, 3)$.

Example 2. For $G(11, 3)$, the processes of the labeling graphs on which $v_f(1) - v_f(0) = 12, 10, \ldots, 0$ are obtained is shown in Figure 2.

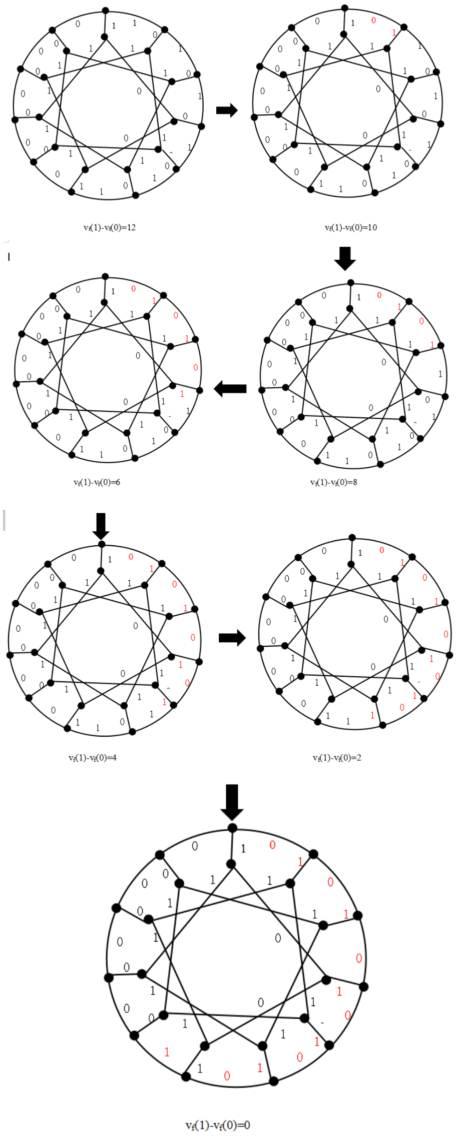

Figure 2. The processes of the labeling graphs obtained on $G(11,3)$.

Combining with the results on the above lemmas, we present the following theorem.

Theorem 2.25. For generalized Petersen graph $G(n,3)$,

$$FEBI(G(n,3)) = \begin{cases} \{0, \ \pm 2, \ \ldots, \ \pm(n-1), \ \pm(n+1)\} & n \text{ is odd}; \\ \{0, \ \pm 2, \ \ldots, \ \pm(n-2), \ \pm n\} & n \equiv 0 \ (mod\ 4); \\ \{0, \ \pm 2, \ \ldots, \ \pm(n-4), \ \pm(n-2)\} & n \equiv 2 \ (mod\ 4). \end{cases}$$

Corollary 2.26. For generalized Petersen graph $G(n,3)$,

$$EBI(G(n,3)) = \begin{cases} \{0, \ 2, \ \ldots, \ n-1, \ n+1\} & n \text{ is odd}; \\ \{0, \ 2, \ \ldots, \ n-2, \ n\} & n \equiv 0 \ (mod\ 4); \\ \{0, \ 2, \ \ldots, \ n-4, \ n-2\} & n \equiv 2 \ (mod\ 4). \end{cases}$$

3. The edge-balance properties of $C(2n; l, m)$ graphs

In this section, we study the cubic graph $C(2n; l, m)$ and determine the edge-balance index sets of $C(2n; l, m)$.

Definition 3.1. Chord: Given a simple graph G, u and v are two not-adjacent vertices on G. Then the edge joining u and v is called a chord.

Given a cycle C_{2n} ($n > 1$), the vertices are denoted successively: u_1, u_2, ..., u_{2n}.

Definition 3.2. $2n$-cycle with warp and weft chords: Given a cycle C_{2n}, there exist l warp chords and m weft chords, $l, m > 0$, $l + m = n$. This cycle is denoted by $C(2n; l, m)$.

In $C(2n; l, m)$, $V(C(2n; l, m)) = V(C_{2n})$ and $E(C(2n; l, m)) = E(C_{2n}) \cup \{(u_i, u_{n+l-i+1}) : 1 \le i \le l\} \cup \{(u_{l+j}, u_{2n-j+1}) : 1 \le j \le m\}$.

$C(2n; l, m)$ is a 3-regular graph. Without loss of generality, let $l \ge m$ in $C(2n; l, m)$. Because $l+m = n$ and $l, m > 0$, $1 \le m \le l \le n - 1$.

$C(8; 3, 1)$ and $C(8; 2, 2)$ are shown in Figure 3.

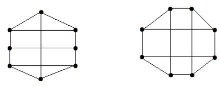

Figure 3. $C(8; 3, 1)$ and $C(8; 2, 2)$.

In $C(2n; l, m)$, $|V| = 2n$, and $|E| = 3n$. The vertices are denoted successively: u_1, u_2, ..., u_l, u_{l+1}, u_{l+2}, ..., and $u_{l+m}(= u_n)$, u_{n+1}, u_{n+2}, ..., u_{n+l}, u_{n+l+1}, u_{n+l+2}, ..., $u_{n+l+m}(= u_{2n})$. The warp chord(s) are denoted successively: (u_1, u_{n+l}), (u_2, u_{n+l-1}), ..., and (u_l, u_{n+1}). The weft chord(s) are denoted successively: (u_{l+1}, u_{2n}), (u_{l+2}, u_{2n-1}), ..., and (u_n, u_{n+l+1}).

Now, we find $FEBI(C(2n; l, m))$. First, we discuss the maximum value of $v(k)$ ($k \in \{0, 1\}$) for any edge-friendly labeling f of $C(2n; l, m)$. Because $l + m = n$ and $1 \le m \le l \le n - 1$, the values of l, m that may be obtained have $n - 1, 1; n - 2, 2; \ldots, \frac{n+1}{2}, \frac{n-1}{2}$ if n is odd, or $n - 1, 1; n - 2, 2; \ldots, \frac{n}{2}, \frac{n}{2}$ if n is even.

Because $|E| = 3n$, if f is an edge-friendly labeling of $C(2n; l, m)$, then $e_f(k) = e_f(1 - k)$ when n is even, or $e_f(k) = e_f(1 - k) + 1$ when n is odd.

When l, m are even, there only exists an even cycle, and when l or m is odd, there exist odd cycle and even cycle, so we determine the maximum value of $v_f(k)$ ($k \in \{0, 1\}$).

Theorem 3.1. For any edge-friendly labeling f of $C(2n; l, m)$, $k \in \{0, 1\}$,

(1) if $n \equiv 0 \ (mod\ 4)$, then $v_f(k) \le \frac{3n}{2}$;
(2) if $n \equiv 1 \ (mod\ 4)$, then $v_f(k) \le \frac{3n+1}{2}$;
(3) if $n \equiv 2 \ (mod\ 4)$, when l, m are odd, then $v_f(k) \le \frac{3n}{2}$; when l, m are even, then $v_f(k) \le \frac{3n-2}{2}$;
(4) if $n \equiv 3 \ (mod\ 4)$, when l is even, then $v_f(k) \le \frac{3n+1}{2}$; when l is odd, then $v_f(k) \le \frac{3n-1}{2}$.

Proof. Let f be an edge-friendly labeling of $C(2n; l, m)$. Because $C(2n; l, m)$ is a cubic graph, by Theorem 2.1, $v_f(*) = 0$.

