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Research article

P_5-factors in line graphs of trees

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Abstract: Let $\mathcal{A}$ be a set of connected graphs. A graph G has an $\mathcal{A}$-factor if G has a spanning subgraph H such that each component of H is isomorphic to one of the members in $\mathcal{A}$. In this paper, we give a necessary and sufficient condition for the existence of P_5-factors in line graphs of trees. Then we present an algorithm to search for a P_5-factor in the line graph of a tree.

Keywords: path factor; line graph; tree; algorithm
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1. Introduction

The inherent topological structure of a parallel computing system can be modeled as a graph, where vertices represent processing elements and edges denote the physical interconnections among these elements—such a graph is designated a communication network. Matching-related parameters play a fundamental role in evaluating the reliability and fault tolerance of communication networks, while graph factors provide natural generalizations of matchings with relevant applications [19, 20, 23]. In this paper, we investigate the path factors problem in graphs, which can be viewed as a relaxation of the classical maximum cardinality matching problem.

The path factor of a graph is a hot topic in graph theory, attracting considerable attention from many researchers. Beyond its theoretical significance, it also demonstrates substantial application value in practical areas, such as network security, operations research, and information science. In particular, several real-world problems—such as file transfer in computer networks, scheduling optimization, and telephone network design—can be effectively modeled and analyzed using path factors in graphs [11, 27].

In this paper, we mainly focus on finite simple graphs, and terminology as well as notation not explicitly defined here are referred to [4]. Let $G = (V(G), E(G))$ denote a graph. For each vertex

$v \in V(G)$, we use $d_G(v)$ to represent the degree of v and $N_G(v)$ to signify the neighborhood of v in G. For $x, y \in V(G)$ with $x \neq y$, we use $P[x, y]$ to denote a path P in G with end vertices x, y, and assume that it has an orientation from x to y. Define $P(x, y] = P[x, y] - \{x\}$ and $P[x, y) = P[x, y] - \{y\}$, respectively. For each $v \in P(x, y]$ and $w \in P[x, y)$, let v^- and w^+ be the predecessor of v and the successor of w on P, respectively. Define recursively $v^{-(k+1)} = (v^{-k})^-$, $v^{+(k+1)} = (v^{+k})^+$ for $k \geq 1$, if they exist. The distance between u and v in $V(G)$, denoted by $dist_G(u, v)$, is defined to be the length of a shortest path joining u and v.

A spanning subgraph of G is a subgraph H of G such that $V(H) = V(G)$ and $E(H) \subseteq E(G)$. Suppose H is a spanning subgraph of G, and let $\mathcal{A}$ be a set of connected graphs. If every component of H is isomorphic to some graph in $\mathcal{A}$, then H is referred to as an $\mathcal{A}$-factor of G. In particular, an F-factor of G is a spanning subgraph H, where each component of H is isomorphic to a fixed graph F—that is, $\mathcal{A} = \{F\}$. Let P_k denote a path with k vertices, where $k \geq 2$. We call H a *P_k-factor* of G if each component of H is isomorphic to a P_k. Specifically, a P_2-factor corresponds to a perfect matching in G. A graph of order $r + 1$ is called a $K_{1,r}$ with center u if it has vertex set $\{u, u_1, u_2, ..., u_r\}$ and edge set $\{uu_1, uu_2, ..., uu_r\}$, where $r \geq 2$.

Since Tutte [22] proposed the well-known Tutte 1-factor theorem, there are many results on graphs with path factors [2, 8–10] and path factors in claw-free graphs [3, 12], cubic graphs [13, 14], square of trees [7, 16], and bipartite graphs [25]. In recent years, research on path-factors has evolved to cover factor-critical graphs [5], factor-critical uniform graphs [18], and the spectral radius characterization of path-factors in graphs [26]. More results on graph factors are referred to the survey papers and books [1, 21, 24].

Kirkpatrick and Hell [15] proved that deciding whether a graph G has an F-factor is NP-complete if F is a graph with at least 3 vertices.

Theorem 1.1. (Hell and Kirkpatrick [15]) *If F is a connected graph with $|V(F)| \geq 3$, then deciding whether a graph G has an F-factor is NP-complete.*

So the chance is small to give a good characterization for general graphs to have P_k-factors with $k \geq 3$. Most of the work on P_k-factors was attempted to find sufficient conditions or necessary conditions for the existence of P_k-factors using various graphic parameters, or to determine special classes of graphs that have P_k-factors. Kaneko et al. [12] and Akiyama et al. [1] considered classes of graphs that admit P_3-factors or P_4-factors, respectively.

Theorem 1.2. (Kaneko et al. [12]) *Let G be a connected claw-free graph of order divisible by 3 with at most two end blocks. Then G admits a P_3-factor.*

Corollary 1.3. (Kaneko et al. [12]) *If G is a 2-connected claw-free graph of order divisible by 3, then G admits a P_3-factor.*

Theorem 1.4. (Akiyama et al. [1]) *Let G be a 3-connected cubic graph of order divisible by 4. Then G admits a P_4-factor.*

The line graph of G is a graph $L(G)$ with vertex set $E(G)$, where two vertices are adjacent in $L(G)$ if the corresponding edges in G have a common end. The size of the graph G refers to the number of edges it contains. Let T be a tree. We call a vertex of T, which has degree one, a leaf of T. The set of leaves of T is denoted by $Leaf(T)$. The subtree $T - Leaf(T)$ of T is called the stem of T, denoted by

$Stem(T)$. Regarding the P_k-factors in the line graphs of trees, we set $C_i(T, v) = \{C|C$ is a component of $T - v$ satisfying $|E(C)| \equiv i \ (mod\ k)\}$, where $i \in \{0, 1, \ldots, k-1\}$. We further define $c_i(T, v) = |C_i(T, v)|$. In [17], a necessary and sufficient condition for the existence of P_3-factors in $L(T)$ was established by Li and Zhang.

Theorem 1.5. (Li et al. [17]) *Let T be a tree such that $|E(T)|$ is divisible by 3. Then the line graph $L(T)$ admits a P_3-factor if and only if $c_0(T, v) \geq c_1(T, v)$ for each $v \in V(T)$.*

Chen et al. [6] recently gave a sharp necessary condition for the existence of P_k-factors in the line graphs of trees, where $k \geq 3$. Additionally, a characterization is provided for the family of trees for which line graphs admit P_4-factors.

