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Research article

Bounds for stop-loss distance of generalized multinomial model of random sum via Stein's method

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Abstract: In 1962, Tallis introduced the generalized multinomial model for a sequence of random variables (X_j). This model generalizes the independent case to a dependent structure resembling equicorrelation, which is meaningful in practice because it captures scenarios in which several risks or variables are jointly driven by a common underlying factor, causing them to move together with comparable strength. Within this model, we denote $W = X_1 + X_2 + \cdots + X_N$ as a random sum with a random index N, and Z as the standard normal random variable. Our goal is to establish a non-uniform bound for the stop-loss distance, $|E(W-k)^+ - E(Z-k)^+|$. In this work, in addition to biasing ideas, we apply Stein's method together with an appropriately chosen test function, which allows us to effectively use the mean value theorem. Moreover, our results are illustrated through a realistic application involving the quantity $E(W-k)^+$, which arises naturally in many financial and insurance settings, as it represents the expected excess of a loss or payoff above a specified threshold k.

Keywords: stop-loss distance; generalized multinomial model; normal approximation; uniform and non-uniform bounds; random sums; Stein's method
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1. Generalized multinomial model for random sums

Consider a sequence of random variables $X_1, X_2, X_3, \ldots$, and let N be a non-negative integer-valued random variable independent of this sequence. The quantity

$$W_N = X_1 + X_2 + \cdots + X_N$$

is referred to as the random sum of the X_j's.

In classical settings, the following assumptions are commonly imposed:

(i) the variables X_j are independent,
(ii) the number of terms is fixed, i.e., $N = n$.

In this work, we aim to relax these assumptions. Before introducing our generalized model, it is helpful to first consider a motivating example presented by Tallis [1] in 1962. In studying the heritability of fertility in livestock, one typically records the first lambing outcomes of the daughters associated with each sire. Let X_j denote the fertility outcome associated with the j^{th} successful mating, typically measured by the number of lambs born in a single birthing event. The total fertility output of the sire's daughter group is then given by the random sum W_N, which collects the reproductive performance within the family. Because all outcomes X_j stem from matings involving the same sire, it is natural to expect dependence among them, thereby motivating a framework capable of accommodating such correlations. In addition, the number of mating events N varies due to biological, environmental, and management factors such as fluctuations in sire availability, differences in mating success rates, or adjustments in herd management practices. This variability makes it natural to treat N as a random index. A convenient modeling choice is to take N to follow a binomial distribution. Specifically, one may take $N \sim Bin(m, r)$, where m represents the total number of mating opportunities and r denotes the probability that a mating results in a surviving female offspring.

Many studies have addressed these limitations. One interesting development is the generalized multinomial (GM) distribution, first introduced by Tallis [1] in the discrete setting. Later, in 2022, Daly [2] extended this framework to accommodate real-valued random variables. The sequence of identically distributed random variables (X_j) is called a GM model with parameter $\rho \in [0, 1]$ if it satisfies

$$E\left[e^{i(t_1 X_1 + \cdots + t_n X_n)}\right] = \rho E\left[e^{iX_1(t_1 + \cdots + t_n)}\right] + (1-\rho)\prod_{j=1}^{n} E\left[e^{it_j X_j}\right],$$

for all $n \in \mathbb{N}$ and for all real t_j $(j = 1, 2, \ldots, n)$. It is straightforward to show that a GM model is equicorrelated, meaning that every pair of distinct random variables (X_i, X_j) for $i \neq j$ shares the same correlation, i.e.,

$$\text{Corr}(X_i, X_j) = \rho \quad \text{for } i \neq j,$$

(for further discussion, see [2–4]). This structural property is one of the key reasons for our interest in the model.

Remark 1.1. We can see that if $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.), then the sequence of $X_j's$ is GM with parameter $\rho = 0$.

Beyond Tallis's classical example, the GM model also appears naturally in finance and insurance. In a financial context, one may interpret X_i as the profit or loss from a single trading day, so that W_N represents the total accumulated profit and loss over N days. Because the number of trading events within a given period can fluctuate due to market conditions, the trader's availability, or varying opportunity levels, it is natural to treat N as a random index, commonly modeled using a Poisson distribution. Additionally, when all trades are made by the same person, the trader's consistent behavior, such as their risk preference, strategy, or reaction to market signals, introduces a common source of dependence across days. This behavioral similarity can induce an approximately uniform level of correlation among daily returns, making it reasonable to assume that $\mathrm{Corr}(X_i, X_j)$ is the same for all $i \neq j$. Under such a structure, the GM framework provides an appropriate modeling tool.

In an insurance setting, W_N may instead represent the aggregate claim amount in a collective risk model, where each X_j denotes the size of the j^{th} individual claim (see [5, 6]). The number of claims N is itself subject to variability, for example due to differences in exposure, driving frequency, or the occurrence of insurable events, making it natural to treat N as a random index. For example, in car insurance, if all claims arise from the same driver, their consistent driving behavior, such as caution, speed habits, or risk-taking tendencies, creates a common source of dependence across multiple trips. This behavioral consistency can generate an approximately uniform correlation among the claim amounts, making the assumption that $\mathrm{Corr}(X_i, X_j)$ equal for all pairs (i, j) both reasonable and practically meaningful, and it once again positions the GM model as a suitable choice for capturing such dependence.

The applications corresponding to these examples are presented in Section 5. In addition, Section 2 provides a brief literature review outlining the historical development of the theory, followed by the main theorems of this paper. Section 3 introduces the technical tools required for our analysis, and the full proofs of our results are given in Section 4.

2. Literature reviews and main results

For $k > 0$, let

$$h_k(x) = (x - k)^+$$

be the call function, where $x^+ = \max\{0, x\}$. This function, known as a call function, plays a central role in both finance and insurance. In finance, it represents the payoff of a European call option with strike price k when the underlying asset price is x at maturity, making it a fundamental object in option pricing theory. In the insurance setting, this function is called the stop-loss function because it measures how much a loss goes beyond a chosen limit k. It is a key tool for calculating excess-of-loss reinsurance costs and assessing tail risk. Consequently, the quantity $Eh_k(W_N) = E(W_N - k)^+$ arises naturally in many applications in both finance and insurance. Several examples are presented in the final section. In practice, however, computing the exact value of $E(W_N - k)^+$ is often difficult. Therefore, to tackle this issue, many researchers turn to estimating the value of $E(W_N - k)^+$ using suitable limiting distributions Z. This motivates the study of how close $E(W_N - k)^+$ is to $E(Z - k)^+$.

The approximation of sums W_n by the standard normal random variable Z has been extensively studied in the context of the central limit theorem. Subsequently, many authors focused on quantifying the accuracy of this approximation by deriving explicit error bounds. Results obtained under this framework, which provide explicit bounds on the approximation error, are commonly referred to as Berry–Esseen type theorems, named after the authors who introduced this concept (see [7] for further details). Beyond the fixed-sample case $N = n$, the study of random sums W_N was initiated by Robbins [8] and Gnedenko and Korolev [9] within the framework of the central limit theorem. More recent works have investigated random sums under alternative probability metrics, such as the Wasserstein distance (which we discuss later), and have provided explicit error bounds; notable contributions in this direction include Döbler [10] and Daly [2]. These works, together with [7] and many others in this line of research, employ Stein's method as a primary tool for obtaining error bounds. We also adopt this approach in the present paper, and provide further details in Section 3.

In this work, we focus on measuring this difference through two versions of the stop-loss distance:

$$d_{\mathrm{SL}}(W_N, Z) = \sup_{k\in\mathbb{R}^+} |E(W_N - k)^+ - E(Z - k)^+|, \tag{2.1}$$

and for a fixed $k \in \mathbb{R}^+$,

$$d^{(k)}_{\mathrm{SL}}(W_N, Z) = |E(W_N - k)^+ - E(Z - k)^+|. \tag{2.2}$$

The distance (2.2) focuses on a single, fixed threshold k, so its resulting error bound therefore typically depends on k, and is known as a non-uniform bound. In contrast, a bound for (2.1) must hold uniformly for all $k \in \mathbb{R}^+$, and is thus referred to as a uniform bound.

Error bounds for the stop-loss distances (2.1) and (2.2) under various assumptions on N and X_j's have been studied extensively in the literature. In the classical setting where the X_j's are independent and $N = n$ is fixed, Neammanee and Jongpreechaharn [11] obtained both uniform and non-uniform error bounds using the normal distribution. In the same framework, Neammanee and Yonghint [12] derived corresponding results based on the Poisson distribution. Moving beyond the case of fixed N, Kammoo et al. [13] (2025) established uniform bounds for the stop-loss distance using a normal approximation. For more general dependence structures, Daly [2] studied the GM model and provided three types of limiting distributions, Gaussian, Poisson, and gamma, together with uniform bounds for the stop-loss distance. However, Daly's analysis did not include non-uniform bounds. The goal of this work is to fill this gap by establishing non-uniform error bounds in the case where the appropriate limiting distribution is Gaussian.

Let us recall the well-known Wasserstein distance, defined by

$$d_W(W_N, Z) = \sup_{h\in\mathcal{H}} |Eh(W_N) - Eh(Z)|,$$

where $\mathcal{H}$ is the class of all Lipschitz continuous functions h on $\mathbb{R}$ with a Lipschitz constant not greater than 1. Since the call function h_k belongs to $\mathcal{H}$, we immediately obtain the useful inequality

$$d_{\mathrm{SL}}(W_N, Z) \le d_W(W_N, Z). \tag{2.3}$$

Motivated by this relation, several works, including that of Daly [2], have investigated the stop-loss distance via the Wasserstein distance. Many authors have analyzed approximations of random sums under various assumptions on the index N. In the absence of specific distributional assumptions for N, Gaussian approximations have been considered in [2, 10, 14]. When N is treated as a known random variable, common assumptions include Poisson [2, 15], binomial [16, 17], negative binomial [15, 18], or mixed Poisson [15]. However, this approach does not yield the non-uniform bound in (2.2). In the present work, to obtain such a non-uniform bound, we instead analyze the stop-loss distance directly using Stein's method. The details of this approach are presented in Section 3.

