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Research article

Generalized derivations and quasi derivations of current Hom-Lie algebras

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Abstract: In this paper, we investigate the structure of generalized derivations and quasiderivations of the tensor product algebra $\mathcal{H}\otimes\mathcal{A}$, where $\mathcal{H}$ is a Hom-Lie algebra and $\mathcal{A}$ is a commutative associative algebra with unity over the field $\mathcal{K}$. We examine whether every generalized derivation (respectively, quasiderivation) of $\mathcal{H}\otimes\mathcal{A}$ can be expressed as a sum of a derivation and a linear map in the centroid of $\mathcal{H}\otimes\mathcal{A}$, provided that this property holds for $\mathcal{H}$. We establish a partial affirmative result under suitable conditions and discuss the algebraic implications of our findings.

Keywords: Hom-Lie algebra; generalized derivations; quasiderivation; centroid
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1. Introduction

The concept of Hom-Lie algebras was first introduced by Hartwig et al. [13], motivated by the study of various deformations of classical Lie algebras [2, 9–11, 26]. These algebraic structures naturally emerge in the context of differential calculus and deformed vector fields, which has led to significant interest and development over the past decade. The main areas of research in Hom-Lie algebra theory include representations, deformations, and (co)homology [1, 3, 23]; structural analysis of semisimple and simple Hom-Lie algebras [4,14,15,24]; geometric generalizations [18,21]; extension theory [8,19]; and integration theory for Hom-Lie algebras [17]. Since Hom-Lie algebras generalize classical Lie algebras, it is natural to extend the classical results and structural properties to this broader setting. In this work, we build on the investigations presented in [5], contributing further to the structural study of Hom-Lie algebras.

Let $\mathcal{A}$ be a commutative associative algebra with unity. Recently, in [5], Benkovič and Eremita proved that every generalized derivation of a current Lie algebra $\mathcal{L} \otimes \mathcal{A}$ can be expressed as the sum of a derivation of $\mathcal{L} \otimes \mathcal{A}$ and an element of the centroid of $\mathcal{L} \otimes \mathcal{A}$, provided the same decomposition holds for the Lie algebra $\mathcal{L}$. Motivated by their ideas, and building upon the foundational results of Sun, Ma, and Chen [22] on commuting maps of Hom-Lie algebras, we extend the ideas of Benkovič and Eremita to the framework of Hom-Lie algebras and establish the corresponding analog results for current Hom-Lie algebras. Given a Hom-Lie algebra $\mathcal{H}$, the tensor product $\mathcal{H} \otimes \mathcal{A}$ admits a natural Hom-Lie algebra structure, referred to as the current Hom-Lie algebra (see Definition 2.7).

In this article, we primarily address the following questions:

(1) If every generalized derivation of a Hom-Lie algebra $\mathcal{H}$ can be written as the sum of a derivation of $\mathcal{H}$ and an element of the centroid of $\mathcal{H}$, does the same property hold for the current Hom-Lie algebra $\mathcal{H} \otimes \mathcal{A}$?

(2) If every quasi-derivation of a Hom-Lie algebra $\mathcal{H}$ can be written as the sum of a derivation of $\mathcal{H}$ and an element of the centroid of $\mathcal{H}$, does the same property hold for the current Hom-Lie algebra $\mathcal{H} \otimes \mathcal{A}$?

We provide a partial affirmative answer to the above questions (Theorems 2.10 and 2.12) and analyze the implications of our results.

The organization of the paper is as follows. In Section 2, we recall the fundamental definitions and preliminary results necessary for our study. In this section, we also state our main results. In Section 3, we present detailed proofs of the main theorems.

2. Hom-Lie algebra

In this section, we recall some basic definitions and results on Hom-Lie algebras and state our main results. Throughout, $\mathcal{K}$ denotes a field.

Definition 2.1. [25] A multiplicative *Hom-Lie algebra* is a triple $(\mathcal{H}, [\cdot,\cdot], \alpha)$, where $\mathcal{H}$ is a vector space over the field $\mathcal{K}$ with the bilinear map $[\cdot,\cdot] : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and a linear map $\alpha : \mathcal{H} \to \mathcal{H}$ satisfying the following conditions for all $\zeta, \xi, \pi \in \mathcal{H}$:

(i) *Hom-Jacobi identity*:

$$[\alpha(\zeta), [\xi, \pi]] + [\alpha(\xi), [\pi, \zeta]] + [\alpha(\pi), [\zeta, \xi]] = 0.$$

(ii) *Hom-skew-symmetry*:

$$[\zeta, \xi] = -[\xi, \zeta].$$

(iii) $\alpha([\zeta, \xi]) = [\alpha(\zeta), \alpha(\xi)]$, for all $\zeta, \xi \in \mathcal{H}$.

The multiplicative Hom-Lie algebra $(\mathcal{H}, [\cdot,\cdot], \alpha)$ is said to be regular if the linear α is bijective.

Definition 2.2. [25] Let $(\mathcal{H}, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra. Then consider the following collection of endomorphisms of $\mathcal{H}$:

$$\mathbb{V} = \{\delta \in \mathrm{End}(\mathcal{H}) \mid \delta\alpha = \alpha\delta\},$$

and a linear map $\sigma : \mathbb{V} \to \mathbb{V}$ by $\sigma(\delta) = \alpha\delta$. Then, $(\mathbb{V}, [.,.], \sigma)$ forms a Hom-Lie algebra over $\mathcal{K}$ with the bracket $[D_1, D_2] = D_1D_2 - D_2D_1$ for all $D_1, D_2 \in \mathbb{V}$.

Definition 2.3. [25] Let $k \in \mathbb{N}$. An α^k*-derivation* d on the Hom-Lie algebra $(\mathcal{H}, [.,.], \alpha)$ is a linear map on $\mathcal{H}$ such that

$$d[\zeta, \xi] = [d(\zeta), \alpha^k(\xi)] + [\alpha^k(\zeta), d(\xi)], \text{ for all } \zeta, \xi \in \mathcal{H} \text{ and } d\alpha = \alpha d.$$

The collection of all α^k-derivations on $\mathcal{H}$ is denoted by $\mathrm{Der}_{\alpha^k}(\mathcal{H})$. The set $\mathrm{Der}(\mathcal{H}) = \bigoplus_{k\geq 0} \mathrm{Der}_{\alpha^k}(\mathcal{H})$ is a Hom-Lie subalgebra of $\mathbb{V}$.

Definition 2.4. [25] Let $k \in \mathbb{N}$. A linear map $\delta : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be an α^k*-quasiderivation* of $\mathcal{H}$ if there exists a linear map γ of $\mathcal{H}$ such that

$$[\delta(\zeta), \alpha^k(\xi)] + [\alpha^k(\zeta), \delta(\xi)] = \gamma([\zeta, \xi])$$

for all $\zeta, \xi \in \mathcal{H}$, and $\alpha\delta = \delta\alpha, \gamma\alpha = \alpha\gamma$.

Definition 2.5. [25] Let $k \in \mathbb{N}$. A linear map $\delta : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be a α^k*-generalized derivation* of $\mathcal{H}$ if there exist linear maps γ and ϕ of $\mathcal{H}$ such that

$$[\delta(\zeta), \alpha^k(\xi)] + [\alpha^k(\zeta), \gamma(\xi)] = \phi([\zeta, \xi])$$

for all $\zeta, \xi \in \mathcal{H}$ and $\alpha\delta = \delta\alpha, \gamma\alpha = \alpha\gamma, \phi\alpha = \alpha\phi$.

The notion of a derivation has several generalizations. In this article, we study the quasiderivations and generalized derivations of Hom-Lie algebras. Let $\mathrm{GDer}_{\alpha^k}(\mathcal{H})$ and $\mathrm{QDer}_{\alpha^k}(\mathcal{H})$ denote the collection of all α^k-generalized derivations and α^k-quasi-derivations of $\mathcal{H}$, respectively. Then,

$$\mathrm{GDer}(\mathcal{H}) = \bigoplus_{k\geq 0} \mathrm{GDer}_{\alpha^k}(\mathcal{H}) \text{ and } \mathrm{QDer}(\mathcal{H}) = \bigoplus_{k\geq 0} \mathrm{QDer}_{\alpha^k}(\mathcal{H}).$$

It is easy to verify that $\mathrm{GDer}(\mathcal{H})$ and $\mathrm{QDer}(\mathcal{H})$ are Hom-Lie subalgebras of $\mathbb{V}$ such that

$$\mathrm{Der}(\mathcal{H}) \subseteq \mathrm{QDer}(\mathcal{H}) \subseteq \mathrm{GDer}(\mathcal{H}) \subseteq \mathbb{V}.$$

Definition 2.6. [25] Let $(\mathcal{H}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra. Then, the centroid of $\mathcal{H}$ is defined by $\mathrm{Cent}(\mathcal{H}) = \bigoplus_{k\geq 0} \mathrm{Cent}_{\alpha^k}(\mathcal{H})$, where $\mathrm{Cent}_{\alpha^k}(\mathcal{H})$ consists of a linear map $\delta \in End(\mathcal{H})$ that satisfies $\delta([\zeta, \xi]) = [\alpha^k(\zeta), \delta(\xi)] = [\delta(\zeta), \alpha^k(\xi)]$ for all $\zeta, \xi \in \mathcal{H}$ and $\alpha\delta = \delta\alpha$.

