Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(2): 4966–4984.
DOI: 10.3934/math.2026203
Received: 11 December 2025
Revised: 31 January 2026
Accepted: 09 February 2026
Published: 27 February 2026

Research article

Lifespan of the non-resistive Hall-MHD system with small magnetic gradient

Linbin Yang and Taoran Zhou*

School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

* **Correspondence:** Email: zhoutaoran@nuist.edu.cn.

Abstract: In this paper, we studied the non-resistive axially symmetric Hall-Magnetohydrodynamics (MHD) system. We showed that the lifespan of their strong solutions can be arbitrarily large if their initial magnetic gradient was small enough. Precise lifespan lower bounds for both viscid and inviscid cases were given.

Keywords: non-resistive; Hall-MHD equations; axially symmetric; small magnetic gradient; lifespan
Mathematics Subject Classification: 35Q35, 76B03, 76D03

1. Introduction and main results

The 3D non-resistive Hall-MHD (HMHD) system reads

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p - \mu \Delta \boldsymbol{u} = \frac{1}{\mu_0} \boldsymbol{h} \cdot \nabla \boldsymbol{h}, \\ \partial_t \boldsymbol{h} + \boldsymbol{u} \cdot \nabla \boldsymbol{h} + \nu_0 \nabla \times [(\nabla \times \boldsymbol{h}) \times \boldsymbol{h}] = \boldsymbol{h} \cdot \nabla \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{u} = 0, \\ \nabla \cdot \boldsymbol{h} = 0, \end{cases} \tag{HMHD}$$

with initial data

$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0(x), \text{ and } \boldsymbol{h}(0,x) = \boldsymbol{h}_0(x). \tag{1.1}$$

Here, $(\boldsymbol{u}, \boldsymbol{h}) : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ is the velocity and the magnetic field, respectively. $p : \mathbb{R}^3 \to \mathbb{R}$ represents the pressure. μ, μ_0, ν_0 stand for the constant viscosity, vacuum permeability, and ratio for the Hall effect.

The equations of MHD are widely used in describing many physical phenomena, such as fusion plasmas and star formation (see [5, 19]), as they play a crucial role in describing the large-scale interaction between conducting fluids and magnetic fields. The HMHD differs from the classical incompressible MHD, due to the Hall term $\nabla \times [(\nabla \times \boldsymbol{h}) \times \boldsymbol{h}]$ which accounts for the decoupling of electron and ion motions. Global well-posedness for the axisymmetric incompressible viscous and resistive HMHD equations was established by Fan et al. [6], and later Li and Liu established a global regularity for the viscous and resistive HMHD equations with low regularity axisymmetric data [11]. Recently, kinds of ill-posedness results of Hall- and electron-MHD systems were shown in [7] and Li provided the local well-posedness of a 3D ideal HMHD system with an azimuthal magnetic field [14].

In this paper, the coefficients μ_0, ν_0 do not play an essential role in the proof, so without loss of generality, we set $\mu_0 = \nu_0 = 1$. We consider the HMHD with the following cylindrical coordinates:

$$r = \sqrt{x_1^2 + x_2^2},\ \theta = \arctan\frac{x_2}{x_1},\ z = x_3.$$

Assume both $\boldsymbol{u}$ and $\boldsymbol{h}$ are axially symmetric, which is

$$\begin{cases} \boldsymbol{u} = u_r(t,r,z)\boldsymbol{e_r} + u_\theta(t,r,z)\boldsymbol{e_\theta} + u_z(t,r,z)\boldsymbol{e_z}, \\ \boldsymbol{h} = h_r(t,r,z)\boldsymbol{e_r} + h_\theta(t,r,z)\boldsymbol{e_\theta} + h_z(t,r,z)\boldsymbol{e_z}, \end{cases}$$

where the basis vectors $\boldsymbol{e_r}, \boldsymbol{e_\theta}, \boldsymbol{e_z}$ are

$$\boldsymbol{e_r} = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right),\ \boldsymbol{e_\theta} = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right),\ \boldsymbol{e_z} = (0,0,1).$$

From the local well-posedness result [14], it is clear that if the initial velocity $\boldsymbol{u_0} \cdot \boldsymbol{e_\theta}$ vanishes, then u_θ will vanish for all time (i.e., the no-swirl case). That is, by assuming $\boldsymbol{u_0} \cdot \boldsymbol{e_\theta} = h_r = h_z \equiv 0$, then $\boldsymbol{u}$ and $\boldsymbol{h}$ are constrained to the form:

$$\boldsymbol{u} = u_r\boldsymbol{e_r} + u_z\boldsymbol{e_z}, \quad \boldsymbol{h} = h_\theta\boldsymbol{e_\theta}.$$

Combining with the following result of $\nabla \times [(\nabla \times \boldsymbol{f}) \times \boldsymbol{f}]$ in cylindrical coordinate:

$$\begin{aligned} \nabla \times [(\nabla \times \boldsymbol{f}) \times \boldsymbol{f}] =& \partial_z(j_z f_r - j_r f_z)\boldsymbol{e_r} + (\partial_z(j_\theta f_z - j_z f_\theta) - \partial_r(j_r f_\theta - j_\theta f_r))\,\boldsymbol{e_\theta} \\ &+ \frac{1}{r}\partial_r\left(r(j_z f_r - j_r f_z)\right)\boldsymbol{e_z}, \end{aligned}$$

where

$$\begin{cases} \boldsymbol{f} = f_r(t,r,z)\boldsymbol{e_r} + f_\theta(t,r,z)\boldsymbol{e_\theta} + f_z(t,r,z)\boldsymbol{e_z}, \\ \boldsymbol{j} = \nabla \times \boldsymbol{f} = j_r(t,r,z)\boldsymbol{e_r} + j_\theta(t,r,z)\boldsymbol{e_\theta} + j_z(t,r,z)\boldsymbol{e_z}, \end{cases}$$

and

$$j_r = -\partial_z f_\theta, \qquad j_\theta = \partial_z f_r - \partial_r f_z, \qquad j_z = \frac{1}{r}\partial_r(r f_\theta),$$

one can easily rewrite the system (HMHD) in the following way:

$$\begin{cases} \partial_t u_r + (u_r\partial_r + u_r\partial_z)u_r + \partial_r p = \mu(\Delta - \frac{1}{r^2})u_r - \frac{(h_\theta)^2}{r}, \\ \partial_t u_z + (u_r\partial_r + u_z\partial_z)u_z + \partial_z p = \mu\Delta u_z, \\ \partial_t h_\theta + (u_r\partial_r + u_z\partial_z)h_\theta - \frac{h_\theta u_r}{r} = \frac{\partial_z(h_\theta)^2}{r}, \\ \nabla \cdot \boldsymbol{u} = \partial_r u_r + \frac{u_r}{r} + \partial_z u_z = 0, \end{cases} \tag{1.2}$$

where $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$.

In this case, the vorticity $\boldsymbol{w}$ of the axially symmetric vector $\boldsymbol{u}$ is given by

$$\boldsymbol{w} = \operatorname{curl} \boldsymbol{u} = w_\theta \boldsymbol{e_\theta},$$

where

$$w_\theta = \partial_z u_r - \partial_r u_z.$$

From (1.2) and the initial data (1.1), one can deduce that $\boldsymbol{w}$ satisfies:

$$\partial_t w_\theta + (u_r\partial_r + u_z\partial_z)w_\theta = \frac{u_r}{r}w_\theta - \frac{1}{r}\partial_z(h_\theta)^2 + \mu(\Delta - \frac{1}{r^2})w_\theta. \tag{1.3}$$

Recently, the HMHD equations have received significant attention from mathematicians due to their crucial role in various plasma phenomena. Significant efforts have been dedicated to understanding their well-posedness and regularity. Chae, Degond, and Liu in [2] established the global existence of weak solutions and the local well-posedness for smooth solutions in Sobolev space $H^s(\mathbb{R}^3)$ with $s > 5/2$. Subsequently, Dai [4] extended the local well-posedness theory to n-dimensional case, in the space $H^s(\mathbb{R}^n)$ with $s > n/2$, for $n \geq 2$. For the non-resistive HMHD system, Chae and Weng in [3] showed it is not globally well-posed in any Sobolev space $H^s(\mathbb{R}^3)$ with $s > 7/2$, even for smooth initial data. More recently, Li and Yang in [15] provided a single-component regularity criterion for the non-resistive axially symmetric HMHD system, demonstrating that the regularity is correlated with a Beale-Kato-Majda (BKM) type condition on the reformulated magnetic quantity $\mathcal{H} := \frac{h_\theta}{r}$.

Previous studies have established fundamental and significant insights into both resistive and non-resistive HMHD equations, establishing criteria for either global regularity or finite-time blowup. However, quantitative estimates for the lifespan of strong solutions remain a relatively underdeveloped area. In the past few years, research related to the lifespan for PDEs (Partial Differential Equations) of fluid dynamics has attracted considerable attention. Recently, Li-Zhou [16] gave an exact lifespan lower bound for axisymmetric incompressible Euler equations with a small swirl. For more related topics, see [1, 8, 13] and references therein.