Throughout this paper, we denote $W := W_N$. Let

$$\tilde{W} = \frac{W}{\sqrt{\mathrm{Var}(W)}}$$

denote the standardized version of W, and $Z \sim N(0,1)$ has the standard normal distribution. A recent result on the stop-loss distance was presented in [13], which can be stated as follows.

Theorem 2.1. *[13] let $X_1, X_2, \ldots$ be i.i.d. random variables with a zero mean and finite absolute third moment, and let N be a non-negative, integer-valued random variable with finite variance that is independent of the X_j's. Then,*

$$d_{SL}(\tilde{W}, Z) \leq \sqrt{\frac{2}{\pi}}\left(\frac{\sqrt{\mathrm{Var}(N)}}{EN}\right) + \frac{10.24E|X_1|^3}{\mathrm{Var}(X_1))^{\frac{3}{2}}\sqrt{EN}}.$$

Moreover, for $k \geq 3$,

$$d_{SL}^{(k)}(\tilde{W}, Z) \leq \frac{1}{1+k}\left(\frac{1.38\sqrt{\mathrm{Var}(N)}}{EN} + \frac{26.78E|X_1|^3}{(\mathrm{Var}(X_1))^{\frac{3}{2}}\sqrt{EN}} + \frac{1.68E|X_1|^3 EN^2}{(\mathrm{Var}(X_1))^{\frac{3}{2}}(EN)^{\frac{5}{2}}}\right).$$

In the identically distributed case, we remark that this result applies when $X_1, X_2, \ldots$ are independent, corresponding to a special instance of the GM model studied by Daly [2]. Although both [2] and [13] employ Stein's method as the primary analytical tool, the test functions used in the respective Stein equations differ. In Daly's work, the test function h is taken from the class of absolutely continuous functions, whereas [13] uses the class of call functions h_k. Consequently, Daly obtained an error bound only for (2.1) by applying relation (2.3) together with the well-known inequality by Goldstein [19],

$$d_{\mathrm{W}}(\tilde{W}, Z) \leq 2E|\tilde{W}^z - \tilde{W}| \tag{2.4}$$

where the zero-bias version W^z of W exists, and is obtained in the following theorem.

Theorem 2.2. *[2] Let N be a non-negative, integer-valued random variable, and let $X_1, X_2, \ldots$ be identically distributed random variables with zero mean and positive variance which satisfy a GM model with parameter ρ and are independent of N. If $E|X_1|^3 < \infty$, then*

$$d_{SL}(\tilde{W}, Z) \leq 2\tau\delta_1 + 2(1-\tau)\delta_2,$$

where τ, δ_1, and δ_2 are defined as follows:

$$\tau = \frac{\rho EN^2}{E[N] + \rho E[N(N-1)]},$$

$$\delta_1 = 1 + \frac{EN^3 E|X_1|^3}{2EN^2\,\mathrm{Var}(X_1)\sqrt{\mathrm{Var}(W)}}, \tag{2.5}$$

$$\delta_2 = \frac{1}{\sqrt{\mathrm{Var}(W)}}\left(\frac{E|X_1|^3}{2\,\mathrm{Var}(X_1)} + E\left|\sum_{j=1}^{N^s-1} X_j' - \sum_{k=1}^{N} X_k\right|\right), \tag{2.6}$$

with X_j''s denoting i.i.d. random variables having the same law as X_1, and N^s having the size-bias distribution of N.

This theorem relies on the concept of the zero-bias and size-bias transformations, which are introduced in Section 3.2.

While Goldstein's bound (2.4) leads to the uniform bound (2.1), the theorem below provides the corresponding non-uniform bound (2.2).

Theorem 2.3. *Suppose that the zero-bias version W^z of W exists. For $k \geq 4$,*

$$d_{SL}^{(k)}(\tilde{W}, Z) \leq \left(\frac{17.79}{k}\right)(E|\tilde{W}^z - \tilde{W}|^2)^{\frac{1}{2}}.$$

Applying Hölder's inequality to (2.4) yields

$$d_{\mathrm{SL}}^{(k)}(\tilde{W}, Z) \leq 2(E|\tilde{W}^z - \tilde{W}|^2)^{\frac{1}{2}}.$$

Comparing this bound with our result in Theorem 2.4, we see that, for sufficiently large k, our estimate is sharper.

Theorem 2.4. *Under the assumptions of Theorem 2.2. If $E|X_1|^4 < \infty$, then for $k \geq 4$,*

$$d_{SL}^{(k)}(\tilde{W}, Z) \leq \frac{35.58}{k}\left(\sqrt{\tau\delta_3} + \sqrt{(1-\tau)\delta_4}\right),$$

where

$$\delta_3 = 1 + \frac{E|NX_1|^4}{3EN^2 \operatorname{Var}(X_1) \operatorname{Var}(W)}, \tag{2.7}$$

and

$$\delta_4 = \frac{1}{\operatorname{Var}(W)}\left(\frac{E|X_1|^4}{3\operatorname{Var}(X_1)} + E\left(\sum_{j=1}^{N^s-1} X_j' - \sum_{k=1}^{N} X_k\right)^2\right). \tag{2.8}$$

Remark 2.1. The independence between the random index N and the summands $\{X_j\}_{j\geq 1}$ is not required for the validity of Theorem 2.3. However, when N and $\{X_j\}_{j\geq 1}$ are assumed to be independent, as in the framework of Daly (2022), this additional structure allows us to derive an explicit non-uniform bound. The resulting bound is presented in Theorem 2.4.

The following result is a direct consequence of Theorems 2.2 and 2.4, but it is presented for the case where the X_i are not required to have zero mean.

Corollary 2.1. *Let N be a non-negative, integer-valued random variable, and let $Y_1, Y_2, \ldots$ be identically distributed random variables with mean μ and positive variance which satisfy a GM model with parameter ρ and are independent of N. Let $X_i = Y_i - \mu$ and $S = \sum_{i=1}^{N} Y_i$. Assume that $E|X_1|^3$ exists. Then,*

$$d_{SL}(\tilde{S}, Z) \leq 2\tau\delta_1 + 2(1-\tau)\delta_2.$$

Furthermore, if $E|X_1|^4$ exists, then for $k \geq 4$,

$$d_{SL}^{(k)}(\tilde{S}, Z) \leq \frac{35.58}{k}\left(\sqrt{\tau\delta_3} + \sqrt{(1-\tau)\delta_4}\right),$$

where δ_1, δ_2, δ_3, and δ_4 are defined in (2.5)–(2.8)*, respectively, and*

$$\tilde{S} = \frac{S - \mu N}{\sqrt{\operatorname{Var}(S) - \mu^2 \operatorname{Var}(N)}}$$

represents the standardized version of S.

3. Stein's method and bias couplings

3.1. Stein's method

In this section, we review the foundational tools for our work, beginning with the ingenious approach developed by Charles Stein in 1972 [20], commonly referred to as Stein's method. Let C_{bd} be the set of continuous and piecewise continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with $E|f'(Z)| < \infty$. Stein's method begins with the Stein equation for normal approximation,

$$xf(x) - f'(x) = h(x) - Eh(Z) \tag{3.1}$$

for a given function h and $f \in C_{bd}$. The solution of (3.1) is

$$\begin{aligned} f_h(x) &= -\frac{1}{\varphi(x)} \int_{-\infty}^{x} [h(t) - Eh(Z)]\varphi(t)\, \mathrm{d}t \\ &= \frac{1}{\varphi(x)} \int_{x}^{\infty} [h(t) - Eh(Z)]\varphi(t)\, \mathrm{d}t, \end{aligned} \tag{3.2}$$

where φ is the density function of Z (see [21, p. 15]).

In this work, we apply the Stein equation (3.1) with a call function h_k defined by

$$h_k(x) = (x - k)^+ \tag{3.3}$$

for fix $k > 0$. Then, we have

$$xf(x) - f'(x) = (x - k)^+ - E(Z - k)^+. \tag{3.4}$$

The solution of (3.4) is

$$f_k(x) = \begin{cases} \sqrt{2\pi} e^{\frac{x^2}{2}} E(Z - k)^+ \Phi(x), & \text{if } x \le k, \\ 1 - \sqrt{2\pi} e^{\frac{x^2}{2}} \Big(k + E(Z - k)^+\Big)\Phi(-x), & \text{if } x > k, \end{cases}$$

where Φ is the distribution function of the standard normal random variable Z, and the expression for the first derivative, denoted as f_k', is as follows:

$$f_k'(x) = \begin{cases} E(Z - k)^+\Big(1 + \sqrt{2\pi} x e^{\frac{x^2}{2}} \Phi(x)\Big), & \text{if } x \le k, \\ \Big(k + E(Z - k)^+\Big)\Big(1 - \sqrt{2\pi} x e^{\frac{x^2}{2}} \Phi(-x)\Big), & \text{if } x > k, \end{cases}$$

(see [11, 13, 23] for additional background). A crucial component of Stein's method is identifying the properties of the solution f_k. From Lemma 2.4 of Chen et al. [7] and some observations shown by Jongpreechaharn and Neammanee [22, pp. 210–211], we have that

$$0 \le f_k'(x) \le \sqrt{\frac{2}{\pi}}, \quad \text{for } x \in \mathbb{R}.$$

Additionally, the second derivative, denoted as f_k'', is expressed as follows:

$$f_k''(x) = \begin{cases} E(Z-k)^+\Big[x + \Big(\sqrt{2\pi}e^{\frac{x^2}{2}}\Phi(x)\Big)\Big(x^2+1\Big)\Big], & \text{if } x < k, \\ \Big(k + E(Z-k)^+\Big)\Big[x - \Big(\sqrt{2\pi}e^{\frac{x^2}{2}}\Phi(-x)\Big)\Big(x^2+1\Big)\Big], & \text{if } x > k. \end{cases}$$

From Stein's equation (3.1) and the fact that h_k is not differentiable at k, we observe that f_k'' does not exist at $x = k$. This lack of differentiability prevents the direct application of the mean value theorem on any interval $[a, b]$ such that $k \in (a, b)$, as the theorem requires differentiability throughout the entire open interval (a, b). The uniform bound for f_k'' was observed by [11, p. 3501], indicating that

$$|f_k''(x)| \le 2, \quad \text{for } x \in \mathbb{R} \setminus \{k\}.$$

To overcome this limitation, we avoid working directly with h_k when applying Stein's method. Instead, following the idea introduced by Daly [2], we employ a smoother test function whose corresponding Stein solution is twice differentiable. For fixed $\varepsilon > 0$ and $k > 0$, let

$$h_{k,\varepsilon}(x) = Eh_k(x + U_\varepsilon) = E(x + U_\varepsilon - k)^+, \tag{3.5}$$

where U_ε follows the uniform distribution $\mathcal{U}(0, \varepsilon)$.