Theorem 1.6. (Chen et al. [6]) *Let T be a tree such that $|E(T)|$ is divisible by k, where $k \geq 3$. If the line graph $L(T)$ admits a P_k-factor, then $c_0(T, v) \geq c_{k-2}(T, v)$ for each $v \in V(T)$.*

Theorem 1.7. (Chen et al. [6]) *Let T be a tree such that $|E(T)|$ is divisible by 4. Then the line graph $L(T)$ admits a P_4-factor if and only if $c_0(T, v) \geq c_2(T, v)$ for each $v \in V(T)$.*

In Section 2, we give a necessary and sufficient condition for the existence of P_5-factors in the line graphs of trees. In Section 3, we present an algorithm to search for a P_5-factor in the line graph of a tree.

2. Characterization of trees whose line graphs admit P_5-factors

Theorem 2.1. *Let T be a tree such that $|E(T)|$ is divisible by 5. Then the line graph $L(T)$ admits a P_5-factor if and only if $c_0(T, v) \geq c_3(T, v) + p|c_2(T, v) - c_1(T, v)| + q$ for each $v \in V(T)$, where*

$$p = \begin{cases} 2, & \text{if } c_2(T, v) \geq c_1(T, v), \\ \frac{1}{2}, & \text{if } c_2(T, v) < c_1(T, v). \end{cases} \quad and \quad q = \begin{cases} \frac{5}{2}, & \text{if } c_1(T, v) - c_2(T, v) \text{ is positive and odd }, \\ 0, & \text{Otherwise }. \end{cases}$$

Proof. Let T be a tree such that $|E(T)|$ is divisible by 5. For each $v \in V(T)$, define $f_T(v) = c_3(T, v) + p|c_2(T, v) - c_1(T, v)| + q$.

First, we establish sufficiency via induction on $|V(T)|$. For $|V(T)| = 6$, $|V(L(T))| = |E(T)| = 5$, and the result is trivially true. Assume $k \geq 2$ and Theorem 2.1 holds for any tree of order $5t + 1$ ($t \leq k - 1$). Let T be a tree of order $5k + 1$. The following claim is immediately derived.

Claim 2.1. *Suppose T_1 and T_2 be two nontrivial subtrees of T satisfying $|V(T_1) \cap V(T_2)| = 1$, $E(T_1) \cup E(T_2) = E(T)$ and $|E(T_1)| \equiv 0 \pmod 5$ (whence $|E(T_2)| \equiv 0 \pmod 5$). If $c_0(T_i, v) \geq f_{T_i}(v)$ for each $v \in V(T_i)$, where $i \in \{1, 2\}$, then the line graph $L(T)$ admits a P_5-factor.* □

Assume for contradiction that the line graph $L(T)$ does not admit a P_5-factor. It is obvious that T cannot be a path. Otherwise, $L(T)$ is a P_{5k}, which necessarily admits a P_5-factor, a contradiction. Now, we choose a vertex $v \in V(T)$ such that $d_T(v) \geq 3$. Then, we can deduce the following claim.

Claim 2.2. (i) $c_4(T, v) = 0$;

(ii) $c_3(T, v) = 0$;

(iii) $c_1(T, v)c_2(T, v) = 0$;

(iv) *For each component $C \in C_0(T, v)$, C is an isolated vertex.*

Proof. (*i*) Assume $c_4(T, v) > 0$. Let C_1 be a component in $C_4(T, v)$. Define T_1 to be the subtree of T induced by $V(C_1) \cup \{v\}$. For convenience, write $T_1 = C_1 \cup \{v\}$(Analogous symbols apply hereinafter). Let $T_2 = T - C_1$. We have $c_i(T_1, v) = 0$ and $c_i(T_2, v) = c_i(T, v)$, where $i \in \{0, 1, 2, 3\}$. It follows that $f_{T_1}(v) = 0$ and $f_{T_2}(v) = f_T(v)$. Thus, $c_0(T_1, v) = f_{T_1}(v)$ and $c_0(T_2, v) = c_0(T, v) \geq f_T(v) = f_{T_2}(v)$. Note that $|E(T_1)| \equiv 0 \pmod 5$ and $|E(T_2)| \equiv 0 \pmod 5$. For each $x \in V(T_1 - v)$, $c_i(T_1, x) = c_i(T, x)$ for $i \in \{0, 1, 2, 3\}$. It follows that $f_{T_1}(x) = f_T(x)$. Hence, $c_0(T_1, x) = c_0(T, x) \geq f_T(x) = f_{T_1}(x)$. Similarly, $c_0(T_2, y) = c_0(T, y) \geq f_T(y) = f_{T_2}(y)$ for each $y \in V(T_2 - v)$. By Claim 2.1, $L(T)$ admits a P_5-factor, a contradiction.

(*ii*) Assume $c_3(T, v) > 0$. Since $c_0(T, v) \geq c_3(T, v)$, we deduce that $c_0(T, v) > 0$. Let C_2 and C_3 be two components in $C_3(T, v)$ and $C_0(T, v)$, respectively. Denote $T_1 = C_2 \cup C_3 \cup \{v\}$ and $T_2 = T - C_2 - C_3$. Then $c_i(T_1, v) = 1$ for $i \in \{0, 3\}$ and $c_i(T_1, v) = 0$ for $i \in \{1, 2\}$. It follows that $f_{T_1}(v) = 1$. Thus $c_0(T_1, v) = f_{T_1}(v)$. We also have $c_3(T_2, v) = c_3(T, v) - 1$ and $c_i(T_2, v) = c_i(T, v)$ for $i \in \{1, 2\}$. It follows that $f_{T_2}(v) = f_T(v) - 1$. Thus, $c_0(T_2, v) = c_0(T, v) - 1 \geq f_T(v) - 1 = f_{T_2}(v)$. Observe that $|E(T_1)| \equiv 0 \pmod 5$ and $|E(T_2)| \equiv 0 \pmod 5$. Utilizing an analogous argument to that in (*i*), we obtain $c_0(T_1, x) \geq f_{T_1}(x)$ for each $x \in V(T_1 - v)$ and $c_0(T_2, y) \geq f_{T_2}(y)$ for each $y \in V(T_2 - v)$. By Claim 2.1, $L(T)$ admits a P_5-factor, a contradiction.