Case 1. When $n \equiv 0 \ (mod\ 4)$, $|E| = 3n$ is even, $e_f(k) = e_f(1 - k) = \frac{3n}{2}$ is even, and the degree of each vertex is 3. So, if the label of the vertex u_i is k, then there are at least two k-edges incident on u_i. Thus, when all k-edges are on some cycles, the maximum value of $v_f(k)$ is obtained. In $C(2n; l, m)$, there are even cycles, so $v_f(k) \leq \frac{3n}{2}$.

Case 2. When $n \equiv 1 \ (mod\ 4)$, $|E| = 3n$ is odd, $e_f(k) = e_f(1-k)+1$, $e_f(k)+e_f(1-k) = 2e_f(k)-1 = 3n$, and $e_f(k) = \frac{3n+1}{2}$ is even. Following the discussion in Case 1, in $C(2n; l, m)$, there are even cycles, so $v_f(k) \leq \frac{3n+1}{2}$.

Case 3. When $n \equiv 2 \ (mod\ 4)$, $|E| = 3n$ is even, and $e_f(k) = e_f(1-k) = \frac{3n}{2}$ is odd. Because $l + m = n$, l, m are odd, or l, m are even. If l, m are odd in $C(2n; l, m)$, there are odd cycles, so $v_f(k) \leq \frac{3n}{2}$. If l, m are even in $C(2n; l, m)$, there are only even cycles. Following the discussion in Case 1, $v_f(k) < \frac{3n}{2}$, so $v_f(k) \leq \frac{3n-2}{2}$.

Case 4. When $n \equiv 3 \ (mod\ 4)$, $|E| = 3n$ is odd, $e_f(k) = e_f(1-k)+1$, $e_f(k)+e_f(1-k) = 2e_f(k)-1 = 3n$, and $e_f(k) = \frac{3n+1}{2}$ is odd. Because n is odd, $l > m$, so one is odd and the other is even in l, m.

If l is even, then m is odd. There exists odd cycle $C_{\frac{3n+1}{2}}$ on which one edge is a warp chord and some edges are on C_{2n}.

If l is odd, then m is even. Then on an odd cycle, there must be weft chord(s). Thus, if $e(k) = \frac{3n+1}{2}$, the cycles on which the edges are labeled by k, the edges belong weft chords, and some edges on C_{2n}. The odd cycle for the minimum length is C_{l+2}, and the odd cycle for the maximum length is C_{l+2m}. Because $l + 2m = n + m$, $\frac{3n+1}{2} - n - m = \frac{n+1}{2} - m > 0$, so in the cycles on which the edges are labeled by k, there is one odd cycle and some even cycles. But if the length of the odd cycle is $a\ l+2 \leq a \leq l+2m$, then the length of even cycles is $b\ 2m-2 \geq b \geq 0$. Thus, there does not exist an odd cycle and some even cycles on which the edges are labeled by k such that $v_f(k) = \frac{3n+1}{2}$, so $v_f(k) \leq \frac{3n-1}{2}$. □

In the following discussions, we can see that the maximum value of $v_f(k)$ will be obtained in the above cases.

Lemma 3.2. $FEBI(C(8; l, m)) = \{0, \pm 2, \pm 4\}$.

Proof. Because $n = 4$, then l, m may be $3, 1; 2, 2$.

Case 1. $l = 3$.

First, define the labels of edges on the closed trail $u_1u_2u_3u_5u_6u_7u_1$ as 1, and the labels of remaining edges are 0. Then $e_f(1) = 6 = e_f(0)$, $v_f(1) = 6$, and $v_f(0) = 2$. The labeling graph on which $v_f(1) - v_f(0) = 4$ is obtained.

In the labeling graph on which $v_f(1) - v_f(0) = 4$, exchange the labels of (u_1, u_2) and (u_2, u_6). Then $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1. The labeling graph on which $v_f(1) - v_f(0) = 2$ is obtained.

In the labeling graph on which $v_f(1) - v_f(0) = 2$, exchange the labels of (u_2, u_3) and (u_3, u_4). Then $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so the labeling graph on which $v_f(1) - v_f(0) = 0$ is obtained.

Case 2. $l = 2$.

First, define the labels of edges on the closed trail $u_1u_2u_3u_4u_5u_6u_1$ as 1, and the labels of remaining edges are 0. Then $e_f(1) = 6 = e_f(0)$, $v_f(1) = 6$, and $v_f(0) = 2$. The labeling graph on which $v_f(1) - v_f(0) = 4$ is obtained.

In the labeling graph on which $v_f(1) - v_f(0) = 4$, exchange the labels of (u_1, u_2) and (u_2, u_5). Then $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so the labeling graph on which $v_f(1) - v_f(0) = 2$ is obtained.

In the labeling graph on which $v_f(1) - v_f(0) = 2$, exchange the labels of (u_1, u_6) and (u_1, u_8). Then $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1, so the labeling graph on which $v_f(1) - v_f(0) = 0$ is obtained.

By Theorems 1.3 and 3.1, combining with the above results, it is obtained that $FEBI(C(8; l, m)) = \{0, \pm2, \pm4\}$. □

Lemma 3.3. When $n \equiv 0\ (mod\ 4)$, $n > 4$, $FEBI(C(2n; l, m)) = \{0, \pm2, \ldots, \pm(n-2), \pm n\}$.

Proof. Because $n \equiv 0\ (mod\ 4)$, for any edge-labeling, $e_f(1) = \frac{3n}{2} = e_f(0)$ and $v(1) \leq \frac{3n}{2}$, so $v_f(1) - v_f(0) \leq 2 \times \frac{3n}{2} - 2n = n$ and $1 \leq m \leq \frac{n}{2} \leq l \leq n-1$.

Case 1. $\frac{3n}{4} \leq l \leq n-1$.

Define the labels of edges on the closed trail $u_1u_2 \cdots u_{\frac{3n}{4}}u_{l+\frac{n}{4}+1}\ u_{l+\frac{n}{4}+2} \ldots u_{n+l}u_1$ as 1, and the labels of remaining edges are 0. Then $e_f(1) = (\frac{3n-4}{4}) + 1 + (\frac{3n-4}{4}) + 1 = \frac{3n}{2}$, $e_f(0) = 3n - \frac{3n}{2} = \frac{3n}{2}$, and $v_f(1) - v_f(0) = 2 \times \frac{3n}{2} - 2n = n$, so the labeling graph on which $v_f(1) - v_f(0) = n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, u_{n+l-i}) for $1 \leq i \leq \frac{n}{2}$. After each exchange, $v_f(1)$ is decreased by 1 and $v_f(0)$ is increased by 1. There are $\frac{n}{2}$ exchanges. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n-2, n-4, \ldots, 0$ are obtained.

Case 2. $\frac{n}{2} \leq l < \frac{3n}{4}$.

Subcase 2.1. l, m are odd.

Define the labels of edges on the closed trails $u_1u_2 \cdots u_{\frac{3n-4}{4}-m}u_{\frac{5n}{4}+2}\ u_{\frac{5n}{4}+3} \cdots u_{2n}u_1$ and $u_lu_{l+1} \cdots u_nu_{n+1}u_l$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = (\frac{3n-4}{4} - m - 1) + 1 + (\frac{3n-4}{4} - m - 1) + m + 1 + m + 1 + 1 = \frac{3n}{2} = e_f(0)$, $v_f(1) = 2 \times (\frac{3n-4}{4} - m) + m + m + 2 = \frac{3n}{2}$, and $v_f(1) - v_f(0) = 2 \times \frac{3n}{2} - 2n = n$. The labeling graph on which $v_f(1) - v_f(0) = n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n$, successively exchange the labels of $(u_{n+l+i-1}, u_{n+l+i})$ and (u_{n+l+i}, u_{n+1-i}) for $1 \leq i \leq m$ and the labels of (u_j, u_{j+1}) and (u_{j+1}, u_{n+l-j}) for $1 \leq j \leq \frac{n}{2} - m$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are $\frac{n}{2}$ exchanges. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n-2$, $n-4, \ldots, 0$ are obtained.