Proposition 3.1. *For the function $h_{k,\varepsilon}$ defined above, we have*

$$h_{k,\varepsilon}(x) = \begin{cases} 0, & \text{if } x \le k - \varepsilon, \\ \frac{1}{2\varepsilon}(\varepsilon - k + x)^2, & \text{if } k - \varepsilon < x < k, \\ (x-k) + \frac{\varepsilon}{2}, & \text{if } x \ge k. \end{cases}$$

Proof. Note that

$$E(x + U_\varepsilon - k)^+ = \frac{1}{\varepsilon}\int_0^\varepsilon (x + s - k)^+ ds.$$

If $x \le k - \varepsilon$, then $x + s - k \le 0$ for all $s \in [0, \varepsilon]$. Consequently, $E(x + U_\varepsilon - k)^+ = 0$. For $k - \varepsilon < x < k$, we have

$$E(x + U_\varepsilon - k)^+ = \frac{1}{\varepsilon}\int_{k-x}^\varepsilon (x + s - k) ds = \frac{1}{2\varepsilon}(\varepsilon - k + x)^2$$

and for $x \ge k$, we have

$$E(x + U_\varepsilon - k)^+ = \frac{1}{\varepsilon}\int_0^\varepsilon (x + s - k) ds = (x - k) + \frac{\varepsilon}{2}.$$

Hence, we obtain

$$E(x + U_\varepsilon - k)^+ = \begin{cases} 0, & \text{if } x \le k - \varepsilon, \\ \frac{1}{2\varepsilon}(\varepsilon - k + x)^2, & \text{if } k - \varepsilon < x < k, \\ (x-k) + \frac{\varepsilon}{2}, & \text{if } x \ge k. \end{cases}$$

□

Given the explicit representation of $h_{k,\varepsilon}$, the derivatives follow by straightforward calculation, leading to

$$h'_{k,\varepsilon}(x) = \begin{cases} 0, & \text{if } x \leq k-\varepsilon, \\ \frac{1}{\varepsilon}(\varepsilon - k + x), & \text{if } k-\varepsilon < x < k, \\ 1, & \text{if } x \geq k, \end{cases}$$

and

$$\begin{aligned} h''_{k,\varepsilon}(x) &= \begin{cases} 0, & \text{if } x < k-\varepsilon, \\ \frac{1}{\varepsilon}, & \text{if } k-\varepsilon < x < k, \\ 0, & \text{if } x > k, \end{cases} \\ &= \frac{1}{\varepsilon} I(k-\varepsilon < x < k). \end{aligned}$$

Let $f_{k,\varepsilon}$ be the solution of Stein's equation (3.1) corresponding to $h_{k,\varepsilon}$ which is defined in (3.5). From Proposition 3.1 and the fact that, for any differentiable function h, the bounds $\|f'_h\| \leq \sqrt{2/\pi}\|h'\|$ and $\|f''_h\| \leq 2\|h'\|$ hold, where $\|g\| = \sup_{x\in\mathbb{R}} |g(x)|$ (see Lemma 2.4 in [7]), we obtain

$$\|f'_{k,\varepsilon}\| \leq \sqrt{\frac{2}{\pi}} \quad \text{and} \quad \|f''_{k,\varepsilon}\| \leq 2. \tag{3.6}$$

In order to establish the non-uniform bound for $f''_{k,\varepsilon}$ given in Proposition 3.3, we need the following property.

Proposition 3.2. *Let* $0 < \varepsilon < k$ *and* U_ε *be independent of* Z. *Then,*

$$E(Z + U_\varepsilon - k)^+ \leq \frac{1}{\sqrt{2\pi} e^{\frac{(k-\varepsilon)^2}{2}}}.$$

Proof. Let f_{Z,U_ε} be the joint density function of Z and U_ε. Recall that the density function of Z is

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

while the density of $U_\varepsilon \sim \mathcal{U}(0,\varepsilon)$ is given by

$$f_{U_\varepsilon}(y) = \begin{cases} \dfrac{1}{\varepsilon}, & 0 < y < \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Since U_ε is independent of Z, the joint density factorizes as $f_{Z,U_\varepsilon}(x,y) = \varphi(x) f_{U_\varepsilon}(y)$. Consequently,

$$\begin{aligned} E(Z + U_\varepsilon - k)^+ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y - k)^+ f_{Z,U_\varepsilon}(x,y)\,dxdy \\ &= \frac{1}{\varepsilon} \int_0^{\varepsilon} \int_{k-y}^{\infty} (x + y - k)\varphi(x)\,dxdy. \end{aligned}$$

Since $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, it follows that $\varphi'(x) = -\frac{x}{\sqrt{2\pi}}e^{-x^2/2} = -x\varphi(x)$. Then,

$$\int_{k-y}^{\infty} x\varphi(x)dx = EZ - \int_{-\infty}^{k-y} x\varphi(x)dx = \varphi(k-y),$$

we obtain

$$\begin{aligned}\int_0^{\varepsilon}\int_{k-y}^{\infty}(x+y-k)\varphi(x)dxdy &= \int_0^{\varepsilon}\Big(\varphi(k-y)-(k-y)(1-\Phi(k-y))\Big)dy\\ &= \int_{k-\varepsilon}^{k}(\varphi(t)-t(1-\Phi(t)))\,dt\\ &\le \max\left\{\int_{k-\varepsilon}^{k}\varphi(t)dt, \int_{k-\varepsilon}^{k}t(1-\Phi(t))dt\right\}.\end{aligned}$$

To bound the first term, observe that

$$\int_{k-\varepsilon}^{k}\varphi(t)dt \le \varepsilon\varphi(k-\varepsilon) = \frac{\varepsilon}{\sqrt{2\pi}e^{\frac{(k-\varepsilon)^2}{2}}}.$$

For the second term, applying the Gaussian tail bound, $1-\Phi(t) \le \frac{1}{t\sqrt{2\pi}}e^{-\frac{t^2}{2}}$ for $t > 0$, yields

$$\int_{k-\varepsilon}^{k}t(1-\Phi(t))dt \le \int_{k-\varepsilon}^{k}\frac{1}{\sqrt{2\pi}e^{\frac{t^2}{2}}}dt \le \frac{\varepsilon}{\sqrt{2\pi}e^{\frac{(k-\varepsilon)^2}{2}}}.$$

Hence,

$$0 \le E(Z+U_{\varepsilon}-k)^+ \le \frac{1}{\sqrt{2\pi}e^{\frac{(k-\varepsilon)^2}{2}}}.$$

□

Proposition 3.3. *Let $f_{k,\varepsilon}$ be the solution of the Stein's equation* (3.1) *corresponding to $h_{k,\varepsilon}$. For* $0 < \varepsilon < 1$, $x \le \frac{k}{2}$, *and* $k \ge 4$, *we have*

$$|f''_{k,\varepsilon}(x)| \le \frac{1.68}{k}.$$

Proof. Since $E(t+U_{\varepsilon}-k)^+ = 0$ for $t \le k-\varepsilon$ and $x \le \frac{k}{2} \le k-\varepsilon$, it follows from (3.2) that

$$\begin{aligned}f_{k,\varepsilon}(x) &= -\frac{1}{\varphi(x)}\int_{-\infty}^{x}[E(t+U_{\varepsilon}-k)^+ - E(Z+U_{\varepsilon}-k)^+]\varphi(t)\,\mathrm{d}t\\ &= E(Z+U_{\varepsilon}-k)^+\frac{\Phi(x)}{\varphi(x)}.\end{aligned}$$

Note that

$$\frac{d}{dx}\left(\frac{\Phi(x)}{\varphi(x)}\right) = \frac{\varphi^2(x)+x\Phi(x)\varphi(x)}{\varphi^2(x)} = 1+\frac{x\Phi(x)}{\varphi(x)},$$

and

$$\frac{d}{dx}\left(\frac{x\Phi(x)}{\varphi(x)}\right) = \frac{x\varphi^2(x) + \Phi(x)\varphi(x) + x^2\Phi(x)\varphi(x)}{\varphi^2(x)} = x + \frac{\Phi(x)}{\varphi(x)} + \frac{x^2\Phi(x)}{\varphi(x)}.$$

Hence,

$$\begin{aligned} f''_{k,\varepsilon}(x) &= E(Z + U_\varepsilon - k)^+ \left(x + \frac{\Phi(x)}{\varphi(x)} + \frac{x^2\Phi(x)}{\varphi(x)}\right) \\ &= E(Z + U_\varepsilon - k)^+ \left(x + \sqrt{2\pi}e^{\frac{x^2}{2}}\Phi(x)(1 + x^2)\right). \end{aligned} \tag{3.7}$$

To establish an upper bound of $\left|f''_{k,\varepsilon}(x)\right|$ for $x \le \frac{k}{2}$, we separately bound the two factors in (3.7). For the first factor, Proposition 3.2 yields

$$E(Z + U_\varepsilon - k)^+ \le \frac{1}{\sqrt{2\pi}e^{\frac{(k-\varepsilon)^2}{2}}} < \frac{1}{\sqrt{2\pi}e^{\frac{9k^2}{32}}}, \tag{3.8}$$

where we use the fact that $0 < k - 1 < k - \varepsilon$ and $k - 1 \ge (3/4)k$ for $k \ge 4$ in the last inequality. For the second factor, we use the bound

$$\left|x + \sqrt{2\pi}e^{\frac{x^2}{2}}\Phi(x)(1 + x^2)\right| \le 2, \quad \text{for } x < 0$$

