*Mathematics*

https://www.aimspress.com/journal/Math

*Research article*

# Spectral radius of biased random walks on regular trees

**He Song and Meng Liu**[*]

School of Mathematics and Statistics, Huaiyin Normal University, Huai'an 223300, China

* **Correspondence:** Email: liumeng@hytc.edu.cn.

**Abstract:** For the biased random walk on $d$-regular trees, we obtain the spectral radius, first return probability and $n$-step transition probability, speed. We prove that the spectral radius depends continuously on the bias parameter and is strictly increasing with respect to it, while the speed remains strictly positive and decreases monotonically as the bias increases.

## 1. Introduction

Biased random walks, defined by directionally weighted transition probabilities that induce a systematic drift on an underlying structure, constitute a fundamental model in probability theory, statistical physics, computer science, and ecology. They effectively capture asymmetric transport phenomena in physical systems [4, 14, 15] and disordered media [17, 20, 34], directional movement and foraging behavior in ecology [2, 3], and guided exploration and sampling mechanisms in Monte Carlo algorithms [11] and refined in a series of subsequent studies [22, 31, 37], thereby sustaining broad interdisciplinary interest. The specific model, biased random walk $RW_\lambda$, where a parameter $\lambda > 0$ quantifies the strength of a bias away from a fixed origin $o$, serves as a canonical framework for studying how drift influences long-term asymptotic behavior, including escape speed, return probabilities, and spectral characteristics. Furthermore, this model has been extensively employed in the analysis of transport phenomena on percolation clusters [4, 10, 14] and random graph structures [23, 24, 33].

The rigorous study of $RW_\lambda$ on graphs was profoundly advanced in the 1990s by Lyons [23–25], Pemantle, Peres, and collaborators [26, 27], who established fundamental connections between random walk transience and recurrence, graph growth rate, and geometric properties such as Hausdorff dimension. A key geometric determinant is the growth rate $\mathrm{gr}(G)$ of the graph $G$. In particular, for

Cayley graphs and regular trees, a sharp transition occurs at $\lambda = \mathrm{gr}(G)$: the walk is transient for $\lambda < \mathrm{gr}(G)$ and recurrent for $\lambda > \mathrm{gr}(G)$ [24, 25]. More recently, $RW_\lambda$ has continued to attract significant attention, as seen in works such as [1, 5–7, 21]. The transience and recurrence of Galton–Watson trees conditioned on non-extinction were studied by Lyons [23, 24]. Two quantitative measures of central interest are the spectral radius $\rho_G(\lambda)$, defined as the exponential decay rate $\limsup_{n\to\infty} p_\lambda^{(n)}(o, o)^{1/n}$ of the return probability, and the speed $\mathcal{S}(\lambda) = \lim_{n\to\infty} |X_n|/n$, which measures the linear rate of escape. Understanding the exact dependence of $\rho_G(\lambda)$ and $\mathcal{S}(\lambda)$ on $\lambda$ reveals a deep interplay between graph geometry and stochastic dynamics.

For the lamplighter group, Lyons et al., [26] proved that the speed of $RW_\lambda$ is either zero or strictly positive depending on the underlying structural and bias conditions. Bergera et al., [9] investigated the monotonicity of the speed for biased random walks on multi-dimensional integer lattices and derived explicit criteria for determining when the speed is strictly positive. Peter explored the influence of the bias parameter $\lambda$ on the properties of random walks, specifically their spectral radius and speed (see [30], Chapter 9). Song et al., [35] established the continuity of the spectral radius of $RW_\lambda$ for free product graphs. Liu et al., [29] studied the large deviation principles and invariance principles for biased random walks on multi-dimensional integer lattices. Gantert et al., [19] gave an explicit formula for the speed of the biased random walk on spanning trees of the ladder graph. These developments are also closely connected to advances in spectral methods, including spectral clustering, as explored in works such as [12, 13, 39].

A long-standing question, posed by Lyons et al., [26, Questions], asks whether the speed $\mathcal{S}(\lambda)$ of $RW_\lambda$ is well-defined and strictly positive for all parameters $\lambda$ strictly between 1 and the growth rate $\mathrm{gr}(G)$ of a Cayley graph $G$. This inquiry, often called the Lyons–Pemantle–Peres monotonicity problem, highlights the need for a precise understanding of how the bias parameter influences the asymptotic dynamics. While explicit formulas for $\rho_G(\lambda)$ and $\mathcal{S}(\lambda)$ have been obtained in several special settings, such as integer lattices [34] and Galton–Watson trees [1, 34, 36], a complete picture for the highly symmetric yet non-amenable family of $d$-regular trees $\mathbb{T}_d$ has remained incomplete. A clear, closed-form description of these quantities, together with a rigorous analysis of their monotonicity and continuity properties, is essential both for intrinsic mathematical interest and for providing benchmarks in more complex disordered environments.

In this paper, we provide a comprehensive analysis of $RW_\lambda$ on $d$-regular trees $\mathbb{T}_d$ ($d \geq 2$). We derive explicit expressions for the spectral radius, first-return probabilities, $n$-step transition probabilities, and the speed. Furthermore, we establish their key analytic properties, including continuity, monotonicity, and precise asymptotic decay rates. Our results not only fill a significant gap in the understanding of biased random walks on homogeneous trees but also offer an affirmative resolution of the Lyons–Pemantle–Peres monotonicity problem in this setting.