Subcase 2.2. l, m are even.

Define the labels of edges on the closed trail $u_1u_2 \cdots u_{\frac{3n-2m}{4}}u_{\frac{n+2m}{4}+l+1}\ u_{\frac{n+2m}{4}+l+2} \cdots u_{2n}u_1$ as 1, the labels of the remaining edges are 0, then $e_f(1) = 2 \times (\frac{3n-2m}{4} - 1) + m + 1 + 1 = \frac{3n}{2} = e_f(0)$, $v_f(1) = 2 \times \frac{3n-2m}{4} + m = \frac{3n}{2}$, and $v_f(1) - v_f(0) = n$. The labeling graph on which $v_f(1) - v_f(0) = n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n$, successively exchange the labels of (u_{2n}, u_1) and (u_1, u_{n+l}), (u_i, u_{i+1}) and (u_{i+1}, u_{n+l-i}) for $1 \leq i \leq \frac{n-4}{2}$; and $(u_{\frac{3n-2m-4}{4}}, u_{\frac{3n-2m}{4}})$ and $(u_{\frac{3n-2m}{4}}, u_{\frac{3n-2m}{4}+1})$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are $\frac{n}{2}$ exchanges. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n-2$, $n-4, \ldots, 0$ are obtained.

Hence, by Theorems 1.3 and 3.1, combining with the above results, $FEBI(C(2n; l, m)) = \{0, \pm 2, \ldots, \pm(n-2), \pm n\}$ when $n \equiv 0 \ (mod\ 4)$, $n > 4$. □

Lemma 3.4. $FEBI(C(10; l, m)) = \{0, \pm 2, \pm 4, \pm 6\}$.

Proof. **Case 1.** $l = 4, m = 1$.

Define the labels of edges on the closed trail $u_1u_2u_3u_4u_6u_7u_8u_9u_1$ as 1, and the labels of the remaining edges are 0, then $e_f(1) = 8$, $e_f(0) = 7$, $v_f(1) = 8$, and $v_f(1) - v_f(0) = 6$, so the labeling graph on which $v_f(1) - v_f(0) = 6$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = 6$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, u_{9-i}) for $i = 1, 2$; and (u_3, u_4) and (u_4, u_5). After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are 3 exchanges. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = 4, 2, 0$ are obtained.

Case 2. $l = 3, m = 2$.

Define the labels of edges on the closed trail $u_1u_2u_3u_6u_7u_8u_9u_{10}u_1$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = 8$, $e_f(0) = 7$, $v_f(1) = 8$, and $v_f(1) - v_f(0) = 6$. The labeling graph on which $v_f(1) - v_f(0) = 6$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = 6$, successively exchange the labels of (u_{10}, u_1) and (u_1, u_8); (u_1, u_2) and (u_2, u_7); and (u_2, u_3) and (u_3, u_4). After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are 3 exchanges. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = 4, 2, 0$ are obtained.

By Theorems 1.3 and 3.1, combining with the above results, it is obtained that $FEBI(C(10; l, m)) = \{0, \pm 2, \pm 4, \pm 6\}$. □

Lemma 3.5. When $n \equiv 1 \ (mod\ 4)$, $n > 5$, $FEBI(C(2n; l, m)) = \{0, \pm 2, \ldots, \pm(n-1), \pm(n+1)\}$.

Proof. Because $n \equiv 1 (mod 4)$, for any edge-labeling, $e_f(1) = \frac{3n+1}{2} = e_f(0) + 1$, and $v_f(1) \le \frac{3n+1}{2}$, so $v_f(1) - v_f(0) \le 2 \times \frac{3n+1}{2} - 2n = n$ and $1 \le m \le l \le n - 1$.

Case 1. $\frac{3n+1}{4} \le l \le n - 1$.

Define the labels of edges on the closed trail $u_1u_2 \cdots u_{\frac{3n+1}{4}} u_{l+\frac{n+3}{4}}\ u_{l+\frac{n+7}{4}} \cdots u_{n+l}u_1$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = (\frac{3n-3}{4}) + 1 + (\frac{3n-3}{4}) + 1 = \frac{3n+1}{2} = e_f(0) + 1$, $v_f(1) = 2 \times \frac{3n+1}{4} = \frac{3n+1}{2}$, and $v_f(1) - v_f(0) = 2 \times \frac{3n+1}{2} - 2n = n + 1$. The labeling graph on which $v_f(1) - v_f(0) = n + 1$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n + 1$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, u_{n+l-i}) for $1 \le i \le \frac{n+1}{2}$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are $\frac{n+1}{2}$ exchanges. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n - 1, n - 3, \ldots, 0$ are obtained.

Case 2. $\frac{n+1}{2} \le l < \frac{3n+1}{4}$.

Define the labels of edges on the closed trails $u_1u_2 \cdots u_{\frac{3(n-1)}{4}-m} u_{\frac{5n+7}{4}}\ u_{\frac{5n+11}{4}} \cdots u_{2n}u_1$ and $u_lu_{l+1} \cdots u_{n+1}u_l$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = m + 1 + 1 + (\frac{3(n-1)}{4} - m - 1) + 1 + (\frac{3(n-1)}{4} - m - 1) + m + 1 = \frac{3(n-1)}{2} + 2 = \frac{3n+1}{2} = e_f(0) + 1$, $v_f(1) = m + 2 + 2 \times (\frac{3(n-1)}{4} - m) + m = \frac{3n+1}{2}$, and $v_f(1) - v_f(0) = 3n + 1 - 2n = n + 1$, so the labeling graph on which $v_f(1) - v_f(0) = n + 1$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n + 1$, successively exchange the labels of (u_i, u_{i+1}) and (u_{i+1}, u_{n+l-i}) for $1 \le i \le \frac{3(n-1)}{4} - m - 2$; $(u_{\frac{3n-7}{4}-m}, u_{\frac{3(n-1)}{4}-m})$ and $(u_{\frac{3(n-1)}{4}-m}, u_{\frac{3n+1}{4}-m})$; and

(u_j, u_{j+1}) and (u_{j+1}, u_{2n+l-j}) for $l \leq j \leq \frac{3n+5}{4}$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. The number of exchanges is $\frac{3(n-1)}{4} - m - 2 + 1 + (\frac{3n+5}{4} + 1 - l) = \frac{n+1}{2}$. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n - 1, n - 3, \ldots, 0$ are obtained.

By Theorem 1.3, this completes the proof. □

Lemma 3.6. $FEBI(C(4; 1, 1)) = \{0, \pm 2\}$.

Proof. Define the labels of edges on the closed trail $u_1u_2u_4u_1$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = 3 = e_f(0)$, $v_f(1) = 3$, and $v_f(0) = 1$. The labeling graph on which $v_f(1) - v_f(0) = 2$ is obtained.

On the labeling graph on which $v_f(1) - v_f(0) = 2$, exchange the labels of (u_2, u_4) and (u_3, u_4). Then $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. The labeling graph on which $v_f(1) - v_f(0) = 0$ is obtained.