(*iii*) Assume $c_1(T, v)c_2(T, v) > 0$. Then $c_1(T, v) > 0$ and $c_2(T, v) > 0$. Let C_4 and C_5 be two components in $C_1(T, v)$ and $C_2(T, v)$, respectively. Set $T_1 = C_4 \cup C_5 \cup \{v\}$ and $T_2 = T - C_4 - C_5$. Then $c_i(T_1, v) = 1$ for $i \in \{1, 2\}$ and $c_i(T_1, v) = 0$ for $i \in \{0, 3\}$. It follows that $f_{T_1}(v) = 0$. Thus, $c_0(T_1, v) = f_{T_1}(v)$. We also have $c_i(T_2, v) = c_i(T, v) - 1$ for $i \in \{1, 2\}$ and $c_i(T_2, v) = c_i(T, v)$ for $i \in \{0, 3\}$. It follows that $f_{T_2}(v) = f_T(v)$. Thus, $c_0(T_2, v) = c_0(T, v) \geq f_T(v) = f_{T_2}(v)$. Note that $|E(T_1)| \equiv 0 \pmod 5$ and $|E(T_2)| \equiv 0 \pmod 5$. Utilizing an analogous argument to that in (*i*), we obtain $c_0(T_1, x) \geq f_{T_1}(x)$ for each $x \in V(T_1 - v)$ and $c_0(T_2, y) \geq f_{T_2}(y)$ for each $y \in V(T_2 - v)$. By Claim 2.1, $L(T)$ admits a P_5-factor, a contradiction.

(*iv*) Assume there exists a nontrivial component C_6 in $C_0(T, v)$. Choose a vertex u in $V(C_6)$ such that $uv \in E(T)$. Set $T_1 = C_6$ and $T_2 = (T - C_6) \cup \{u\}$. As $C_6 \in C_0(T, v)$, we obtain $|E(T_1)| \equiv 0 \pmod 5$ and $|E(T_2)| \equiv 0 \pmod 5$. Then for each $x \in V(T_1)$, we have $c_i(T_1, x) = c_i(T, x)$ for $i \in \{0, 1, 2, 3\}$. It follows that $f_T(x) = f_{T_1}(x)$. Thus, $c_0(T_1, x) = c_0(T, x) \geq f_T(x) = f_{T_1}(x)$ for each $x \in V(T_1)$. Similarly, $c_0(T_2, y) = c_0(T, y) \geq f_T(y) = f_{T_2}(y)$ for each $y \in V(T_2)$. By Claim 2.1, $L(T)$ admits a P_5-factor, a contradiction. □

It follows from (i) and (ii) of Claim 2.2 that $c_3(T, v) = c_4(T, v) = 0$. As $|E(T)|$ is divisible by 5, note that there are $c_i(T, v)$ edges connecting v and the set $C_i(T, v)$ for $i \in \{0, 1, 2\}$. We obtain

$$\sum_{i=0}^{2} \sum_{C \in C_i(T,v)} |E(C)| + c_0(T, v) + c_1(T, v) + c_2(T, v) \equiv 0 \pmod 5 \tag{2.1}$$

Since $|E(C)| \equiv i \pmod 5$ for each $C \in C_i(T, v)$, where $i \in \{0, 1, 2\}$. We can derive that

$$c_0(T, v) + 2c_1(T, v) + 3c_2(T, v) \equiv 0 \pmod 5 \tag{2.2}$$

By (*iii*) of Claim 2.2, $c_1(T, v)c_2(T, v) = 0$.

If $c_1(T, v) = c_2(T, v) = 0$, then $f_T(v) = 0$ and $c_0(T, v) \geq 0$. By (1), we deduce $c_0(T, v) \equiv 0 \pmod 5$. If $c_0(T, v) = 0$, then $|E(T)| = 0$, yielding a contradiction. If $c_0(T, v) \geq 5$, from (*iv*) of Claim 2.2, $T - v$ contains at least five isolated vertices, which we designate as u_1, u_2, u_3, u_4, and u_5. Let T_1 denote the subtree of T induced by $\{v, u_1, u_2, u_3, u_4, u_5\}$ and $T_2 = T - \{u_1, u_2, u_3, u_4, u_5\}$. It is straightforward to verify that $c_0(T_1, v) = 5$ and $f_{T_1}(v) = c_3(T_1, v) = 0$, which implies that $c_0(T_1, v) \geq f_{T_1}(v)$. Since $c_3(T, v) = 0$, we have $c_3(T_2, v) = 0$. Then $f_{T_2}(v) = 0$. It follows that $c_0(T_2, v) = c_0(T, v) - 5 \geq f_{T_2}(v)$. Observe that $|E(T_1)| = 5$ and $|E(T_2)| \equiv 0 \pmod 5$. Utilizing an analogous argument to that in (i) of Claim 2.2, we have $c_0(T_1, x) \geq f_{T_1}(x)$ for each $x \in V(T_1 - v)$ and $c_0(T_2, y) \geq f_{T_2}(y)$ for each $y \in V(T_2 - v)$. By Claim 2.1, $L(T)$ admits a P_5-factor, a contradiction.

If $c_1(T, v) = 0$ and $c_2(T, v) > 0$, then $f_T(v) = 2c_2(T, v)$ and $c_0(T, v) \geq 2c_2(T, v)$. By (1), we have $c_0(T, v) + 3c_2(T, v) \equiv 0 \pmod 5$. Let u_1 and u_2 be two isolated vertices in $C_0(T, v)$, and C_7 be a component in $C_2(T, v)$. Define $T_1 = C_7 \cup \{u_1, u_2, v\}$ and $T_2 = T - C_7 - \{u_1, u_2\}$. It is straightforward to verify that $c_0(T_1, v) = 2$ and $f_{T_1}(v) = 2$, which implies that $c_0(T_1, v) \geq f_{T_1}(v)$. Since $c_2(T_2, v) = c_2(T, v) - 1$ and $c_i(T_2, v) = c_i(T, v) = 0$ for $i \in \{1, 3\}$, we have $f_{T_2}(v) = 2c_2(T_2, v) = 2(c_2(T, v) - 1)$. It follows that $c_0(T_2, v) = c_0(T, v) - 2 \geq f_T(v) - 2 = 2c_2(T, v) - 2 = f_{T_2}(v)$. Note that $|E(T_1)| \equiv 0 \pmod 5$ and $|E(T_2)| \equiv 0 \pmod 5$. As a similar argument to that in (i) of Claim 2.2, we have $c_0(T_1, x) \geq f_{T_1}(x)$ for each $x \in V(T_1 - v)$ and $c_0(T_2, y) \geq f_{T_2}(y)$ for each $y \in V(T_2 - v)$. By Claim 2.1, $L(T)$ admits a P_5-factor, a contradiction.