[11, p. 3502]. Combining this with (3.8), we obtain for $x < 0$,

$$\left|f''_{k,\varepsilon}(x)\right| \le \frac{2}{\sqrt{2\pi}e^{\frac{9k^2}{32}}} < \frac{1.68}{k},$$

where the last inequality uses the fact that the function $ke^{-\frac{9k^2}{32}}$ attains its maximum at $k = 4/3$, which implies that $e^{-\frac{9k^2}{32}} < 1/k$ for $k > 0$. For $0 \le x \le \frac{k}{2}$, we apply (3.7) and (3.8), together with the estimate

$$g(k) = \frac{k^2}{2\sqrt{2\pi}e^{\frac{9k^2}{32}}} + \frac{k(4 + k^2)}{4e^{\frac{5k^2}{32}}} < 1.68, \quad \text{for } k \ge 4,$$

to obtain

$$\begin{aligned} 0 \le f''_{k,\varepsilon}(x) &\le \frac{1}{\sqrt{2\pi}e^{\frac{9k^2}{32}}}\left[\frac{k}{2} + \frac{\sqrt{2\pi}}{4}e^{\frac{k^2}{8}}\left(4 + k^2\right)\right] \\ &= \frac{1}{k}\left[\frac{k^2}{2\sqrt{2\pi}e^{\frac{9k^2}{32}}} + \frac{k(4 + k^2)}{4e^{\frac{5k^2}{32}}}\right] \\ &\le \frac{1.68}{k}. \end{aligned}$$

where the estimate is obtained by observing that $g(k)$ is decreasing for $k \ge 4$. □

For further study of Stein's method, one may consult [27–29] for additional details.

3.2. Bias couplings

In this section, we introduce the concepts of size-bias and zero-bias distributions, which was given by Goldstein and Reinert [30] in 1977. These constructions are particularly useful for reformulating certain expressions that arise in Stein's equation into more tractable forms. The foundation of Stein's method lies in the following characterization of a normal random variable Z with variance σ^2. The random variable $X \sim N(0, \sigma^2)$ if and only if

$$Ef'(X) = \frac{E[Xf(X)]}{\sigma^2} \tag{3.9}$$

for all smooth f (see [7] for more detail).

While the equality of the left and right-hand sides in (3.9) is specific to the normal case, one can create an identity in the same spirit that holds more generally. Let X be a zero mean and non-zero, finite variance random variable. Goldstein and Reinert [30] introduced a new distribution related to the distribution of X according to the following definition.

Definition 3.1. For a random variable X with zero mean and finite positive variance σ^2, the zero-bias distribution of X, denoted by X^z, satisfies

$$Ef'(X^z) = \frac{E[Xf(X)]}{\sigma^2} \tag{3.10}$$

for all absolutely continuous functions f for which this expectation exists.

To clarify the definition further, Chen et al. [7, p. 27] provided a simple example of the zero-bias distribution in the case $X = I - p$, where $I \sim Ber(p)$. Applying the definition yields $X^z \sim \mathcal{U}(-p, 1-p)$. Moreover, the identity, $(cX)^z \overset{d}{=} cX^z$ for any random variable X and $c \neq 0$ follows directly from a straightforward application of the definition.

Remark 3.1. In the original work (1997), the identity (3.10) was shown to hold for smooth functions. Chen et al. [7] later demonstrated that the same identity continues to hold in the more general class of absolutely continuous functions.

This definition was first developed for use with Stein's method for Gaussian approximation, motivated by the fact that X has a standard normal if and only if X^z has the same distribution as X. See [7] for both historical background and numerous applications of zero biasing in Gaussian approximation.

Another transformation used in this work is the size-biased distribution, defined as follows.

Definition 3.2. For a non-negative random variables X with non-zero mean, the size-biased distribution of X, denoted by X^s, is defined by

$$Ef(X^s) = \frac{E[Xf(X)]}{EX}$$

for all functions f for which $E[Xf(X)]$ exists.

Remark 3.2. For any non-negative integer-valued random variable X, the size-biased X^s is defined by the identity

$$\mathbb{P}(X^s = k) = \frac{k\mathbb{P}(X = k)}{EX}, \quad \text{for } k = 0, 1, 2, \ldots, \tag{3.11}$$

where $\mathbb{P}(X = x)$ denotes the probability mass function of X [31, p. 232].

In the context of Stein's method, size biasing appears most often in a Poisson approximation (see [12, 32, 33] for examples). A key tool in this setting is the well-known characterization that a random variable X has a Poisson distribution if and only if X^s shares the same distribution as $X + 1$. Indeed, by applying the identity (3.11) to the case $X \sim Poi(\lambda)$, we obtain

$$\mathbb{P}(X^s = k) = \frac{k\mathbb{P}(X = k)}{EX} = \frac{ke^{-\lambda}\lambda^k}{k!\lambda} = e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} = \mathbb{P}(X = k - 1).$$

Hence,

$$X^s \stackrel{d}{=} X + 1. \tag{3.12}$$

The size-bias distribution arises as a random index in the following lemma. Therefore, to apply the lemma effectively, it is essential to understand the size-bias distribution of commonly-used random variables N. We now illustrate this concept by calculating the size-bias distribution for a binomial random variable N.

Example 3.1. Let $N \sim Bin(m, r)$, i.e.,

$$\mathbb{P}(N = k) = \binom{m}{k} r^k (1 - r)^{m-k} \qquad \text{for } k = 0, 1, \dots, m.$$

According to Remark 3.2, it follows that the size-biased distribution N^s has the probability mass function for $k = 1, 2, \dots, m$,

$$\mathbb{P}(N^s = k) = \frac{k\binom{m}{k} r^k (1 - r)^{m-k}}{mr} = \binom{m-1}{k-1} r^{k-1} (1 - r)^{m-k}.$$

Let $B_1, B_2, \dots, B_m$ be i.i.d. Bernoulli random variables with parameter r such that $N = \sum_{i=1}^m B_i$, and let U be a random index chosen uniformly from $\{1, 2, \dots, m\}$, independent of B_i's. Note that for $k = 1, 2, \dots, m$,

$$\begin{aligned}
\mathbb{P}(N - B_U = k - 1) &= \sum_{i=1}^m \mathbb{P}(U = i)\mathbb{P}\left(\sum_{j\neq i} B_j = k - 1\right) \\
&= \frac{1}{m}\sum_{i=1}^m \mathbb{P}\Big(Bin(m - 1, r) = k - 1\Big) \\
&= \binom{m-1}{k-1} r^{k-1} (1 - r)^{m-k}.
\end{aligned}$$

Hence, we can write

$$N^s - 1 \stackrel{d}{=} N - B_U. \tag{3.13}$$

Both transformation distributions are particularly useful for reformulating certain expressions to align with the framework of Stein's method. However, deriving their explicit forms is generally nontrivial, as illustrated even in simple cases. In 2022, Daly [2] derived an exact formula for the zero-bias distribution of W by utilizing its characteristic function together with the concept of size-biasing, as follows.

Lemma 3.1. *Let N be a non-negative, integer-valued random variable, and let $X_1, X_2, \ldots$ be identically distributed random variables satisfying a GM model with zero mean and positive variance, independent of N. Let*

$$\tau = \frac{\rho E[N^2]}{E[N] + \rho E[N(N-1)]}, \tag{3.14}$$

and let $I_\tau \sim Ber(\tau)$ be a Bernoulli random variable, independent of all else. Then,

$$W^z \stackrel{d}{=} I_\tau (NX_1)^z + (1 - I_\tau)\left(X_1^z + \sum_{j=1}^{N^s-1} X'_j\right),$$

where the X'_j's are i.i.d. random variables having the same law as X_1, all random variables on right-hand are independent, and N^s is the size-bias distribution of N.

In addition to the zero-bias and size-bias distributions discussed in this section, the reader may refer to [2, 10] for more details of other bias transformation concepts.

Before presenting the proofs of the main results in the next section, we require the following proposition, which can be derived by applying Lemma 3.1.

Proposition 3.4. *Let N be a non-negative, integer-valued random variable, and let $X_1, X_2, \ldots$ be identically distributed random variables satisfying a GM model with zero mean and positive variance, independent of N. Let τ be as defined in* (3.14)*. Then, we have*

(*i*) $E|\tilde{W}^z - \tilde{W}| \le \tau\delta_1 + (1-\tau)\delta_2$,
(*ii*) $E|\tilde{W}^z - \tilde{W}|^2 \le 4\tau\delta_3 + 4(1-\tau)\delta_4$.

where δ_1, δ_2, δ_3, and δ_4 are defined in (2.5)–(2.8)*, respectively.*

Proof. (*i*) This has already been established in [2, p. 476].
(*ii*) By Lemma 3.1 and the inequality $(a+b)^2 \le 2(a^2+b^2)$ for all $a, b \in \mathbb{R}$, we obtain that

$$\begin{aligned} E|\tilde{W}^z - \tilde{W}|^2 &= \frac{1}{\mathrm{Var}(W)} E\left[I_\tau\Big((NX_1)^z - W\Big) + (1-I_\tau)\Big(X_1^z + \sum_{j=1}^{N^s-1} X'_j - \sum_{k=1}^{N} X_k\Big)\right]^2 \\ &\le \frac{2}{\mathrm{Var}(W)} E\left[I_\tau^2\Big((NX_1)^z - W\Big)^2 + (1-I_\tau)^2\Big(X_1^z + \sum_{j=1}^{N^s-1} X'_j - \sum_{k=1}^{N} X_k\Big)^2\right]. \end{aligned}$$

By the definition of the zero-bias distribution, for any random variable X with zero mean, we have

$$E[X^z]^2 = \frac{E[X(X)^3]}{3\,\mathrm{Var}(X)} = \frac{EX^4}{3\,\mathrm{Var}(X)}. \tag{3.15}$$