Our main results include the following. For $\lambda \in (0, d-1]$, the exact spectral radius of $\mathbb{T}_d$ is continuous and strictly increasing. Asymptotics for the even-step return probabilities revealing a universal $n^{-3/2}$ decay modulated by $\rho_{\mathbb{T}_d}(\lambda)^{2n}$. The speed formula $S(\lambda)$ for $\lambda \in (0, d-1)$, which is positive and strictly decreasing, thereby affirmatively resolving the Lyons–Pemantle–Peres monotonicity problem for regular trees.

These findings not only extend known results for symmetric random walks but also provide a benchmark for analogous studies of biased random walks on less regular structures, such as Galton-Watson trees or percolation clusters. The explicit dependence of $\rho_{\mathbb{T}_d}(\lambda)$ on both the branching number

$d - 1$ and the bias $\lambda$ underscores a competition between graph expansion and drift.

The remainder of this paper is organized as follows. Section 2 presents our main theorems. Section 3 provides the detailed proofs, employing generating functions, Catalan number enumerations, and the Darboux method for asymptotic analysis. Section 4 presents numerical simulations that validate the asymptotic formulas and lists several open problems and conjectures motivated by our results.

## 2. Main results

We now briefly introduce the necessary notation. Let $G = (V(G), E(G))$ be a locally finite, connected infinite graph with a distinguished root vertex $o$. For a vertex $x$, write $|x|$ for the graph distance from $x$ to $o$. For any edge $e = \{u, v\}$ of $G$, $u$ and $v$ are its endpoints. Define the graph distance from $e$ to $o$ as $|e| = \min\{|u|, |v|\}$, where $|u|$ and $|v|$ are the graph distances from $u$ and $v$ to $o$, respectively. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{Z}_+$, we define the ball of radius $n$ centered at $o$ as

$$B_G(n) = \{x \in V(G) : |x| \leq n\};$$

The boundary of $B_G(n)$,
$$\partial B_G(n) = \{x \in V(G) : |x| = n\}.$$

Let $M_n = |\partial B_G(n)|$ be the cardinality of $\partial B_G(n)$ for any $n \in \mathbb{Z}_+$. Define the growth rate of $G$ as

$$\text{gr}(G) = \liminf_{n \to \infty} \sqrt[n]{M_n}.$$

Since the sequence $\{M_n\}_{n=0}^{\infty}$ is submultiplicative, the limit $\text{gr}(G) = \lim_{n \to \infty} \sqrt[n]{M_n}$ exists indeed. It is easy to see that for $\mathbb{T}_d$, $M_n = |\partial B_{\mathbb{T}_d}(n)| = d * (d - 1)^{n-1}$. Hence, $\text{gr}(\mathbb{T}_d) = d - 1$.

The biased random walk $RW_\lambda = (X_n)_{n \geq 0}$ is the nearest-neighbor Markov chain whose transition probabilities are determined by assigning conductance $\lambda^{-n}$ to every edge at distance $n$ from $o$ (see below for the explicit form).

$$p(v, u) := p_\lambda(v, u) = \begin{cases} 1/d_v & \text{if } v = o, \\ \frac{\lambda}{d_v + (\lambda - 1)d_v^-} & \text{if } u \in \partial B_G(|v| - 1) \text{ and } v \neq o, \\ \frac{1}{d_v + (\lambda - 1)d_v^-} & \text{otherwise.} \end{cases}$$

Here $d_v$ is the degree of vertex $v$; and $d_v^-, d_v^0$ and $d_v^+$ are the numbers of edges connecting $v$ to $\partial B_G(|v|-1)$, $\partial B_G(|v|)$ and $\partial B_G(|v| + 1)$ respectively. Note

$$d_v^+ + d_v^0 + d_v^- = d_v, \ d_v^- \geq 1, \ d_o^- = d_o^0 = 0;$$

Its $n$-step transition probability is denoted by

$$p_\lambda^{(n)}(x, y) = \mathbb{P}_x(X_n = y),$$

where $\mathbb{P}_x$ is the law of the walk started at $x$. The spectral radius is defined as

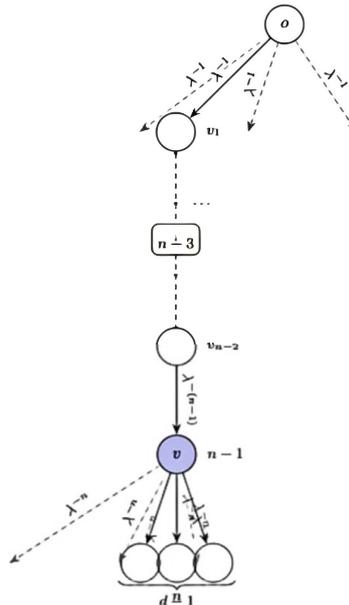$$\rho_G(\lambda) = \limsup_{n \to \infty} p_\lambda^{(n)}(o, o)^{1/n}.$$

Define

$$\tau_y^+ = \tau_y^+(\lambda) = \inf\{n \geq 1 | X_n = y\},$$

where $\tau_y^+$ is the first positive time when $RW_\lambda$ hits $y$. And the $n$-step first hitting probability of $RW_\lambda$ is defined as follows:

$$f_\lambda^{(n)}(x, y) = \mathbb{P}_x(\tau_y^+ = n).$$

For the definition of $RW_\lambda$ on $\mathbb{T}_d$, we have the local structure of $\mathbb{T}_d$, see Figure 1.