If $c_2(T, v) = 0$ and $c_1(T, v) > 0$, where $c_1(T, v) \equiv 0 \pmod 2$, then $f_T(v) = \frac{1}{2}c_1(T, v)$ and $c_0(T, v) \geq \frac{1}{2}c_1(T, v)$. By (1), we have $c_0(T, v) + 2c_1(T, v) \equiv 0 \pmod 5$. Let C_8 and C_9 be two components in $C_1(T, v)$, and let u_1 be an isolated vertex in $C_0(T, v)$. Set $T_1 = C_8 \cup C_9 \cup \{u_1, v\}$ and $T_2 = T - C_8 - C_9 - \{u_1\}$. It is straightforward to verify that $c_0(T_1, v) = 1$ and $f_{T_1}(v) = 1$, which implies that $c_0(T_1, v) \geq f_{T_1}(v)$. Since $c_1(T_2, v) = c_1(T, v) - 2$ and $c_i(T_2, v) = c_i(T, v) = 0$ for $i \in \{2, 3\}$, we have $f_{T_2}(v) = \frac{1}{2}c_1(T_2, v) = \frac{1}{2}(c_1(T, v) - 2)$. It follows that $c_0(T_2, v) = c_0(T, v) - 1 \geq f_T(v) - 1 = \frac{1}{2}c_1(T, v) - 1 = f_{T_2}(v)$. Note that $|E(T_1)| \equiv 0 \pmod 5$ and $|E(T_2)| \equiv 0 \pmod 5$. Utilizing an analogous argument to that in (i) of Claim 2.2, we have $c_0(T_1, x) \geq f_{T_1}(x)$ for each $x \in V(T_1 - v)$ and $c_0(T_2, y) \geq f_{T_2}(y)$ for each $y \in V(T_2 - v)$. By Claim 2.1, $L(T)$ admits a P_5-factor, also a contradiction.

If $c_2(T, v) = 0$ and $c_1(T, v) > 0$, where $c_1(T, v) \equiv 1 \pmod 2$, then $f_T(v) = \frac{1}{2}(c_1(T, v) + 5)$ and $c_0(T, v) \geq \frac{1}{2}(c_1(T, v) + 5) \geq 3$. By (1), we have $c_0(T, v) + 2c_1(T, v) \equiv 0 \pmod 5$. Let u_1, u_2 and u_3 be three isolated vertices in $C_0(T, v)$ and C_{10} be a component in $C_1(T, v)$. Define $T_1 = C_{10} \cup \{u_1, u_2, u_3, v\}$ and $T_2 = T - C_{10} - \{u_1, u_2, u_3\}$. It is straightforward to verify that $c_0(T_1, v) = 3$ and $f_{T_1}(v) = 3$, which implies that $c_0(T_1, v) \geq f_{T_1}(v)$. Since $c_1(T_2, v) = c_1(T, v) - 1$ and $c_i(T_2, v) = c_i(T, v) = 0$ for $i \in \{2, 3\}$, we have $f_{T_2}(v) = \frac{1}{2}c_1(T_2, v) = \frac{1}{2}(c_1(T, v) - 1)$. It follows that $c_0(T_2, v) = c_0(T, v) - 3 \geq f_T(v) - 3 = \frac{1}{2}(c_1(T, v) + 5) - 3 = \frac{1}{2}(c_1(T, v) - 1) = f_{T_2}(v)$. Observe that $|E(T_1)| \equiv 0 \pmod 5$ and $|E(T_2)| \equiv 0 \pmod 5$. Utilizing an analogous argument to that in (i) of Claim 2.2, we have $c_0(T_1, x) \geq f_{T_1}(x)$ for each $x \in V(T_1 - v)$ and $c_0(T_2, y) \geq f_{T_2}(y)$ for each $y \in V(T_2 - v)$. By Claim 2.1, $L(T)$ admits a P_5-factor, also a contradiction. Thus, the sufficiency is proved.

Next, we will prove necessity. Let H denote a P_5-factor of $L(T)$. For any $v \in V(T)$, let $N_T(v) = \{y_1, y_2, \ldots, y_d\}$, where $d = d_T(v)$. For each $i \in \{1, 2, \ldots, d\}$, let T_i be the component of $T - v$ that contains y_i, and let $v_i \in V(L(T))$ correspond to edge vy_i. For any $T_i \in C_j(T, v)$ with $1 \leq j \leq 4$, there exists a P_5-component H_i of H such that $1 \leq |V(H_i) \cap V(L(T_i))| \leq j$.

For any $u \in V(L(T_i))$, observe that

$$N_H(u) \subseteq N_{L(T)}(u) \subseteq V(L(T_i)) \cup \{v_i\} \tag{1}$$

From (2), we derive that

$$v_i \in V(H_i) \quad \text{and} \quad |V(H_i) \cap V(L(T_i))| = j \tag{2}$$

It follows that $|V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\})| \leq 5 - (j+1) = 4 - j$. We say T_i is *related to* T_{j_i} if $v_{j_i} \in V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\})$.

(a) For each $T_i \in C_4(T, v)$, by (3), $|V(H_i) \cap V(L(T_i))| = 4$ and $v_i \in V(H_i)$, since $|V(H_i) \cap V(L(T_i \cup \{v_i\}))| = 5$, thus $H \cap L(T_i \cup \{v\})$ is a P_5-factor of $L(T_i \cup \{v\})$.

(b) For each $T_i \in C_3(T, v)$, $|V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\})| = 1$. Let $V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\}) = \{v_{j_i}\}$. Then by (2), $T_{j_i} \in C_0(T, v)$. Obviously, $j_i \neq j_{i'}$ if T_i and $T_{i'}$ are distinct components in $C_3(T, v)$.

(c) For each $T_i \in C_2(T, v)$, $|V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\})| = 1$ or 2.

If $|V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\})| = 1$, let $V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\}) = \{v_{j_i}\}$, then by (2), $T_{j_i} \in C_1(T, v)$. Obviously, $j_i \neq j_{i'}$ if T_i and $T_{i'}$ are distinct components in $C_2(T, v)$.

If $|V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\})| = 2$, let $V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\}) = \{v_{j_{i_1}}, v_{j_{i_2}}\}$, then by (2), $T_{j_{i_m}} \in C_0(T, v)$ for $m \in \{1, 2\}$. Obviously, $j_{i_m} \neq j_{i'_m}$ if T_{i_m} and $T_{i'_m}$ are distinct components in $C_2(T, v)$.

(d) For each $T_i \in C_1(T, v)$, $|V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\})| = 1$ or 2 or 3.

If $|V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\})| = 1$, let $V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\}) = \{v_{j_i}\}$, then by (2), $T_{j_i} \in C_2(T, v)$. Obviously, $j_i \neq j_{i'}$ if T_i and $T_{i'}$ are distinct components in $C_1(T, v)$.