Hence, by Theorem 1.3, $FEBI(C(4; 1, 1)) = \{0, \pm 2\}$. □

Lemma 3.7. When $n \equiv 2 \ (mod\ 4)$, $n > 2$, $FEBI(C(2n; l, m)) =$

(1) $\{0, \pm 2, \ldots, \pm(n-2), \pm n\}$ when l, m are odd;
(2) $\{0, \pm 2, \ldots, \pm(n-4), \pm(n-2)\}$ when l, m are even.

Proof. **Case 1.** l, m are odd.

Define the labels of edges on the closed trail $u_1u_2 \cdots u_{\frac{3n-2}{4}-\frac{m-1}{2}}\ u_{\frac{3n+2}{4}+\frac{l+1}{2}}u_{\frac{3n+2}{4}+\frac{l+3}{2}} \cdots u_{2n}u_1$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = 2 \times (\frac{3(n-2)}{4} - \frac{m-3}{2} - 1) + m + 1 + 1 = \frac{3n}{2} = e_f(0)$, $v_f(1) = \frac{3n}{2}$, and $v_f(1) - v_f(0) = n$. The labeling graph on which $v_f(1) - v_f(0) = n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n$, successively exchange the labels of (u_1, u_{2n}) and (u_1, u_{n+l}), (u_i, u_{i+1}) and (u_{i+1}, u_{n+l-i}) for $1 \leq i \leq \frac{n}{2} - 2$; and $(u_{\frac{3(n-2)}{4}-\frac{m-1}{2}}, u_{\frac{3(n-2)}{4}-\frac{m-3}{2}})$ and $(u_{\frac{3(n-2)}{4}-\frac{m-3}{2}}, u_{\frac{3(n-2)}{4}-\frac{m-5}{2}})$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. The number of exchanges is $\frac{n}{2}$. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n - 2, n - 4, \ldots, 0$ are obtained.

Case 2. l, m are even.

Because l, m are even, by Theorem 3.1, $v_f(1) \leq \frac{3n-2}{2}$. Define the labels of edges on the closed trail $u_1u_2 \cdots u_{\frac{3n-2}{4}-\frac{m}{2}}u_{\frac{3n+2}{4}+\frac{l}{2}+1}u_{3n+24+\frac{l}{2}+2} \cdots u_{2n}u_1$ and edge (u_n, u_{n+1}) are 1, and the labels of the remaining edges are 0. Then $e_f(1) = 2\times(\frac{3n-2}{4} - \frac{m}{2} - 1) + 1 + m + 1 + 1 = \frac{3n}{2} = e_f(0)$, $v_f(1) = 2\times(\frac{3n-2}{4} - \frac{m}{2}) + m = \frac{3n-2}{2}$, and $v_f(1) - v_f(0) = 2 \times \frac{3n-2}{2} - 2n = n - 2$. The labeling graph on which $v_f(1) - v_f(0) = n - 2$ is obtained.

After the manner of the discussions in Case 1, the labeling graphs on which $v_f(1) - v_f(0) = n - 4$, $n - 6, \ldots, 0$ are obtained.

By Theorems 1.3 and 3.1, combining with the above results, the conclusions are correct. □

Lemma 3.8. $FEBI(C(6; l, m)) = \{0, \pm 2, \pm 4\}$.

Proof. For $C(6; l, m)$, we can know that $n = 3$, $l = 2$, and $m = 1$. By Theorem 3.1, $v(1) \leq 5$.

Define the labels of edges on the closed trail $u_1u_2u_4u_5u_6u_1$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = 5 = e_f(0) + 1$, $v_f(1) = 5$, and $v_f(1) - v_f(0) = 2 \times 5 - 6 = 4$. The labeling graph on which $v_f(1) - v_f(0) = 4$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = 4$, successively exchange the labels of

(u_1, u_6) and (u_1, u_5); and (u_1, u_2) and (u_2, u_3). After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. Once 2 exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = 2, 0$ are obtained.

By Theorems 1.3 and 3.1, $FEBI(C(6; l, m)) = \{0, \pm 2, \pm 4\}$. □

Lemma 3.9. When $n \equiv 3\ (mod\ 4)$, $n > 3$, $FEBI(C(2n; l, m)) =$

(1) $\{0, \pm 2, \ldots, \pm(n-1), \pm(n+1)\}$ when l is even;
(2) $\{0, \pm 2, \ldots, \pm(n-3), \pm(n-1)\}$ when l is odd.

Proof. Because $n \equiv 3(mod 4)$, so, in l, m, one is odd, the other is even, $|E|$ is odd.

Case 1. l is even, and m is odd.

Define the labels of edges on the closed trail $u_1 u_2 \cdots\ u_{\frac{3n-1}{4}-\frac{m-1}{2}} u_{\frac{3n-1}{4}+\frac{l}{2}+1} u_{\frac{3n-1}{4}+\frac{l}{2}+2} \cdots u_{2n} u_1$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = 2 \times (\frac{3n-1}{4} - \frac{m-1}{2} - 1) + 1 + m + 1 = \frac{3n+1}{2} = e_f(0) + 1$, $v_f(1) = 2 \times (\frac{3n-1}{4} - \frac{m-1}{2}) + m = \frac{3n+1}{2}$, and $v_f(1) - v_f(0) = n + 1$, so the labeling graph on which $v_f(1) - v_f(0) = n + 1$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n + 1$, successively exchange the labels of (u_{2n}, u_1) and (u_1, u_{n+l}); and (u_i, u_{i+1}) and (u_{i+1}, u_{n+l-i}) for $1 \le i \le \frac{n-3}{2}$ and $(u_{\frac{3n-1}{4}-\frac{m-1}{2}-1}, u_{\frac{3n-1}{4}-\frac{m-1}{2}})$ and $(u_{\frac{3n-1}{4}-\frac{m-1}{2}}, u_{\frac{3n-1}{4}-\frac{m-1}{2}+1})$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. The number of exchanges is $\frac{n}{2}$. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n - 1, n - 3, \ldots, 0$ are obtained.

Case 2. l is odd, m is even.

Define the labels of edges on the closed trail $u_1 u_2 \cdots u_{\frac{3n-1}{4}-\frac{m}{2}} u_{\frac{3n-1}{4}+\frac{l+3}{2}}\ u_{\frac{3n-1}{4}+\frac{l+3}{2}+1} \cdots u_{2n} u_1$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = 2 \times (\frac{3n-1}{4} - \frac{m}{2} - 1) + m + 1 + 1 = \frac{3n-1}{2} = e_f(0) - 1$, $v_f(1) = 2 \times (\frac{3n-1}{4} - \frac{m}{2}) + m = \frac{3n-1}{2}$, and $v_f(1) - v_f(0) = n - 1$. The labeling graph on which $v_f(1) - v_f(0) = n - 1$ is obtained.

After the manner of the discussions in Case 1, the labeling graphs on which $v_f(1) - v_f(0) = n - 3$, $n - 5, \ldots, 0$ are obtained.

By Theorems 1.3 and 3.1, combining with the above discussions, the theorem holds. □

Combining with the results on Lemmas 3.2–3.9, we present the following theorem.