Our proof of main theorems is carried out mainly in the following two quantities:

$$\Omega := \frac{w_\theta}{r}, \quad \mathcal{H} := \frac{h_\theta}{r}.$$

Using the first three equations of (1.2), one finds that the $(\Omega, \mathcal{H})$-system satisfies

$$\begin{cases} \partial_t\Omega + \boldsymbol{u}\cdot\nabla\Omega = \mu\left(\Delta + \frac{2}{r}\partial_r\right)\Omega - \partial_z\mathcal{H}^2, \\ \partial_t\mathcal{H} + (u_r\partial_r + u_z\partial_z)\mathcal{H} - 2\mathcal{H}\partial_z\mathcal{H} = 0. \end{cases} \tag{1.4}$$

1.1. Notations

For the derivation that follows, we list the notations that will be used throughout the paper.

- We use standard notations for Lebesgue and Sobolev functional spaces in $\mathbb{R}^3$: for $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, L^p denotes the Lebesgue space with norm

$$\|f\|_{L^p} := \begin{cases} \left(\int_{\mathbb{R}^3} |f(x)|^p \, \mathrm{d}x\right)^{1/p} & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^3} |f(x)| & p = \infty, \end{cases}$$

which can be rewritten by:

$$\|f\|_{L^p} := \begin{cases} \left(2\pi \int_0^\infty \int_{-\infty}^\infty |f(r,z)|^p r \, \mathrm{d}r \, \mathrm{d}z\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{(r,z)\in\mathbb{R}^+\times\mathbb{R}} |f(r,z)|, & p = \infty \end{cases}$$

in the cylindrical coordinates (r, z) for axially symmetric functions.

- We write $A \lesssim B$ to indicate that $A \le CB$ for some constant $C > 0$. Similarly, $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$.
- The commutator of two operators $\mathcal{A}$ and $\mathcal{B}$ is defined as $[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$.
- The notation $C_{a,b,...}$ represents a positive constant that depends on the parameters $a, b, \ldots$, and its value may vary from one occurrence to another.
- $\nabla^{\mathfrak{M}}$ denotes the following compound gradient operator in cylindrical coordinates:

$$\nabla^{\mathfrak{M}} := \partial_z^{m_z} \partial_r^{m_r} \left(\frac{\partial_r}{r}\right)^{m_c}, \tag{1.5}$$

  where $\mathfrak{M}$ is a 3D multi-index such that $\mathfrak{M} = (m_c, m_r, m_z)$ with $m_c, m_r, m_z \in \mathbb{N} \cup \{0\}$, and $|\mathfrak{M}| = 2m_c + m_r + m_z$. Also, we denote $\overline{\mathfrak{M}} = (m_r, m_z)$, with $|\overline{\mathfrak{M}}| = m_r + m_z$.
- $W^{k,p}$ denotes the usual Sobolev space with its norm

$$\|f\|_{W^{k,p}} := \sum_{0 \le |\mathfrak{M}| \le k} \left\|\nabla^{\mathfrak{M}} f\right\|_{L^p}.$$

  We also simply denote H^k and $\dot{H}^k$ instead of $W^{k,p}$ and $\dot{W}^{k,p}$, provided $p = 2$.

1.2. Main results

Now we are ready to show the main results of the current paper. First, we consider the viscous case. We set the viscous coefficient $\mu = 1$ without loss of generality. The results are as follows:

Theorem 1.1 (viscid case)**.** *Let $(\boldsymbol{u}, \boldsymbol{h})$ be a local smooth axially symmetric solution of system* (HMHD) *with the initial data $(\boldsymbol{u_0}, \boldsymbol{h_0}, \mathcal{H}_0) \in ((H^3)(\mathbb{R}^3))^7$, satisfied $\nabla \cdot \boldsymbol{u_0} = \boldsymbol{u_0} \cdot \boldsymbol{e_\theta} = h_r = h_z \equiv 0$. Here, $\mathcal{H}_0$ stands for the initial data of $\mathcal{H}$, that is, $\mathcal{H}_0 = \frac{h_{0,\theta}}{r}$. Suppose*

$$\|\nabla \mathcal{H}_0\|_{L^\infty} = \varepsilon \ll 1,$$

then the solution $(\boldsymbol{u}, \boldsymbol{h})$ keeps in H^3 when $t \le T_*$, where T_* satisfies:

$$T_* = \frac{C_*}{(1 + E_0)^{\frac{4}{5}}} (\log(\log(\varepsilon^{-1})))^{4/5}.$$

Here, C_* is a constant, and $E_0 := \|(\boldsymbol{u_0}, \boldsymbol{h_0}, \mathcal{H}_0)\|_{H^3}$.

Remark 1.2. *Roughly speaking, Theorem 1.1 indicates the lifespan of system* (1.2) *can be arbitrarily large if the initial quantity $\mathcal{H}_0$ is close enough to a constant.*

Furthermore, we consider axially symmetric solutions of system (HMHD) without the viscous term ($\mu = 0$). For convenience, we rewrite the system (1.2) as follows:

$$\begin{cases} \partial_t u_r + (u_r\partial_r + u_z\partial_z)\, u_r + \partial_r p = -\frac{(h_\theta)^2}{r}, \\ \partial_t u_z + (u_r\partial_r + u_z\partial_z)\, u_z + \partial_z p = 0, \\ \partial_t h_\theta + (u_r\partial_r + u_z\partial_z)\, h_\theta - \frac{h_\theta u_r}{r} = \frac{\partial_z (h_\theta)^2}{r}, \\ \nabla\cdot \boldsymbol{u} = \partial_r u_r + \frac{u_r}{r} + \partial_z u_z = 0. \end{cases} \tag{1.6}$$

By the first three equations of (1.6), one deduces that $w_\theta = \partial_z u_r - \partial_r u_z$ satisfies

$$\partial_t w_\theta + (u_r\partial_r + u_z\partial_z)\, w_\theta = \frac{u_r}{r} w_\theta - \frac{1}{r}\partial_z\, (h_\theta)^2\,,$$

and the inviscid reformulated $(\Omega, \mathcal{H})$-system follows

$$\begin{cases} \partial_t \Omega + \boldsymbol{u}\cdot\nabla\Omega = -\partial_z \mathcal{H}^2, \\ \partial_t \mathcal{H} + (u_r\partial_r + u_z\partial_z)\mathcal{H} - 2\mathcal{H}\partial_z\mathcal{H} = 0. \end{cases}$$

Compared to the viscid system, the lower bound on lifespan of the inviscid system has more logarithmic factor. The result is given in the following:

Theorem 1.3 (inviscid case)**.** *Let* $(\boldsymbol{u}, \boldsymbol{h})$ *be a local smooth axially symmetric solution of* (HMHD) ($\mu = 0$) *with the initial data* $(\boldsymbol{u_0}, \boldsymbol{h_0}, \mathcal{H}_0) \in ((H^3)(\mathbb{R}^3))^7$, *satisfied* $\nabla\cdot\boldsymbol{u_0} = \boldsymbol{u_0}\cdot\boldsymbol{e_\theta} = h_r = h_z \equiv 0$. *Suppose*

$$\|\nabla \boldsymbol{h}_0\|_{L^\infty} + \|\nabla\mathcal{H}_0\|_{L^\infty} = \varepsilon \ll 1, \tag{1.7}$$

then the solution $(\boldsymbol{u}, \boldsymbol{h})(t, \cdot)$ *keeps in* H^3 *when* $t \le T_*$, *where* T_* *satisfies:*

$$T_* = \frac{C_*}{1+E_0}\log\left(\log\left(\log\left(\log\left(\varepsilon^{-1}\right)\right)\right)\right).$$

Here, C_* *is a constant, and* $E_0 := \|(\boldsymbol{u_0}, \boldsymbol{h_0}, \mathcal{H}_0)\|_{H^3}$.