If $|V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\})| = 2$, let $V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\}) = \{v_{j_{i_1}}, v_{j_{i_2}}\}$, then by (2), T_i is related to exactly one component $T_{j_{i_1}}$ in $C_0(T, v)$ and one component $T_{j_{i_2}}$ in $C_1(T, v)$.

If $|V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\})| = 3$, let $V(H_i) \cap (\{v_1, v_2, \ldots, v_d\} - \{v_i\}) = \{v_{j_{i_1}}, v_{j_{i_2}}, v_{j_{i_3}}\}$, then by (2), $T_{j_{i_n}} \in C_0(T, v)$ for $j \in \{1, 2, 3\}$. Obviously, $j_{i_n} \neq j_{i'_n}$ if T_{i_n} and $T_{i'_n}$ are distinct components in $C_1(T, v)$.

From the discussion above, by (b), each component in $C_3(T, v)$ is related to exactly one component in $C_0(T, v)$.

Case 1. $c_1(T, v) \leq c_2(T, v)$

By (c), each component in $C_2(T, v)$ is related to one component in $C_1(T, v)$ or two components in $C_0(T, v)$. Then the components in $C_2(T, v)$ are related to at least $2(c_2(T, v) - c_1(T, v))$ components in $C_0(T, v)$. Hence,

$$c_0(T, v) \geq c_3(T, v) + 2(c_2(T, v) - c_1(T, v)).$$

Case 2. $c_1(T, v) > c_2(T, v)$

By (d), each component in $C_1(T, v)$ is related to one component in $C_2(T, v)$, or one component in $C_0(T, v)$ and one component in $C_1(T, v)$, or three components in $C_0(T, v)$.

If $c_1(T, v) - c_2(T, v) \equiv 0 \pmod 2$, note that two distinct components in $C_1(T, v)$ can be related to one component in $C_0(T, v)$, then the components in $C_1(T, v)$ are related to at least $\frac{1}{2}(c_1(T, v) - c_2(T, v))$ components in $C_0(T, v)$. Hence,

$$c_0(T, v) \geq c_3(T, v) + \frac{1}{2}(c_1(T, v) - c_2(T, v)).$$

If $c_1(T, v) - c_2(T, v) \equiv 1 \pmod 2$, then the components in $C_0(T, v)$ related to $C_1(T, v)$ should be at least $\frac{1}{2}(c_1(T, v) - c_2(T, v) - 1) + 3$. Hence,

$$c_0(T, v) \geq c_3(T, v) + \tfrac{1}{2}(c_1(T, v) - c_2(T, v) - 1) + 3 = c_3(T, v) + \tfrac{1}{2}(c_1(T, v) - c_2(T, v)) + \tfrac{5}{2}.$$

Therefore, Theorem 2.1 is proved.

3. An algorithm for finding a P_5-factor in the line graph of a tree

Let T be a tree such that $|E(T)|$ is divisible by 5, satisfying $c_0(T, v) \geq f_T(v)$ for each $v \in V(T)$. By Theorem 2.1, the line graph $L(T)$ contains a P_5-factor. First, we apply the following algorithm to obtain a subtree of T, denoted by T_0.

Algorithm **Input:** A tree T.

Step 1 : Set $T_0 := T$ and $H := \emptyset$.

Step 2 : Let v be a leaf of $Stem(T_0)$.

Case 1 If $|N_{T_0}(v) \cap Leaf(T_0)| \geq 5$, let $\{v_1, v_2, v_3, v_4, v_5\} \subseteq N_{T_0}(v) \cap Leaf(T_0)$, then go to Step 3.

Case 2 If $|N_{T_0}(v) \cap Leaf(T_0)| = 4$, let $N_{T_0}(v) = \{w, v_1, v_2, v_3, v_4\}$, where $w \in V(Stem(T_0)) \cap N_{T_0}(v)$ and $v_i \in Leaf(T_0)$ for $i \in [1, 4]$, then go to Step 4.

Otherwise, go to Step 5;

Step 3 : Add $v_{vv_1}v_{vv_2}v_{vv_3}v_{vv_4}v_{vv_5}$ to H, and set $T_0 := T_0 - \{v_1, v_2, v_3, v_4, v_5\}$, go to Step 2;

Step 4 : Add $v_{vv_1}v_{vv_2}v_{vv_3}v_{vv_4}v_{vw}$ to H, and set $T_0 := T_0 - \{v_1, v_2, v_3, v_4, v\}$, go to Step 2;

Step 5 : Output T_0 and H.

There exists a P_5-factor in $L(T_0)$. Let $P_{T_0}[v, u]$ be a path from v to u in T_0. For $x \in V(P)$, denote by $H(T_0, x)$ the set of components in $T_0 - x$ excluding those containing u or v. Let $q(u)$ be the first vertex of degree ≥ 3 met going along the path P starting from u. Let $x \in V(P)$ such that $d_{T_0}(x) \geq 3$. Let $(T_0 - x)_u$ and $(T_0 - x)_v$ be the components in $T_0 - x$ containing u and v, respectively. We can easily obtain the following Claims.

Claim 3.1. *If $(T_0 - x)_u \in C_3(T_0, x)$, then one of the following holds:*
(i) there exists a component $C \in C_0(T_0, x)$ in $H(T_0, x)$;
(ii)$(T_0 - x)_v \in C_0(T_0, x)$ and $c_4(T_0, x) \neq 0$;
(iii) $(T_0 - x)_v \in C_0(T_0, x)$ and $c_1(T_0, x) = c_2(T_0, x) \neq 0$.

Proof. Since $L(T_0)$ has a P_5-factor and $(T_0 - x)_u \in C_3(T_0, x)$, by Theorem 2.1, $c_0(T_0, x) \geq 1$. If there exists no component C in $H(T_0, x)$ such that $C \in C_0(T_0, x)$, then we have $(T_0 - x)_v \in C_0(T_0, x)$ and $c_0(T_0, x) = 1$. If $c_4(T_0, x) = 0$, note that $(T_0 - x)_u \in C_3(T_0, x)$ and $d_{T_0}(x) \geq 3$, then we have $c_3(T_0, x) = 1$ and $c_1(T_0, x) = c_2(T_0, x) \neq 0$. □

Claim 3.2. *If $(T_0 - x)_u \in C_2(T_0, x)$, then there exists a component $C \in C_0(T_0, x) \cup C_1(T_0, x) \cup C_4(T_0, x)$ in $H(T_0, x)$.*