**Figure 1.** Local structure of $\mathbb{T}_d$.

Fix a vertex $o$ of $\mathbb{T}_d$ as the root, $RW_\lambda$ on $\mathbb{T}_d$ has the following transition probabilities: if $u$ and $v$ are adjacent on $\mathbb{T}_d$,

$$p(v, u) := p_\lambda^{\mathbb{T}_d} = \begin{cases} \frac{1}{d} & \text{if } v = o, \\ \frac{\lambda}{d+\lambda-1} & \text{if } u \in \partial B_{\mathbb{T}_d}(|v| - 1) \text{ and } v \neq o, \\ \frac{1}{d+\lambda-1} & \text{otherwise.} \end{cases} \tag{2.1}$$

Notice $RW_1$ is just the simple random walk on $\mathbb{T}_d$. The parameter $d$ controls the branching number of the tree. A larger $d$ corresponds to a faster growth rate, which intuitively makes it harder for $RW_\lambda$ to return to $o$.

We are ready to state our main results. The proofs will be presented in Section 3. The tilde symbol ($\sim$) denotes asymptotic equivalence.

**Theorem 2.1.** *For the d-regular tree* $\mathbb{T}_d$ *and* $\lambda \in (0, \infty)$ *and* $n \to \infty$,

$$f_\lambda^{(2n)}(o, o) \sim \frac{1}{\sqrt{\pi}} \left( \frac{2\sqrt{(d-1)\lambda}}{d-1+\lambda} \right)^{2n} n^{-3/2}. \tag{2.2}$$

*And for the d-regular tree* $\mathbb{T}_d$, *the following holds:*

$$\rho_{\mathbb{T}_d}(\lambda) = \frac{2\sqrt{(d-1)\lambda}}{d-1+\lambda}, \qquad \lambda \in (0, \lambda_c(\mathbb{T}_d)] = (0, d-1]. \tag{2.3}$$

*Moreover, the spectral radius* $\rho_{\mathbb{T}_d}(\lambda)$ *is continuous in* $\lambda \in (0, \infty)$; $\rho_{\mathbb{T}_d}(\lambda)$ *is strictly increasing for* $\lambda \in (0, d-1)$.

**Remark 2.2.** *Although d takes positive integer values, to analyze how the spectral radius* $\rho_{\mathbb{T}_d}(\lambda)$ *changes with d, we can treat d as a real variable. Hence, for* $\lambda \in (0, d-1)$, *we can differentiate* $\rho_{\mathbb{T}_d}(\lambda)$ *with respect to d and obtain*

$$\frac{\partial \rho_{T_d}}{\partial d} = \frac{\sqrt{\lambda}(\lambda - d + 1)}{\sqrt{d-1}(d-1+\lambda)^2} < 0.$$

*It means that* $\rho_{\mathbb{T}_d}(\lambda)$ *is strictly decreasing for d.*

The spectral radius $\rho_{\mathbb{T}_d}(\lambda)$ is a function of the bias parameter $\lambda$. Figure 2 shows several values of the degree $d$ (e.g., $d = 2, 3, 4, 5$) of $\rho_{\mathbb{T}_d}(\lambda)$.
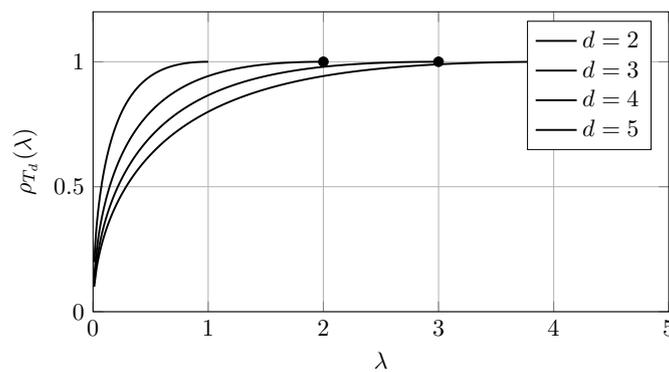


**Figure 2.** Spectral radius on *d*-regular trees, $d = 2, 3, 4, 5$.

**Theorem 2.3.** *For the d-regular tree* $\mathbb{T}_d$ *and* $\lambda \in (0, d-1]$ *and* $n \to \infty$,

$$p^{(2n)}(o, o) \sim \begin{cases} \frac{(d-1-\lambda)^2}{16(\pi\lambda)^{1/2}(d-1)^{3/2}}\rho_{\mathbb{T}_d}(\lambda)^{2n}n^{-3/2} & \text{if } \lambda \in (0, d-1), \\ \frac{1}{\sqrt{\pi n}} & \text{if } \lambda = d-1. \end{cases} \tag{2.4}$$

**Theorem 2.4.** *For the speed* $\mathcal{S}(\lambda)$ *of* $RW_\lambda$ *on d-regular tree* $\mathbb{T}_d$ *and* $\lambda \in (0, d-1)$, *the following hold.*

$$\mathcal{S}(\lambda) = \frac{d-1-\lambda}{d-1+\lambda}. \tag{2.5}$$

*And* $\mathcal{S}(\lambda)$ *is positive and strictly decreasing for* $\lambda \in (0, d-1)$.

**Remark 2.5.** *Recall the Lyons–Pemantle–Peres monotonicity problem [26]: for a Cayley graph G with growth rate* $\mathrm{gr}(G) > 1$, *the speed of a biased random walk exists and is positive for biased parameter* $\lambda \in (1, \mathrm{gr}(G))$.