Remark 1.4. *We give an exact nontrivial example of the initial magnetic field that satisfies* (1.7) *since the condition looks strict. It is inspired by the design of Tokamaks [20]. For* a, $M > 0$, *let* $\boldsymbol{h}_0 = h_{\theta,0}\boldsymbol{e_\theta}$, *where*

$$h_{\theta,0} = M\phi\left(\frac{2r}{a} - 3\right).$$

Here, ϕ *is the standard smooth bump function:*

$$\phi(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| \ge 1, \end{cases}$$

and, thus, $h_{\theta,0}$ *is supported on* $r \in [a, 2a]$. *Direct calculation shows*

$$\begin{aligned} \partial_r h_{\theta,0} &= \frac{2M}{a}\phi'\left(\frac{2r}{a} - 3\right), \\ \partial_r\left(\frac{h_{\theta,0}}{r}\right) &= \frac{2M}{ar}\phi'\left(\frac{2r}{a} - 3\right) - \frac{M}{r^2}\phi\left(\frac{2r}{a} - 3\right). \end{aligned}$$

This indicates

$$\|\nabla\boldsymbol{h_0}\|_{L^\infty} \simeq \frac{M}{a}, \quad \|\nabla\mathcal{H}_0\|_{L^\infty} \lesssim \frac{M}{a^2}.$$

Thus, by choosing $a = CM\varepsilon^{-1}$, *where* $C > 0$ *is a universal constant, the condition* (1.7) *holds.*

Remark 1.5. *By choosing $M >> 1$, the size of magnetic field $\boldsymbol{h_0}$ constructed in Remark 1.4 can be arbitrarily large.*

Remark 1.6. *Inspired by recent work [21] on global solutions to the 3D MHD equations with fractional dissipation, we believe our paper might potentially be extended to compressible axisymmetric systems and incompressible systems with fractional dissipation (such as those involving the fractional Laplacian operator $(-\Delta)^{\alpha}$, $\alpha > 0$). These will be addressed in our future research.*

The rest of this paper is organized as follows. To prepare to give a precise lower bound of lifespan of the non-resistive axially symmetric HMHD system, we first compile some essential preliminaries. These include key lemmas on interpolation inequalities, commutator estimates, and the fundamental energy estimate for the system in Section 2. With these tools in hand, the main result of the viscid case is proved in Section 3, and the main result of the inviscid case is proved in Section 4.

2. Preliminaries

To begin with, we introduce some useful lemmas which will be frequently used in the proof of the main theorem.

We first present the Gagliardo-Nirenberg interpolation inequality, the proof of which can be found in [18].

Lemma 2.1 (Gagliardo-Nirenberg)**.** *Fix $q, r \in [1, \infty]$ and $j, m \in \mathbb{N} \cup \{0\}$ with $j \leq m$. Suppose that $f \in L^q \cap \dot{W}^{m,r}$, and there exists a real number $\alpha \in [j/m, 1]$ such that*

$$\frac{1}{p} = \frac{j}{3} + \alpha\left(\frac{1}{r} - \frac{m}{3}\right) + \frac{1-\alpha}{q}.$$

Then, $f \in \dot{W}^{j,p}$, and there exists a constant $C > 0$ such that

$$\left\|\nabla^j f\right\|_{L^p} \leq C \left\|\nabla^m f\right\|_{L^r}^{\alpha} \left\|f\right\|_{L^q}^{1-\alpha},$$

except the following two cases:

(i) $j = 0, mr < 3$ and $q = \infty$; (In this case it is necessary to assume also that either $f \to 0$ at infinity, or $f \in L^s$ for some $s < \infty$).

(ii) $1 < r < \infty$ and $m - j - 3/r \in \mathbb{N}$; (In this case it is necessary to assume also that $\alpha < 1$).

Next, we present the following estimate for the triple product form that will be frequently used in the final proof.

Lemma 2.2. *Let $m \in \mathbb{N}$ and $m \geq 2, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{k} \in C_0^{\infty}\left(\mathbb{R}^3\right)$. The following estimates hold:*

$$\left|\int_{\mathbb{R}^3} \left[\nabla^m, \boldsymbol{f} \cdot \nabla\right] \boldsymbol{g} \nabla^m \boldsymbol{k} dx\right| \leq C \left\|\nabla^m(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{k})\right\|_{L^2}^2 \left\|\nabla(\boldsymbol{f}, \boldsymbol{g})\right\|_{L^\infty}. \tag{2.1}$$

Proof. We apply the Hölder inequality, and one derives

$$\left|\int_{\mathbb{R}^3} \left[\nabla^m, \boldsymbol{f} \cdot \nabla\right] \boldsymbol{g} \nabla^m \boldsymbol{k} dx\right| \leq \left\|\left[\nabla^m, \boldsymbol{f} \cdot \nabla\right] \boldsymbol{g}\right\|_{L^2} \left\|\nabla^m \boldsymbol{k}\right\|_{L^2}. \tag{2.2}$$

Due to the commutator estimate by Kato-Ponce [9], it follows that

$$\left\|[\nabla^m, \boldsymbol{f}\cdot\nabla]\,\boldsymbol{g}\right\|_{L^2} \le C\left(\|\nabla \boldsymbol{f}\|_{L^\infty}\|\nabla^m \boldsymbol{g}\|_{L^2} + \|\nabla \boldsymbol{g}\|_{L^\infty}\|\nabla^m \boldsymbol{f}\|_{L^2}\right). \tag{2.3}$$

Then, (2.1) follows from substituting (2.3) in (2.2). □

The following is the fundamental energy estimate and the L^p conservation of $\mathcal{H}$.

Lemma 2.3. *Let* $(\boldsymbol{u}, \boldsymbol{h})$ *be a local smooth axially symmetric solution of the system* (HMHD) *on* $t \in [0, T)$ *with the initial data* $(\boldsymbol{u_0}, \boldsymbol{h_0}, \mathcal{H}_0) \in \left((H^3)(\mathbb{R}^3)\right)^7$, *satisfied* $\nabla\cdot\boldsymbol{u_0} = \boldsymbol{u_0}\cdot\boldsymbol{e_\theta} = h_r = h_z \equiv 0$, *then we have the following states in different case: For* $p \in [2, \infty]$ *and* $t \in (0, \infty)$, *the viscid case ensures the following estimate holds:*

$$\|(\boldsymbol{u}, \boldsymbol{h})(t, \cdot)\|_{L^2}^2 + 2\int_0^t \|\nabla \boldsymbol{u}(s, \cdot)\|_{L^2}^2 ds \le \|\boldsymbol{u_0}, \boldsymbol{h_0}\|_{L^2}^2. \tag{2.4}$$

In the inviscid case, (2.4) *reduces to*

$$\|(\boldsymbol{u}, \boldsymbol{h})(t, \cdot)\|_{L^2}^2 \le \|(\boldsymbol{u_0}, \boldsymbol{h_0})\|_{L^2}^2. \tag{2.5}$$

Both viscid case and inviscid case hold

$$\|\mathcal{H}(t, \cdot)\|_{L^p} = \|\mathcal{H}_0\|_{L^p}. \tag{2.6}$$

Proof. Inequalities (2.4) and (2.5) follow from the standard L^2 energy estimate of the system (1.2). By $(1.2)_3$ in different cases, we have the same estimate of L^p conservation of $\mathcal{H}$. We omit all the details here. □

Not influenced by the loss of the viscous term, the next three lemmas are devoted to some basic estimates of the system (HMHD). These lemmas will be used both in the proof of Theorem 1.1 and Theorem 1.3.

Lemma 2.4. *Let* $(\boldsymbol{u}, \boldsymbol{h})$ *be a local smooth axially symmetric solution of the system* (HMHD) *on* $t \in [0, T)$ *with the initial data* $(\boldsymbol{u_0}, \boldsymbol{h_0}, \mathcal{H}_0) \in \left((H^3)(\mathbb{R}^3)\right)^7$, *satisfied* $\nabla\cdot\boldsymbol{u_0} = \boldsymbol{u_0}\cdot\boldsymbol{e_\theta} = h_r = h_z \equiv 0$. *One has the following* $L_t^\infty L^p$ *estimates of* h_θ *and* w_θ, *for all* $p \in (1, \infty]$:

$$\|h_\theta(t, \cdot)\|_{L^p} \le \|\boldsymbol{h_0}\|_{L^p}\exp\left(C\int_0^t \left[\|\frac{u_r}{r}(s, \cdot)\|_{L^\infty} + \|\partial_z\mathcal{H}(s, \cdot)\|_{L^\infty}\right]ds\right),$$

$$\|w_\theta(t, \cdot)\|_{L^p} \le (\|\boldsymbol{w_0}\|_{L^p} + \|h_\theta\|_{L_t^\infty L^p}\|\partial_z\mathcal{H}\|_{L_t^1 L^\infty}) \times \exp\left(\int_0^t \|\frac{u_r}{r}(s, \cdot)\|_{L^\infty} ds\right).$$

The next lemma states the $L^p(1 < p < \infty)$ norm of $\nabla\frac{v_r}{r}$ could be controlled by the L^p norm of $\frac{w_\theta}{r}$, whose proof could be found in ([17], Proposition 2.5) and ([10] Equation A.5).

Lemma 2.5. *Define* $\Omega := \frac{w_\theta}{r}$. *For* $1 < p < \infty$, *there exists an absolute constant* $C_p > 0$ *such that*

$$\left\|\nabla\frac{u_r}{r}(t, \cdot)\right\|_{L^p} \le C_p\|\Omega(t, \cdot)\|_{L^p}.$$

Next, we present several relevant estimates, which will be employed both in the proof of Theorem 1.1 and in the proof of Theorem 1.3.