Theorem 3.10. For $C(2n; l, m)$, $FEBI(C(2n; l, m)) =$

(1) $\{0, \pm 2, \ldots, \pm n\}$ when $n \equiv 0\ (mod\ 4)$;
(2) $\{0, \pm 2, \ldots, \pm(n+1)\}$ when $n \equiv 1\ (mod\ 4)$;
(3) $\{0, \pm 2, \ldots, \pm n\}$ when $n \equiv 2\ (mod\ 4)$ and l, m are odd;
(4) $\{0, \pm 2, \ldots, \pm(n-2)\}$ when $n \equiv 2\ (mod\ 4)$ and l, m are even;
(5) $\{0, \pm 2, \ldots, \pm(n-1)\}$ when $n \equiv 3\ (mod\ 4)$ and l is odd;
(6) $\{0, \pm 2, \ldots, \pm(n+1)\}$ when $n \equiv 3\ (mod\ 4)$ and l is even.

Theorem 3.11. For $C(2n; l, m)$, $EBI(C(2n; l, m)) =$

(1) $\{0, 2, \ldots, n\}$ when $n \equiv 0\ (mod\ 4)$;
(2) $\{0, 2, \ldots, n+1\}$ when $n \equiv 1\ (mod\ 4)$;
(3) $\{0, 2, \ldots, n\}$ when $n \equiv 2\ (mod\ 4)$ and l, m are odd;

(4) $\{0, 2, \ldots, n-2\}$ when $n \equiv 2 \ (mod\ 4)$ and l, m are even;
(5) $\{0, 2, \ldots, n-1\}$ when $n \equiv 3 \ (mod\ 4)$ and l is odd;
(6) $\{0, 2, \ldots, n+1\}$ when $n \equiv 3 \ (mod\ 4)$ and l is even.

4. The full edge-balance properties of $IS(n)$

Issacs [4] was considered a class of odd degree graphs with perfect matching. In this section, we investigate the $FEBI$ of a class of 3-regular graphs that belong to graphs defined as Issacs.

Definition 4.1. For any integer $n \geq 3$, we denote the graph with vertex set $V = \{x_j, c_{i,j} : i = 1, 2, 3,\ j = 1, 2, \ldots, n\}$ such that $c_{i,1}c_{i,2} \cdots c_{i,n}c_{i,1}$ (for $i = 1, 2, 3$) are three disjoint cycles, respectively, and x_j is adjacent to $c_{1,j}$, $c_{2,j}$, and $c_{3,j}$. We call this graph a 3-regular Issacs graph and denote it by $IS(n)$.

The $IS(4)$ is shown in Figure 4.

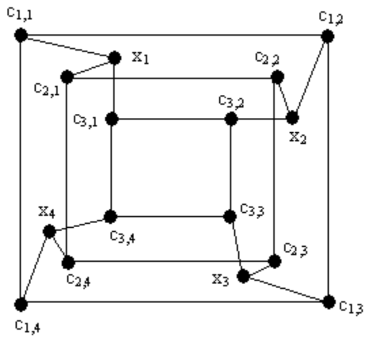

Figure 4. $IS(4)$.

In $IS(n)$, $|V| = 4n$ and $|E| = 6n$.

Now, we find the $FEBI$ of $IS(n)$. First, we discuss the maximum value of $v(k)$ ($k \in \{0, 1\}$) for any edge-friendly labeling f of $IS(n)$.

Theorem 4.1. For any edge-friendly labeling f of $IS(n)$, $v(k) \leq 3n$ ($k \in \{0, 1\}$).

Proof. Assume f is an edge-friendly labeling of $IS(n)$, then $e_f(1) = 3n = e_f(0)$. Since the degree of any vertex v is 3, if $f^+(v) = k$ ($k \in \{0, 1\}$), then there are at least two k-edges incident on v, thereby, when all k-edges are on some cycles, the maximum value of $v_f(k)$ is obtained. So, $v(k) \leq 3n$ for $k \in \{0, 1\}$. □

Next, we determine the $FEBI$ of $IS(n)$.

Theorem 4.2. For any integer $n \geq 3$, $FEBI(IS(n)) = \{0, \pm 2, \ldots, \pm 2n\}$.

Proof. **Case 1.** $n = 3$.

Define the labels of edges on the closed trail $x_1c_{1,1}c_{1,2}x_2c_{2,2}c_{2,3}x_3c_{3,3}c_{3,1}x_1$ as 1, and the labels of the remaining edges on $IS(3)$ are 0. Then $e_f(1) = 9 = e_f(0)$, $v_f(1) = 9$, and $v_f(0) = 3$. The labeling graph on which $v_f(1) - v_f(0) = 6$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = 6$, successively exchange the labels of $(x_1, c_{1,1})$ and $(x_1, c_{2,1})$; $(x_2, c_{2,2})$ and $(x_2, c_{3,2})$; and $(x_3, c_{2,3})$ and $(x_3, c_{1,3})$. $v_f(1)$ is decreased by 1, and

$v_f(0)$ is increased by 1 after each exchange. Once 3 exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = 4, 2, 0$ are obtained.

Case 2. $n > 3$ is odd.

Define the labels of edges on the closed trail $x_1c_{1,1}c_{1,2}x_2c_{2,2}c_{2,3}x_3c_{1,3}$ $c_{1,4}x_4c_{2,4}c_{2,5}\cdots x_{2j-1}c_{1,2j-1}c_{1,2j}$ $x_{2j}c_{2,2j}c_{2,2j+1}\cdots x_{n-2}c_{1,n-2}c_{1,n-1}$ $x_{n-1}c_{2,n-1}c_{2,n}x_nc_{3,n}c_{3,1}x_1$ as 1, and the labels of the remaining edges on $IS(n)$ are 0. Then $e_f(1) = 2n + 2 \times \frac{n-1}{2} + 1 = 3n = e_f(0)$, $v_f(1) = 3n$, and $v_f(0) = n$. The labeling graph on which $v_f(1) - v_f(0) = 2n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = 2n$, successively exchange the labels of $(x_i, c_{2,i})$ and $(x_i, c_{3,i})$ for $1 \leq i \leq n-1$; and $(x_n, c_{2,n})$ and $(x_n, c_{1,n})$, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1 after each exchange. There are n exchanges. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = 2n-2, 2n-4, \ldots, 2, 0$ are obtained.

Case 3. $n > 2$ is even.

Define the labels of edges on the closed trail $x_1c_{1,1}c_{1,2}x_2c_{2,2}c_{2,3}x_3c_{1,3}c_{1,4}x_4$ $c_{2,4}c_{2,5}\cdots x_{2j-1}c_{1,2j-1}c_{1,2j}$ $x_{2j}c_{2,2j}c_{2,2j+1}\cdots$ $x_{n-1}c_{1,n-1}c_{1,n}x_nc_{2,n}c_{2,1}x_1$ as 1, the labels of other edges are 0. Then $e_f(1) = 2n + 2 \times \frac{n}{2} = 3n = e_f(0)$, $v_f(1) = 3n$, and $v_f(0) = n$. The labeling graph on which $v_f(1) - v_f(0) = 2n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = 2n$, successively exchange the labels of $(x_i, c_{2,i})$ and $(x_i, c_{3,i})$ for $1 \leq i \leq n$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are n exchanges. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = 2n-2, 2n-4, \ldots, 2, 0$ are obtained.

By Theorem 1.3, this completes the proof. □

Theorem 4.3. For $IS(n)$ $(n \geq 3)$, $EBI(IS(n)) = \{0, 2, \ldots, 2n\}$.

5. The full edge-balance properties of $XStrip(n-4)$

In this section, we will investigate the $FEBI$ of the graph $XStrip(n-4)$ $(n \geq 4)$, which is a 3-regular graph.