It is easy to see that for every $\zeta, \xi \in \mathcal{H}$, we have

$$[\delta(\zeta), \alpha^k(\xi)] + [\alpha^k(\zeta), \delta(\xi)] = 2\delta([\zeta, \xi].$$

Thus, $\mathrm{Cent}(\mathcal{H}) \subseteq \mathrm{QDer}(\mathcal{H})$, and therefore

$$\mathrm{Der}(\mathcal{H}) + \mathrm{Cent}(\mathcal{H}) \subseteq \mathrm{QDer}(\mathcal{H}).$$

In many cases this inclusion is strict, but in some Lie algebras $(\mathcal{H}, [.,.], \alpha = Id)$,

$$\mathrm{Der}(\mathcal{H}) + \mathrm{Cent}(\mathcal{H}) = \mathrm{QDer}(\mathcal{H}) \tag{2.1}$$

or even

$$\mathrm{Der}(\mathcal{H}) + \mathrm{Cent}(\mathcal{H}) = \mathrm{GDer}(\mathcal{H}). \tag{2.2}$$

In [20], Leger and Luks proved that Eq (2.1) holds for every centerless Lie algebras generated by weight spaces. In [7], one can find examples of Lie algebras that satisfy Eq (2.2).

Definition 2.7. [16] A current Hom-Lie algebra is a tensor product of the form $(\mathcal{H}\otimes A,\ [\cdot,\cdot]_{\mathcal{H}}\otimes\mu,\ \alpha\otimes\beta)$, where $(\mathcal{H},[\cdot,\cdot]_{\mathcal{H}},\alpha)$ is a Hom-Lie algebra and (A,μ,β) is a Hom-associative commutative algebra. The current Hom-Lie algebra is denoted by $(\mathcal{H}\otimes A,\ [\cdot,\cdot]_{\mathcal{H}\otimes A},\ \gamma)$ instead of $(\mathcal{H}\otimes A,\ [\cdot,\cdot]_{\mathcal{H}}\otimes\mu,\ \alpha\otimes\beta)$.

The goal of this paper is to answer the following questions:

(i) If $\mathcal{H}$ satisfies Eq (2.1), does $\mathcal{H}\otimes A$ also satisfy Eq (2.1)?

(ii) If $\mathcal{H}$ satisfies Eq (2.2), does $\mathcal{H}\otimes A$ also satisfy Eq (2.2)?

Our work is inspired by Brešar's [6, 7], who introduced the study of functional identities on tensor products of algebras.

Example 2.8 (Loop Hom-Lie algebras). Let $(\mathcal{H},[\cdot,\cdot],\alpha)$ be a Hom-Lie algebra, define $\tilde{\mathcal{H}}=\mathcal{H}\otimes\mathbb{C}[t,t^{-1}]$, where $\mathbb{C}[t,t^{-1}]$ represents the Laurent polynomials. Define a bracket $[\cdot,\cdot]$ on $\tilde{\mathcal{H}}$ as follows:

$$[\zeta\otimes t^n,\xi\otimes t^m]=[\zeta,\xi]\otimes t^{n+m},\quad \forall\,\zeta,\xi\in\mathcal{H}\ \forall\,m,n\in\mathbb{Z},$$

and an endomorphism $\tilde{\alpha}:\tilde{\mathcal{H}}\to\tilde{\mathcal{H}}$ given by $\tilde{\alpha}=\alpha\otimes Id$, where Id denotes the identity map. Then, $(\tilde{\mathcal{H}},[\cdot,\cdot],\tilde{\alpha})$ is a multiplicative Hom-Lie algebra, which is commonly termed as Loop Hom-Lie algebra.

Now, we give a non-trivial example of Hom-Lie algebra that satisfies Eqs (2.1) and (2.2).

Example 2.9. Let $\mathcal{H}=\mathrm{Span}\{e,f\}$ be a vector space over a field $\mathcal{K}$ with $\mathrm{char}(\mathcal{K})\neq 2$. Define a skew-symmetric bracket $[\cdot,\cdot]:\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ by $[e,f]=f$, $[e,e]=[f,f]=0$, and an automorphism $\alpha:\mathcal{H}\to\mathcal{H}$ by $\alpha(e)=\lambda e$ and $\alpha(f)=\lambda f$. Then, $(\mathcal{H},[\cdot,\cdot],\alpha)$ forms a Hom-Lie algebra.

It is a routine computation to show that

$$\mathrm{Cent}(\mathcal{H})=\left\{\begin{pmatrix}a&0\\0&a\end{pmatrix}\mid a\in\mathcal{K}\right\},$$

$$\mathrm{Der}(\mathcal{H})=\left\{\begin{pmatrix}0&0\\a_1&a_2\end{pmatrix}\mid a_1,a_2\in\mathcal{K}\right\},$$

and

$$\mathrm{QDer}(\mathcal{H})=\mathrm{GDer}(\mathcal{H})=\left\{\begin{pmatrix}a_1&0\\a_2&a_3\end{pmatrix}\mid a_1,a_2,a_3\in\mathcal{K}\right\}.$$

Thus, it is clear that $\mathcal{H}$ satisfies Eqs (2.1) and (2.2).

Let $(\mathcal{H},[.,.],\alpha)$ be a Hom-Lie algebra over a field $\mathcal{K}$. The center of $\mathcal{H}$, denoted by $Z(\mathcal{H})$, is defined as

$$Z(\mathcal{H}):=\{\zeta\in\mathcal{H}\mid[\zeta,\xi]=0\text{ for all }\xi\in\mathcal{H}\}.$$

The derived algebra $[\mathcal{H},\mathcal{H}]$ of $\mathcal{H}$ is given by

$$[\mathcal{H},\mathcal{H}]:=\mathrm{Span}(\{[\zeta,\xi]\mid\xi,\zeta\in\mathcal{H}\}).$$

Both $Z(\mathcal{H})$ and $[\mathcal{H},\mathcal{H}]$ are ideals of $\mathcal{H}$. We say that $\mathcal{H}$ is centerless if $Z(\mathcal{H})=\{0\}$.

For any subset $\mathcal{W}$ of $\mathcal{H}$, the set

$$Z_{\mathcal{H}}(\mathcal{W}):=\{\zeta\in\mathcal{H}\mid[\zeta,w]=0\text{ for all }w\in\mathcal{W}\}$$

is called the annihilator of $\mathcal{W}$ in $\mathcal{H}$. It follows that $Z_{\mathcal{H}}(\mathcal{I})$ is an ideal of $\mathcal{H}$ whenever $\mathcal{I}$ is an ideal of $\mathcal{H}$ and $\alpha(\mathcal{I}) = \mathcal{I}$. Hence, $Z_{\mathcal{H}}([\mathcal{H}, \mathcal{H}])$ is an ideal of $\mathcal{H}$, and

$$Z(\mathcal{H}) = Z_{\mathcal{H}}(\mathcal{H}) \subseteq Z_{\mathcal{H}}([\mathcal{H}, \mathcal{H}]).$$

Observe that for any centerless Hom-Lie algebra $\mathcal{H}$, the sum

$$\mathrm{Der}(\mathcal{H}) + \mathrm{Cent}(\mathcal{H}) = \mathrm{Der}(\mathcal{H}) \oplus \mathrm{Cent}(\mathcal{H})$$

is a direct sum of vector spaces if α is invertible or $[\mathcal{H}, \mathcal{H}] = \mathcal{H}$.

A Hom-Lie algebra $\mathcal{H}$ is said to be prime if $\mathcal{H}$ has no nonzero ideals $\mathcal{I}$ and $\mathcal{J}$ such that $[\mathcal{I}, \mathcal{J}] = 0$. Obviously, all prime Hom-Lie algebras are centerless. If $[\mathcal{H}, \mathcal{H}] = \mathcal{H}$, then, the Hom-Lie algebra $\mathcal{H}$ is said to be perfect.

Let us demonstrate our major findings in the form of quasi-derivations of a current Hom-Lie algebra $\mathcal{H} \otimes \mathcal{A}$.

Theorem 2.10. *Let $\mathcal{H}$ be a centerless regular Hom-Lie algebra over the field $\mathcal{K}$ with* $\mathrm{char}(\mathcal{K}) \neq 2$*. Let $\mathcal{H} \otimes \mathcal{A}$ be a current Hom-Lie algebra over a field $\mathcal{K}$. Suppose that $\mathcal{H}$ is either prime or perfect. If*

$$\mathrm{QDer}(\mathcal{H}) = \mathrm{Der}(\mathcal{H}) \oplus \mathrm{Cent}(\mathcal{H})$$

(i.e., every α^k-quasiderivation can be written as a sum of an α^k-derivation and a α^k-centroid element), then,

$$\mathrm{QDer}(\mathcal{H} \otimes \mathcal{A}) = \mathrm{Der}(\mathcal{H} \otimes \mathcal{A}) \oplus \mathrm{Cent}(\mathcal{H} \otimes \mathcal{A}).$$

A similar kind of result can be established for generalized derivations of Hom-Lie algebras. In [25], the quasicentroid $\mathrm{QCent}(\mathcal{H})$ of a Hom-Lie algebra $\mathcal{H}$ is defined as

$$\mathrm{QCent}(\mathcal{H}) = \bigoplus_{k \geq 0} \mathrm{QCent}_{\alpha^k}(\mathcal{H}),$$

where

$$\mathrm{QCent}_{\alpha^k}(\mathcal{H}) = \{\delta \in \mathbb{V} \mid [\delta(\zeta), \alpha^k(\xi)] = [\alpha^k(\zeta), \delta(\xi)] \text{ for all } \zeta, \xi \in \mathcal{H}, \delta\alpha = \alpha\delta\}.$$

Obviously,

$$\mathrm{Cent}(\mathcal{H}) \subseteq \mathrm{QCent}(\mathcal{H})$$

and

$$\mathrm{GDer}(\mathcal{H}) = \mathrm{QDer}(\mathcal{H}) + \mathrm{QCent}(\mathcal{H}) \text{ (see [25])}. \tag{2.3}$$

Note that a map $\delta : \mathcal{H} \to \mathcal{H}$ is said to be commuting if

$$[\delta(\zeta), \alpha(\zeta)] = 0 \quad \text{for all } \zeta \in \mathcal{H}$$

and

$$\delta(\alpha(\zeta)) = \alpha(\delta(\zeta)) \quad \text{for all } \zeta \in \mathcal{H}.$$

Recall that the set of commuting linear maps $\delta : \mathcal{H} \to \mathcal{H}$ is a subset of $\mathrm{QCent}(\mathcal{H})$. Furthermore, $\mathrm{QCent}(\mathcal{H})$ coincides with the set of all commuting linear maps of $\mathcal{H}$ if $\mathrm{char}(\mathcal{K}) \neq 2$.