Proof. Suppose on the contrary that $C \in C_2(T_0, x) \cup C_3(T_0, x)$ for any $C \in H(T_0, x)$. If there exists a component $C \in C_2(T_0, x)$ in $H(T_0, x)$, note that $(T_0 - x)_u \in C_2(T, x)$, by Theorem 2.1, if $(T_0 - x)_v \in C_1(T_0, x)$, then $c_0(T_0, x) \geq 2$, a contradiction. If $(T_0 - x)_v \notin C_1(T_0, x)$, then $c_0(T_0, x) \geq 4$, also a contradiction. Otherwise, each component C in $H(T_0, x)$ belongs to $C_3(T_0, x)$. By Theorem 2.1, if $(T_0 - x)_v \in C_1(T_0, x)$, then $c_0(T_0, x) \geq 1$, a contradiction. If $(T_0 - x)_v \notin C_1(T_0, x)$, then $c_0(T_0, x) \geq 3$, also a contradiction. □

Let $P := P_{T_0}[v, u]$ is a longest path in T_0 such that
(P1) $dist_{T_0}(u, q(u))$ is as small as possible;
(P2) $d_{T_0}(q(u))$ is as large as possible, subject to (P1);
(P3) $d_{T_0}((q(u)^{-})$ is as large as possible, subject to (P2);
(P4) $d_{T_0}((q(u)^{-2})$ is as large as possible, subject to (P3).

Obviously, $v, u \in Leaf(T_0)$. Our algorithm is based on the following Observations.

(a) For any $C \in H(T_0, u^{-2})$, C is an isolated vertex, an isolated edge, or a $K_{1,t}$ with center adjacent to u^{-2}, where $t \in \{2, 3\}$.

(b) If $d_{T_0}(u^-) = 3$, $d_{T_0}(u^{-2}) = 2$ and $d_{T_0}(u^{-3}) \geq 3$, then by Claim 3.1, there exists a component $C \in \bigcup_{i\in I} C_i(T_0, u^{-3})$ in $H(T_0, u^{-3})$, where $I = \{0, 1, 2, 4\}$. By the choice of P, we have $max_{y\in V(C)} dist_{T_0}(y, u^{-3}) \leq 3$.

If $dist_{T_0}(y, u^{-3}) \leq 2$ for any $y \in V(C)$, then C is an isolated vertex or an isolated edge or a $K_{1,2}$ with center adjacent to u^{-3}. If there exists a vertex $y \in V(C)$ such that $dist_{T_0}(y, u^{-3}) = 3$, by the choice of P, we have $d_{T_0}(y^-) = 2$. If $d_{T_0}(y^{-2}) \geq 4$, then there exist at least two isolated edges yy^-, zz^- and an isolated vertex w(type 1), or exactly one isolated edge yy^- and at least two isolated vertices in $T - y^{-2}$(type 2). If $d_{T_0}(y^{-2}) = 3$, then there are two isolated edges yy^- and zz^- in $T - y^{-2}$(type 3). If $d_{T_0}(y^{-2}) = 2$, then C is a P_3 with one end vertex adjacent to u^{-3}(type 4).

(c) If $q(u) = u^{-2}$, by Observation (a) and the choice of P, each component C in $H(T_0, u^{-2})$ is an isolated vertex or an isolated edge. If there exists an isolated edge in $H(T_0, u^{-2})$, by Theorem 2.1, then $d_{T_0}(u^{-2}) = 3$ or $H(T_0, u^{-2})$ contains at least an isolated vertex. If there exist only isolated vertices in $H(T_0, u^{-2})$, for the case that $d_{T_0}(u^{-2}) = 3$, note that $(T_0 - u^{-3})_u \in C_3(T_0, u^{-3})$, as the similar argument to that in Observation (b), we have $d_{T_0}(u^{-3}) = 2$ or $H(T_0, u^{-3})$ contains a component of type (3), type (4), an isolated vertex, an isolated edge, or a $K_{1,2}$ with center adjacent to u^{-3}.

(d) If $q(u) = u^{-3}$, then by Claim 3.2, there exists a component $C \in \bigcup_{i\in I} C_i(T_0, u^{-3})$ in $H(T_0, u^{-3})$, where $I = \{0, 1, 4\}$. By the choice of P, we have $max_{y\in V(C)} dist_{T_0}(y, u^{-3}) \leq 2$. It follows that C is an isolated edge or an isolated vertex.

(e) If $q(u) = u^{-4}$, then by Claim 3.1, there exists a component $C \in \bigcup_{i\in I} C_i(T_0, u^{-4})$ in $H(T_0, u^{-4})$, where $I = \{0, 1, 2, 4\}$. By the choice of P, we have $max_{y\in V(C)} dist_{T_0}(y, u^{-4}) \leq 3$.

If $dist_{T_0}(y, u^{-4}) \leq 2$ for any $y \in V(C)$, then C is an isolated vertex, an isolated edge or a $K_{1,2}$ with center adjacent to u^{-4}. If there exists a vertex $y \in V(C)$ such that $dist_{T_0}(y, u^{-4}) = 3$, then $d_{T_0}(y^-) \leq 4$. Choose such a vertex y satisfying that $d_{T_0}(y^-)$ is as large as possible. If $d_{T_0}(y^-) = 3$, then $d_{T_0}(y^{-2}) \geq 3$. If $d_{T_0}(y^-) = 2$, as a similar argument to that in Observation (b), the component C is of type (1)–(4).

Algorithm

Input: A tree T_0, set $m = |E(T_0)|$ and $H_0 := \emptyset$.

Output: A P_5-factor H_0 in $L(T_0)$.

Do while $m > 5$

Step 1 Choose a longest path $P := P[v, u]$ satisfying (P1)-(P4) in T_0.

Step 2

Case 1 $q(u) = u^-$

Case 1.1 $d_{T_0}(u^-) = 4$

Set $N_T(u^-) = \{u, u_1, u_2, u^{-2}\}$.

Subcase 1.1.1 $d_{T_0}(u^{-2}) = 2$

Add $v_{uu^-}v_{u_1u^-}v_{u_2u^-}v_{u^-u^{-2}}v_{u^{-2}u^{-3}}$ to H_0, and set $T_0 := T_0 - \{u, u_1, u_2, u^-, u^{-2}\}$.