This implies

$$E\left[(1-I_\tau)^2\Big(X_1^z + \sum_{j=1}^{N^s-1} X'_j - \sum_{k=1}^{N} X_k\Big)^2\right] \le 2(1-\tau)\left(E[X_1^z]^2 + E\Big(\sum_{j=1}^{N^s-1} X'_j - \sum_{k=1}^{N} X_k\Big)^2\right)$$

$$= 2(1-\tau)\bigg(\frac{E|X_1|^4}{3\operatorname{Var}(X_1)} + E\Big(\sum_{j=1}^{N^s-1} X'_j - \sum_{k=1}^{N} X_k\Big)^2\bigg).$$

Using (3.15) again and noting that $\operatorname{Var}(NX_1) = EN^2 \operatorname{Var}(X_1)$, it follows that

$$\begin{aligned} E[I_\tau((NX_1)^z - W)]^2 &\le 2\tau\Big(E[(NX_1)^z]^2 + \operatorname{Var}(W)\Big) \\ &= 2\tau\left(\frac{E|NX_1|^4}{3\operatorname{Var}(NX_1)} + \operatorname{Var}(W)\right) \\ &= 2\tau\left(\frac{E|NX_1|^4}{3EN^2\operatorname{Var}(X_1)} + \operatorname{Var}(W)\right). \end{aligned}$$

Combining the above inequalities, we obtain

$$E|\tilde{W}^z - \tilde{W}|^2 \le \frac{2}{\operatorname{Var}(W)}\Big(2\tau\delta_3 \operatorname{Var}(W) + 2(1-\tau)\delta_4 \operatorname{Var}(W)\Big) = 4\tau\delta_3 + 4(1-\tau)\delta_4.$$

□

4. Proof of main results

4.1. Proof of Theorem 2.3

For $0 < \varepsilon < 1$, let h_k and $h_{k,\varepsilon}$ be defined as in (3.3) and (3.5), respectively, and let $f_{k,\varepsilon}$ denote the solution to the Stein equation (3.1) corresponding to $h_{k,\varepsilon}$. Using the triangle inequality, we have

$$|Eh_k(\tilde{W}) - Eh_k(Z)| \le |Eh_k(\tilde{W}) - Eh_{k,\varepsilon}(\tilde{W})| + |Eh_{k,\varepsilon}(\tilde{W}) - Eh_{k,\varepsilon}(Z)| + |Eh_{k,\varepsilon}(Z)| - Eh_k(Z)|. \quad (4.1)$$

From this inequality, together with the fact that if $E[g(X)] \le E[g(Y)]$ for any random variables X, Y, and any increasing function g on $\mathbb{R}$, then

$$d_{\mathrm{SL}}(X,Y) = \sup_{k\in\mathbb{R}^+} |Eh_k(X) - Eh_k(Y)| = EY - EX, \qquad [34, \text{Corollary } 4],$$

we deduce that

$$|Eh_k(\tilde{W}) - Eh_{k,\varepsilon}(\tilde{W})| = |Eh_k(\tilde{W}) - Eh_k(\tilde{W} + U_\varepsilon)| \le E(U_\varepsilon + \tilde{W}) - E\tilde{W} = EU_\varepsilon$$

and

$$|Eh_{k,\varepsilon}(Z) - Eh_k(Z)| = |Eh_k(Z) - Eh_k(Z + U_\varepsilon)| \le E(U_\varepsilon + Z) - EZ = EU_\varepsilon,$$

which implies that

$$|Eh_k(\tilde{W}) - Eh_k(Z)| \le 2EU_\varepsilon + |Eh_{k,\varepsilon}(\tilde{W}) - Eh_{k,\varepsilon}(Z)| = \varepsilon + |Eh_{k,\varepsilon}(\tilde{W}) - Eh_{k,\varepsilon}(Z)|, \quad (4.2)$$

where we use the fact that $EU_\varepsilon = \varepsilon/2$. By Stein's equation (3.1), we observe that

$$Eh_{k,\varepsilon}(\tilde{W}) - Eh_{k,\varepsilon}(Z)$$

$$
\begin{aligned}
&= E[\tilde{W} f_{k,\varepsilon}(\tilde{W})] - E[f'_{k,\varepsilon}(\tilde{W})] \\
&= E[f'_{k,\varepsilon}(\tilde{W}^z)] - E[f'_{k,\varepsilon}(\tilde{W})] \qquad (4.3)\\
&= E\left[\Big(f'_{k,\varepsilon}(\tilde{W}^z) - f'_{k,\varepsilon}(\tilde{W})\Big)\mathbb{I}\left(|\tilde{W}^z - \tilde{W}| \le \frac{k}{8}\right)\right] + E\left[\Big(f'_{k,\varepsilon}(\tilde{W}^z) - f'_{k,\varepsilon}(\tilde{W})\Big)\mathbb{I}\left(|\tilde{W}^z - \tilde{W}| > \frac{k}{8}\right)\right], \qquad (4.4)
\end{aligned}
$$

where the indicator function $\mathbb{I}(A)$, for an event A, is defined by

$$
\mathbb{I}(A)(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}
$$

To bound the second term on the right-hand side of (4.4), we apply Chebyshev's inequality together with (3.6) to obtain

$$
\begin{aligned}
E\left[\Big|f'_{k,\varepsilon}(\tilde{W}^z) - f'_{k,\varepsilon}(\tilde{W})\Big|\mathbb{I}\left(|\tilde{W}^z - \tilde{W}| > \frac{k}{8}\right)\right] &\le 2\|f'_{k,\varepsilon}\|\mathbb{P}\left(|\tilde{W}^z - \tilde{W}| > \frac{k}{8}\right) \\
&\le \left(\frac{12.77}{k}\right) E|\tilde{W}^z - \tilde{W}|. \qquad (4.5)
\end{aligned}
$$

To bound the first term, we apply the mean value theorem, which gives that for some random variable C lying between $\tilde{W}^z$ and $\tilde{W}$,

$$
\begin{aligned}
E\left[\Big(f'_{k,\varepsilon}(\tilde{W}^z) - f'_{k,\varepsilon}(\tilde{W})\Big)\mathbb{I}\left(|\tilde{W}^z - \tilde{W}| \le \frac{k}{8}\right)\right] &= f''_{k,\varepsilon}(C) E\left[(\tilde{W}^z - \tilde{W})\mathbb{I}\left(|\tilde{W}^z - \tilde{W}| \le \frac{k}{8}\right)\right] \\
&= A + B, \qquad (4.6)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= f''_{k,\varepsilon}(C) E\left[\Big(\tilde{W}^z - \tilde{W}\Big)\mathbb{I}\left(|\tilde{W}^z - \tilde{W}| \le \frac{k}{8}\right)\mathbb{I}\left(C \le \frac{k}{2}\right)\right], \\
B &= f''_{k,\varepsilon}(C) E\left[\Big(\tilde{W}^z - \tilde{W}\Big)\mathbb{I}\left(|\tilde{W}^z - \tilde{W}| \le \frac{k}{8}\right)\mathbb{I}\left(C > \frac{k}{2}\right)\right].
\end{aligned}
$$

Applying Proposition 3.3 yields the following bound for $|A|$:

$$
|A| \le \|f''_{k,\varepsilon}\|_{(-\infty,\frac{k}{2})} E|\tilde{W}^z - \tilde{W}| \le \left(\frac{1.68}{k}\right) E|\tilde{W}^z - \tilde{W}|. \qquad (4.7)
$$

To bound $|B|$, we begin by observing that

$$
\begin{aligned}
\mathbb{I}\left(C > \frac{k}{2}\right)\mathbb{I}\left(|\tilde{W}^z - \tilde{W}| \le \frac{k}{8}\right)\mathbb{I}\Big(\tilde{W} \le C \le \tilde{W}^z\Big) &\le \mathbb{I}\left(\tilde{W}^z > \frac{k}{2}\right)\mathbb{I}\left(\tilde{W}^z - \tilde{W} \le \frac{k}{8}\right) \\
&\le \mathbb{I}\left(\tilde{W} > \frac{3k}{8}\right),
\end{aligned}
$$

and

$$
\mathbb{I}\left(C > \frac{k}{2}\right)\mathbb{I}\left(|\tilde{W}^z - \tilde{W}| \le \frac{k}{8}\right)\mathbb{I}\Big(\tilde{W}^z < C < \tilde{W}\Big) \le \mathbb{I}\left(\tilde{W} > \frac{k}{2}\right).
$$

These inequalities and (3.6) imply that

$$
\begin{aligned}
|B| &\le \|f''_{k,\varepsilon}\| E\left[\left|\tilde{W}^z-\tilde{W}\right|\mathbb{I}\left(|\tilde{W}^z-\tilde{W}|\le\frac{k}{8}\right)\mathbb{I}\left(C>\frac{k}{2}\right)\left(\mathbb{I}\left(\tilde{W}\le C\le\tilde{W}^z\right)+\mathbb{I}\left(\tilde{W}^z<C<\tilde{W}\right)\right)\right]\\
&\le 2E\left[\left|\tilde{W}^z-\tilde{W}\right|\left(\mathbb{I}\left(\tilde{W}>\frac{3k}{8}\right)+\mathbb{I}\left(\tilde{W}>\frac{k}{2}\right)\right)\right]\\
&\le 2\left[\mathbb{P}\left(\tilde{W}>\frac{3k}{8}\right)+\mathbb{P}\left(\tilde{W}>\frac{k}{2}\right)\right]^{\frac{1}{2}}(E|\tilde{W}^z-\tilde{W}|^2)^{\frac{1}{2}}, \qquad (4.8)
\end{aligned}
$$

where the Cauchy–Schwarz inequality is applied to obtain the last inequality. Combining the fact that

$$\mathbb{P}\left(\tilde{W}>\frac{3k}{8}\right)\le\frac{64E\tilde{W}^2}{9k^2}=\frac{64}{9k^2}$$

and

$$\mathbb{P}\left(\tilde{W}>\frac{k}{2}\right)\le\frac{4E\tilde{W}^2}{k^2}=\frac{4}{k^2}$$

with (4.8), we obtain

$$|B|\le\left(\frac{3.34}{k}\right)(E|\tilde{W}^z-\tilde{W}|^2)^{\frac{1}{2}}. \qquad (4.9)$$

Next, taking (4.2), (4.4), (4.5), (4.6), (4.7), and (4.9) together, we get

$$|Eh_k(\tilde{W})-Eh_k(\tilde{Z})|\le\varepsilon+\left(\frac{14.45}{k}\right)E|\tilde{W}^z-\tilde{W}|+\left(\frac{3.34}{k}\right)(E|\tilde{W}^z-\tilde{W}|^2)^{\frac{1}{2}}.$$

Letting $\varepsilon\to 0$ and using this bound along with Hölder's inequality completes the proof.