Let $\mathcal{H}$ be a centerless Hom-Lie algebra over a field $\mathcal{K}$ with $\text{char}(\mathcal{K}) \neq 2$. Assume that $\mathcal{H}$ is either perfect or prime. Then, it follows that $Z_{\mathcal{H}}([\mathcal{H}, \mathcal{H}]) = 0$. By applying the result of B. Sun, Y. Ma, and L. Chen [22, Theorem 4.3], we conclude that the centroid $\text{Cent}(\mathcal{H})$ coincides with the set of all commuting linear maps on $\mathcal{H}$. Hence, $\text{Cent}(\mathcal{H}) = \text{QCent}(\mathcal{H}) \subseteq \text{QDer}(\mathcal{H})$. Finally, using Eq (2.3), we deduce that $\text{GDer}(\mathcal{H}) = \text{QDer}(\mathcal{H})$. Furthermore, since $Z_{\mathcal{H}\otimes\mathcal{A}}([\mathcal{H} \otimes \mathcal{A}, \mathcal{H} \otimes \mathcal{A}]) = 0$, we conclude that $\text{GDer}(\mathcal{H} \otimes \mathcal{A}) = \text{QDer}(\mathcal{H} \otimes \mathcal{A})$. Thus, the following corollary is an immediate consequence of Theorem 2.10.

Corollary 2.11. *Let $\mathcal{H} \otimes \mathcal{A}$ be a current Hom-Lie algebra over a field $\mathcal{K}$, where $\mathcal{H}$ be regular , centerless, and* $\text{char}(\mathcal{K}) \neq 2$. *Suppose that $\mathcal{H}$ is perfect or prime. Then,*

$$\text{GDer}(\mathcal{H}) = \text{Der}(\mathcal{H}) \oplus \text{Cent}(\mathcal{H})$$

implies

$$\text{GDer}(\mathcal{H} \otimes \mathcal{A}) = \text{Der}(\mathcal{H} \otimes \mathcal{A}) \oplus \text{Cent}(\mathcal{H} \otimes \mathcal{A}).$$

If $\mathcal{H} \otimes \mathcal{A}$ forms a current Hom-Lie algebra with $\mathcal{A}$ finite-dimensional, then, the same conclusion holds under the weaker assumption that $\mathcal{H}$ is centerless.

Theorem 2.12. *Let $\mathcal{A}$ be a finite dimensional commutative algebra, $\mathcal{H}$ is regular, centerless Hom-Lie algebra and $\mathcal{H} \otimes \mathcal{A}$ be a current Hom-Lie algebra over a field $\mathcal{K}$ with* $\text{char}(\mathcal{K}) \neq 2$. *Then,*

(i) If $\text{Der}(\mathcal{H}) \oplus \text{Cent}(\mathcal{H}) = \text{GDer}(\mathcal{H})$, *then,* $\text{Der}(\mathcal{H} \otimes \mathcal{A}) \oplus \text{Cent}(\mathcal{H} \otimes \mathcal{A}) = \text{GDer}(\mathcal{H} \otimes \mathcal{A})$.

(ii) If $\text{QDer}(\mathcal{H}) = \text{Der}(\mathcal{H}) \oplus \text{Cent}(\mathcal{H})$, *then,* $\text{QDer}(\mathcal{H} \otimes \mathcal{A}) = \text{Der}(\mathcal{H} \otimes \mathcal{A}) \oplus \text{Cent}(\mathcal{H} \otimes \mathcal{A})$.

3. The proofs

Throughout this section, we denote by $\mathcal{H}$ the Hom-Lie algebra $(\mathcal{H}, [\cdot, \cdot], \alpha)$, and by $\mathcal{H} \otimes \mathcal{A}$ the current Hom-Lie algebra $(\mathcal{H}\otimes\mathcal{A}, [\cdot, \cdot], \bar{\alpha})$, where α is an automorphism of $\mathcal{H}$ and $\bar{\alpha}$ is the automorphism on $\mathcal{H} \otimes \mathcal{A}$ defined by $\bar{\alpha}(\zeta \otimes a) = \alpha(\zeta) \otimes a$ for all $\zeta \in \mathcal{H}$ and $a \in \mathcal{A}$.

Let $C = \{c_i : i \in I\}$ be a basis of $\mathcal{A}$. Thus, every element of $\mathcal{H} \otimes \mathcal{A}$ can be uniquely represented in the form $\zeta_{i_1} \otimes c_{i_1} + \zeta_{i_2} \otimes c_{i_2} + \cdots + \zeta_{i_n} \otimes c_{i_n}$, where $\zeta_i \in \mathcal{H}$ and $n \geq 1$. Let δ represent a linear map on $\mathcal{H} \otimes \mathcal{A}$. There exists a unique element $\delta_i(\zeta) \in \mathcal{H}, i \in I$, for each element $\zeta \in \mathcal{H}$ such that

$$\delta(\zeta \otimes 1) = \sum_{i \in I} \delta_i(\zeta) \otimes c_i,$$

where $\delta_i(\zeta) = 0$ for all but finitely many $i \in I$. The map δ_i on $\mathcal{H}$, defined as $\delta_i : \zeta \mapsto \delta_i(\zeta)$ is evidently linear for each $i \in I$. Let δ_C denotes a linear map on $\mathcal{H} \otimes \mathcal{A}$ such that

$$\delta_C(\zeta \otimes a) = \sum_{i \in I} \delta_i(\zeta) \otimes ac_i \tag{3.1}$$

for every simple tensor $\zeta \otimes a \in \mathcal{H} \otimes \mathcal{A}$. Clearly, δ_C is well-defined because for each $\zeta \in \mathcal{H}$, $\delta_i(\zeta)$ is nonzero for only a finitely many number of elements $i \in I$. Moreover, observe that $\delta(\zeta \otimes 1) = \delta_C(\zeta \otimes 1)$ for all $\zeta \in \mathcal{H}$.

We first establish some preliminary results that will be instrumental in proving our main theorems.

Proposition 3.1. *Let $\mathcal{H} \otimes \mathcal{A}$ denote current Hom-Lie algebra over a field $\mathcal{K}$. Then, for any basis C of $\mathcal{A}$, the following statements are true:*

(i) If $\delta \in \mathrm{GDer}(\mathcal{H} \otimes \mathcal{A})$, then, $\delta_C \in \mathrm{GDer}(\mathcal{H} \otimes \mathcal{A})$.

(ii) If $\delta \in \mathrm{QDer}(\mathcal{H} \otimes \mathcal{A})$, then, $\delta_C \in \mathrm{QDer}(\mathcal{H} \otimes \mathcal{A})$.

Proof. First, assume that $\delta \in \mathrm{GDer}(\mathcal{H} \otimes \mathcal{A})$. Then, from the definition, there exist two linear maps γ, ϕ on $\mathcal{H} \otimes \mathcal{A}$ such that

$$[\delta(\zeta), \bar{\alpha}^k(\xi)] + [\bar{\alpha}^k(\zeta), \gamma(\xi)] = \phi([\zeta, \xi]), \ \forall\, \zeta, \xi \in \mathcal{H} \otimes \mathcal{A}. \tag{3.2}$$

Choose a basis $C = \{c_i \mid i \in I\}$ of $\mathcal{A}$. Then, from Eq (3.1), the linear maps $\gamma_C, \phi_C : \mathcal{H} \otimes \mathcal{A} \to \mathcal{H} \otimes \mathcal{A}$ are given by

$$\gamma_C(\zeta \otimes a) = \sum_{i \in I} \gamma_i(\zeta) \otimes ac_i \quad \text{and} \quad \phi_C(\zeta \otimes a) = \sum_{i \in I} \phi_i(\zeta) \otimes ac_i$$

for every simple tensor $\zeta \otimes a \in \mathcal{H} \otimes \mathcal{A}$. To prove that $\delta_C \in \mathrm{GDer}(\mathcal{H} \otimes \mathcal{A})$, we need to show that

$$[\delta_C(\zeta), \bar{\alpha}^k(\xi)] + [\bar{\alpha}^k(\zeta), \gamma_C(\xi)] = \phi_C([\zeta, \xi]), \ \forall\, \zeta, \xi \in \mathcal{H} \otimes \mathcal{A}. \tag{3.3}$$

Setting simple tensors $\zeta \otimes 1$ and $\xi \otimes 1$ in Eq (3.2) and using $[\zeta \otimes 1, \xi \otimes 1] = [\zeta, \xi] \otimes 1$, we get

$$\phi([\zeta, \xi] \otimes 1) = [\delta(\zeta \otimes 1), \bar{\alpha}^k(\xi \otimes 1)] + [\bar{\alpha}^k(\zeta \otimes 1), \gamma(\xi \otimes 1)].$$

According to Eq (3.1), this identity can be rewritten as

$$\begin{aligned}
\sum_{i \in I} \varphi_i([\zeta, \xi]) \otimes c_i &= \left[\sum_{i \in I} \delta_i(\zeta) \otimes c_i,\ \bar{\alpha}^k(\xi \otimes 1)\right] + \left[\bar{\alpha}^k(\zeta \otimes 1), \sum_{i \in I} \gamma_i(\xi) \otimes c_i\right] \\
&= \left[\sum_{i \in I} \delta_i(\zeta) \otimes c_i,\ \alpha^k(\xi) \otimes 1\right] + \left[\alpha^k(\zeta) \otimes 1, \sum_{i \in I} \gamma_i(\xi) \otimes c_i\right] \\
&= \sum_{i \in I} \left[\delta_i(\zeta), \alpha^k(\xi)\right] \otimes c_i + \sum_{i \in I} \left[\alpha^k(\zeta), \gamma_i(\xi)\right] \otimes c_i,
\end{aligned}$$

and consequently,

$$\sum_{i \in I} \left([\delta_i(\zeta), \alpha^k(\xi)] + [\alpha^k(\zeta), \gamma_i(\xi)] - \phi_i([\zeta, \xi])\right) \otimes c_i = 0, \ \forall\, \zeta, \xi \in \mathcal{H}.$$