Subcase 1.1.2 $d_{T_0}(u^{-2}) \geq 3$

By Claim 3.1 and Observation (a), if $H(T_0, u^{-2})$ contains an isolated vertex u_3, then add $v_{uu^-}v_{u_1u^-}v_{u_2u^-}v_{u^-u^{-2}}\ v_{u^{-2}u_3}$ to H_0, and set $T_0 := T_0 - \{u, u_1, u_2, u_3, u^-\}$. If $H(T_0, u^{-2})$ contains an isolated edge u_4u_5 and a component $C_1 = u_6u_7u_8$, where $u_4, u_7 \in N_{T_0}(u^{-2})$, then add $v_{u_5u_4}v_{u_4u^{-2}}v_{u^{-2}u_7}v_{u_7u_6}v_{u_7u_8}$ to H_0, and set $T_0 := T_0 - \{u_4, u_5, u_6, u_7, u_8\}$.

Case 1.2 $d_{T_0}(u^-) = 3$

Set $N_{T_0}(u^-) = \{u, u_1, u^{-2}\}$.

Subcase 1.2.1 $d_{T_0}(u^{-2}) \geq 3$

By Claim 3.2 and Observation (a), if $H(T_0, u^{-2})$ contains an isolated edge u_2u_3, where $u_2 \in N_{T_0}(u^{-2})$, then add $v_{uu^-}v_{u_1u^-}v_{u^-u^{-2}}v_{u^{-2}u_2}v_{u_2u_3}$ to H_0, and set $T_0 := T_0 - \{u_1, u, u^-, u_2, u_3\}$. If $H(T_0, u^{-2})$ contains at least two isolated vertices, denoted by u_4 and u_5, then add $v_{uu^-}v_{u_1u^-}v_{u^-u^{-2}}\ v_{u^{-2}u_4}v_{u^{-2}u_5}$ to H_0, and set $T_0 := T_0 - \{u_1, u, u^-, u_4, u_5\}$. Otherwise, $d_{T_0}(u^{-2}) = 3$ and $H(T_0, u^{-2})$ contains exactly an isolated vertex denoted by u_6. Then add $v_{uu^-}v_{u_1u^-}v_{u^-u^{-2}}v_{u^{-2}u_6}\ v_{u^{-2}u^{-3}}$ to H_0, and set $T_0 := T_0 - \{u_1, u, u^-, u^{-2}, u_6\}$.

Subcase 1.2.2 $d_{T_0}(u^{-2}) = 2$ and $d_{T_0}(u^{-3}) \geq 3$

Let C be a component in $C_i(T_0, u^{-3})$ in $H(T_0, u^{-3})$, where $i \in \{0, 1, 2, 4\}$. By Claim 3.1 and Observation (b), if there exists an isolated vertex z in $H(T_0, u^{-3})$, then add $v_{uu^-}v_{u^-u_1}v_{u^-u^{-2}}v_{u^{-2}u^{-3}}\ v_{u^{-3}z}$ to H_0, and set $T_0 := T_0 - \{u, u^-, u_1, u^{-2}, z\}$. If there exist $C_1 = z_1z_2$ and $C_2 = w_1w_2w_3$ in $H(T_0, u^{-3})$, where $z_1, w_2 \in N_{T_0}(u^{-3})$. Then add $v_{z_2z_1}v_{z_1u^{-3}}v_{u^{-3}w_2}v_{w_2w_1}v_{w_2w_3}$ to H_0, and set $T_0 := T_0 - \{z_2, z_1, w_1, w_2, w_3\}$.

Otherwise, there exists a component C such that $max_{y \in V(C)} dist_{T_0}(y, u^{-3}) = 3$. If C belongs to type (1), then add $v_{yy^-}v_{y^-y^{-2}}v_{wy^{-2}}v_{y^{-2}z^-}v_{z^-z}$ to H_0, and set $T_0 = T_0 - \{y, y^-, z^-, z, w\}$. If C belongs to type (2), for the case that there exist at least three isolated vertices w_1, w_2 and w_3 in $T_0 - y^{-2}$, then add $v_{yy^-}v_{y^-y^{-2}}v_{y^{-2}w_1}v_{y^{-2}w_2}v_{y^{-2}w_3}$ to H_0, and set $T_0 := T_0 - \{y, y^-, w_1, w_2, w_3\}$; for the case that there exist exactly two isolated vertices w_1 and w_2 in $T_0 - y^{-2}$, then $v_{yy^-}v_{y^-y^{-2}}\ v_{y^{-2}w_1}v_{y^{-2}w_2}v_{y^{-2}u^{-3}}$ to H_0, and set $T_0 := T_0 - \{y, y^-, , y^{-2}, w_1, w_2\}$. If C belongs to type (3), then add $v_{yy^-}v_{y^-y^{-2}}v_{y^{-2}u^{-3}}v_{y^{-2}z^-}v_{z^-z}$ to H_0, and set $T_0 := T_0 - \{y, y^-, y^{-2}, z^-, z\}$. If C belongs to type (4), by Claim 3.1, there exists an isolated edge w^-w in $H(T_0, u^{-3})$, where $w^- \in N_{T_0}(u^{-3})$, then add $v_{yy^-}v_{y^-y^{-2}}v_{y^{-2}u^{-3}}v_{u^{-3}w^-}v_{w^-w}$ to H_0, and set $T_0 := T_0 - \{y, y^-, y^{-2}, w^-, w\}$.

Subcase 1.2.3 $d_{T_0}(u^{-2}) = d_{T_0}(u^{-3}) = 2$

Add $v_{u_1u^-}v_{uu^-}v_{u^-u^{-2}}v_{u^{-2}u^{-3}}v_{u^{-3}u^{-4}}$ to H, and set $T_0 := T_0 - \{u, u_1, u^-, u^{-2}, u^{-3}\}$.

Case 2 $q(u) = u^{-2}$

Let C be a component in $H(T_0, u^{-2})$.

Case 2.1 $d_{T_0}(u^{-2}) = 3$

By Observation (c), if C is an isolated edge w_1w_2 such that $w_1 \in N_{T_0}(u^{-2})$, then add $v_{uu^-}v_{u^-u^{-2}}v_{u^{-2}u^{-3}}v_{u^{-2}w_1}v_{w_1w_2}$ to H_0, and set $T_0 := T_0 - \{u, u^-, u^{-2}, w_1, w_2\}$. If C is an isolated vertex w, for the case that $d_{T_0}(u^{-3}) = 2$, then add $v_{uu^-}v_{u^-u^{-2}}v_{u^{-2}w}v_{u^{-2}u^{-3}}v_{u^{-3}u^{-4}}$ to H_0, and set $T_0 := T_0 - \{u, u^-, u^{-2}, w, u^{-3}\}$; for the case that $d_{T_0}(u^{-3}) \geq 3$, note that $(T - u^{-3})_u \in C_3(T, u^{-3})$, using the same method in Subcase 1.2.2, we can add a P_5 to H_0 and obtain a new tree T_0.