Remark 4.1. Following this argument, we can offer an alternative proof of a uniform version

$$d_{\mathrm{SL}}\left(\tilde{W},Z\right)\le 2E\left|\tilde{W}^z-\tilde{W}\right|. \qquad (4.10)$$

To establish this claim, we combine (4.2) with (4.3) and then apply the mean value theorem to obtain that

$$d_{\mathrm{SL}}(\tilde{W},Z)\le\varepsilon+\sup_{k\in\mathbb{R}^+}\left|Ef'_{k,\varepsilon}(\tilde{W}^z)-Ef'_{k,\varepsilon}(\tilde{W})\right|\le\varepsilon+\|f''_{k,\varepsilon}\|E\left|\tilde{W}^z-\tilde{W}\right|\le\varepsilon+2E\left|\tilde{W}^z-\tilde{W}\right|,$$

where we use (3.6) in the last inequality. Finally, we let $\varepsilon\to 0$ to complete the proof.

4.2. Proof of Theorem 2.4

Together, Theorem 2.3 and Proposition 3.4(*ii*) provide the desired bound.

4.3. Proof of Corollary 2.1

Denote that

$$X_i = Y_i - \mu.$$

We first claim that the sequence $X_1, X_2, \ldots$ satisfies the GM model with parameter ρ. Note that

$$\begin{aligned} E\left[e^{i(t_1X_1+\cdots+t_nX_n)}\right] &= e^{-i\mu(t_1+\cdots+t_n)}E\left[e^{i(t_1Y_1+\cdots+t_nY_n)}\right] \\ &= e^{-i\mu(t_1+\cdots+t_n)}\left[\rho E\left[e^{iY_1(t_1+\cdots+t_n)}\right] + (1-\rho)\prod_{j=1}^{n} E\left[e^{it_jY_j}\right]\right] \\ &= \rho E\left[e^{iX_1(t_1+\cdots+t_n)}\right] + (1-\rho)\prod_{j=1}^{n} E\left[e^{it_jX_j}\right] \end{aligned}$$

for all $n \in \mathbb{N}$ and for all real t_j ($j = 1, 2, \ldots, n$). Nevertheless, we need to determine the expression for $\mathrm{Var}(S - \mu N)$. Now, we have

$$\mathrm{Var}(S - \mu N) = \mathrm{Var}(S) + \mu^2 \mathrm{Var}(N) - 2\,\mathrm{Cov}(S, \mu N).$$

Since

$$E[NS] = \sum_{n=0}^{\infty} E\left(n\sum_{j=1}^{n} Y_j\right)\mathbb{P}(N = n) = \mu\sum_{n=0}^{\infty} n^2\mathbb{P}(N = n) = \mu EN^2$$

we obtain

$$\mathrm{Cov}(S, \mu N) = \mu E[NS] - \mu ENES = \mu^2 EN^2 - \mu^2 E^2N = \mu^2 \mathrm{Var}(N).$$

This implies that

$$\mathrm{Var}(S - \mu N) = \mathrm{Var}(S) - \mu^2 \mathrm{Var}(N).$$

Because X_j's have zero mean, Theorems 2.2 and 2.4 apply directly by noting that

$$\tilde{W} = \frac{W}{\sqrt{\mathrm{Var}(W)}} = \frac{S - \mu N}{\sqrt{\mathrm{Var}(S - \mu N)}} = \frac{S - \mu N}{\sqrt{\mathrm{Var}(S) - \mu^2 \mathrm{Var}(N)}}.$$

5. Applications

5.1. Original examples by Tallis

Tallis [1] first introduced the concept of the GM model and demonstrated its application in estimating the heritability and repeatability of fertility traits. When the random variables are interpreted as fertility records, the presence of common genetic or environmental influences naturally leads to correlated observations.

Example 5.1 (Heritability of Fertility Among Half-Siblings)**.** A classical application of the GM model appears in the estimation of the heritability of fertility in livestock. Suppose we observe the first lambing records resulting from multiple matings of a given sire. Let Y_j denote the fertility record of the j^{th} mating, typically representing the number of lambs born in a single birthing event. Because all fertility records Y_j arise from matings involving the same sire and are conducted under similar environmental and management conditions, each mating is governed by essentially the same genetic and environmental mechanisms. Consequently, the distribution of the fertility outcome does not vary across matings, making it natural to model the Y_j's as identically distributed. The total fertility output associated with that sire is then represented by the random sum S, which aggregates the reproductive performance across all matings.

To illustrate, consider a simple fertility model where each ewe may produce 0, 1, or 2 lambs with probabilities 0.3, 0.6, and 0.1, respectively, reflecting the fact that single births are most common, twins occur less frequently, and barren outcomes are possible. This discrete distribution captures the essential structure of fertility traits in sheep, where litters larger than two are rare. In this context, the number of mating N produced by a sire can vary due to biological factors such as mating success, conception rates, or chance of male versus female offspring. Modeling N as a binomial random variable, $N \sim Bin(m, r)$, is natural because each of m potential mating opportunities can independently result in a female offspring with probability r. This choice allows the GM model to realistically reflect both the variability in family size and the correlation structure among the daughters' fertility outcomes.

To apply Corollary 2.1, we define

$$X_i = Y_i - EY_i = Y_i - 0.8, \quad \text{and} \quad W = \sum_{i=1}^{N} X_i.$$

Then, we observe that $EX_1 = 0$, $E|X_1| = 0.48$, $EX_1^2 = \mathrm{Var}(X_1) = \mathrm{Var}(Y_1) = 0.36$, $E|X_1|^3 = 0.3312$, $E|X_1|^4 = 0.3312$, and $\mathrm{Cov}(X_1, X_2) = \mathrm{Cov}(Y_1, Y_2) = \rho\,\mathrm{Var}(Y_1) = 0.36\rho$. Since $EX_1 = 0$, we compute

$$\begin{aligned}\mathrm{Var}(W) = EW^2 &= EX_1^2 EN + \mathrm{Cov}(X_1, X_2)E[N(N-1)]\\ &= 0.36mr + 0.36m(m-1)r^2\rho.\end{aligned}$$

Let $X_1', X_2', \ldots$ be i.i.d. random variables with the same distribution as X_1, independent of N, and denote

$$W' = \sum_{i=1}^{N} X_i'.$$

Then, we have

$$\mathrm{Var}(W') = \mathrm{Var}(X_1')EN = \mathrm{Var}(X_1)EN = 0.36mr.$$

Moreover, let $B_1, B_2, \ldots, B_m$ be i.i.d. Bernoulli random variables with parameter r such that $N = \sum_{i=1}^{m} B_i$. Then, by (3.1), $N^s - 1 = N - B_U$, for some index U uniformly from $\{1, 2, \ldots, m\}$ independent of B_i's (see Example 3.1 for further details). Then, we note that

$$E\left|\sum_{j=1}^{N^s-1} X_j' - W'\right| = E\left[E\left[\left|\sum_{j=1}^{N-B_U} X_j' - W'\right|\middle| B_U\right]\right] = \mathbb{P}(B_U = 1)E|X_1| = 0.48r,$$

$$E\left|\sum_{j=1}^{N^s-1} X_j' - W'\right|^2 = E\left[E\left[\left|\sum_{j=1}^{N-B_U} X_j' - W'\right|^2\middle| B_U\right]\right] = \mathbb{P}(B_U = 1)E|X_1|^2 = 0.36r,$$

and if we assume that $\mathrm{Cov}(X'_j, X_k) = \frac{\mathrm{Var}(W')}{EN^2}$ for all j, k, we have

$$\begin{aligned} E[W'W] &= \sum_{n=0}^{\infty} \mathbb{P}(N=n) E\left(\sum_{j=1}^{n} \sum_{k=1}^{n} X'_j X_k \right) \\ &= \Big(\mathrm{Cov}(X'_1, X_1) + EX'_1 EX_1 \Big) \sum_{n=0}^{\infty} n^2 \mathbb{P}(N=n) \\ &= \mathrm{Cov}(X'_1, X_1) EN^2 \\ &= \mathrm{Var}(W') \end{aligned}$$

which implies that

$$E|W' - W|^2 = \mathrm{Var}(W') - 2E[W'W] + \mathrm{Var}(W) = \mathrm{Var}(W) - \mathrm{Var}(W') = 0.36m(m-1)r^2\rho.$$

Therefore, using the triangle inequality, we obtain

$$E\left| \sum_{j=1}^{N^s - 1} X'_j - W \right| \leq E\left| \sum_{j=1}^{N^s - 1} X'_j - W' \right| + E\left| W' - W \right| \leq 0.48r + \sqrt{0.36m(m-1)r^2\rho},$$

and

$$E\left| \sum_{j=1}^{N^s - 1} X'_j - W \right|^2 \leq 2E\left| \sum_{j=1}^{N^s - 1} X'_j - W' \right|^2 + 2E\left| W' - W \right|^2 = 0.72r + 0.72m(m-1)r^2\rho.$$