*Regular trees are a special class of Cayley graphs, and our results prove that the Lyons–Pemantle–Peres monotonicity problem has an affirmative answer on regular trees.*

**Remark 2.6.** *Similar to the analysis in Remark 2.2, for $\lambda \in (0, d - 1)$, we can differentiate $\mathcal{S}(\lambda)$ with respect to d and obtain*

$$\frac{\partial \mathcal{S}}{\partial d} = \frac{2\lambda}{(d - 1 + \lambda)^2} > 0.$$

*It means that $\mathcal{S}(\lambda)$ is strictly increasing for d.*

## 3. Proof of Theorems 2.1–2.3

We first introduce the notion of biased random walks on general infinite graphs. The Green function of $n$-step transition probability, $p^{(n)}(x, y)$, is given by

$$\mathbb{G}(x, y \,|\, z) := \mathbb{G}_\lambda(x, y \,|\, z) = \sum_{n=0}^{\infty} p^{(n)}(x, y) z^n, \quad x, y \in V(G), \, z \in \mathbb{C}, \, |z| < R_{\mathbb{G}}, \tag{3.1}$$

where $R_{\mathbb{G}} = R_{\mathbb{G}}(\lambda)$ is its convergence radius and $\mathbb{C}$ denotes the set of complex numbers.
   Define

$$\tau_y^+ = \tau_y^+(\lambda) = \inf\{n \geq 1 \,|\, X_n = y\},$$

$$f_\lambda^{(n)}(x, y) = \mathbb{P}_x(\tau_y^+ = n),$$

the associated generating function

$$\mathbb{U}(x, y|z) = \sum_{n=1}^{\infty} f_\lambda^{(n)}(x, y) z^n, \quad x, y \in V(G), \, z \in \mathbb{C}. \tag{3.2}$$

To continue, let's recall some background about Catalan numbers. For more information about Catalan numbers, readers can refer to the following textbooks, An introduction to Catalan numbers ( [32], Chapter 3–4) or Analytic combinatorics ( [18], Chapter 1 and Chapter 7). The Catalan numbers form a sequence of natural numbers that occur in various counting problems, especially in computer science and combinatorics. Catalan numbers count the number of valid or balanced sequences or structures in which two elements (like opening and closing parentheses) are matched without exceeding or violating a constraint. And the $n$-th ($n \in \mathbb{N}$) Catalan number, $c_n$ is given directly in terms of binomial coefficients by $c_k = \frac{1}{k+1}\binom{2k}{k}$, with the associated related generating function

$$C(x) := \sum_{k=0}^{\infty} c_k x^k = \frac{1 - \sqrt{1 - 4x}}{2x}, \qquad x \in \left[-\frac{1}{4}, \frac{1}{4}\right]. \tag{3.3}$$

The asymptotic behavior of the generating function near its singularity is translated into the asymptotic behavior of the coefficients of the sequence. This yields the well-known result:

$$c_n \sim \frac{4^n}{n^{3/2} \sqrt{\pi}}. \tag{3.4}$$

Since the regular tree $\mathbb{T}_d$ is a bipartite graph, its vertices can be partitioned into two sets according to the distances to the root $o$. Each step of $RW_\lambda$ crosses between these two sets. Therefore, starting from $o$, the walk can only return to $o$ at even steps, and for all odd $n$, we have $p^{(n)}(o, o) = 0$. Hence, we only need to consider the even-step return probabilities $f_\lambda^{(2n)}(o, o)$ and $p^{(2n)}(o, o)$.

### 3.1. Proof of Theorem 2.1

*Proof.* Assume $\lambda > 0$, for $n \geq 1$

$$f_\lambda^{(2n)}(o, o) = \mathbb{P}_o\left(\tau_o^+ = 2n\right), \qquad f_\lambda^{(2n-1)}(o, o) = 0, \ \lambda \in (0, \infty),$$

To compute $f^{(2n)}(o, o)$, we can regard $RW_\lambda$ as a random walk on $\mathbb{Z}_+$ since the symmetry of $\mathbb{T}_d$. And $\{Y_n\}_{n=0}^\infty = \{|X_n|\}_{n=0}^\infty$ with $|X_0| = 0$ is a Markov chain on $\mathbb{Z}_+$ with transition probabilities given by

$$p(x, y) = \begin{cases} 1 & \text{if } x = 0, \ y = 1 \\ \frac{\lambda}{d-1+\lambda} & \text{if } y = x - 1 \text{ and } x \neq 0, \\ \frac{d-1}{d-1+\lambda} & \text{otherwise.} \end{cases}$$

Let

$$T = \inf\{n \geq 1|\ Y_n = 0\}.$$

Notice that $Y_n = 0$ means that $RW_\lambda$ returns to its starting point $o$. Hence,

$$\mathbb{P}(\tau = 2n) = \mathbb{P}(T = 2n) = \sharp\{\text{paths returning } 0 \text{ with } T = 2n\}\left(\frac{d-1}{d+\lambda-1}\right)^{n-1}\left(\frac{\lambda}{d+\lambda-1}\right)^n.$$

Note the number of all $2n$-length nearest-neighbor paths $\gamma = w_0 w_1 \cdots w_{2n}$ on $\mathbb{Z}_+$ such that

$$w_0 = w_{2n} = 0, \qquad w_j \geq 1, \ 1 \leq j \leq 2n - 1$$

is precisely $c_{n-1}$. Hence, for any $\lambda > 0$,

$$f_\lambda^{(2n)}(o, o) = c_{n-1}\left(\frac{d-1}{d-1+\lambda}\right)^{n-1}\left(\frac{\lambda}{d-1+\lambda}\right)^n, \ n \in \mathbb{N}. \tag{3.5}$$

Combining (3.4), we have

$$f_\lambda^{(2n)}(o, o) \sim \frac{4^{n-1}}{(n-1)^{3/2}\sqrt{\pi}}\left(\frac{d-1}{d-1+\lambda}\right)^{n-1}\left(\frac{\lambda}{d-1+\lambda}\right)^n$$

$$\sim \frac{1}{\sqrt{\pi}}\left(\frac{2\sqrt{(d-1)\lambda}}{d-1+\lambda}\right)^{2n} n^{-3/2}, \ \forall \lambda \in (0, \infty).$$

Hence, we obtain (2.2) by means of Stirling's formula.