Thus, for any $i \in I$, we have

$$[\delta_i(\zeta), \alpha^k(\xi)] + [\alpha^k(\zeta), \gamma_i(\xi)] = \phi_i([\zeta, \xi]), \ \forall\, \zeta, \xi \in \mathcal{H}. \tag{3.4}$$

Using Eq (3.4), we have

$$\begin{aligned}
\phi_C([\zeta\otimes a,\xi\otimes c]) &= \phi_C([\zeta,\xi]\otimes ac)\\
&= \sum_{i\in I}\phi_i([\zeta,\xi])\otimes acc_i\\
&= \sum_{i\in I}\left([\delta_i(\zeta),\alpha^k(\xi)]+[\alpha^k(\zeta),\gamma_i(\xi)]\right)\otimes acc_i\\
&= \sum_{i\in I}[\delta_i(\zeta),\alpha^k(\xi)]\otimes acc_i+\sum_{i\in I}[\alpha^k(\zeta),\gamma_i(\xi)]\otimes acc_i\\
&= \sum_{i\in I}[\delta_i(\zeta)\otimes ac_i,\alpha^k(\xi)\otimes c]+\sum_{i\in I}[\alpha^k(\zeta)\otimes a,\gamma_i(\xi)\otimes cc_i]\\
&= \left[\sum_{i\in I}\delta_i(\zeta)\otimes ac_i,\bar{\alpha}^k(\xi\otimes c)\right]+\left[\bar{\alpha}^k(\zeta\otimes a),\sum_{i\in I}\gamma_i(\xi)\otimes cc_i\right]\\
&= [\delta_C(\zeta\otimes a),\bar{\alpha}^k(\xi\otimes c)]+[\bar{\alpha}^k(\zeta\otimes a),\gamma_C(\xi\otimes c)]
\end{aligned}$$

for all simple tensors $\zeta\otimes a,\xi\otimes c\in\mathcal{H}\otimes\mathcal{A}$. Since δ_C,γ_C, and ϕ_C are linear maps, it follows that Eq (3.3) holds. Hence, $\delta_C\in\mathrm{GDer}(\mathcal{H}\otimes\mathcal{A})$, and the proof of Part (i) is complete.

Observe that Part (ii) follows analogously by taking $\delta=\gamma$ in the preceding arguments. □

Proposition 3.2. *Let $\mathcal{H}\otimes\mathcal{A}$ be a current Hom-Lie algebra over a field $\mathcal{K}$, and let $C=\{c_i\mid i\in I\}$ be a basis of $\mathcal{A}$. Suppose that $\{\delta_i:\mathcal{H}\to\mathcal{H}\mid i\in I\}$ represents a family of linear maps such that for any $\zeta\in\mathcal{H}$, $\delta_i(\zeta)=0$ for only finitely many elements $i\in I$. Define a linear map $\delta_C:\mathcal{H}\otimes\mathcal{A}\to\mathcal{H}\otimes\mathcal{A}$ as in Eq (3.1).*

(i) If $\delta_i\in\mathrm{Der}(\mathcal{H})$ for all $i\in I$, then, $\delta_C\in\mathrm{Der}(\mathcal{H}\otimes\mathcal{A})$.

(ii) If $\delta_i\in\mathrm{Cent}(\mathcal{H})$ for all $i\in I$, then, $\delta_C\in\mathrm{Cent}(\mathcal{H}\otimes\mathcal{A})$.

Proof. (i) For all $i\in I$, let $\delta_i\in\mathrm{Der}(\mathcal{H})$. Thus, for every $i\in I$, we have:

$$[\delta_i(\zeta),\alpha^k(\xi)]+[\alpha^k(\zeta),\delta_i(\xi)]=\delta_i([\zeta,\xi]),\ \forall\,\zeta,\xi\in\mathcal{H}.$$

Let $\zeta\otimes a$, and $\xi\otimes c\in\mathcal{H}\otimes\mathcal{A}$ be arbitrary simple tensors. Then, we have

$$\begin{aligned}
\delta_C([\zeta\otimes a,\xi\otimes c]) &= \delta_C([\zeta,\xi]\otimes ac)=\sum_{i\in I}\delta_i([\zeta,\xi])\otimes acc_i\\
&= \sum_{i\in I}\Big([\delta_i(\zeta),\alpha^k(\xi)]+[\alpha^k(\zeta),\delta_i(\xi)]\Big)\otimes acc_i\\
&= \sum_{i\in I}[\delta_i(\zeta),\alpha^k(\xi)]\otimes acc_i+\sum_{i\in I}[\alpha^k(\zeta),\delta_i(\xi)]\otimes acc_i\\
&= \sum_{i\in I}[\delta_i(\zeta)\otimes ac_i,\alpha^k(\xi)\otimes c]+\sum_{i\in I}[\alpha^k(\zeta)\otimes a,\delta_i(\xi)\otimes cc_i]\\
&= \left[\sum_{i\in I}\delta_i(\zeta)\otimes ac_i,\bar{\alpha}^k(\xi\otimes c)\right]+\left[\bar{\alpha}^k(\zeta\otimes a),\sum_{i\in I}\delta_i(\xi)\otimes cc_i\right]\\
&= [\delta_C(\zeta\otimes a),\bar{\alpha}^k(\xi\otimes c)]+[\bar{\alpha}^k(\zeta\otimes a),\delta_C(\xi\otimes c)].
\end{aligned}$$

Thus, it implies that $\delta_C \in \mathrm{Der}(\mathcal{H} \otimes \mathcal{A})$, since δ_C is linear.

(ii) Now, let $\delta_i \in \mathrm{Cent}(\mathcal{H})$ for each $i \in I$. Thus, for every $i \in I$,

$$[\delta_i(\zeta), \alpha^k(\xi)] = \delta_i([\zeta, \xi]), \ \forall \ \zeta, \xi \in \mathcal{H}.$$

Let $\zeta \otimes a$, and $\xi \otimes c \in \mathcal{H} \otimes \mathcal{A}$ be an arbitrary simple tensors, then, we have

$$\begin{aligned}\delta_C([\zeta \otimes a, \xi \otimes c]) = \delta_C([\zeta, \xi] \otimes ac) &= \sum_{i \in I} \delta_i([\zeta, \xi]) \otimes acc_i \\ &= \sum_{i \in I} [\delta_i(\zeta), \alpha^k(\xi)] \otimes acc_i \\ &= \sum_{i \in I} [\delta_i(\zeta) \otimes ac_i, \alpha^k(\xi) \otimes c] \\ &= [\delta_C(\zeta \otimes a), \bar{\alpha}^k(\xi \otimes c)].\end{aligned}$$

Again, since δ_C is linear, it implies that $\delta_C \in \mathrm{Cent}(\mathcal{H} \otimes \mathcal{A})$. □

Proposition 3.3. *Let $\mathcal{H} \otimes \mathcal{A}$ be a current Hom-Lie algebra over a field $\mathcal{K}$, and let $C = \{c_i \mid i \in I\}$ be a basis of $\mathcal{A}$. Suppose that $\mathcal{H}$ is either prime or perfect and* $\mathrm{QDer}(\mathcal{H}) = \mathrm{Der}(\mathcal{H}) \oplus \mathrm{Cent}(\mathcal{H})$. *If* $\delta \in \mathrm{QDer}(\mathcal{H} \otimes \mathcal{A})$*, then,* $\delta_C \in \mathrm{Der}(\mathcal{H} \otimes \mathcal{A}) \oplus \mathrm{Cent}(\mathcal{H} \otimes \mathcal{A})$.

Proof. Let $\delta \in \mathrm{QDer}(\mathcal{H} \otimes \mathcal{A})$ be an arbitrary quasi-derivation. Then, we have a linear map $h : \mathcal{H} \otimes \mathcal{A} \to \mathcal{H} \otimes \mathcal{A}$ such that

$$h([\zeta, \xi]) = [\delta(\zeta), \bar{\alpha}^k(\xi)] + [\bar{\alpha}^k(\zeta), \delta(\xi)]$$

for all $\zeta, \xi \in \mathcal{H} \otimes \mathcal{A}$. We know that the sets $\{\delta_i : \mathcal{H} \to \mathcal{H} | i \in I\}$ and $\{h_i : \mathcal{H} \to \mathcal{H} | i \in I\}$ are the families of linear maps such that

$$\delta(\zeta \otimes 1) = \sum_{i \in I} \delta_i(\zeta) \otimes c_i \quad \text{and} \quad h(\zeta \otimes 1) = \sum_{i \in I} h_i(\zeta) \otimes c_i, \ \forall \ \zeta \in \mathcal{H},$$

where for all $\zeta \in \mathcal{H}$, $\delta_i(\zeta) = 0$ and $h_i(\zeta) = 0$ for finitely many $i \in I$. Similarly from the arguments used in the proof of Proposition 3.1, we get that

$$[\delta_i(\zeta), \alpha^k(\xi)] + [\alpha^k(\zeta), \delta_i(\xi)] = h_i([\zeta, \xi]) \tag{3.5}$$

for any $i \in I$ and for all $\zeta, \xi \in \mathcal{H}$. Thus, each δ_i is a quasi-derivation of $\mathcal{H}$. Now, from our hypothesis, we get that $\delta_i \in \mathrm{Der}(\mathcal{H}) \oplus \mathrm{Cent}(\mathcal{H})$ for all $i \in I$. Therefore, for every $i \in I$, there exist two maps $\psi_i \in \mathrm{Der}(\mathcal{H})$ and $\eta_i \in \mathrm{Cent}(\mathcal{H})$ such that $\delta_i = \psi_i + \eta_i$. Hence, Eq (3.5) can be rewritten as

$$\begin{aligned}h_i([\zeta, \xi]) &= [\psi_i(\zeta) + \eta_i(\zeta), \alpha^k(\xi)] + [\alpha^k(\zeta), \psi_i(\xi) + \eta_i(\xi)] \\ &= [\psi_i(\zeta), \alpha^k(\xi)] + [\alpha^k(\zeta), \psi_i(\xi)] + [\eta_i(\zeta), \alpha^k(\xi)] + [\alpha^k(\zeta), \eta_i(\xi)] \\ &= \psi_i([\zeta, \xi]) + \eta_i([\zeta, \xi]) + \eta_i([\zeta, \xi]) \\ &= \delta_i([\zeta, \xi]) + \eta_i([\zeta, \xi])\end{aligned} \tag{3.6}$$

for each $\zeta, \xi \in \mathcal{H}$ and $i \in I$.