By Corollary 2.1, we have

$$d_{\mathrm{SL}}(\tilde{S}, Z) \leq 2\tau\delta_1 + 2(1-\tau)\delta_2,$$

and for $k \geq 4$,

$$d_{\mathrm{SL}}^{(k)}(\tilde{S}, Z) \leq \frac{35.58}{k} \left(\sqrt{\tau\delta_3} + \sqrt{(1-\tau)\delta_4} \right),$$

where

$$\begin{aligned} \tau &= \frac{\rho[mr(1-r) + (mr)^2]}{mr + \rho m(m-1)r^2} = \frac{\rho(1-r+mr)}{1+\rho(m-1)r}, \\ \delta_1 &= 1 + \frac{0.3312\Big(m(m-1)(m-2)r^3 + 3m(m-1)r^2 + mr\Big)}{0.72\Big(mr(1-r) + (mr)^2\Big)\sqrt{0.36mr + 0.36m(m-1)r^2\rho}}, \\ \delta_2 &= \frac{0.46 + 0.48r + \sqrt{0.36m(m-1)r^2\rho}}{\sqrt{0.36mr + 0.36m(m-1)r^2\rho}}, \\ \delta_3 &= 1 + \frac{0.3312\Big(m(m-1)(m-2)(m-3)r^4 + 6m(m-1)(m-2)r^3 + 7m(m-1)r^2\Big)}{1.08\Big(mr(1-r) + (mr)^2\Big)\Big(0.36mr + 0.36m(m-1)r^2\rho\Big)}, \end{aligned}$$

$$\delta_4 = \frac{0.3067 + 0.72r + 0.72m(m-1)r^2\rho}{0.36mr + 0.36m(m-1)r^2\rho}.$$

To illustrate a specific case, letting $\rho = \frac{1}{m^3}$ and $r = 0.5$, we have for $m \geq 2$,

$$\begin{aligned}
\tau &= \frac{m+1}{m^3 + 0.5(m-1)} < \frac{m}{m^3} + \frac{1}{m^3} \leq \frac{1}{m^2} + \frac{0.5}{m^2} = \frac{1.5}{m^2},\\
1-\tau &= \frac{m^3 - 0.5m - 1.5}{m^3 + 0.5(m-1)} \leq 1,\\
\delta_1 &= 1 + \frac{0.23m^2(m+3)}{(m+1)\sqrt{0.18m^3 + 0.09(m-1)}}\\
&< 1 + \frac{0.23}{m^2\sqrt{0.18m}} + \frac{0.69}{m^2\sqrt{0.18m}} \leq 1 + 0.5422\sqrt{m} + \frac{1.6264}{\sqrt{m}},\\
\delta_2 &= \frac{0.47m + 0.3\sqrt{m-1}}{\sqrt{0.18m^3 + 0.09(m-1)}} < \frac{0.47m}{m\sqrt{0.18m}} + \frac{0.3\sqrt{m}}{m\sqrt{0.18m}} \leq \frac{1.1079}{\sqrt{m}} + \frac{0.7072}{m},\\
\delta_3 &= 1 + \frac{23(m-1)(m^2+7m+10)m^2}{300\big(m+1\big)\big(0.18m^3 + 0.09(m-1)\big)} \leq 1 + 0.0138m + 0.0966 + \frac{0.138}{m},\\
\delta_4 &= \frac{0.6667m^2 - 0.18(m-1)}{0.18m^3 + 0.09(m-1)} \leq \frac{0.6667m^2}{0.18m^3} \leq \frac{3.7039}{m}.
\end{aligned}$$

Hence,

$$d_{\mathrm{SL}}(\tilde{S}, Z) \leq 2\left(\frac{1.1079}{\sqrt{m}} + \frac{0.7072}{m} + \frac{0.8133}{m\sqrt{m}} + \frac{1.5}{m^2} + \frac{2.4396}{m^2\sqrt{m}}\right),$$

and for $k \geq 4$,

$$d_{\mathrm{SL}}^{(k)}(\tilde{S}, Z) \leq \frac{35.58}{k}\sqrt{\frac{3.7246}{m} + \frac{1.6449}{m^2} + \frac{0.207}{m^3}}.$$

Furthermore, for $m \geq 2$, we may simplify it as follows:

$$d_{\mathrm{SL}}(\tilde{S}, Z) \leq \frac{6.3038}{\sqrt{m}} \quad \text{and} \quad d_{\mathrm{SL}}^{(k)}(\tilde{S}, Z) \leq \frac{76.3214}{k\sqrt{m}} \quad \text{for } k \geq 4.$$

This observation clearly indicates that when k is sufficiently large, the error bound for $d_{\mathrm{SL}}^{(k)}(\tilde{S}, Z)$ improves upon the bound for $d_{\mathrm{SL}}(\tilde{S}, Z)$.

5.2. *Examples in finance and insurance*

Let us connect the generalized multinomial (GM) model to finance and insurance, where correlated random variables naturally appear. Below are two strong examples, each illustrating how the GM structure (equicorrelated variables) can arise.

Example 5.2. Think of X_i as the profit or loss from a single trading day, where the position moves by $\pm m$ units for some $m \in \mathbb{R}^+$. Then,

$$W = X_1 + X_2 + \cdots + X_N$$

represents the total accumulated profit and loss after N days. The quantity $E(W-k)^+$ is the expected payoff of a contract that pays the amount by which the cumulative P&L exceeds a specified threshold k, and pays nothing if the cumulative P&L is below that level. For a more concrete illustration, suppose a trader holds 1 Bitcoin, and each day its price fluctuates such that the trader either gains $+1$ unit or loses -1 unit. Define

$$X_i = \begin{cases} +1, & \text{with probability 0.5 (a profitable day)}, \\ -1, & \text{with probability 0.5 (a losing day)}. \end{cases}$$

The trader is entitled to a bonus equal to the amount by which their cumulative profit exceeds a threshold of k unit. If the total profit does not exceed this level, no bonus is paid. The payoff can therefore be written as Bonus $= (W-k)^+$. In many real-world situations, the number of trades or price-changing events within a given period is itself random. To model this randomness in trading activity, it is common to assume that the number of events follows a Poisson distribution, which is widely used to describe the arrival pattern of trades or jumps in financial markets. In this context, we let $N \sim Poi(\lambda)$. Since the X_i all correspond to the same trader, it is reasonable to assume a constant correlation between them, i.e., $\mathrm{Corr}(X_i, X_j) = \rho$ for all $i \neq j$, and that they may follow a GM model. Before applying our result, we note that $EX_1 = 0$, $E|X_1| = EX_1^2 = \mathrm{Var}(X_1) = E|X_1|^3 = E|X_1|^4 = 1$, $\mathrm{Cov}(X_1, X_2) = \rho\,\mathrm{Var}(X_1) = \rho$, and

$$\mathrm{Var}(W) = EW^2 = EX_1^2 EN + \mathrm{Cov}(X_1, X_2)E[N(N-1)] = \lambda + \lambda^2\rho.$$

Let $X_1', X_2', \ldots$ be i.i.d. random variables with the same distribution as X_1, independent of N. If we assume that $\mathrm{Cov}(X_j', X_k) = \frac{1}{\lambda}$ for all j and $k \neq 1$, we have

$$\begin{aligned} E\left(\sum_{j=1}^{N} X_j' - \sum_{k=1}^{N} X_k\right)^2 &= E\left(\sum_{j=1}^{N} X_j'\right)^2 - 2E\left(\sum_{j=1}^{N}\sum_{k=1}^{N} X_j' X_k\right) + E\left(\sum_{j=1}^{N} X_j\right)^2 \\ &\leq \sqrt{2\,\mathrm{Var}(X_1)EN + \mathrm{Cov}(X_1, X_2)E[N(N-1)] - 2\,\mathrm{Cov}(X_1', X_2)E[N(N-1)]} \\ &= \sqrt{2\lambda + \lambda^2\rho - 2\lambda} \\ &= \lambda\sqrt{\rho}. \end{aligned}$$

Then, by Theorems 2.2 and 2.4, the expected bonus can be approximated as follows:

$$|E(\tilde{W}-k)^+ - E(Z-k)^+| \leq 2\tau\delta_1 + 2(1-\tau)\delta_2, \tag{5.1}$$

and for $k \geq 4$,

$$|E(\tilde{W}-k)^+ - E(Z-k)^+|p \leq \frac{35.58}{k}\left(\sqrt{\tau\delta_3} + \sqrt{(1-\tau)\delta_4}\right), \tag{5.2}$$

where

$$
\begin{aligned}
\tau =& \frac{(1+\lambda)\rho}{1+\lambda\rho},\\
\delta_1 =& 1+\frac{\lambda^2+3\lambda+1}{2(\lambda+1)\sqrt{\lambda(1+\lambda\rho)}}, \quad \delta_2 = \frac{1}{2\sqrt{\lambda(1+\lambda\rho)}}+\sqrt{\frac{\lambda\rho}{1+\lambda\rho}},\\
\delta_3 =& 1+\frac{\lambda^3+6\lambda^2+7\lambda+1}{3\lambda(\lambda+1)(1+\lambda\rho)}, \quad \delta_4 = \frac{1}{3\lambda(1+\lambda\rho)}+\frac{\lambda\rho}{1+\lambda\rho}.
\end{aligned}
$$