Case 2.2 $d_{T_0}(u^{-2}) \geq 4$

If there exist an isolated edge w_1w_2 and an isolated vertex w in $H(T_0, u^{-2})$, where $w_1 \in N_{T_0}(u^{-2})$, then add $v_{uu^-}v_{u^-u^{-2}}v_{u^{-2}w}v_{u^{-2}w_1}v_{w_1w_2}$ to H_0, and set $T_0 := T_0 - \{u, u^-, w_1, w_2, w\}$. If each component C is an isolated vertex, for the case that $d_{T_0}(u^{-2}) \geq 5$, let w_1, w_2 and w_3 be three isolated vertices in $H(T_0, u^{-2})$, then add $v_{uu^-}v_{u^-u^{-2}}v_{u^{-2}w_1}v_{u^{-2}w_2}v_{u^{-2}w_3}$ to H_0, and set $T_0 := T_0 - \{u, u^-, w_1, w_2, w_3\}$; for the case that $d_{T_0}(u^{-2}) = 4$, let w_1 and w_2 be two isolated vertices in $H(T_0, u^{-2})$, then add $v_{uu^-}v_{u^-u^{-2}}v_{u^{-2}w_1}v_{u^{-2}w_2}v_{u^{-2}u^{-3}}$ to H_0, and set $T_0 := T_0 - \{u, u^-, w_1, w_2, u^{-2}\}$.

Case 3 $q(u) = u^{-3}$

Let C be a component in $C_i(T_0, u^{-3})$ in $H(T_0, u^{-3})$, where $i \in \{0, 1, 4\}$. By Observation (d), if there exists a component $C = w_1w_2$ such that $w_1 \in N_{T_0}(u^{-3})$, then add $v_{uu^-}v_{u^-u^{-2}}v_{u^{-2}u^{-3}}\ v_{u^{-3}w_1}v_{w_1w_2}$ to H_0, and set $T_0 := T_0 - \{u, u^-, u^{-2}, w_1, w_2\}$. If $H(T_0, u^{-3})$ contains at least two isolated vertices w_1 and w_2, then add

$v_{uu^-}v_{u^-u^{-2}}v_{u^{-2}u^{-3}}v_{u^{-3}w_1}v_{u^{-3}w_2}$ to H_0, and set $T_0 := T_0 - \{u, u^-, u^{-2}, w_1, w_2\}$. Otherwise, $H(T_0, u^{-3})$ contains exactly one isolated vertex w. For the case that $d_{T_0}(u^{-3}) = 3$, then add $v_{uu^-}v_{u^-u^{-2}}v_{u^{-2}u^{-3}}v_{u^{-3}w}v_{u^{-3}u^{-4}}$ to H_0, and set $T_0 := T_0 - \{u, u^-, u^{-2}, u^{-3}, w\}$; for the case that $d_{T_0}(u^{-3}) \geq 4$, by Theorem 2.1, there exists a component C' isomorphic to a $K_{1,3}$ in $H(T_0, u^{-3})$, where $V(C) = \{z_1, z_2, z_3, z_4\}$ with $z_4 \in N_{T_0}(u^{-3})$, then add $v_{z_1z_4}v_{z_2z_4}v_{z_3z_4}v_{z_4w}$ to H_0, and set $T_0 := T_0 - \{z_1, z_2, z_3, z_4, w\}$.

Case 4 $q(u) = u^{-4}$

Let C be a component in $C_i(T_0, u^{-4})$ in $H(T_0, u^{-4})$, where $i \in \{0, 1, 2, 4\}$. By Claim 3.1 and Observation (e), if there exists an isolated vertex z in $H(T_0, u^{-4})$, then add $v_{uu^-}v_{u^-u^{-2}}v_{u^{-2}u^{-3}}v_{u^{-3}u^{-4}}\ v_{u^{-4}z}$ to H_0, and set $T_0 := T_0 - \{u, u^-, u^{-2}, u^{-3}, z\}$. If there exist $C_1 = z_1z_2$ and $C_2 = w_1w_2w_3$ in $H(T_0, u^{-4})$, where $z_1, w_2 \in N_{T_0}(u^{-3})$, then add $v_{z_2z_1}v_{z_1u^{-3}}v_{u^{-3}w_2}v_{w_2w_1}\ v_{w_2w_3}$ to H_0, and set $T_0 := T_0 - \{z_2, z_1, w_1, w_2, w_3\}$.

Otherwise, there exists a component C such that $max_{y \in V(C)} dist_{T_0}(y, u^{-4}) = 3$. Choose such a vertex y satisfying that $d_{T_0}(y^-)$ is as large as possible. If $d_{T_0}(y^-) = 4$, using the same method in Subcase 1.1, we can add a P_5 to H_0 and obtain a new tree T_0. If $d_{T_0}(y^-) = 3$, by Observation (e), $d_{T_0}(y^{-2}) \geq 3$, using the same method in Subcase 1.2.1, we can add a P_5 to H_0 and obtain a new tree T_0. If $d_{T_0}(y^-) = 2$ and $d_{T_0}(y^{-2}) \geq 3$, using the same method in Case 2, we can add a P_5 to H_0 and obtain a new tree T_0. Otherwise, we have $d_{T_0}(y^-) = 2$ and $d_{T_0}(y^{-2}) = 2$. By Claim 3.1, there exists an isolated edge w_1w_2 in $H(T_0, u^{-4})$, where $w_1 \in N_{T_0}(u^{-4})$. Then add $v_{w_2w_1}v_{w_1u^{-4}}v_{u^{-4}y^{-2}}v_{y^{-2}y^-}v_{y^-y}$ to H_0, and set $T_0 := T_0 - \{w_2, w_1, y^{-2}, y^-, y\}$.

Case 5 $q(u) \in V(P[v, u^{-5}])$

Add $v_{uu^-}v_{u^-u^{-2}}v_{u^{-2}u^{-3}}v_{u^{-3}u^{-4}}v_{u^{-4}u^{-5}}$ to H, set $T_0 := T_0 - \{u, u^-, u^{-2}, u^{-3}, u^{-4}\}$.

$m = m - 5$. End while.

Output H_0.

Therefore, $H \cup H_0$ is a desired P_5-factor in $L(T)$.

Author contributions

Yuan Chen: Conceptualization, Investigation, Methodology, Validation, Writing–original draft, Writing–review & editing; Chi Fan: Conceptualization, Investigation, Methodology, Validation, Writing–original draft; Pei Sun: Conceptualization, Investigation.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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