Next, suppose that $\mathcal{H}$ is prime. From Eq (3.6), it is evident that for any $\zeta, \xi \in \mathcal{H}$, we have

$$\eta_i([\zeta, \xi]) = 0 \quad \text{for all but finitely many} \quad i \in I.$$

If $\mathcal{H}$ is the zero module, then, no further argument is needed. Thus, assume that $\mathcal{H}$ is nonzero. Since $\mathcal{H}$ is prime, we know that $[\mathcal{H}, \mathcal{H}] \neq \{0\}$. Consequently, there exist elements $\zeta_0, \xi_0 \in \mathcal{H}$ such that $[\zeta_0, \xi_0] \neq 0$. Given that $\mathcal{H}$ is torsion-free over $\text{Cent}(\mathcal{H})$ (see [12, Theorem 1.1]) and that $\eta_i([\zeta_0, \xi_0]) = 0$ for all but finitely many $i \in I$; it follows that $\eta_i = 0$ for all but finitely many $i \in I$. Thus, for any $\zeta \in \mathcal{H}$, we also conclude that

$$\psi_i(\zeta) = \delta_i(\zeta) - \eta_i(\zeta) = 0 \quad \text{for all but finitely many} \quad i \in I.$$

Define the linear maps $\psi_C, \eta_C : \mathcal{H} \otimes \mathcal{A} \to \mathcal{H} \otimes \mathcal{A}$ as follows:

$$\psi_C(\zeta \otimes a) = \sum_{i \in I} \psi_i(\zeta) \otimes ac_i \quad \text{and} \quad \eta_C(\zeta \otimes a) = \sum_{i \in I} \eta_i(\zeta) \otimes ac_i \tag{3.7}$$

for every simple tensor $\zeta \otimes a \in \mathcal{H} \otimes \mathcal{A}$. Since both the sums in Eq (3.7) are finite when $\mathcal{H}$ is perfect or prime, it implies that ψ_C and η_C are well-defined. From Proposition 3.2, we get that $\psi_C \in \text{Der}(\mathcal{H} \otimes \mathcal{A})$ and $\eta_C \in \text{Cent}(\mathcal{H} \otimes \mathcal{A})$. Thus, we conclude that

$$\begin{aligned} \delta_C(\zeta \otimes a) &= \sum_{i \in I} \delta_i(\zeta) \otimes ac_i = \sum_{i \in I} (\psi_i(\zeta) + \eta_i(\zeta)) \otimes ac_i \\ &= \sum_{i \in I} \psi_i(\zeta) \otimes ac_i + \sum_{i \in I} \eta_i(\zeta) \otimes ac_i \\ &= \psi_C(\zeta \otimes a) + \eta_C(\zeta \otimes a) \end{aligned}$$

for every simple tensor $\zeta \otimes a \in \mathcal{H} \otimes \mathcal{A}$. Since the maps δ_C, ψ_C, and η_C are linear, it implies that $\delta_C = \psi_C + \eta_C \in \text{Der}(\mathcal{H} \otimes \mathcal{A}) \oplus \text{Cent}(\mathcal{H} \otimes \mathcal{A})$. □

We can eliminate the assumption that $\mathcal{H}$ must be perfect or prime if we assume that $\mathcal{A}$ is a finite-dimensional algebra in Proposition 3.3. Specifically, in this case, both sums in Eq (3.7) will be finite, ensuring that the maps ψ_C and γ_C are well-defined. Consequently, we can obtain the following proposition by applying reasoning similar to that used in the proof of Proposition 3.3.

Proposition 3.4. *Let $\mathcal{H} \otimes \mathcal{A}$ be a current Hom-Lie algebra over a field $\mathcal{K}$ and let C be a basis of $\mathcal{A}$. Suppose that $\mathcal{A}$ is a finite-dimensional algebra over $\mathcal{K}$.*

(i) If $\delta \in \text{GDer}(\mathcal{H} \otimes \mathcal{A})$ and $\text{GDer}(\mathcal{H}) = \text{Der}(\mathcal{H}) \oplus \text{Cent}(\mathcal{H})$, then, $\delta_C \in \text{Der}(\mathcal{H} \otimes \mathcal{A}) \oplus \text{Cent}(\mathcal{H} \otimes \mathcal{A})$.

(ii) If $\delta \in \text{QDer}(\mathcal{H} \otimes \mathcal{A})$ and $\text{QDer}(\mathcal{H}) = \text{Der}(\mathcal{H}) \oplus \text{Cent}(\mathcal{H})$, then, $\delta_C \in \text{Der}(\mathcal{H} \otimes \mathcal{A}) \oplus \text{Cent}(\mathcal{H} \otimes \mathcal{A})$.

Now, we prove the following propositions that will be used in the proof of Theorem 2.3.

Proposition 3.5. *Let $\mathcal{H}$ be a centerless regular Hom-Lie algebra over a field $\mathcal{K}$ with char$(\mathcal{K}) \neq 2$. If δ is a commuting α-quasi-derivation, then, $\delta \in \text{Cent}(\mathcal{H})$.*

Proof. From the hypothesis that δ is a α-quasi-derivation, thus there exists a linear map h on $\mathcal{H}$ such that

$$h([\zeta_1, \zeta_2]) = [\delta(\zeta_1), \alpha(\zeta_2)] + [\alpha(\zeta_1), \delta(\zeta_2)], \ \forall\, \zeta_1, \zeta_2 \in \mathcal{H}. \tag{3.8}$$

Since δ is commuting, we have

$$[\delta(\zeta), \alpha(\zeta)] = 0, \ \forall\, \zeta \in \mathcal{H}. \tag{3.9}$$

Now, replacing ζ by $\zeta_1 + \zeta_2$ in Eq (3.9), we get

$$[\delta(\zeta_1), \alpha(\zeta_2)] = [\alpha(\zeta_1), \delta(\zeta_2)], \ \forall\, \zeta_1, \zeta_2 \in \mathcal{H}.$$

Thus, from Eq (3.8), we have

$$g([\zeta_1, \zeta_2]) = [\delta(\zeta_1), \alpha(\zeta_2)] = [\alpha(\zeta_1), \delta(\zeta_2)], \tag{3.10}$$

for all $\zeta_1, \zeta_2 \in \mathcal{H}$, where $h = 2g$.

Observe that for all $\zeta_1, \zeta_2, \zeta_3 \in \mathcal{H}$,

$$[g([\zeta_1, \zeta_2]), \alpha(\zeta_3)] = [[\delta(\zeta_1), \alpha(\zeta_2)], \alpha(\zeta_3)]. \tag{3.11}$$

By the Hom-Lie Jacobi identity, we get

$$[[\delta(\zeta_1), \alpha(\zeta_2)], \alpha(\zeta_3)] = [\alpha(\delta(\zeta_1)), [\alpha(\zeta_2), \zeta_3]] - [\alpha^2(\zeta_2), [\delta(\zeta_1), \zeta_3]]. \tag{3.12}$$

Since $\delta(\alpha(\zeta)) = \alpha(\delta(\zeta))$ for all $\zeta \in \mathcal{H}$, from Eq (3.12), we have

$$[[\delta(\zeta_1), \alpha(\zeta_2)], \alpha(\zeta_3)] = [\delta(\alpha(\zeta_1)), \alpha([\zeta_2, \alpha^{-1}(\zeta_3)])] - [\alpha^2(\zeta_2), [\delta(\zeta_1), \zeta_3]].$$

Now, using Eq (3.10), we get

$$[[\delta(\zeta_1), \alpha(\zeta_2)], \alpha(\zeta_3)] = [\alpha^2(\zeta_1), \delta([\zeta_2, \alpha^{-1}(\zeta_3)])] - [\alpha^2(\zeta_2), [\delta(\zeta_1), \zeta_3]].$$

Replacing ζ_3 by $\alpha(\zeta_3)$, we get

$$[[\delta(\zeta_1), \alpha(\zeta_2)], \alpha^2(\zeta_3)] = [\alpha^2(\zeta_1), \delta([\zeta_2, \zeta_3])] - [\alpha^2(\zeta_2), [\delta(\zeta_1), \alpha(\zeta_3)]].$$

Thus, from Eq (3.11), we have

$$[g([\zeta_1, \zeta_2]), \alpha^2(\zeta_3)] = [\alpha^2(\zeta_1), \delta([\zeta_2, \zeta_3])] - [\alpha^2(\zeta_2), [\delta(\zeta_1), \alpha(\zeta_3)]].$$

This implies

$$\begin{aligned} [g([\zeta_1, \zeta_2]), \alpha^2(\zeta_3)] + [g([\zeta_3, \zeta_1]), \alpha^2(\zeta_2)] &= [\alpha^2(\zeta_1), \delta([\zeta_2, \zeta_3])] \\ &= g([\alpha(\zeta_1), [\zeta_2, \zeta_3]]). \end{aligned} \tag{3.13}$$