By definition (3.2), for $\lambda > 0$, which, in view of (3.3), implies that for $|z| \leq \frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}}$,

$$\mathbb{U}(o, o|z) = \sum_{n=0}^\infty f^{(n)}(o, o)z^n$$

$$= \sum_{n=1}^\infty f^{(2n)}(o, o)z^{2n}$$

$$= \sum_{n=1}^\infty c_{n-1}\left(\frac{d-1}{d+\lambda-1}\right)^{n-1}\left(\frac{\lambda}{d+\lambda-1}\right)^n z^{2n}$$

$$= \frac{\lambda}{d+\lambda-1}z^2 \sum_{n=1}^{\infty} c_{n-1} \left(\frac{d-1}{d+\lambda-1}\right)^{n-1} \left(\frac{\lambda}{d+\lambda-1}\right)^{n-1} z^{2(n-1)}$$

$$= \frac{\lambda}{d+\lambda-1}z^2 \sum_{n=0}^{\infty} c_n \left(\frac{d-1}{d+\lambda-1}\right)^{n} \left(\frac{\lambda}{d+\lambda-1}\right)^{n} z^{2n}.$$

$$= \frac{\lambda}{d+\lambda-1}z^2 \sum_{n=0}^{\infty} c_n \left(\left(\frac{d-1}{d+\lambda-1}\right)\left(\frac{\lambda}{d+\lambda-1}\right)z^2\right)^{n}.$$

Combining (3.3), we have that

$$\mathbb{U}(o,o|z) = \frac{\lambda}{d+\lambda-1}z^2 C\left(\frac{\lambda(d-1)z^2}{(d+\lambda-1)^2}\right)$$

$$= \frac{(d-1+\lambda) - \sqrt{(d-1+\lambda)^2 - 4\lambda(d-1)z^2}}{2(d-1)}.$$

and $|\frac{\lambda(d-1)z^2}{(d+\lambda-1)^2}| \le \frac{1}{4}$, which implies that for $|z| \le \frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}}$.
Thus,

$$\mathbb{U}(o,o\,|\,z) = \frac{(d-1+\lambda) - \sqrt{(d-1+\lambda)^2 - 4\lambda(d-1)z^2}}{2(d-1)}. \tag{3.6}$$

Notice from (3.6) that when $0 < \lambda \le d-1$,

$$\mathbb{U}\left(o, o \,\Big|\, \frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}}\right) = \frac{d-1+\lambda}{2(d-1)} \le 1.$$

Hence, for $n \ge 1$, $|z| < \frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}}$ and $0 < \lambda \le d-1$, we have

$$p^{(n)}(o,o) = \sum_{k=1}^{n} \mathbb{P}_o[\tau_o^+ = k] p^{n-k}(o,o),$$

from which we can obtain

$$p^{(n)}(o,o)z^n = \sum_{k=1}^{n} \mathbb{P}_o[\tau_o^+ = k]z^k p^{n-k}(o,o)z^{n-k},$$

$$\sum_{n=1}^{\infty} p^{(n)}(o,o)z^n = \mathbb{U}(o,o|z)\mathbb{G}(o,o|z),$$

$$\mathbb{G}(o,o|z) - 1 = \mathbb{U}(o,o|z)\mathbb{G}(o,o|z).$$

Hence,

$$\mathbb{G}(o,o\,|\,z) = \frac{1}{1 - U_\lambda(o,o\,|\,z)}$$

$$= \frac{2(d-1)}{2(d-1) - (d-1+\lambda) + \sqrt{(d-1+\lambda)^2 - 4\lambda(d-1)z^2}}. \tag{3.7}$$

This implies that the convergence radius for $\mathbb{G}_\lambda(o, o \,|\, z)$ is $\frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}}$. Recall [28] *(Exercise 1.2)*, the convergence radius

$$R_{\mathbb{T}_d} = R_{\mathbb{T}_d}(\lambda) = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{p^{(n)}(x, y)}}$$

is independent of $x$, $y$ when $RW_\lambda$ is irreducible, i.e., $\lambda > 0$ (for more details, readers can refer to [30], Chapter 1). In other words,

$$\rho(\lambda) := \rho_{\mathbb{T}_d}(\lambda) = \frac{1}{R_{\mathbb{T}_d}(\lambda)} = \frac{2\sqrt{\lambda(d-1)}}{d-1+\lambda}, \qquad 0 < \lambda \le d-1.$$

So we obtain (2.3).

For $\lambda \in (0, d-1)$, the derivative of $\rho_{\mathbb{T}_d}(\lambda)$:

$$\frac{d\rho_{\mathbb{T}_d}(\lambda)}{d\lambda} = \frac{\sqrt{(d-1)/\lambda}(d-1+\lambda)}{(d-1-\lambda)^2} > 0.$$

It means that $\rho_{\mathbb{T}_d}(\lambda)$ is strictly increasing for $\lambda \in (0, d-1)$.

Because for $0 < \lambda \le d-1$, $\lambda^2 + (d-1)^2 \ge 2\lambda(d-1)$. So $\rho(\lambda) \le 1$. And for the case $\lambda > \lambda_c(\mathbb{T}_d) = d-1$, $RW_\lambda$ is recurrent, which means that $\rho_{\mathbb{T}_d}(\lambda) = 1$. Hence, the spectral radius $\rho_\lambda$ is continuous in $\lambda \in (0, \infty)$.