Lemma 2.6. *Under the same assumptions as Lemma 2.4, one has the following estimates of* $\|\boldsymbol{u}, \boldsymbol{h}, \mathcal{H}\|_{H^3}^2$, $\|\nabla \boldsymbol{h}\|_{L^\infty}$, *and* $\|\nabla\mathcal{H}\|_{L^\infty}$*:*

$$\|\nabla \boldsymbol{h}(t,\cdot)\|_{L^\infty} \le \|\nabla \boldsymbol{h_0}\|_{L^\infty} \exp\left(C\int_0^t (\|\nabla \boldsymbol{u}(s,\cdot)\|_{L^\infty} + \|\partial_z\mathcal{H}(s,\cdot)\|_{L^\infty})ds\right), \tag{2.7}$$

$$\|\nabla \mathcal{H}(t,\cdot)\|_{L^\infty} \le \|\nabla \mathcal{H}_0\|_{L^\infty} \exp\left(C\int_0^t (\|\nabla \boldsymbol{u}(s,\cdot)\|_{L^\infty} + \|\partial_z\mathcal{H}(s,\cdot)\|_{L^\infty})ds\right), \tag{2.8}$$

$$\|(\boldsymbol{u}, \boldsymbol{h}, \mathcal{H})(t,\cdot)\|_{H^3}^2 \le \|\boldsymbol{u_0}, \boldsymbol{h_0}, \mathcal{H}_0\|_{H^3}^2 \exp\left(C\int_0^t \|\nabla(\boldsymbol{u}, \boldsymbol{h}, \mathcal{H})(t,\cdot)\|_{L^\infty} ds\right). \tag{2.9}$$

Proof. At the beginning, we notice that

$$|\nabla \boldsymbol{h}| \simeq |\partial_r h_\theta| + |\partial_z h_\theta| + |\mathcal{H}|.$$

Since the $L_t^\infty L^\infty$ -estimate of $\mathcal{H}$ has already derived in Lemma 2.3, one only focuses on the rest two above. Acting $\bar\nabla = (\partial_r, \partial_z)$ on $(1.2)_3$, respectively, and performing L^p $(2 \le p < \infty)$ energy estimates on each resulting equation, by integration by parts and Hölder's inequality, one deduces

$$\frac{d}{dt}\left\|\bar\nabla h_\theta(t,\cdot)\right\|_{L^p}^p \le Cp\left\|\bar\nabla h_\theta(t,\cdot)\right\|_{L^p}^{p-1} (\|\nabla \boldsymbol{u}(t,\cdot)\|_{L^\infty} + \|\partial_z\mathcal{H}(t,\cdot)\|_{L^\infty}) \times \left(\left\|\bar\nabla h_\theta(t,\cdot)\right\|_{L^p} + \|\mathcal{H}(t,\cdot)\|_{L^p}\right). \tag{2.10}$$

Canceling $p\left\|\bar\nabla h_\theta(t,\cdot)\right\|_{L^p}^{p-1}$ on each sides of (2.10), noting $\frac{d}{dt}\|\mathcal{H}(t,\cdot)\|_{L^p} \equiv 0$ from Lemma 2.3, and applying one Grönwall's inequality, one arrives at

$$\|\nabla \boldsymbol{h}(t,\cdot)\|_{L^p} \le \|\nabla \boldsymbol{h_0}\|_{L^p} \exp\left(C\int_0^t (\|\nabla \boldsymbol{u}(s,\cdot)\|_{L^\infty} + \|\partial_z\mathcal{H}(s,\cdot)\|_{L^\infty})\, ds\right).$$

Note that the constant C above is independent with $p \in [2,\infty)$. Let $p \to \infty$, and one concludes the estimate (2.7).

Next, we present a concise derivation of the estimate (2.8). Acting ∂_r on both sides of equation

$$\partial_z\mathcal{H} + (u_r\partial_r + u_z\partial_z)\mathcal{H} - 2\mathcal{H}\partial_z\mathcal{H} = 0,$$

followed by multiplication with $p\partial_r\mathcal{H}|\mathcal{H}|^{p-2}$, and subsequent integration over $\mathbb{R}^3$ combining with integration by parts, one derives at

$$\frac{d}{dt}\|\partial_r\mathcal{H}(t,\cdot)\|_{L^p}^p \le 2(p-1)\int_{\mathbb{R}^3} \partial_z\mathcal{H}|\partial_r\mathcal{H}|^p dx + Cp\int_{\mathbb{R}^3} |\nabla \boldsymbol{u}||\nabla\mathcal{H}||\partial_r\mathcal{H}|^{p-1}dx. \tag{2.11}$$

Applying Hölder's inequality to the above equality, one arrives

$$\frac{d}{dt}\|\partial_r\mathcal{H}(t,\cdot)\|_{L^p}^p \lesssim p(\|\nabla \boldsymbol{u}(t,\cdot)\|_{L^\infty} + \|\partial_z\mathcal{H}(t,\cdot)\|_{L^\infty})\|\nabla\mathcal{H}(t,\cdot)\|_{L^p}^p. \tag{2.12}$$

Similarly, one has

$$\frac{d}{dt}\|\partial_z\mathcal{H}(t,\cdot)\|_{L^p}^p \lesssim p(\|\nabla\boldsymbol{u}(t,\cdot)\|_{L^\infty} + \|\partial_z\mathcal{H}(t,\cdot)\|_{L^\infty})\|\nabla\mathcal{H}(t,\cdot)\|_{L^p}^p. \tag{2.13}$$

Combining (2.12) and (2.13), cancelling $p\|\nabla\mathcal{H}(t,\cdot)\|_{L^p}^{p-1}$ on each side of the inequality, one concludes the estimate (2.8).

Finally, we give a concise proof of the higher-order bounds of (1.2) and (1.6). The key of the proof is performing energy estimates of the system together with the equation of $\mathcal{H}$; in the viscid case, it means

$$\begin{cases} \partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p - \Delta\boldsymbol{u} = \boldsymbol{h}\cdot\nabla\boldsymbol{h}, \\ \partial_t\boldsymbol{h} + \boldsymbol{u}\cdot\nabla\boldsymbol{h} - \boldsymbol{h}\cdot\nabla\boldsymbol{u} = 2\mathcal{H}\partial_z\boldsymbol{h}, \\ \partial_t\mathcal{H} + \boldsymbol{u}\cdot\nabla\mathcal{H} - 2\mathcal{H}\partial_z\mathcal{H} = 0. \end{cases} \tag{2.14}$$

Applying ∇^3 to three equations in (2.14), performing the L^2 inner product of the resulting equations with $\nabla^3\boldsymbol{u}$, $\nabla^3\boldsymbol{h}$, and $\nabla^3\mathcal{H}$, respectively, noting that $\boldsymbol{h}$ is divergence-free, and applying Lemma 2.2 to the remaining terms of the functions, we can finally arrive at

$$\frac{d}{dt}\|(\boldsymbol{u},\boldsymbol{h},\mathcal{H})(t,\cdot)\|_{\dot{H}^3}^2 \leq C\|\nabla(\boldsymbol{u},\boldsymbol{h},\mathcal{H})(t,\cdot)\|_{L^\infty}\|(\boldsymbol{u},\boldsymbol{h},\mathcal{H})(t,\cdot)\|_{\dot{H}^3}^2.$$

Combining this with the fundamental energy estimate (2.4) and the L^p conservation law of $\mathcal{H}$ (2.6), thus the energy estimate of the system is proved by applying Grönwall's inequality, which indicates

$$\|(\boldsymbol{u},\boldsymbol{h},\mathcal{H})(t,\cdot)\|_{H^3}^2 \leq \|(\boldsymbol{u_0},\boldsymbol{h_0},\mathcal{H}_0)\|_{H^3}^2 \exp\left(C\int_0^t \|\nabla(\boldsymbol{u},\boldsymbol{h},\mathcal{H})(s,\cdot)\|_{L^\infty}ds\right).$$

In the inviscid case, the only difference that the missing of Δu brings is we lose a positive term $\left\|\nabla^4 u(t,\cdot)\right\|_{L^2}^2$ on the left-hand side of the energy estimate equation, which makes no influence to the result of the higher-order bound estimate. More details of the proof refers to [15]. □

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. First, we establish the $L_t^\infty L^\infty$ -boundedness of the quantity $\frac{u_r}{r}$. Second, we derive an L^∞ estimate for $\nabla\boldsymbol{u}$. Having completed these preliminary analyses, we proceed to present the proof of Theorem 1.1, which constitutes the final component of our argument.

3.1. $L_t^\infty L^\infty$ -Boundedness of $\frac{u_r}{r}$

We first present the estimate for $\frac{u_r}{r}$ in the form of a proposition and provide a brief proof.