Now, cyclically permuting ζ_1, ζ_2, and ζ_3 in Eq (3.13), we obtain the following two equations

$$[g([\zeta_2, \zeta_3]), \alpha^2(\zeta_1)] + [g([\zeta_1, \zeta_2]), \alpha^2(\zeta_3)] = g([\alpha(\zeta_2), [\zeta_3, \zeta_1]]) \tag{3.14}$$

and

$$[g([\zeta_3,\zeta_1]),\alpha^2(\zeta_2)]+[g([\zeta_2,\zeta_3]),\alpha^2(\zeta_1)]=g([\alpha(\zeta_3),[\zeta_1,\zeta_2]]). \tag{3.15}$$

Now, adding Eqs (3.13)–(3.15), we get

$$2([g([\zeta_3,\zeta_1]),\alpha^2(\zeta_2)]+[g([\zeta_2,\zeta_3]),\alpha^2(\zeta_1)]+[g([\zeta_1,\zeta_2]),\alpha^2(\zeta_3)])=0.$$

Thus, for all $\zeta_1,\zeta_2,\zeta_3\in\mathcal{H}$,

$$[g([\zeta_3,\zeta_1]),\alpha^2(\zeta_2)]+[g([\zeta_1,\zeta_2]),\alpha^2(\zeta_3)]=[\alpha^2(\zeta_1),g([\zeta_2,\zeta_3])]. \tag{3.16}$$

Comparing Eqs (3.13) and (3.16), we get

$$[\alpha^2(\zeta_1),g([\zeta_2,\zeta_3])]=[\alpha^2(\zeta_1),[\delta(\zeta_2),\alpha(\zeta_3)]]=[\alpha^2(\zeta_1),\delta([\zeta_2,\zeta_3])].$$

Since $Z(\mathcal{H})=0$, it follows that

$$\delta([\zeta_2,\zeta_3])=[\delta(\zeta_2),\alpha(\zeta_3)]=[\alpha(\zeta_2),\delta(\zeta_3)]$$

for all $\zeta_2,\zeta_3\in\mathcal{H}$. Thus, $\delta\in\mathrm{Cent}(\mathcal{H})$. □

Proposition 3.6. *Let $\mathcal{H}$ be a centerless regular Hom-Lie algebra over a field $\mathcal{K}$ with char$(\mathcal{K})\neq 2$. If δ is an α^k-quasi-derivation and $[\delta(\zeta),\alpha^k(\zeta)]=0$ for all $\zeta\in\mathcal{H}$, then, $\delta\in\mathrm{Cent}(\mathcal{H})$.*

Proof. Since δ is an α^k-quasi-derivation, there exists a linear map h on $\mathcal{H}$ such that

$$h([\zeta_1,\zeta_2])=[\delta(\zeta_1),\alpha^k(\zeta_2)]+[\alpha^k(\zeta_1),\delta(\zeta_2)]$$

for all $\zeta_1,\zeta_2\in\mathcal{H}$. Applying α^{-k+1} to both sides yields

$$\alpha^{-k+1}h([\zeta_1,\zeta_2])=[\alpha^{-k+1}\delta(\zeta_1),\alpha(\zeta_2)]+[\alpha(\zeta_1),\alpha^{-k+1}\delta(\zeta_2)]$$

for all $\zeta_1,\zeta_2\in\mathcal{H}$.

Moreover, since $[\delta(\zeta),\alpha^k(\zeta)]=0$ for all $\zeta\in\mathcal{H}$, it follows that

$$[\alpha^{-k+1}\delta(\zeta),\alpha(\zeta)]=0$$

for all $\zeta\in\mathcal{H}$. This shows that $\alpha^{-k+1}\delta$ is a commuting α-quasi-derivation.

Now, by Proposition 3.5, we conclude that $\alpha^{-k+1}\delta\in\mathrm{Cent}(\mathcal{H})$, and hence $\delta\in\mathrm{Cent}(\mathcal{H})$. □

Now, we proceed to prove our main results, namely Theorems 2.10 and 2.12.

Proof of Theorem 2.10. Suppose that $\mathrm{QDer}(\mathcal{H})=\mathrm{Der}(\mathcal{H})\oplus\mathrm{Cent}(\mathcal{H})$. Choose any basis $C=\{c_i\mid i\in I\}$ of $\mathcal{A}$. Let $\delta\in\mathrm{QDer}(\mathcal{H}\otimes\mathcal{A})$ be any arbitrary quasi-derivation. By Proposition 3.1, the map δ_C is a quasi-derivation of $\mathcal{H}\otimes\mathcal{A}$. Furthermore, by Proposition 3.3, we have

$$\delta_C\in\mathrm{Der}(\mathcal{H}\otimes\mathcal{A})\oplus\mathrm{Cent}(\mathcal{H}\otimes\mathcal{A}).$$

Define $\mathcal{F}:=\delta-\delta_C$. Clearly, $\mathcal{F}\in\mathrm{QDer}(\mathcal{H}\otimes\mathcal{A})$, and for all $\zeta\in\mathcal{H}$, we have

$$\mathcal{F}(\zeta\otimes 1)=\delta(\zeta\otimes 1)-\delta_C(\zeta\otimes 1)=0.$$

Since $\mathcal{F} \in \mathrm{QDer}(\mathcal{H} \otimes \mathcal{A})$, there exists a linear map $\mathcal{G}\colon \mathcal{H} \otimes \mathcal{A} \to \mathcal{H} \otimes \mathcal{A}$ such that

$$\mathcal{G}([\xi, \eta]) = [\mathcal{F}(\xi), \bar{\alpha}(\eta)] + [\bar{\alpha}(\xi), \mathcal{F}(\eta)] \tag{3.17}$$

for all $\xi, \eta \in \mathcal{H} \otimes \mathcal{A}$. There exist unique elements $\mathcal{F}_i(\zeta \otimes a) \in \mathcal{H}$, $i \in I$ for each simple tensor $\zeta \otimes a \in \mathcal{H} \otimes \mathcal{A}$, such that

$$\mathcal{F}(\zeta \otimes a) = \sum_{i \in I} \mathcal{F}_i(\zeta \otimes a) \otimes c_i,$$

where $\mathcal{F}_i(\zeta \otimes a) = 0$ for only finitely many $i \in I$. We define a map $\mathcal{F}_{a,i} : \mathcal{H} \to \mathcal{H}$ by $\mathcal{F}_{a,i}(\zeta) = \mathcal{F}_i(\zeta \otimes a)$ for each $a \in \mathcal{A}$ and each $i \in I$. The map $\mathcal{F}_{a,i}$ is linear. First, we show that for any $a \in \mathcal{A}$ and any $i \in I$, the element $\mathcal{F}_{a,i} \in \mathrm{Cent}(\mathcal{H})$. Let us fix any arbitrary $a \in \mathcal{A}$. Choosing simple tensors $\zeta \otimes a$ and $\zeta \otimes 1$ in Eq (3.17) and using $\mathcal{F}(\zeta \otimes 1) = 0$, we get

$$[\mathcal{F}(\zeta \otimes a), \bar{\alpha}^k(\zeta \otimes 1)] = \mathcal{G}([\zeta \otimes a, \zeta \otimes 1]) = \mathcal{G}([\zeta, \zeta] \otimes a) = 0$$

for all $\zeta \in \mathcal{H}$. Thus, $0 = [\mathcal{F}(\zeta \otimes a), \bar{\alpha}^k(\zeta \otimes 1)] = \sum_{i \in I}[\mathcal{F}_i(\zeta \otimes a) \otimes c_i, \alpha^k(\zeta) \otimes 1] = \sum_{i \in I}[\mathcal{F}_{a,i}(\zeta), \alpha^k(\zeta)] \otimes c_i$ for all $\zeta \in \mathcal{H}$. Thus, for any $\zeta \in \mathcal{H}$ and for all $i \in I$, we have $[\mathcal{F}_{a,i}(\zeta), \alpha^k(\zeta)] = 0$, which implies $[\alpha^{-k+1}\mathcal{F}_{a,i}(\zeta), \alpha(\zeta)] = 0$. Hence, the map $\alpha^{-k+1}\mathcal{F}_{a,i}$ is commuting for each $a \in \mathcal{A}$ and each $i \in I$. By the hypothesis, either $\mathcal{H}$ is a prime Hom-Lie algebra or it is perfect and centerless. In both cases, we have $Z_{\mathcal{H}}([\mathcal{H}, \mathcal{H}]) = \{0\}$. Therefore, it follows from [22, Theorem 4.3] that for all $i \in I$ and $a \in \mathcal{A}$, the map $\alpha^{-k+1}\mathcal{F}_{a,i} \in \mathrm{Cent}(\mathcal{H})$, and hence $\mathcal{F}_{a,i} \in \mathrm{Cent}(\mathcal{H})$.