Definition 5.1. For graph $P_n \times P_2 (n \geq 4)$, the vertex set $V = \{u_{i,j}$: $i = 1, 2;\ 1 \leq j \leq n\}$. The graph is denoted by $XStrip(n-4)$, is such a graph: the vertex set $V(XStrip(n-4)) = V(P_n \times P_2)$, the edge set $E(XStrip(n-4)) = (E(P_n \times P_2) - \{(u_{1,2}, u_{2,2}), (u_{1,n-1}, u_{2,n-1})\}) \cup \{(u_{1,1}, u_{2,2}), (u_{2,1}, u_{1,2}), (u_{1,n}, u_{2,n-1}), (u_{1,n-1}, u_{2,n})\}$.

The $XStrip(2)$ is shown in Figure 5.

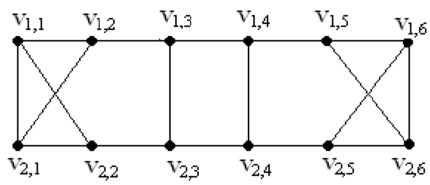

Figure 5. $XStrip(2)$.

In $XStrip(n-4)$, $|V| = 2n$, $|E| = 3n$.

Theorem 5.1. For any edge-balance labeling f of $XStrip(n-4)$ $(n \geq 4)$, then $v_f(k) \leq \frac{3n}{2}$ for n is even and $v_f(k) \leq \frac{3n+1}{2}$ for n is odd, where $k \in \{0,1\}$.

Proof. **Case 1.** n is even.

Assume f is an edge-friendly labeling, then $e_f(1) = \frac{3n}{2} = e_f(0)$. For any vertex u, $d(u) = 3$. If $f^+(u) = k$ $(k \in \{0,1\})$, then there are at least two k-edges incident on u. When all k-edges are on some cycles, the maximum value of $v_f(k)$ is obtained, thereby $v_f(k) \leq \frac{3n}{2}$ for $k \in \{0,1\}$.

Case 2. n is odd.

Assume f is an edge-friendly labeling, then $e_f(1) = \frac{3n+1}{2}$ or $e_f(1) = \frac{3n-1}{2}$. For any vertex u, $d(u) = 3$. If $f^+(u) = k$ $(k \in \{0,1\})$, then there are at least two k-edges incident on v. Following the discussion in Case 1, $v_f(k) \leq \frac{3n+1}{2}$. □

Theorem 5.2. For any even integer $n \geq 4$, $FEBI(XStrip(n-4)) = \{0, \pm 2, \ldots, \pm n\}$.

Proof. **Case 1.** $n \equiv 0$ $(mod\ 4)$.

Subcase 1.1. $n > 4$.

Define the labels of edges on the closed trail $u_{1,1}u_{1,2}\ldots u_{1,\frac{3n}{4}}u_{2,\frac{3n}{4}}\ u_{2,\frac{3n}{4}-1}\ldots u_{2,1}u_{1,1}$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = \frac{3n}{2} = e_f(0)$, $v_f(1) = \frac{3n}{2}$, $e_f(0) = \frac{n}{2}$, and $v_f(1) - v_f(0) = n$. The labeling graph on which $v_f(1) - v_f(0) = n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n$, successively exchange the labels of $(u_{2,1}, u_{2,2})$ and $(u_{1,1}, u_{2,2})$; and $(u_{2,j}, u_{2,j+1})$ and $(u_{1,j+1}, u_{2,j+1})$ for $j = 2, 3, \ldots, \frac{n}{2}$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are n exchanges. Once all exchanges are completed, the labeling graphs with $v_f(1) - v_f(0) = n-2, n-4, \ldots, 0$ are obtained.

Subcase 1.2. $n = 4$.

Define the labels of edges $(u_{1,1}, u_{1,2})$, $(u_{1,1}, u_{2,1})$, $(u_{1,2}, u_{2,1})$, $(u_{1,3}, u_{1,4})$, $(u_{1,3}, u_{2,4})$, $(u_{1,4}, u_{2,4})$ as 1, and the labels of the remaining edges are 0. Then the labeling graphs on which $v_f(1) - v_f(0) = 4$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = 4$, successively exchange the labels of $(u_{1,2}, u_{2,1})$ and $(u_{2,1}, u_{2,2})$; and $(u_{1,3}, u_{2,4})$ and $(u_{2,3}, u_{2,4})$. Once two exchanges are completed, the labeling graphs with $v_f(1) - v_f(0) = 2, 0$ are obtained.

Case 2. $n \equiv 2$ $(mod\ 4)$.

Define the labels of edges on two closed trails $u_{1,1}u_{1,2}\ldots u_{1,\frac{3(n-2)}{4}}u_{2,\frac{3(n-2)}{4}}\ u_{2,\frac{3(n-2)}{4}-1}\ldots u_{2,1}u_{1,1}$ and $u_{1,n-1}u_{1,n}u_{2,n}u_{1,n-1}$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = \frac{3(n-2)}{2}+3 = \frac{3n}{2} = e_f(0)$, $v_f(1) = \frac{3n}{2}$, $v_f(0) = \frac{n}{2}$, and $v_f(1) - v_f(0) = n$. The labeling graph with $v_f(1) - v_f(0) = n$ is obtained.

Subcase 2.1. $n > 10$.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n$, successively exchange the labels of $(u_{2,1}, u_{2,2})$ and $(u_{1,1}, u_{2,2})$; and $(u_{2,j}, u_{2,j+1})$ and $(u_{1,j+1}, u_{2,j+1})$ for $j = 2, 3, \ldots, \frac{n}{2}$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are n exchanges. Once all exchanges are completed, the labeling graphs with $v_f(1) - v_f(0) = n-2, n-4, \ldots, 0$ are obtained.

Subcase 2.2. $n = 10$.

Starting with the labeling graph on which $v_f(1) - v_f(0) = 10$, successively exchange the labels of

$(u_{2,1}, u_{2,2})$ and $(u_{1,1}, u_{2,2})$; $(u_{2,j}, u_{2,j+1})$ and $(u_{1,j+1}, u_{2,j+1})$ for $2 \le j \le 4$; the labels of $(u_{1,10}, u_{2,10})$ and $(u_{1,10}, u_{2,9})$. Once all exchanges are completed, the labeling graphs with $v_f(1) - v_f(0) = 8, 6, \ldots, 0$ are obtained.

Subcase 2.3. $n = 6$.

Starting with the labeling graph on which $v_f(1) - v_f(0) = 6$, successively exchange the labels of $(u_{2,1}, u_{2,2})$ and $(u_{1,1}, u_{2,2})$; $(u_{2,2}, u_{2,3})$ and $(u_{2,3}, u_{2,4})$; $(u_{1,6}, u_{2,6})$ and $(u_{1,6}, u_{2,5})$; and $(u_{1,10}, u_{2,10})$ and $(u_{1,10}, u_{2,9})$. Once all exchanges are completed, the labeling graphs with $v_f(1) - v_f(0) = 4, 2, 0$ are obtained.

Hence, by Theorem 1.3, $FEBI(XStrip(n-4)) = \{0, \pm 2, \ldots, \pm n\}$ when $n \ge 4$ is even. □

Theorem 5.3. For any odd integer $n \ge 5$, $FEBI(XStrip(n-4)) = \{0, \pm 2, \ldots, \pm(n+1)\}$.

Proof. **Case 1.** $n \equiv 1\ (mod\ 4)$.

Subcase 1.1. $n > 5$.