Proposition 3.1. *Define $\Omega := \frac{w_\theta}{r}$. Assume that $\nabla\cdot\boldsymbol{u_0} = \boldsymbol{u_0}\cdot\boldsymbol{e_\theta} = h_r = h_z \equiv 0$. Let $(\boldsymbol{u},\boldsymbol{h})$ be the unique local axially symmetric solution of* (HMHD) *on $t\in[0,T)$ with the initial data $(\boldsymbol{u_0},\boldsymbol{h_0})\in H^m (m\geq 3)$. Then, the following estimate of $\frac{u_r}{r}$ holds uniformly:*

$$\int_0^t \|\frac{u_r}{r}(s,\cdot)\|_{L^\infty}ds \lesssim t^{3/4}(\|\Omega_0\|_{L^2} + t^{1/2}\|\mathcal{H}_0\|_{L^4}^2). \tag{3.1}$$

Proof. Applying the L^2 energy estimate on the equation of $\Omega = \frac{w_\theta}{r}$:

$$\partial_r\Omega + \boldsymbol{u}\cdot\nabla\Omega = (\Delta + \frac{2}{r}\partial_r)\Omega - \partial_z\mathcal{H}^2, \tag{3.2}$$

which can be derived from $(1.3)_1$ by direct calculations. Performing the L^2 estimates for (3.2), one arrives at

$$\frac{1}{2}\frac{d}{dt}\|\Omega(t,\cdot)\|_{L^2}^2 + \|\nabla\Omega(t,\cdot)\|_{L^2}^2 = \underbrace{-\frac{1}{2}\int_{\mathbb{R}^3}\boldsymbol{u}\cdot\nabla\Omega^2 dx}_{I_1} + \underbrace{\int_{\mathbb{R}^3}\frac{\partial_r\Omega^2}{r}dx}_{I_2} - \underbrace{\int_{\mathbb{R}^3}\Omega\partial_z\mathcal{H}^2 dx}_{I_3}. \tag{3.3}$$

By the divergence-free property of $\boldsymbol{u}$ and using integration by parts, one can derive the vanishing of I_1. Writing in the cylindrical coordinates, one can deduce that I_2 satisfies

$$I_2 = 2\pi\int_{\mathbb{R}}\int_0^\infty \partial_r\Omega^2 drdz = -2\pi\int_{\mathbb{R}}|\Omega(t,0,z)|^2 dz \le 0.$$

Using integration by parts together with Hölder's inequality and Young's inequality, one arrives at

$$|I_3| = \left|\int_{\mathbb{R}^3}\partial_z\Omega\mathcal{H}^2 dx\right| \le \|\partial_z\Omega(t,\cdot)\|_{L^2}\|\mathcal{H}(t,\cdot)\|_{L^4}^2 \le \frac{1}{2}\|\nabla\Omega(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\|\mathcal{H}(t,\cdot)\|_{L^4}^4.$$

Substituting above estimates for $I_1 - I_3$ in (3.3), and integrating over $(0,t)$, one can deduce

$$\|\Omega(t,\cdot)\|_{L^2}^2 + \int_0^t\|\nabla\Omega(s,\cdot)\|_{L^2}^2 ds \le \|\Omega_0\|_{L^2}^2 + \int_0^t\|\mathcal{H}(s,\cdot)\|_{L^4}^4 ds. \tag{3.4}$$

Finally, using the Gagliardo-Nirenberg inequality, together with the Sobolev inequality and (2.5), one can deduce

$$\|\frac{u_r}{r}(t,\cdot)\|_{L^\infty} \le C\|\nabla\frac{u_r}{r}(t,\cdot)\|_{L^2}^{1/2}\|\nabla\frac{u_r}{r}(t,\cdot)\|_{L^6}^{1/2} \le C\|\Omega(t,\cdot)\|_{L^2}^{1/2}\|\nabla\Omega(t,\cdot)\|_{L^2}^{1/2}. \tag{3.5}$$

Then, by integrating on $(0,t)$ and recalling (3.4), one concludes the estimate (3.1). □

3.2. L^∞ Estimate of $\nabla\boldsymbol{u}$

In this subsection we present an estimate for $\nabla\boldsymbol{u}$ by applying the maximal regularity of heat flows. Before stating this result, we first introduce the following lemma, which states the standard maximal regularity of heat flows in $L^q_T L^p$ type spaces.

Lemma 3.2 (Maximal $L^q_T L^p$ -regularity for the heat flow)**.** *Let us define the operator $\mathcal{A}$ by the formula*

$$\mathcal{A}: \qquad f \longmapsto \int_0^t \nabla^2 e^{(t-s)\Delta} f(s,\cdot)ds,$$

where $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$, and the specific form of ∇^2 refers to (1.5)*.*

Then, $\mathcal{A}$ is bounded from $L^q(0,T;L^p(\mathbb{R}^d))$ to itself every $T \in (0,\infty]$ and $1 < p,\ q < \infty$. Moreover, there holds:

$$\|\mathcal{A}f\|_{L^q(0,T;L^p(\mathbb{R}^d))} \le C\|f\|_{L^q(0,T;L^p(\mathbb{R}^d))}.$$

Lemma 3.3. *Under the system* (HMHD)*, by using Lemma 3.2, one has the following $L_t^2L^4$ estimate of $\nabla \boldsymbol{w}$:*

$$\|\nabla \boldsymbol{w}\|_{L_t^2L^4} \le Ct^{1/2}\left(\|w_\theta\|_{L_t^\infty L^4}\|w_\theta\|_{L^\infty(L^2\cap L^6)} + \|\mathcal{H}\|_{L_t^\infty L^\infty}\|h_\theta\|_{L_t^\infty L^4}\right).$$

Here, $C > 0$ is a universal constant.

Proof. Recalling $\boldsymbol{w} = \operatorname{curl}\boldsymbol{u}$ and rewriting the equation $(1.3)_1$ in the vector form, one deduces

$$\partial_t \boldsymbol{w} - \Delta \boldsymbol{w} = -\nabla \times (\boldsymbol{w} \times \boldsymbol{u}) - \partial_z(\mathcal{H}\boldsymbol{h}). \tag{3.6}$$

Similarly as (3.5) and combining Lemma 2.5, one deduces

$$\|\boldsymbol{u}(t,\cdot)\|_{L^\infty} \le C\|\nabla \boldsymbol{u}(t,\cdot)\|_{L^2}^{1/2}\|\nabla \boldsymbol{u}(t,\cdot)\|_{L^6}^{1/2} \le C\|\boldsymbol{w}(t,\cdot)\|_{(L^2\cap L^6)}. \tag{3.7}$$

By Hölder's inequality and (3.7), one can deduce

$$\|(\boldsymbol{w}\times\boldsymbol{u})(t,\cdot)\|_{L^4} \le C\|w_\theta(t,\cdot)\|_{L^4}\|w_\theta(t,\cdot)\|_{(L^2\cap L^6)},$$

thus, one has

$$\|(\boldsymbol{w}\times\boldsymbol{u})\|_{L_t^\infty L^4} \le C\|w_\theta\|_{L_t^\infty L^4}\|w_\theta\|_{L_t^\infty(L^2\cap L^6)}.$$

Similarly, using Hölder's inequality, Lemma 2.3, and Lemma 2.4, one derives

$$\|\mathcal{H}\boldsymbol{h}\|_{L_t^\infty L^4} \le \|\mathcal{H}\|_{L_t^\infty L^4}\|h_\theta\|_{L_t^\infty L^4}.$$

By employing the maximal regularity property of the heat flow described in equation (3.6), one can deduce

$$\begin{aligned}\|\nabla \boldsymbol{w}\|_{L_t^2L^4} &\le C\left(\|\boldsymbol{w}\times\boldsymbol{u}\|_{L_t^2L^4} + \|\mathcal{H}\boldsymbol{h}\|_{L_t^2L^4}\right)\\ &\le Ct^{1/2}\left(\|\omega_\theta\|_{L_t^\infty L^4}\|w_\theta\|_{L_t^\infty(L^2\cap L^6)} + \|\mathcal{H}\|_{L_t^\infty L^\infty}\|h_\theta\|_{L_t^\infty L^4}\right).\end{aligned}$$

□

Next, we state the estimate of $\nabla\boldsymbol{u}$ and present a concise demonstration.

Corollary 3.4. *Under the same assumptions as Proposition 3.1, one has the following estimate of $\nabla\boldsymbol{u}$:*

$$\int_0^t \|\nabla \boldsymbol{u}(s,\cdot)\|_{L^\infty} ds \lesssim t^{4/7}\|\boldsymbol{w}\|_{L_t^\infty L^2}^{1/7}\|\nabla\boldsymbol{w}\|_{L_t^2L^4}^{6/7}. \tag{3.8}$$

Proof. By invoking Lemma 2.5 in conjunction with the Gagliardo-Nirenberg inequality, this estimate can be rigorously substantiated. □

3.3. End of the proof

Since the non-resistive HMHD system with an azimuthal magnetic field is locally well-posed in H^3 (see [14] for the main result) *, there exists $T_* > 0$ such that

$$\sup_{0\le s\le t} s\|\nabla\mathcal{H}(s,\cdot)\|_{L^\infty} \le 1, \qquad \text{for all } t \in [0, T_*). \tag{3.9}$$

*Although the main result of [14] gives the local well-posedness for the 3D inviscid HMHD system, it is sufficient to guarantee the local well-posedness for the viscid system via the same approach.