So far, we have finished proving Theorem 2.1. □

### 3.2. Proof of Theorem 2.3

*Proof.* It remains to show (2.4) for $\lambda \in (0, d-1)$. To simplify the expression of the Green's function, we introduce $a(\lambda) = \frac{2(d-1)}{d-1+\lambda}$ and $b(\lambda) = \frac{d-1-\lambda}{d-1+\lambda}$. Then for any $|z| \le R_{\mathbb{T}_d}(\lambda) = \frac{1}{\rho(\lambda)}$,

$$\mathbb{G}(o, o \,|\, z) = \frac{2(d-1)}{d-1+\lambda}\frac{1}{\frac{d-1-\lambda}{d-1+\lambda} + \sqrt{1-\rho(\lambda)^2 z^2}} = \frac{a(\lambda)}{b(\lambda) + \sqrt{1-\rho(\lambda)^2 z^2}}.$$

The Darboux method is a classical technique in asymptotic analysis, used to derive the asymptotic behavior of a sequence from the singularities of its generating function. It is particularly effective when the generating function has algebraic singularities. In the following, the Darboux method is used to derive the asymptotic probability $p^{(2n)}(o, o)$. The relevant theorem (see Theorem 5 of [8]) is recalled below.

**Theorem 3.1.** *Assume that the power series $w(z) = \sum a_n z^n$ with nonnegative coefficients satisfies $F(z, w) \equiv 0$. Suppose there exist real numbers $r > 0$ and $s > a_0$ such that*

*(i) for some $\delta > 0$, $F(z, w)$ is analytic whenever $|z| < r + \delta$ and $|w| < s + \delta$;*
*(ii) $F(r, s) = \frac{\partial F}{\partial w}(r, s) = 0$;*
*(iii) $\frac{\partial F}{\partial z}(r, s) \ne 0$, and $\frac{\partial^2 F}{\partial w^2}(r, s) \ne 0$; and*
*(iv) if $|z| \le r, |w| \le s$, and $F(z, w) = \frac{\partial F}{\partial w}(z, w) = 0$, then $z = r$ and $w = s$.*

*Then*

$$a_n \sim \left(\frac{r\frac{\partial F}{\partial z}}{2\pi\frac{\partial^2 F}{\partial w^2}}\right)^{1/2} n^{-3/2} r^{-n},$$

*where the partial derivatives are evaluated at $(z, w) = (r, s)$.*

Let

$$\Phi(t) := \Phi_\lambda(t) = \frac{-a(\lambda)b(\lambda) + \sqrt{a(\lambda)^2 + \rho(\lambda)^2(1 - b(\lambda)^2)t^2}}{1 - b(\lambda)^2}, \qquad t \in \mathbb{R}.$$

Then for any $|z| \le R_\mathbb{G}(\lambda)$,

$$\mathbb{G}(o, o \,|\, z) = \Phi\left(z \, \mathbb{G}(o, o \,|\, z)\right).$$

Define

$$\Psi(u, v) := \Phi(uv) - v, \qquad u, v \in \mathbb{R}.$$

Then

$$\Psi\left(z, \mathbb{G}(o, o|z)\right) = 0, \; |z| \le R_\mathbb{G}(\lambda);$$

and for any $u, v \in \mathbb{R}$,

$$\frac{\partial \Psi(u, v)}{\partial v} = \frac{\rho(\lambda)^2 u^2 v}{\sqrt{a(\lambda)^2 + \rho(\lambda)^2 \left(1 - b(\lambda)^2\right) u^2 v^2}} - 1,$$

$$\frac{\partial^2 \Psi(u, v)}{\partial v^2} = \frac{\rho(\lambda)^2 a(\lambda)^2 u^2}{\left(a(\lambda)^2 + \rho(\lambda)^2 \left(1 - b(\lambda)^2\right) u^2 v^2\right)^{3/2}},$$

$$\frac{\partial \Psi(u, v)}{\partial u} = \frac{\rho(\lambda)^2 u v^2}{\sqrt{a(\lambda)^2 + \rho(\lambda)^2 \left(1 - b(\lambda)^2\right) u^2 v^2}}.$$

Then

$$\left. \frac{\partial \Psi(u, \, v)}{\partial v} \right|_{(u, v) = \left(\frac{1}{\rho(\lambda)}, \, \mathbb{G}\left(o, o \,|\, \frac{1}{\rho(\lambda)}\right)\right)} = 0,$$

$$\left. \frac{\partial^2 \Psi(u, \, v)}{\partial v^2} \right|_{(u, v) = \left(\frac{1}{\rho(\lambda)}, \, \mathbb{G}\left(o, o \,|\, \frac{1}{\rho(\lambda)}\right)\right)}$$

$$= \frac{a(\lambda)^2}{\left(a(\lambda)^2 + (1 - b(\lambda)^2) \, \mathbb{G}\left(o, o \left|\frac{1}{\rho(\lambda)}\right.\right)^2\right)^{3/2}}$$

$$= \frac{(d - 1 - \lambda)^3}{2(d - 1)(d - 1 + \lambda)^2} \neq 0,$$

$$\left. \frac{\partial \Psi(u, \, v)}{\partial u} \right|_{(u, v) = \left(\frac{1}{\rho(\lambda)}, \, \mathbb{G}\left(o, o \,|\, \frac{1}{\rho(\lambda)}\right)\right)}$$