Define the labels of edges on the closed trail $u_{1,1}u_{1,2}\ldots u_{1,\frac{3n+1}{4}}u_{2,\frac{3n+1}{4}}\ u_{2,\frac{3n-3}{4}}\ldots u_{2,1}u_{1,1}$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = \frac{3n+1}{2} = e_f(0) + 1$, $v_f(1) = \frac{3n+1}{2}$, $v_f(0) = \frac{n-1}{2}$, and $v_f(1) - v_f(0) = n + 1$. The labeling graph with $v_f(1) - v_f(0) = n + 1$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n+1$, successively exchange the labels of $(u_{2,1}, u_{2,2})$ and $(u_{1,1}, u_{2,2})$; and $(u_{2,j}, u_{2,j+1})$ and $(u_{1,j+1}, u_{2,j+1})$ for $j = 2, 3, \ldots, \frac{n-1}{2}$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are $n+1$ exchanges. Once all exchanges are completed, the labeling graphs with $v_f(1) - v_f(0) = n-1, n-3, \ldots, 0$ are obtained.

Subcase 1.2. $n = 5$.

Define the labels of edges on the closed trail $u_{1,1}u_{1,2}u_{1.3}u_{2,3}u_{2,2}u_{1,1}$ and $(u_{1,4}, u_{1,5})$, $(u_{1,5}, u_{2,5})$, $(u_{1,4}, u_{2,5})$ as 1, and the labels of the remaining edges are 0. The labeling graph on which $v_f(1) - v_f(0) = 6$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = 6$, exchange the labels of $(u_{1,1}, u_{2,2})$ and $(u_{1,1}, u_{2,1})$; $(u_{2,2}, u_{2,3})$ and $(u_{1,1}, u_{2,2})$; and $(u_{1,5}, u_{2,5})$ and $(u_{1,5}, u_{2,4})$. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = 4, 2, 0$ are obtained.

Case 2. $n \equiv 3\ (mod\ 4)$.

Define the labels of edges on two closed trails $u_{1,1}u_{1,2}\ldots u_{1,\frac{3n-5}{4}}u_{2,\frac{3n-5}{4}}\ u_{2,\frac{3n-9}{4}}\ldots u_{2,1}u_{1,1}$, and $u_{1,n-1}\ u_{1,n}u_{2,n}u_{1,n-1}$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = \frac{3n-5}{2} + 3 = \frac{3n+1}{2} = e_f(0) + 1$, $v_f(1) = \frac{3n+1}{2}$, $v_f(0) = \frac{n-1}{2}$, and $v_f(1) - v_f(0) = n + 1$. The labeling graph on which $v_f(1) - v_f(0) = n + 1$ is obtained.

Subcase 2.1. $n > 7$.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n + 1$, successively exchange the labels of $(u_{2,1}, u_{2,2})$ and $(u_{1,1}, u_{2,2})$; $(u_{2,j}, u_{2,j+1})$ and $(u_{1,j+1}, u_{2,j+1})$ for $j = 2, 3, \ldots, \frac{n-1}{2}$; and $(u_{1,n-1}, u_{2,n})$ and $(u_{2,n-1}, u_{2,n})$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are $n + 1$ exchanges. Once all exchanges are completed, the labeling graphs with $v_f(1) - v_f(0) = n - 1, n - 3, \ldots, 0$ are obtained.

Subcase 2.2. $n = 7$.

Starting with the labeling graph on which $v_f(1) - v_f(0) = 8$, successively exchange the labels of $(u_{2,1}, u_{2,2})$ and $(u_{1,1}, u_{2,2})$; $(u_{2,2}, u_{2,3})$ and $(u_{1,3}, u_{2,3})$; $(u_{2,3}, u_{2,4})$ and $(u_{2,4}, u_{2,5})$; and $(u_{1,6}, u_{2,7})$ and $(u_{2,6}, u_{2,7})$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are 8 exchanges. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = 6, 4, 2, 0$ are obtained.

Hence, by Theorem 1.3, $FEBI(XStrip(n-4)) = \{0, \pm 2, \ldots, \pm(n+1)\}$ when $n \geq 5$ is odd. □

Theorem 5.4. For $n \geq 4$, $EBI(XStrip(n-4)) =$
(1) $\{0, 2, \ldots, n\}$ *for n is even;*
(2) $\{0, 2, \ldots, n+1\}$ *for n is odd.*

6. The full edge-balance properties of Möbius ladder graphs

In this section, we study the $FEBI$ of Möbius ladder graphs, a class of 3-regular graphs.

Definition 6.1. For graph $P_n \times P_2$ $(n \geq 3)$, the vertex set $V = \{u_{1,1}, u_{1,2}, \ldots, u_{1,n}, u_{2,1}, u_{2,2}, \ldots, u_{2,n}\}$. The Möbius ladder graph is denoted by M_n,

$V(M_n) = V(P_n \times P_2)$,

$E(M_n)) = (E(P_n \times P_2)) \cup \{(u_{1,1}, u_{2,n}), (u_{2,1}, u_{1,n})\}$.

M_6 is shown in Figure 6.

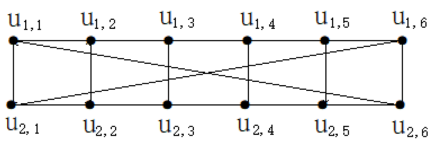

Figure 6. M_6.

In M_n $(n \geq 3)$, $|V| = 2n$, and $|E| = 3n$.

Theorem 6.1. For any edge-friendly labeling f of M_n $(n \geq 3)$, then

(1) $v(k) \leq \frac{3n}{2}$ for n is even;
(2) $v_f(k) \leq \frac{3n+1}{2}$ for $n \equiv 1$ $(mod\ 4)$;
(3). $v_f(k) \leq \frac{3n-1}{2}$ for $n \equiv 3$ $(mod\ 4)$, where $k \in \{0, 1\}$.

Proof. Assume f is an edge-friendly labeling of M_n $(n \geq 3)$.

Case 1. n is even.

$e_f(1) = \frac{3n}{2} = e_f(0)$. Since the degree of any vertex v is 3, if $f^+(v) = k$ $(k \in \{0, 1\})$, then there are at least two k-edges incident on v, thereby most $\frac{3n}{2}$ k-vertices will be obtained by $\frac{3n}{2}$ k-edges. When all k-edges are on some cycles, the maximum value of $v_f(k)$ is obtained. So, $v_f(k) \leq \frac{3n}{2}$.

Case 2. $n \equiv 1$ $(mod\ 4)$.

$e(k) = \frac{3n+1}{2}$ or $\frac{3n-1}{2}$. Since the degree of any vertex v is 3, if $f^+(v) = k$ $(k \in \{0, 1\})$, then there are at least two k-edges incident on v, thereby most $\frac{3n+1}{2}$ k-vertices will be obtained by $\frac{3n+1}{2}$ k-edges. Following the discussion in Case 1, $v_f(k) \leq \frac{3n+1}{2}$.

Case 3. $n \equiv 3$ $(mod\ 4)$.

$e_f(k) = \frac{3n+1}{2}$ or $\frac{3n-1}{2}$, and $\frac{3n+1}{2}$ is odd. Since the degree of any vertex v is 3, if $f^+(v) = k$ ($k \in \{0, 1\}$), then there are at least two k-edges incident on v, thereby most $\frac{3n+1}{2}$ k-vertices will be obtained by $\frac{3n+1}{2}$ k-edges. Following the discussion in Case 1, $v_f(k) \leq \frac{3n+1}{2}$. But there does not exist odd cycle $C_{\frac{3n+1}{2}}$, so $v_f(k) \leq \frac{3n-1}{2}$. □

Theorem 6.2. For any even integer $n \geq 4$, $FEBI(M_n) = \{0, \pm 2, \ldots, \pm n\}$.