In the following, our argument will be carried out before this T_*. Substituting the estimate (3.1) into Lemma 2.4, one deduces

$$\begin{aligned}\|h_\theta(t,\cdot)\|_{L^p} &\le \|\boldsymbol{h_0}\|_{L^p}\exp\left\{C\int_0^t\left[\|\frac{u_r}{r}(s,\cdot)\|_{L^\infty}+\|\partial_z\mathcal{H}(s,\cdot)\|_{L^\infty}\right]ds\right\}\\ &\lesssim \|\boldsymbol{h_0}\|_{L^p}\exp\left\{C\|\mathcal{H}_0\|_{L^4}^2t^{5/4}+C\|\Omega_0\|_{L^2}t^{3/4}+C\|\partial_z\mathcal{H}\|_{L^1_tL^\infty}\right\}\end{aligned}$$

for all $t\in[0,T_*)$. By integrating (3.9), the above estimate can be further simplified to

$$\|h_\theta(t,\cdot)\|_{L^p}\le E_0\exp\left(CE_0(t^{5/4}+t^{3/4})+1\right), \tag{3.10}$$

where $E_0:=\|\boldsymbol{u_0},\boldsymbol{h_0},\mathcal{H}_0\|_{H^3}$. Similarly as (3.10), substituting the estimate (3.1) into Lemma 2.4 and using (3.9), one has

$$\|w_\theta(t,\cdot)\|_{L^p}\le CE_0\exp\left(CE_0(t^{5/4}+t^{3/4})+1\right).$$

Therefore, by the L^p conservation of $\mathcal{H}$ and Corollary 3.3 , one can deduce that $\|\nabla\boldsymbol{w}\|_{L^2_tL^4}$ satisfies

$$\begin{aligned}\|\nabla\boldsymbol{w}\|_{L^2_tL^4} &\le Ct^{1/2}\left(\|w_\theta\|_{L^\infty_tL^4}\|w_\theta\|_{L^\infty_t(L^2\cap L^6)}+\|\mathcal{H}\|_{L^\infty_tL^\infty}\|h_\theta\|_{L^\infty_tL^4}\right)\\ &\le CE_0^2t^{1/2}\exp\left(CE_0(t^{5/4}+t^{3/4})+1\right),\end{aligned}$$

that is to say, the inequality (3.8) can be written as

$$\int_0^t\|\nabla\boldsymbol{u}(s,\cdot)\|_{L^\infty}ds\le t^{4/7}\|\boldsymbol{w}\|_{L^\infty_tL^2}^{1/7}\|\nabla\boldsymbol{w}\|_{L^2_tL^4}^{6/7}\le C(1+E_0)^2t\exp\left\{CE_0(t^{5/4}+t^{3/4})+1\right\}. \tag{3.11}$$

Substituting the inequality (3.11) into (2.7) and (2.8), one derives:

$$\|\nabla\boldsymbol{h}(t,\cdot)\|_{L^\infty}\lesssim E_0\exp\left\{C(1+E_0)^2t\exp\left(CE_0(t^{5/4}+t^{3/4})+1\right)+1\right\}, \tag{3.12}$$

$$\|\nabla\mathcal{H}(t,\cdot)\|_{L^\infty}\lesssim E_0\exp\left\{C(1+E_0)^2t\exp\left(CE_0(t^{5/4}+t^{3/4})+1\right)+1\right\}. \tag{3.13}$$

To ensure the a priori assumption (3.9) holds, one deduces the following restriction: when $t=T_*$, by (3.13), we have

$$\varepsilon E_0T_*\exp\left\{C(1+E_0)^2T_*\exp\left(CE_0(T_*^{5/4}+T_*^{3/4})+1\right)+1\right\}\le\frac{1}{2}. \tag{3.14}$$

Without loss of generality, we assume that ε is sufficiently small, therefore, $T_*>1$. By $T_*^{5/4}+T_*^{3/4}\le CT_*^{5/4}$ for all $T_*>1$, one has

$$\begin{aligned}&\varepsilon E_0T_*\exp\left\{C(1+E_0)^2T_*\ \exp\left(CE_0(T_*^{5/4}+T_*^{3/4})+1\right)+1\right\}\\ &\le\varepsilon E_0T_*\exp\left\{C(1+E_0)^2T_*\exp\left\{C(1+E_0)T_*^{5/4}\right\}\right\}.\end{aligned}$$

A basic inequality $\log(1+t)\le t\le \mathrm{e}^t-1$, for all $t\ge0$, implies that

$$\begin{aligned}E_0T_*\exp\left\{C(1+E_0)^2T_*\exp\{C(1+E_0)T_*^{5/4}\}\right\}&=\exp\left\{\log E_0T_*+C(1+E_0)^2T_*\exp\{C(1+E_0)T_*^{5/4}\}\right\}\\ &\le\frac{1}{2}\exp\{\exp(C(1+E_0)T_*^{5/4})\}.\end{aligned}$$

Thus,

$$\varepsilon E_0 T_* \exp\left\{C(1+E_0)T_* \exp\{C(1+E_0)T_*^{5/4}\}\right\} \leq \frac{\varepsilon}{2}\exp\{\exp(C(1+E_0)T_*^{5/4})\}.$$

Therefore, the condition (3.9) is satisfied, provided

$$\exp\left\{\exp(C_*^{-1}(1+E_0)T_*^{5/4})\right\} = \frac{1}{\varepsilon},$$

for sufficiently small $\varepsilon > 0$ and some constant $C_* > 0$, and one finds

$$T_* = \frac{C_*}{(1+E_0)^{\frac{4}{5}}}(\log(\log(\varepsilon^{-1})))^{4/5}.$$

This gives the desired lower bound of the lifespan in Theorem 1.1.

4. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. First, we re-establish the $L_t^\infty L^p$ -estimate for Ω and the $L_t^\infty L^\infty$ -estimate for w_θ. We then focus on the L^∞ estimate for ∇u. Finally, on the basis of the higher-order bound of the system (1.6), we verify the a priori assumption to show the lifespan of the inviscid axially symmetric HMHD system.

We first show the definition of Bounded Mean Oscillations (BMO) space.

Definition 4.1. *The space $BMO\left(\mathbb{R}^d\right)$ is the set of locally integrable functions f such that*

$$\|f\|_{BMO} \overset{def}{=} \sup_B \frac{1}{|B|}\int_B |f - f_B|\,dx < \infty \quad \text{with} \quad f_B \overset{def}{=} \frac{1}{|B|}\int_B f dx.$$

The above supremum is taken over the set of Euclidean balls.

The following logarithm inequality is key to the higher-order estimate of the solution. We refer readers to ([12], Corollary 2.8) for a detailed proof.

Lemma 4.2. *For all divergence free vector field $\boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\boldsymbol{g} \in H^3\left(\mathbb{R}^3\right)$, the following estimate holds:*

$$\|\nabla \boldsymbol{g}(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \lesssim 1 + \|\nabla \times \boldsymbol{g}(t,\cdot)\|_{BMO(\mathbb{R}^3)} \log\left(e + \|\boldsymbol{g}(t,\cdot)\|_{H^3(\mathbb{R}^3)}\right).$$

As we have shown in the beginning of Section 3.3, since the local well-posedness of the non-resistive axially symmetric HMHD system in H^m for $m \geq 3$, there exists $T_* > 0$ such that:

$$\sup_{0\leq s\leq t} s\left(\|\nabla \boldsymbol{h}(s,\cdot)\|_{L^\infty} + \|\nabla \mathcal{H}(s,\cdot)\|_{L^\infty}\right) \leq 1, \quad \text{for all } t \in [0, T_*). \tag{4.1}$$

In the following, our argument will be carried out before this T_*. Noticing that the estimates of $\nabla \boldsymbol{h}$ and $\nabla \mathcal{H}$ in Lemma 2.6 are not influenced by the loss of viscous term, we still have the validity of (2.7) and (2.8).