$$= \frac{\rho(\lambda) \mathbb{G}\left(o, o \left|\frac{1}{\rho(\lambda)}\right.\right)^2}{\sqrt{a(\lambda)^2 + (1 - b(\lambda)^2) \, \mathbb{G}\left(o, o \left|\frac{1}{\rho(\lambda)}\right.\right)^2}}$$

$$= \frac{2\rho(\lambda)(d - 1)}{d - 1 - \lambda} \neq 0.$$

Applying the method of Darboux (see [8] Theorem 5), we obtain that

$$p^{(2n)}(o, o) \sim \left( \frac{c_1(\lambda)}{2\pi\rho(\lambda)c_2(\lambda)} \right)^{1/2} \rho(\lambda)^{2n}(2n)^{-3/2}$$

$$\sim \frac{(d-1-\lambda)^2}{16(\pi\lambda)^{1/2}(d-1)^{3/2}} \rho(\lambda)^{2n} n^{-3/2}.$$

The idea of using the method of Darboux to establish the asymptotics for $p_\lambda^{(2n)}(o, o)$ is not new. For example, in Woess [38], Chapter III, Section 17, pp. 181–189, examples of random walks on groups are given such that $p^{(n)}(o, o) \sim c\rho^n n^{-3/2}$ for some constant $c > 0$. The exact value of $c$ is not known in general.

For $z \in (-1, 1)$, $\lambda = d - 1$, $\mathbb{G}(o, o \,|\, z) = \frac{1}{\sqrt{1-z^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} z^{2n}$. Thus

$$p^{(2n)}(o, o) = \frac{(2n)!}{2^{2n}(n!)^2} \sim \frac{1}{\sqrt{\pi n}}.$$

Hence, we prove Theorem 2.3. □

### 3.3. Proof of Theorem 2.4

*Proof.* Now assume $\lambda \in (0, d - 1)$. Then the following holds for $\lambda < d - 1 = \lambda_c(\mathbb{T}_d)$ due to transience.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} I_{\{X_i=o\}} = 0 \quad a.s.. \tag{3.8}$$

By the ergodic theorem for irreducible Markov chains ( [16] Theorem 6.6.1),

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} I_{\{X_i=o\}} = \frac{1}{\mathbb{E}[\tau_o^+]} = 0 \quad a.s..$$

To continue, define the following function $g$ on $\mathbb{T}_d$ :

$$g(x) = \begin{cases} 1 & \text{if } x = o, \\ \frac{d-1-\lambda}{d-1+\lambda} & \text{if } x \neq o. \end{cases}$$

Then $(|X_n| - |X_{n-1}| - g(X_{n-1}))_{n=1}^{\infty}$ is a martingale-difference sequence. By the strong Law of Large Numbers for uncorrelated random variables ( [28] Theorem 13.1), the strong Law of Large Numbers holds for this sequence:

$$\lim_{n \to \infty} \frac{1}{n} \left( |X_n| - \sum_{i=1}^{n-1} g(X_i) \right) = 0 \quad a.s.. \tag{3.9}$$

When $x \neq o$,

$$g(x) = \frac{d-1-\lambda}{d-1+\lambda}.$$

Applying 3.8, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} I_{\{X_i \neq o\}} = 1 \quad a.s..$$

Thus

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(X_i) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(X_i) I_{\{X_i \neq o\}} + \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(X_i) I_{\{X_i = o\}}$$

$$= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(X_i) I_{\{X_i \neq o\}}$$

$$= \frac{d - 1 - \lambda}{d - 1 + \lambda} = \mathcal{S}(\lambda).$$

Combining with (3.8) and (3.9), we yield (2.5).

For $\lambda \in (0, d-1)$, the derivative of $\mathcal{S}(\lambda)$:

$$\frac{d\mathcal{S}(\lambda)}{d\lambda} = \frac{-2(d-1)}{(d-1+\lambda)^2} < 0.$$

It means that $\rho_{\mathbb{T}_d}(\lambda)$ is strictly decreasing for $\lambda \in (0, d-1)$.

So far, we have proved Theorem 2.4. □

## 4. Simulation and open problems

### 4.1. Simulation

To test the effectiveness of the estimate, we take $f_\lambda^{(2n)}(o, o)$ as an example. Recall (2.2) and (3.5), $f_\lambda^{(2n)}(o, o)$ has exact value

$$f_\lambda^{(2n)}(o, o) = c_{n-1} \left( \frac{d-1}{d-1+\lambda} \right)^{n-1} \left( \frac{\lambda}{d-1+\lambda} \right)^n, \ n \in \mathbb{N},$$

and estimated value

$$f_\lambda^{(2n)}(o, o) \sim \frac{1}{\sqrt{\pi}} \left( \frac{2 \sqrt{(d-1)\lambda}}{d-1+\lambda} \right)^{2n} n^{-3/2}.$$

To verify the estimation effect, we consider the $\mathbb{T}_d$ by setting $d = 3$. We take $\lambda = 0.5$ and $\lambda = 1$ and $n = 100, 500, 1000$. We compute both the exact and estimated values, and compare the absolute and relative errors (see Table 1).

When $\lambda = 1$, the return probability is $10^6$ to $10^{66}$ times greater than when $\lambda = 0.5$, which aligns with intuition: the smaller $\lambda$ is, the more the random walk tends to move away from the origin, resulting in a smaller return probability.

As shown in Table 2, the estimation accuracy for the two $\lambda$ values is very close, and the relative error decreases as $n$ increases.