Proof. **Case 1.** $n \equiv 0$ $(mod\ 4)$.

Define the labels of edges on the closed trail $u_{1,1}u_{1,2}\cdots u_{1,\frac{3n}{4}}u_{2,\frac{3n}{4}}u_{1,\frac{3n-4}{4}}\cdots u_{2,1}u_{1,1}$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = \frac{3n}{2} = e_f(0)$, $v_f(1) = \frac{3n}{2}$, $v_f(0) = \frac{n}{2}$, and $v_f(1) - v_f(0) = n$. The labeling graph on which $v_f(1) - v_f(0) = n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n$, successively exchange the labels of $(u_{1,\frac{3n}{4}}, u_{2,\frac{3n}{4}})$ and $(u_{1,\frac{3n}{4}}, u_{2,\frac{3n+4}{4}})$; and $(u_{2,j}, u_{2,j+1})$ and $(u_{1,j+1}, u_{2,j+1})$ for $j = 1, 2, \ldots, \frac{n-2}{2}$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are n exchanges. Once all exchanges are completed, the labeling graphs with $v_f(1) - v_f(0) = n-2, n-4, \ldots, 0$ are obtained.

Case 2. $n \equiv 2$ $(mod\ 4)$.

Define the labels of edges on the closed trail $u_{1,1}u_{2,1}u_{1,n}u_{1,n-1}\cdots u_{1,\frac{n}{2}}u_{2,\frac{n}{2}}u_{2,\frac{n-2}{2}}u_{1,\frac{n-2}{2}}\ u_{1,\frac{n-4}{2}}u_{2,\frac{n-4}{2}}\cdots u_{2,2}$ $u_{1,2}u_{1,1}$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = \frac{3n}{2} = e_f(0)$, $v_f(1) = \frac{3n}{2}$, and $v_f(0) = \frac{n}{2}$. The labeling graph on which $v_f(1) - v_f(0) = n$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n$, successively exchange the labels of $(u_{1,\frac{n}{2}+j-1}, u_{1,\frac{n}{2}+j})$ and $(u_{1,\frac{n}{2}+j}, u_{2,\frac{n}{2}+j})$ for $1 \leq j \leq \frac{n}{2}$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are n exchanges. Once all exchanges are completed, the labeling graphs with $v_f(1) - v_f(0) = n-2, n-4, \ldots, 0$ are obtained.

By Theorem 1.3, this completes the proof. □

Theorem 6.3. *For any odd integer $n \geq 3$, $FEBI(M_n) =$*

(1) $\{0, \pm 2, \ldots, \pm(n+1)\}$ for $n \equiv 1$ $(mod\ 4)$;
(2) $\{0, \pm 2, \ldots, \pm(n-1)\}$ for $n \equiv 3$ $(mod\ 4)$.

Proof. **Case 1.** $n \equiv 1$ $(mod\ 4)$.

Define the labels of edges on the closed trail $u_{1,1}u_{1,2}\cdots u_{1,\frac{3n+1}{4}}u_{2,\frac{3n+1}{4}}u_{2,\frac{3n-3}{4}}\cdots u_{2,1}u_{1,1}$ as 1, and the labels of the remaining edges are 0. Then $e_f(1) = \frac{3n+1}{2} = e_f(0) + 1$, $v_f(1) = \frac{3n+1}{2}$, $v_f(0) = \frac{n-1}{2}$, and $v_f(1) - v_f(0) = n+1$. The labeling graphs on which $v_f(1) - v_f(0) = n+1$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n+1$, successively exchange the labels of $(u_{1,\frac{3n+1}{4}}, u_{2,\frac{3n+1}{4}})$ and $(u_{1,\frac{3n+1}{4}}, u_{1,\frac{3n+5}{4}})$; and $(u_{2,j}, u_{2,j+1})$ and $(u_{1,j+1}, u_{2,j+1})$ for $j = 1, 2, \ldots, \frac{n-1}{2}$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are $n+1$ exchanges. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n-1, n-3, \ldots, 0$ are obtained.

Case 2. $n \equiv 3$ $(mod\ 4)$.

Subcase 2.1. $n > 3$.

Define the labels of edges on the closed trail $u_{1,1}u_{1,2}\cdots u_{1,\frac{3n-1}{4}}u_{2,\frac{3n-1}{4}}u_{1,\frac{3n-5}{4}}\cdots u_{2,1}u_{1,1}$, and ($u_{1,n}, u_{2,n}$) as 1, and the labels of the remaining edges are 0. Then $e_f(1) = \frac{3n+1}{2} = e_f(0) + 1$, $v_f(1) = \frac{3n-1}{2}$, $v_f(0) = \frac{n+1}{2}$, and $v_f(1) - v_f(0) = n - 1$. The labeling graph on which $v_f(1) - v_f(0) = n - 1$ is obtained.

Starting with the labeling graph on which $v_f(1) - v_f(0) = n - 1$, successively exchange the labels of $(v_{2,j}, v_{2,j+1})$ and $(v_{1,j+1}, v_{2,j+1})$ for $j = 1, 2, \ldots, \frac{n-1}{2}$. After each exchange, $v_f(1)$ is decreased by 1, and $v_f(0)$ is increased by 1. There are $n - 1$ exchanges. Once all exchanges are completed, the labeling graphs on which $v_f(1) - v_f(0) = n - 3, n - 5, \ldots, 0$ are obtained.

Subcase 2.2. $n = 3$.

Define the labels of edges on the closed trail $u_{1,1}u_{1,2}u_{2,2}u_{2,1}u_{1,1}$ as 1, and the labels of the remaining edges are 0. The labeling graph on which $v_f(1)-v_f(0) = 2$ is obtained. On the labeling graph on which $v_f(1) - v_f(0) = 2$, exchange the labels of $(u_{2,1}, u_{2,2})$ and $(u_{2,2}, u_{2,3})$, and the labeling graph on which $v_f(1) - v_f(0) = 0$ is obtained.

By Theorem 1.3, this completes the proof. □

Theorem 6.4. For M_n ($n \geq 3$), $EBI(M_n) =$

(1) $\{0, 2, \ldots, n\}$ when $n \geq 4$ is even;
(2) $\{0, 2, \ldots, n + 1\}$ when $n \equiv 1$ $(mod\ 4)$;
(3) $\{0, 2, \ldots, n - 1\}$ when $n \equiv 3$ $(mod\ 4)$.

7. Conclusions

In this paper, we studied five classes of cubic graphs, by finding some closed trails and exchanging the labels of edges method. We obtained the full edge-balance index sets of these graphs and knew that these graphs are edge-balance. Thus, there are the following problems:

Problem 1. For any cubic graph, is it edge-balance?
Problem 2. For any odd-regular graph, under what conditions is it edge-balance?
Problem 3. For any odd-regular graph G, in its edge-balance index set, does there only exist even numbers?

Author contributions

Zhen-Bin Gao participated in the research work of all sections. Juan Chen participated in the research work of the edge-balance properties of generalized Petersen graphs. Feng-Xia Chen participated in the research work of the edge-balance properties of the edge-balance properties of $C(2n; l, m)$ graphs. Feng Liang participated in the research work of the full edge-balance properties of $IS(n)$. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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