4.1. $L_t^\infty L^p$ -Estimate of Ω and $L_t^\infty L^\infty$ -Estimate of w_θ

Our first step is to obtain an a priori bound for $\|\Omega\|_{L_t^\infty L^p}$. We will apply the L^p energy estimate on the equation of $\Omega = \frac{w_\theta}{r}$:

$$\partial_t \Omega + \boldsymbol{u} \cdot \nabla \Omega = -\partial_z \mathcal{H}^2. \tag{4.2}$$

The detailed result is stated as follows:

Proposition 4.3. *Under the same assumptions as Theorem 1.3, the following estimate of Ω holds for all $1 \le p \le \infty$:*

$$\|\Omega(t,\cdot)\|_{L^p} \le \|\Omega_0\|_{L^p} + 2\|\mathcal{H}_0\|_{L^p}, \quad \textit{for all} \quad t \in [0, T_*)\,.$$

Proof. Performing the L^p estimates for (4.2), one arrives at

$$\frac{d}{dt}\|\Omega(t,\cdot)\|_{L^p}^p = -p \int_{\mathbb{R}^3} \partial_z \mathcal{H}^2 \cdot |\Omega|^{p-2} \Omega dx. \tag{4.3}$$

Applying the Hölder inequality, we find

$$\begin{aligned} p \left| \int_{\mathbb{R}^3} \partial_z \mathcal{H}^2 \cdot |\Omega|^{p-2}\, \Omega dx \right| &\le p \left\| \partial_z \mathcal{H}^2(t,\cdot) \right\|_{L^p} \|\Omega(t,\cdot)\|_{L^p}^{p-1} \\ &\le 2p \|\mathcal{H}(t,\cdot)\|_{L^p} \|\nabla \mathcal{H}(t,\cdot)\|_{L^\infty} \|\Omega(t,\cdot)\|_{L^p}^{p-1}. \end{aligned}$$

Recalling the L^p conservation of $\mathcal{H}$ in Lemma 2.3, one concludes from (4.3) that

$$\frac{d}{dt}\|\Omega(t,\cdot)\|_{L^p}^p \le 2p \|\mathcal{H}_0\|_{L^p} \|\nabla \mathcal{H}(t,\cdot)\|_{L^\infty} \|\Omega(t,\cdot)\|_{L^p}^{p-1}.$$

Canceling $p\|\Omega(t,\cdot)\|_{L^p}^{p-1}$ on both sides and integrating over $[0,t]$, one deduces

$$\|\Omega(t,\cdot)\|_{L^p} \le \|\Omega_0\|_{L^p} + 2\|\mathcal{H}_0\|_{L^p} \int_0^t \|\nabla \mathcal{H}(s,\cdot)\|_{L^\infty} ds.$$

Thus, one concludes the proposition by using the a priori condition (4.1). □

Corollary 4.4. *Let the assumptions of Proposition 4.3 be fulfilled. Then, the following estimate of w_θ holds uniformly for all $t \in [0, T_*)$:*

$$\|w_\theta(t,\cdot)\|_{L^\infty} \le C \|(\boldsymbol{w_0}, \boldsymbol{h_0})\|_{H^3} \exp\left(C \|(\boldsymbol{u_0}, \mathcal{H}_0)\|_{H^3} t\right). \tag{4.4}$$

Proof. By using Lemma 2.1 and Sobolev embedding, one has

$$\left\| \frac{u_r}{r}(t,\cdot) \right\|_{L^\infty} \le C \left\| \nabla \frac{u_r}{r}(t,\cdot) \right\|_{L^2}^{1/2} \left\| \nabla \frac{u_r}{r}(t,\cdot) \right\|_{L^6}^{1/2},$$

then by applying Lemma 2.5 and Proposition 4.3, one arrives at

$$\left\| \frac{u_r}{r}(t,\cdot) \right\|_{L^\infty} \le C \|\Omega(t,\cdot)\|_{L^2}^{1/2} \|\Omega(t,\cdot)\|_{L^6}^{1/2} \le C \left(\|\Omega_0\|_{L^2} + 2\|\mathcal{H}_0\|_{L^2}\right)^{1/2} \left(\|\Omega_0\|_{L^6} + 2\|\mathcal{H}_0\|_{L^6}\right)^{1/2},$$

which indicates

$$\int_0^t \left\| \frac{u_r}{r}(s,\cdot) \right\|_{L^\infty} ds \le C\|(\boldsymbol{u_0}, \mathcal{H}_0)\|_{H^3} t.$$

Using Lemma 2.4, one deduces, for all $t \in [0, T_*)$:

$$\begin{aligned}\|w_\theta(t,\cdot)\|_{L^\infty} &\le (\|\boldsymbol{w_0}\|_{L^\infty} + \|\boldsymbol{h_0}\|_{L^\infty}) \exp\left(C\|(\boldsymbol{u_0}, \mathcal{H}_0)\|_{H^3} t + C\right) \\ &\le C\|(\boldsymbol{w_0}, \boldsymbol{h_0})\|_{H^3} \exp\left(C\|(\boldsymbol{u_0}, \mathcal{H}_0)\|_{H^3} t\right).\end{aligned}$$

This completes the proof. □

4.2. L^∞ Estimate of $\nabla \boldsymbol{u}$

In this subsection, we show an estimate of $\nabla \boldsymbol{u}$ by using Lemma 4.2 and the divergence-free condition $\nabla \cdot \boldsymbol{u} = 0$.

Lemma 4.5. *For all $t < T_*$, the L^∞ estimate of $\nabla \boldsymbol{u}$ holds :*

$$\|\nabla \boldsymbol{u}(t,\cdot)\|_{L^\infty} \lesssim 1 + \|(\boldsymbol{w_0}, \boldsymbol{h_0})\|_{H^3} \exp\left(C\|(\boldsymbol{u_0}, \mathcal{H}_0)\|_{H^3} t\right) \log\left(e + \|\boldsymbol{u}(t,\cdot)\|_{H^3}\right). \tag{4.5}$$

Proof. Applying Lemma 4.2 to $\nabla \boldsymbol{u}$, for $\boldsymbol{u}$ is divergence-free and belongs to $C\left([0, T_*); H^3\left(\mathbb{R}^3\right)\right)$, we get

$$\begin{aligned}\|\nabla \boldsymbol{u}(t,\cdot)\|_{L^\infty} &\lesssim 1 + \|\nabla \times \boldsymbol{u}(t,\cdot)\|_{BMO} \log\left(e + \|\boldsymbol{u}(t,\cdot)\|_{H^3}\right) \\ &\lesssim 1 + \|w_\theta(t,\cdot)\|_{L^\infty} \log\left(e + \|\boldsymbol{u}(t,\cdot)\|_{H^3}\right), \quad \forall t \in [0, T_*).\end{aligned}$$

Using (4.4), we arrive at the L^∞ estimate of $\nabla \boldsymbol{u}$ (4.5). □

4.3. End of the proof

Finally, we arrive at higher-order estimates of the non-resistive inviscid HMHD system (1.6). Define, for $t \in [0, T_*)$,

$$E^2(t) := \|(\boldsymbol{u}, \boldsymbol{h}, \mathcal{H})(t,\cdot)\|_{H^3}^2, \quad \text{with} \quad E_0^2 := \|(\boldsymbol{u_0}, \boldsymbol{h_0}, \mathcal{H}_0)\|_{H^3}^2.$$

In this way, the higher-order bound we have given in Lemma 2.6 reads

$$E^2(t) \le E_0^2 \exp\left(C \int_0^t \|\nabla(\boldsymbol{u}, \boldsymbol{h}, \mathcal{H})(s,\cdot)\|_{L^\infty} ds\right).$$

Based on the estimate of $\|\nabla \boldsymbol{u}(t,\cdot)\|_{L^\infty}$ derived in (4.5), and recalling the a priori assumption (4.1), we have

$$E^2(t) \le E_0^2 \exp\left(C \int_0^t \left\{1 + \|(\boldsymbol{w_0}, \boldsymbol{h_0})\|_{H^3} \exp\left(C\|(\boldsymbol{u_0}, \mathcal{H}_0)\|_{H^3} s\right) \log\left(e + \|\boldsymbol{u}(s,\cdot)\|_{H^3}\right)\right\} ds + C\right).$$

This indicates

$$\log(e + E^2(t)) \le \log(e + E_0^2) + C + C \int_0^t \left\{1 + \|(\boldsymbol{w}_0, \boldsymbol{h}_0)\|_{H^3} \times \exp(C\|(\boldsymbol{u}_0, \mathcal{H}_0)\|_{H^3} s) \log(e + E^2(s))\right\} ds.$$

Applying Grönwall's inequality, one finds that

$$E^2(t) \leq \left(C\left(e+E_0^2\right)\right)^{\exp\left(Ct+CE_0\exp(CE_0t)\right)}, \quad \forall t\in[0,T_*). \tag{4.6}$$

The last inequality follows from the a priori assumption (4.1). This shows that as long as (4.1) holds on $[0,T_*)$, the solution $(\boldsymbol{u},\boldsymbol{h})$ to the initial value problem (1.6) would keep in H^3 before $t=T_*$.