Next, we assert that $\mathcal{F} \in \mathrm{Der}(\mathcal{H} \otimes \mathcal{A})$. Indeed, choosing simple tensors $\zeta \otimes a$ and $\xi \otimes b$ in Eq (3.17), we get

$$\mathcal{G}([\zeta \otimes a, \xi \otimes b]) = [\mathcal{F}(\zeta \otimes a), \bar{\alpha}^k(\xi \otimes b)] + [\bar{\alpha}^k(\zeta \otimes a), \mathcal{F}(\xi \otimes b)]. \tag{3.18}$$

However, we know that $[\zeta \otimes a, \xi \otimes b] = [\zeta \otimes ab, \xi \otimes 1]$ and $\mathcal{F}(\xi \otimes 1) = 0$, thus we get

$$\mathcal{G}[\zeta \otimes a, \xi \otimes b] = [\mathcal{F}(\zeta \otimes ab), \bar{\alpha}^k(\xi \otimes 1)] = \Big[\sum_{i \in I} \mathcal{F}_{ab,i}(\zeta) \otimes c_i, \bar{\alpha}^k(\xi \otimes 1)\Big]$$

for all $\zeta, \xi \in \mathcal{H}$ and $a, b \in \mathcal{A}$. Since $\mathcal{F}_{ab,i} \in \mathrm{Cent}(\mathcal{H})$, it implies that

$$\mathcal{G}([\zeta \otimes a, \xi \otimes b]) = \sum_{i \in I}[\mathcal{F}_{ab,i}(\zeta), \alpha^k(\xi)] \otimes c_i = \sum_{i \in I} \mathcal{F}_{ab,i}([\zeta, \xi]) \otimes c_i = \mathcal{F}([\zeta, \xi] \otimes ab)$$

for all $\zeta, \xi \in \mathcal{H}$ and $a, b \in \mathcal{A}$. Hence,

$$\mathcal{G}([\zeta \otimes a, \xi \otimes b]) = \mathcal{F}([\zeta \otimes a, \xi \otimes b])$$

for all $\zeta \otimes a, \xi \otimes b \in \mathcal{H} \otimes \mathcal{A}$. Thus, Eq (3.18) can be rephrased as

$$\mathcal{F}(\zeta \otimes a, \xi \otimes b) = [\mathcal{F}(\zeta \otimes a), \bar{\alpha}^k(\xi \otimes b)] + [\bar{\alpha}^k(\zeta \otimes a), \mathcal{F}(\xi \otimes b)]$$

for every $\zeta \otimes a, \xi \otimes b \in \mathcal{H} \otimes \mathcal{A}$. Since $\mathcal{F}$ is linear, it implies that $\mathcal{F} \in \mathrm{Der}(\mathcal{H} \otimes \mathcal{A})$. Thus, we conclude that

$$\delta = \delta_C + \mathcal{F},$$

where $\delta_C \in \mathrm{Der}(\mathcal{H} \otimes \mathcal{A}) \oplus \mathrm{Cent}(\mathcal{H} \otimes \mathcal{A})$ and $\mathcal{F} \in \mathrm{Der}(\mathcal{H} \otimes \mathcal{A})$. Thus, $\delta \in \mathrm{Der}(\mathcal{H} \otimes \mathcal{A}) \oplus \mathrm{Cent}(\mathcal{H} \otimes \mathcal{A})$. This completes the proof. □

Proof of Theorem 2.12 (i). Suppose that $\mathrm{GenDer}(\mathcal{H}) = \mathrm{Der}(\mathcal{H}) \oplus \mathrm{Cent}(\mathcal{H})$. Choose any basis $C = \{c_i \mid i \in I\}$ of $\mathcal{A}$. Since $\mathcal{A}$ is finite-dimensional so, here $I = \{1, 2, \ldots, n\}$. Let $\delta \in \mathrm{GenDer}(\mathcal{H} \otimes \mathcal{A})$. Then, there exist linear maps $\gamma, \phi : \mathcal{H} \otimes \mathcal{A} \to \mathcal{H} \otimes \mathcal{A}$ such that

$$[\delta(\zeta), \bar{\alpha}^k(\xi)] + [\bar{\alpha}^k(\zeta), \gamma(\xi)] = \phi([\zeta, \xi]) \tag{3.19}$$

for all $\zeta, \xi \in \mathcal{H} \otimes \mathcal{A}$. From Proposition 3.1, we have $\delta_C \in \mathrm{GenDer}(\mathcal{H} \otimes \mathcal{A})$. Moreover,

$$[\delta_C(\zeta), \bar{\alpha}^k(\xi)] + [\bar{\alpha}^k(\zeta), \gamma_C(\xi)] = \phi_C([\zeta, \xi]) \tag{3.20}$$

for all $\zeta, \xi \in \mathcal{H} \otimes \mathcal{A}$. From Proposition 3.4, we have that $\delta_C \in \mathrm{Der}(\mathcal{H} \otimes \mathcal{A}) \oplus \mathrm{Cent}(\mathcal{H} \otimes \mathcal{A})$. Now, define the maps: $\mathcal{F} = \delta - \delta_C$, $\Gamma = \gamma - \gamma_C$, and $T = \phi - \phi_C$. Clearly, $\mathcal{F}$, Γ, and T are linear maps. From Eqs (3.19) and (3.20), we get

$$[\mathcal{F}(\zeta), \bar{\alpha}^k(\xi)] + [\bar{\alpha}^k(\zeta), \Gamma(\xi)] = T([\zeta, \xi]) \tag{3.21}$$

for all $\zeta, \xi \in \mathcal{H} \otimes \mathcal{A}$. Additionally, $\mathcal{F}(\zeta \otimes 1) = 0 = \Gamma(\zeta \otimes 1)$ for every $\zeta \in \mathcal{H}$. For any simple tensor $\zeta \otimes a \in \mathcal{H} \otimes \mathcal{A}$, there exist unique elements $\mathcal{F}_i(\zeta \otimes a), \Gamma_i(\zeta \otimes a), T_i(\zeta \otimes a) \in \mathcal{H}, i \in I$, such that

$$\begin{aligned} \mathcal{F}(\zeta \otimes a) &= \sum_{i \in I} \mathcal{F}_i(\zeta \otimes a) \otimes c_i, \\ \Gamma(\zeta \otimes a) &= \sum_{i \in I} \Gamma_i(\zeta \otimes a) \otimes c_i, \\ T(\zeta \otimes a) &= \sum_{i \in I} T_i(\zeta \otimes a) \otimes c_i. \end{aligned} \tag{3.22}$$

We define maps $\mathcal{F}_{a,i}, \Gamma_{a,i}, T_{a,i} : \mathcal{H} \to \mathcal{H}$ by $\mathcal{F}_{a,i}(\zeta) \mapsto \mathcal{F}_i(\zeta \otimes a)$, $\Gamma_{a,i}(\zeta) \mapsto \Gamma_i(\zeta \otimes a)$, and $T_{a,i}(\zeta) \mapsto T_i(\zeta \otimes a)$ for each $a \in \mathcal{A}$ and each $i \in I$. It is evident that the maps $\mathcal{F}_{a,i}, \Gamma_{a,i},$ and $T_{a,i}$ are linear. Our goal is to show that $\mathcal{F}$ is a derivation. First, we will prove that $\mathcal{F}_{a,i} \in \mathrm{Cent}(\mathcal{H})$ for every $a \in \mathcal{A}$ and $i \in I$. Since $\Gamma(\xi \otimes 1) = 0$, then, from (3.21) we get

$$[\mathcal{F}(\zeta \otimes a), \bar{\alpha}^k(\xi \otimes 1)] = T([\zeta \otimes a, \xi \otimes 1]) = T([\zeta, \xi]) \otimes a \tag{3.23}$$

for each $\zeta, \xi \in \mathcal{H}$ and $a \in \mathcal{A}$. Fix an arbitrary element $a \in \mathcal{A}$. From Eq (3.22), we can rewrite Eq (3.23) as

$$\begin{aligned} 0 &= [\mathcal{F}(\zeta \otimes a), \bar{\alpha}^k(\xi \otimes 1)] - T([\zeta, \xi]) \otimes a \\ &= \left[\sum_{i \in I} \mathcal{F}_i(\zeta \otimes a) \otimes c_i, \bar{\alpha}^k(\xi \otimes 1) \right] - \sum_{i \in I} T_i([\zeta, \xi] \otimes a) \otimes c_i \\ &= \sum_{i \in I} \Big([\mathcal{F}_i(\zeta \otimes a), \alpha^k(\xi)] - T_i([\zeta, \xi]) \Big) \otimes c_i, \end{aligned}$$

for all $\zeta, \xi \in \mathcal{H}$. Thus,

$$[\mathcal{F}_{a,i}(\zeta), \alpha^k(\xi)] = T_{a,i}([\zeta, \xi]) \tag{3.24}$$

for all $\zeta, \xi \in \mathcal{H}$. As a result,

$$[\mathcal{F}_{a,i}(\zeta), \alpha^k(\zeta)] = 0$$

for all $\zeta \in \mathcal{H}$. By interchanging ζ and ξ in Eq (3.24) and using the fact $[\zeta, \xi] = -[\xi, \zeta]$, we get

$$[\alpha^k(\zeta), \mathcal{F}_{a,i}(\xi)] = T_{a,i}([\zeta, \xi]),$$

and thus

$$[\mathcal{F}_{a,i}(\zeta), \alpha^k(\xi)] + [\alpha^k(\zeta), \mathcal{F}_{a,i}(\xi)] = 2T_{a,i}([\zeta, \xi])$$

for all $\zeta, \xi \in \mathcal{H}$. This implies that $\mathcal{F}_{a,i} \in \mathrm{QDer}(\mathcal{H})$ and hence from Proposition 3.6, we get that $\mathcal{F}_{a,i} \in \mathrm{Cent}(\mathcal{H})$ for every $a \in \mathcal{A}$ and every $i \in I$. Now, we will prove that $\mathcal{F} = \Gamma$. Since $\Gamma(\xi \otimes 1) = 0 = \mathcal{F}(\zeta \otimes 1)$, then, from Eq (3.21), we get that

$$[\mathcal{F}(\zeta \otimes a), \bar{\alpha}^k(\xi \otimes 1)] = T([\zeta \otimes a, \xi \otimes 1]) = T([\zeta \otimes 1, \xi \otimes a])$$

$$= [\bar{\alpha}^k(\zeta \otimes 1), \Gamma(\xi \otimes a)]$$

for all $\zeta, \xi \in \mathcal{H}$ and $a \in \mathcal{A}$. Fix an arbitrary $a \in \mathcal{A}$. From (3.22), we can rephrased the last identity as

$$0 = \sum_{i \in I} \Big([\mathcal{F}_{a,i}(\zeta), \alpha^k(\xi)] \otimes c_i - [\alpha^k(\zeta), \Gamma_{a,i}(\xi)] \otimes c_i \Big)$$