We observe that if

$$\lambda\sqrt{\rho} \to 0 \text{ and } \lambda \to \infty,$$

then $\rho \to 0$, $\lambda\rho \to 0$. These imply the bounds in (5.1) and (5.2) tend to 0. Furthermore, for clarity, we now set $\rho = \frac{1}{n^3}$ and $\lambda = n$ for some $n \in \mathbb{N}$. Under this choice, we obtain, for any $k > 0$,

$$
\begin{aligned}
&|E(\tilde{W}-k)^+ - E(Z-k)^+|\\
\leq& \frac{2(n+1)}{n(n^2+1)}\left(1+\frac{n^2+3n+1}{2(n+1)}\sqrt{\frac{n}{n^2+1}}\right)+\frac{2(n^3-1)}{n(n^2+1)}\left(\frac{1}{2}\sqrt{\frac{n}{n^2+1}}+\sqrt{\frac{1}{n^2+1}}\right)\\
\leq& \frac{2(1+1/n)}{n^2}\left(1+\frac{1}{2}\left(n+2-\frac{1}{n+1}\right)\frac{1}{\sqrt{n}}\right)+2\left(\frac{1}{2\sqrt{n}}+\frac{1}{n}\right)\\
\leq& \frac{4}{n^2}\left(1+\frac{\sqrt{n}}{2}+\frac{1}{\sqrt{n}}\right)+\frac{1}{\sqrt{n}}+\frac{2}{n}\\
<& \frac{1}{\sqrt{n}}+\frac{2}{n}+\frac{2}{n\sqrt{n}}+\frac{4}{n^2}+\frac{4}{n^2\sqrt{n}},
\end{aligned}
$$

and for $k \geq 4$,

$$
\begin{aligned}
&|E(\tilde{W}-k)^+ - E(Z-k)^+|\\
\leq& \frac{35.58}{k}\left\{\left(\frac{n+1}{n(n^2+1)}\right)^{\frac{1}{2}}\left(1+\frac{n^4+6n^3+7n^2+n}{3(n+1)(n^2+1)}\right)^{\frac{1}{2}}+\left(\frac{n^3-1}{n(n^2+1)}\right)^{\frac{1}{2}}\left(\frac{1}{3}\left(\frac{n}{n^2+1}\right)+\frac{1}{n^2+1}\right)^{\frac{1}{2}}\right\}\\
\leq& \frac{35.58}{k}\left\{\left(\frac{2}{n^2}\right)^{\frac{1}{2}}\left(1+\frac{n+6+7/n+1/n^2}{3}\right)^{\frac{1}{2}}+\left(\frac{1}{3n}+\frac{1}{n^2}\right)^{\frac{1}{2}}\right\}\\
\leq& \frac{35.58}{k}\left\{\left(\frac{2}{3n}+\frac{34}{3n^2}\right)^{\frac{1}{2}}+\left(\frac{1}{3n}+\frac{1}{n^2}\right)^{\frac{1}{2}}\right\}\\
<& \frac{35.58}{k}\left(\frac{1.3939}{\sqrt{n}}+\frac{4.3666}{n}\right).
\end{aligned}
$$

Example 5.3 (The collective risk model)**.** As noted earlier, we may interpret $S := \sum_{j=1}^{N} Y_j$ as the total aggregate claim, where each Y_j represents the size of the j^{th} claim. For example, in a car insurance contract, every Y_j corresponds to a payment made when the policyholder files a claim. The collective risk model is commonly used to describe such aggregate losses over a fixed period, typically one year. Moreover, if all claims originate from the same driver, their consistent driving behavior, such as caution, speed patterns, or risk-taking tendencies, introduces a common source of dependence across trips. This behavioral consistency can induce an approximately uniform correlation among claim amounts, making the GM model a natural way to capture the correlation structure in this context. The stop-loss (or call) function describes the amount paid by a reinsurer above a specified retention level. Specifically, for a retention $k > 0$, a stop-loss contract pays $(S-k)^+$ to the insured (or reinsurer) at the end of the year; that is, the insurer covers losses up to k, and the reinsurer pays any excess beyond that.

Let $Y_j \sim Exp(1)$. The exponential distribution offers a convenient and analytically tractable model for the size of car insurance claims: minor repair costs occur with high frequency, whereas severe accidents leading to large claim amounts are less common and naturally receive exponentially declining probability. We also let $N \sim Poi(\lambda)$ denote the number of claims, which is reasonable because claims are relatively rare events. Before applying our result, let $X_i = Y_i - EY_i$ and observe that $EX_1^2 = \mathrm{Var}(X_1) = \mathrm{Var}(Y_1) = 1$, $E|X_1|^3 < 2.42$, $EX_1^4 = 9$, and

$$\mathrm{Var}(W) = EW^2 = EX_1^2 EN + \mathrm{Cov}(X_1, X_2)E[N(N-1)] = \lambda + \lambda^2\rho.$$

Moreover, if we assume that $\mathrm{Cov}(X'_j, X_k) = \frac{1}{\lambda}$ for all j and $k \neq 1$, we have

$$E\left|\sum_{j=1}^{N^s-1} X'_j - \sum_{k=1}^{N} X_k\right| = E\left|\sum_{j=1}^{N}(X'_j - X_j)\right| = \lambda\sqrt{\rho}.$$

By Corollary 2.1, we have

$$d_{\mathrm{SL}}(\tilde{S}, Z) \leq 2\tau\left(1 + \frac{1.21(\lambda^2+3\lambda+1)}{(\lambda+1)\sqrt{\lambda(1+\lambda\rho)}}\right) + 2(1-\tau)\left(\frac{1.21}{\sqrt{\lambda(1+\lambda\rho)}} + \sqrt{\frac{\lambda\rho}{1+\lambda\rho}}\right),$$

and for $k \geq 4$,

$$d_{\mathrm{SL}}^{(k)}(\tilde{S}, Z) \leq \frac{35.58}{k}\left\{\sqrt{\tau}\left(1 + \frac{9(\lambda^3+6\lambda^2+7\lambda+1)}{3\lambda(\lambda+1)(1+\lambda\rho)}\right)^{\frac{1}{2}} + \sqrt{1-\tau}\left(\frac{9}{3\lambda(1+\lambda\rho)} + \frac{\lambda\rho}{1+\lambda\rho}\right)^{\frac{1}{2}}\right\},$$

where

$$\tau = \frac{(1+\lambda)\rho}{1+\lambda\rho}.$$

By an argument analogous to that in Example 5.2, we see that these error bounds converge to zero as $\lambda\sqrt{\rho} \to 0$ and $\lambda \to \infty$. As a specific illustration, consider the case $\rho = \frac{1}{n^3}$ and $\lambda = n$ for some $n \in \mathbb{N}$. In this setting, we have

$$\begin{aligned}
d_{\mathrm{SL}}(\tilde{S}, Z) &\leq \frac{2(n+1)}{n(n^2+1)}\left(1 + \frac{1.21(n^2+3n+1)}{(n+1)}\sqrt{\frac{n}{n^2+1}}\right) + \frac{2(n^3-1)}{n(n^2+1)}\left(1.21\sqrt{\frac{n}{n^2+1}} + \sqrt{\frac{1}{n^2+1}}\right) \\
&\leq \frac{4}{n^2}\left(1 + 1.21\left(n + 3 + \frac{1}{n}\right)\frac{1}{\sqrt{n}}\right) + 2\left(\frac{1.21}{\sqrt{n}} + \frac{1}{n}\right) \\
&\leq \left(\frac{1.21}{n\sqrt{n}} + \frac{4}{n^2} + \frac{19.36}{n^2\sqrt{n}}\right) + \left(\frac{2.42}{\sqrt{n}} + \frac{2}{n}\right)
\end{aligned}$$

$$\leq \frac{24.57}{n\sqrt{n}} + \frac{4.42}{\sqrt{n}},$$

and for $k \geq 4$,

$$\begin{aligned}
d_{\mathrm{SL}}^{(k)}(\tilde{S}, Z) &\leq \frac{35.58}{k}\left\{\left(\frac{n+1}{n(n^2+1)}\right)^{\frac{1}{2}}\left(1+\frac{3(n^4+6n^3+7n^2+n)}{(n+1)(n^2+1)}\right)^{\frac{1}{2}} + \left(\frac{n^3-1}{n(n^2+1)}\right)^{\frac{1}{2}}\left(3\left(\frac{n}{n^2+1}\right)+\frac{1}{n^2+1}\right)^{\frac{1}{2}}\right\} \\
&\leq \frac{35.58}{k}\left\{\left(\frac{2}{n^2}\right)^{\frac{1}{2}}\left(1+3(n+6+7/n+1/n^2)\right)^{\frac{1}{2}} + \left(\frac{3}{n}+\frac{1}{n^2}\right)^{\frac{1}{2}}\right\} \\
&\leq \frac{35.58}{k}\left\{\left(\frac{6}{n}+\frac{84}{n^2}\right)^{\frac{1}{2}} + \left(\frac{3}{n}+\frac{1}{n^2}\right)^{\frac{1}{2}}\right\} \\
&\leq \frac{35.58}{k}\left(\frac{9}{\sqrt{n}}+\frac{9.22}{n}\right).
\end{aligned}$$

6. Conclusions

In this work, we investigated both uniform and non-uniform bounds for the stop-loss distance between a random sum

$$W = X_1 + X_2 + \cdots + X_N$$

and the standard normal random variable Z. The stop-loss distance, expressed through the quantity

$$\left|E(W-k)^+ - E(Z-k)^+\right|,$$

plays a central role in finance and insurance, as $E(W-k)^+$ represents the expected excess of a loss or payoff above a specified threshold k. Accurate approximations of this quantity are therefore essential in applications such as risk assessment, pricing, and capital allocation.

Beyond the classical setting where the summands (X_j) are independent, our analysis extends to sequences (X_j) satisfying the GM model. This framework captures realistic forms of dependence, and we provide examples illustrating how the GM structure arises naturally in practical situations, thereby demonstrating its relevance.

Our approach relies heavily on Stein's method combined with biasing techniques, including the zero-bias and size-bias transformations. A key step in our analysis is the construction of an appropriate test function $h_{k,\varepsilon}$, which enables the application of the mean value theorem and leads to sharp non-uniform bounds. This technique also allows us to recover a special case of the classical result of Goldstein (2.4), thereby connecting our findings with established theory.

Finally, we present an applied example that highlights the practical significance of our bounds. In particular, we demonstrate how non-uniform bounds can yield substantially more informative estimates than uniform bounds when the threshold k varies, underscoring their importance in financial and insurance contexts.

Author contributions

Punyapat Kammoo: Visualization, writing – original draft, funding acquisition, formal analysis, validation , writing – review & editing; Kritsana Neammanee: Conceptualization, methodology, project administration, visualization, writing – original draft, formal analysis, validation, writing – review & editing; Kittipong Laipaporn: Conceptualization, methodology, project administration, funding acquisition, formal analysis, validation , writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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