Now it remains to verify the a priori assumption (4.1). Recalling the estimates of $\nabla\boldsymbol{h}$ and $\nabla\mathcal{H}$ in Lemma 2.6, using (4.6), one deduces

$$\begin{aligned}\|\nabla(\boldsymbol{h},\mathcal{H})(t,\cdot)\|_{L^\infty} &\leq \|\nabla(\boldsymbol{h}_0,\mathcal{H}_0)\|_{L^\infty}\exp\left(C\int_0^t\left(\|\nabla\boldsymbol{u}(s,\cdot)\|_{L^\infty}+\|\partial_z\mathcal{H}(s,\cdot)\|_{L^\infty}\right)ds\right)\\ &\leq \|\nabla(\boldsymbol{h}_0,\mathcal{H}_0)\|_{L^\infty}\exp\left(C\int_0^t E(s)ds\right)\\ &\leq \varepsilon\exp\left(Ct\left(C\left(e+E_0\right)\right)^{\exp\left(Ct+CE_0\exp(CE_0t)\right)}\right).\end{aligned}$$

Therefore, to ensure the a priori assumption (4.1) holds, one deduces the following restriction of T_*

$$\varepsilon T_*\exp\left(CT_*\left(C\left(e+E_0\right)\right)^{\exp\left(CT_*+CE_0\exp(CE_0T_*)\right)}\right)\leq\frac{1}{3}. \tag{4.7}$$

Without loss of generality, we assume that ε is sufficiently small, therefore $T_*>1$. Noticing that $\log(1+t)\leq t\leq e^t-1$ for all $t\geq 0$, one deduces

$$\exp(CT_*+CE_0\exp\left(CE_0T_*\right))\leq\exp\left(\exp\left(CE_0(T_*+1)\right)\right),$$

and

$$\begin{aligned}CT_*\left(C\left(e+E_0\right)\right)^B &\leq \exp\left(B\log\left(CT_*\left(C\left(e+E_0\right)\right)\right)\right)\\ &\leq \exp\left(CB\left((1+E_0)T_*\right)\right)\end{aligned}$$

for any $B>0$. Thus, one derives that

$$T_*\exp\left(CT_*\left(C\left(e+E_0\right)\right)^{\exp\left(CT_*+CE_0\exp(CE_0T_*)\right)}\right)\leq\frac{1}{3}\exp\left(\exp\left(\exp\left(\exp\left(C_*^{-1}\left(1+E_0\right)T_*\right)\right)\right)\right)$$

for some constant $C_*>0$. Therefore, the condition (4.7) is satisfied, provided

$$\frac{1}{\varepsilon}=\exp\left(\exp\left(\exp\left(\exp\left(C_*^{-1}\left(1+E_0\right)T_*\right)\right)\right)\right),$$

for sufficiently small $\varepsilon>0$. In this way, one finds

$$T_*=\frac{C_*}{1+E_0}\log\left(\log\left(\log\left(\log\left(\varepsilon^{-1}\right)\right)\right)\right).$$

This gives the desired lower bound of the lifespan in Theorem 1.3.

5. Conclusions

In the current paper, we established precise lifespan bounds for the non-resistive axially symmetric HMHD in both viscid and inviscid case, proving that strong solutions can be arbitrarily large if their initial magnetic gradient was small enough. The key ingredient of the proof lies in the use of the a prior estimate, the Gagliardo–Nirenberg interpolation inequality, and the higher-order energy estimate, which together enabled us to control the nonlinear interactions between velocity and magnetic fields.

In the viscid case, the lifespan was derived by an L^∞ estimate of $\nabla \boldsymbol{u}$ which derived by maximal regularity estimates for the heat flow. In the inviscid case, however, an additional logarithmic factor appeared in the lifespan due to the failure of maximal regularity for the heat flow. To overcome this difficulty, we employed BMO estimates and commutator estimate. These results highlight the crucial role of the magnetic gradient in determining the lifespan of strong solutions. Moreover, the constructive example of initial data satisfying the small-gradient condition further supports the applicability of our theorems. Our future work may extend these ideas to compressible settings or systems with fractional dissipation.

Author contributions

Linbin Yang: Conceptualization; Writing–original draft; Writing–review & editing. Taoran Zhou: Conceptualization; Writing–original draft; Writing–review & editing; Funding Acquisition; Project Administration. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

T. Zhou is supported by Postgraduate Research & Practice Innovation Program of Jiangsu Province (No. KYCX25_1562).

Conflict of interest

The authors declare no conflict of interest.

References

1. H. Bae, A. Mazzucato, H. Lopes, M. Filho, Long-time existence for the 2D ideal Boussinesq and the 2D density-dependent Euler equations, preprint paper, 2025. https://doi.org/10.48550/arXiv.2507.18244
2. D. Chae, M. Schonbek, On the temporal decay for the Hall-magnetohydrodynamic equations, *J. Differ. Equ.*, **255** (2013), 3971–3982. https://doi.org/10.1016/j.jde.2013.07.059

3. D. Chae, S. Weng, Singularity formation for the incompressible Hall-MHD equations without resistivity, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **33** (2016), 1009–1022. https://doi.org/10.1016/j.anihpc.2015.03.002
4. M. Dai, Local well-posedness of the Hall-MHD system in $H^s(\mathbb{R}^n)$ with $s > n/2$, *Math. Nachr.*, **293** (2020), 67–78. https://doi.org/10.1002/mana.201800107
5. T. Forbes, Magnetic reconnection in solar flares, *Geophys. Astrophys. Fluid Dyn.*, **62** (1991), 15–36. https://doi.org/10.3367/UFNe.0180.201009j.0997
6. J. Fan, S. Huang, G. Nakamura, Well-posedness for the axisymmetric incompressible viscous Hall-magnetohydrodynamic equations, *Appl. Math. Lett.*, **26** (2013), 963–967. https://doi.org/10.1016/j.aml.2013.04.008
7. I. Jeong, S. Oh, On the cauchy problem for the Hall and electron magnetohydrodynamic equations without resistivity I: Illposedness near degenerate stationary solutions, *Ann. PDE*, **8** (2022), 15. https://doi.org/10.1007/s40818-022-00134-5
8. T. Hao, Y. Liu, The influence of viscous coefficients on the lifespan of 3-D anisotropic Navier–Stokes system, *SIAM J. Math. Anal.*, **55** (2023), 2186–2210. https://doi.org/10.1137/21M1461459
9. T. Kato, G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Comm. Pure Appl. Math.*, **41** (2010), 891–907. https://doi.org/10.1002/cpa.3160410704
10. Z. Lei, On axially symmetric incompressible magnetohydrodynamics in three dimensions, *J. Differ. Equ.*, **259** (2015), 3202–3215. https://doi.org/10.1016/j.jde.2015.04.017
11. Z. Li, P. Liu, Global regularity for the 3D Hall-MHD equations with low regularity axisymmetric data, *Monatsh. Math.*, **201** (2023), 173–195. https://doi.org/10.1007/s00605-022-01795-x
12. Z. Li, Critical conditions on w^θ imply the regularity of axially symmetric MHD-Boussinesq systems, *J. Math. Anal. Appl.*, **505** (2022), 2333–2353. https://doi.org/10.1016/j.jmaa.2021.125451
13. Z. Li, A refined long time asymptotic bound for 3D axially symmetric Boussinesq system with zero thermal diffusivity, *J. Differ. Equ.*, **374** (2023), 737–760. https://doi.org/10.1016/j.jde.2023.08.011
14. Z. Li, On local well-posedness of 3D ideal Hall–MHD system with an azimuthal magnetic field, *Acta Math. Sin. (Engl. Ser.)*, **41** (2025), 2921–2940. https://doi.org/10.1007/s10114-025-4107-4
15. Z. Li, M. Yang, On a single-component regularity criterion for the non-resistive axially symmetric Hall-MHD system, *Acta Appl. Math.*, **181** (2022), 1. https://doi.org/10.1007/s10440-022-00519-5
16. Z. Li, T. Zhou, On the lifespan of axisymmetric incompressible Euler equations with a small initial swirl, *Z. Angew. Math. Phys.*, **75** (2024), 219. https://doi.org/10.1007/s00033-024-02355-z
17. C. Miao, X. Zheng, On the global Well-posedness for the Boussinesq system with Horizontal dissipation, *Commun. Math. Phys.*, **321** (2013), 33–67. https://doi.org/10.1007/s00220-013-1721-2
18. L. Nirenberg, On elliptic partial differential equations, In: *Faedo, S. (eds) Il principio di minimo e sue applicazioni alle equazioni funzionali*, Berlin: Springer, 2011, 1–48. https://doi.org/10.1007/978-3-642-10926-3_1

19. D. Shalybkov, V. Urpin, The Hall effect and the decay of magnetic fields, *Astron. Astrophys.*, **321** (1997), 685–690.

20. J. Wesson, *Tokamaks*, 4 Eds., Oxford: Oxford University Press, 2011.

21. X. Zhai, Global solutions to 3D MHD equations with fractional dissipation, *Appl. Math. Lett.*, **176** (2026), 109873. https://doi.org/10.1016/j.aml.2026.109873