$$= \sum_{i \in I} \Big([\mathcal{F}_{a,i}(\zeta), \alpha^k(\xi)] - [\alpha^k(\zeta), \Gamma_{a,i}(\xi)] \Big) \otimes c_i$$

for all $\zeta, \xi \in \mathcal{H}$. Consequently, for every $i \in I$, we get

$$[\mathcal{F}_{a,i}(\zeta), \alpha^k(\xi)] = [\alpha^k(\zeta), \Gamma_{a,i}(\xi)]$$

for all $\zeta, \xi \in \mathcal{H}$. Since $\mathcal{F}_{a,i} \in \mathrm{Cent}(\mathcal{H})$, it implies that

$$[\alpha^k(\zeta), \Gamma_{a,i}(\xi)] = [\mathcal{F}_{a,i}(\zeta), \alpha^k(\xi)] = \mathcal{F}_{a,i}([\zeta, \xi]) = [\alpha^k(\zeta), \mathcal{F}_{a,i}(\xi)]$$

and thus

$$[\alpha^k(\zeta), \Gamma_{a,i}(\xi) - \mathcal{F}_{a,i}(\xi)] = 0$$

for every $\zeta, \xi \in \mathcal{H}$ and $i \in I$. Since $\mathcal{Z}(\mathcal{H}) = 0$, therefore, $\Gamma_{a,i} = \mathcal{F}_{a,i}$ for all $i \in I$ and all $a \in \mathcal{A}$. Similarly, from Eq (3.22), we have $\mathcal{F}(\zeta \otimes a) = \Gamma(\zeta \otimes a)$ for every simple tensor $\zeta \otimes a \in \mathcal{H} \otimes \mathcal{A}$. From the linearity of $\mathcal{F}$ and Γ, it follows that $\mathcal{F} = \Gamma$. By similar arguments used in the proof of Theorem 2.1, we can show that $\mathcal{F}$ is a derivation. Now, since $\delta = \delta_C + \mathcal{F}$, where $\delta_C \in \mathrm{Der}(\mathcal{H} \otimes \mathcal{A}) \oplus \mathrm{Cent}(\mathcal{H} \otimes \mathcal{A})$ and $\mathcal{F} \in \mathrm{Der}(\mathcal{H} \otimes \mathcal{A})$, it implies that $\delta \in \mathrm{Der}(\mathcal{H} \otimes \mathcal{A}) \oplus \mathrm{Cent}(\mathcal{H} \otimes \mathcal{A})$. This completes the proof of (i). □

Note that (ii) can be proved analogously by setting $\delta = \gamma$ in the above arguments.

4. Conclusions

In this paper, we investigated generalized derivations and quasi-derivations of the current Hom-Lie algebra $\mathcal{H} \otimes A$, where A is a finite-dimensional commutative algebra over a field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$.

Assuming that $\mathcal{H}$ is a regular, centerless Hom-Lie algebra, we proved that the decomposition properties of generalized derivations and quasi-derivations are preserved under current extension. More precisely,

$$\mathrm{Der}(\mathcal{H}) \oplus \mathrm{Cent}(\mathcal{H}) = \mathrm{GDer}(\mathcal{H})$$

implies

$$\mathrm{Der}(\mathcal{H} \otimes A) \oplus \mathrm{Cent}(\mathcal{H} \otimes A) = \mathrm{GDer}(\mathcal{H} \otimes A),$$

and

$$\mathrm{QDer}(\mathcal{H}) = \mathrm{Der}(\mathcal{H}) \oplus \mathrm{Cent}(\mathcal{H})$$

implies

$$\mathrm{QDer}(\mathcal{H} \otimes A) = \mathrm{Der}(\mathcal{H} \otimes A) \oplus \mathrm{Cent}(\mathcal{H} \otimes A).$$

Thus, both motivating questions are answered affirmatively, showing that these structural decomposition properties remain stable when passing to current Hom-Lie algebras.

Author contributions

All authors contributed to the conception and design of the study. Ashutosh Pandey: contributed to the conceptualization and formulation of the problem, conducted a significant portion of the mathematical analysis, and participated in drafting the manuscript; Mani Shankar Pandey: played a central role in developing the theoretical framework, performing key computations, and revising the manuscript; Omaima Alshanqiti: provided guidance on the research methodology, assisted with the mathematical proofs, and was involved in reviewing and refining the manuscript. All authors read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest or competing interests that could have influenced the results and/or discussion presented in this paper.

References

1. B. Agrebaoui, K. Benali, A. Makhlouf, Representations of simple Hom-Lie algebras, *J. Lie Theory*, **29** (2019), 1119–1135.
2. N. Aizawa, H. Sato, q-deformation of the Virasoro algebra with central extension, *Phys. Lett. B*, **256** (1991), 185–190. https://doi.org/10.1016/0370-2693(91)90671-C
3. M. A. Alvarez, F. Cartes, Cohomology and deformations for the Heisenberg Hom-Lie algebras, *Linear Multilinear Algebra*, **67** (2019), 2209–2229. https://doi.org/10.1080/03081087.2018.1487379
4. F. Ammar, M. Makhlouf, Hom-Lie superalgebras and Hom-Lie admissible superalgebras, *J. Algebra*, **324** (2010), 1513–1528. https://doi.org/10.1016/j.jalgebra.2010.06.014
5. D. Benkovič, D. Eremita, Generalized derivations of current Lie algebras, *Commun. Algebra*, **52** (2024), 4603–4611. https://doi.org/10.1080/00927872.2024.2354423
6. M. Brešar, Functional identities on tensor products of algebras, *J. Algebra*, **455** (2016), 108–136. https://doi.org/10.1016/j.jalgebra.2016.02.012
7. M. Brešar, Near-derivations in Lie algebras, *J. Algebra*, **320** (2008), 3765–3772. https://doi.org/10.1016/j.jalgebra.2008.09.007
8. J. M. Casas, X. García-Martínez, Abelian extensions and crossed modules of Hom-Lie algebras, *J. Pure Appl. Algebra*, **224** (2020), 987–1008. https://doi.org/10.1016/j.jpaa.2019.06.018
9. M. Chaichian, P. Kulish, J. Lukierski, q-deformed Jacobi identity, q-oscillators and q-deformed infinite dimensional algebras, *Phys. Lett. B*, **237** (1990), 401–406. https://doi.org/10.1016/0370-2693(90)91196-I
10. T. L. Curtright, C. K. Zachos, Deforming maps for quantum algebras, *Phys. Lett. B*, **243** (1990), 237–244. https://doi.org/10.1016/0370-2693(90)90845-W
11. Y. Chen, S. Ren, J. Shan, R. Zhang, Generalized derivations of ω-Lie algebras, *J. Algebra Appl.*, **25** (2026), 1–16. https://doi.org/10.1142/S0219498826502063
12. T. S. Erickson, W. S. Martindale, J. M. Osborn, Prime nonassociative algebras, *Pac. J. Math.*, **60** (1975), 49–63. https://doi.org/10.2140/pjm.1975.60.49
13. J. T. Hartwig, D. Larsson, S. D. Silvestrov, Deformations of Lie algebras using σ-derivations, *J. Algebra*, **295** (2006), 314–361. https://doi.org/10.1016/j.jalgebra.2005.07.036
14. Z. Hao, J. Zhang, L. Chen, On the classification of (simple) regular Hom-Lie groups, 2023. Preprint. https://doi.org/10.13140/RG.2.2.27894.31048
15. Q. Jin, X. Li, Hom-Lie algebra structures on semisimple Lie algebras, *J. Algebra*, **319** (2008), 1398–1408. https://doi.org/10.1016/j.jalgebra.2007.12.005
16. T. B. Jmaa, A. Makhlouf, N. Saadaoui, Current Hom-Lie algebras, *Acta Comment. Univ. Tartu. Math.*, **26** (2022), 103–127. https://doi.org/10.12697/ACUTM.2022.26.08
17. C. Laurent-Gengoux, A. Makhlouf, J. Teles, Universal algebra of a Hom-Lie algebra and group-like elements, *J. Pure Appl. Algebra*, **222** (2018), 1139–1163. https://doi.org/10.1016/j.jpaa.2017.06.012

18. C. Laurent-Gengoux, J. Teles, Hom-Lie algebroids, *J. Geom. Phys.*, **68** (2013), 69–75. https://doi.org/10.1016/j.geomphys.2013.02.003

19. D. Larsson, S. D. Silvestrov, Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities, *J. Algebra*, **288** (2005), 321–344. https://doi.org/10.1016/j.jalgebra.2005.02.032

20. G. F. Leger, E. M. Luks, Generalized derivations of Lie algebras, *J. Algebra*, **228** (2000), 165–203. https://doi.org/10.1006/jabr.1999.8250

21. E. Peyghan, L. Nourmohammadifar, Para-Kähler Hom-Lie algebras, *J. Algebra Appl.*, **18** (2019), 1950044. https://doi.org/10.1142/S0219498819500440

22. B. Sun, Y. Ma, L. Chen, Biderivations and commuting linear maps on Hom-Lie algebras, *Algebra Colloq.*, **32** (2025), 527–540. https://doi.org/10.1142/S1005386725000380

23. Y. Sheng, Representations of Hom-Lie algebras, *Algebr. Represent. Theory*, **15** (2012), 1081–1098. https://doi.org/10.1007/s10468-011-9280-8

24. W. Xie, W. Liu, Hom-structures on simple graded Lie algebras of finite growth, *J. Algebra Appl.*, **16** (2017), 1750154. https://doi.org/10.1142/S0219498817501547

25. J. Zhou, Y. J. Niu, L. Y. Chen, Generalized derivations of Hom-Lie algebras, *Acta Math. Sin. (Chin. Ser.)*, **58** (2015), 583–594. https://doi.org/10.12386/A2015sxxb0056

26. R. Zhang, Y. Zhang, Generalized derivations of Lie superalgebras, *Commun. Algebra*, **38** (2010), 3737–3751. https://doi.org/10.1080/00927870903236228

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