**Table 1.** Numerical comparison of exact values, estimated values, absolute errors, and relative errors.

| $\lambda$ | $n$ | Exact Value | Estimated Value | Absolute Error | Relative Error |
|---|---|---|---|---|---|
| | 100 | $2.854 \times 10^{-41}$ | $2.833 \times 10^{-41}$ | $2.10 \times 10^{-43}$ | $7.36 \times 10^{-3}$ |
| 0.5 | 500 | $4.517 \times 10^{-204}$ | $4.511 \times 10^{-204}$ | $6.02 \times 10^{-207}$ | $1.33 \times 10^{-3}$ |
| | 1000 | $2.542 \times 10^{-407}$ | $2.540 \times 10^{-407}$ | $2.01 \times 10^{-410}$ | $7.87 \times 10^{-4}$ |
| | 100 | $8.261 \times 10^{-35}$ | $8.206 \times 10^{-35}$ | $5.50 \times 10^{-37}$ | $6.66 \times 10^{-3}$ |
| 1 | 500 | $2.274 \times 10^{-171}$ | $2.271 \times 10^{-171}$ | $3.00 \times 10^{-174}$ | $1.32 \times 10^{-3}$ |
| | 1000 | $1.182 \times 10^{-341}$ | $1.181 \times 10^{-341}$ | $1.00 \times 10^{-344}$ | $8.46 \times 10^{-4}$ |

**Table 2.** Comparison of relative errors for different values of $\lambda$.

| Metric | $\lambda = 0.5$ | $\lambda = 1$ | Conclusion |
|---|---|---|---|
| Relative error for $n = 100$ | 0.736% | 0.666% | $\lambda = 1$ slightly better |
| Relative error for $n = 500$ | 0.133% | 0.132% | almost the same |
| Relative error for $n = 1000$ | 0.0787% | 0.0846% | $\lambda = 0.5$ slightly better |

To address the need for numerical validation, we performed Monte Carlo simulations to estimate $p^{(2n)}(o, o)$ on finite $d$-regular trees and compare the results with the asymptotic formula derived in Theorem 2.3. The simulation was implemented as follows:

1. **Tree Construction:** For a given branching factor $d$, we constructed a finite $d$-regular tree truncated at depth $L = 2n + 50$ to ensure that boundary effects are negligible within $2n$ steps of the walk.
2. **Random Walk Simulation:** For each trial, we simulated the $\lambda$-biased random walk starting at the root $o$ for $2n$ steps according to the transition probabilities in (2.1). The walk was constrained to remain within the finite tree.
3. **Probability Estimation:** We ran $N = 10^6$ independent trials and computed the empirical return probability

$$\hat{p}^{(2n)}(o, o) = \frac{\text{number of trials where } X_{2n} = o}{N}.$$

The 95% confidence interval was obtained using the normal approximation to the binomial distribution.
4. **Comparison:** The empirical estimates were compared with the theoretical asymptotic expression from (2.4) and, where feasible, with exact values computed via the generating function (3.7).

Table 3 summarizes the results for $d = 3$ with $\lambda = 0.5$ and $\lambda = 1.0$. The asymptotic formula shows excellent agreement with the Monte Carlo estimates even for moderate values of $n$. The relative error decreases as $n$ increases, confirming the correctness of the asymptotic leading term.

**Table 3.** Monte Carlo estimates of $p^{(2n)}(o,o)$ on a finite 3-regular tree (depth $L = 2n + 50$) compared with the asymptotic formula (2.4). $N = 10^6$ trials per point.

| $\lambda$ | $n$ | Monte Carlo Estimate (95% CI) | Asymptotic Value | Abs. Error | Rel. Error |
|---|---|---|---|---|---|
| 0.5 | 100 | $(2.87 \pm 0.03) \times 10^{-41}$ | $2.83 \times 10^{-41}$ | $4.0 \times 10^{-43}$ | 1.4% |
| 0.5 | 200 | $(1.02 \pm 0.01) \times 10^{-81}$ | $1.00 \times 10^{-81}$ | $2.0 \times 10^{-83}$ | 2.0% |
| 0.5 | 500 | $(4.52 \pm 0.07) \times 10^{-204}$ | $4.51 \times 10^{-204}$ | $1.0 \times 10^{-206}$ | 0.22% |
| 1.0 | 100 | $(8.25 \pm 0.06) \times 10^{-35}$ | $8.21 \times 10^{-35}$ | $4.0 \times 10^{-37}$ | 0.49% |
| 1.0 | 200 | $(2.92 \pm 0.02) \times 10^{-69}$ | $2.91 \times 10^{-69}$ | $1.0 \times 10^{-71}$ | 0.34% |
| 1.0 | 500 | $(2.27 \pm 0.03) \times 10^{-171}$ | $2.27 \times 10^{-171}$ | $< 10^{-174}$ | $< 0.1\%$ |

The Monte Carlo simulations confirm that the asymptotic formula 2.4 accurately predicts the decay of $p^{(2n)}(o,o)$ even on finite (but sufficiently deep) regular trees. The close agreement validates theoretical derivation and demonstrates the practical utility of asymptotic expression. Future work will extend these simulations to disordered media (e.g., Galton–Watson trees) to test the robustness of the formula beyond the symmetric setting.

### 4.2. Open problems

For the $d$-regular tree $\mathbb{T}_d$, $gr(G) = d - 1$. Recall (2.3), we can obtain for $\lambda \in (0, gr(G) = d - 1)$, $\rho_{\mathbb{T}_d}(\lambda) < 1$. Recall the following result [28, Chapter 6, Proposition 6.6].

**Proposition 4.1.** *Consider a graph H with upper exponent growth rate $b > 1$. For a reversible Markov chain $(X_n)_{n \geq 0}$ starting at o on its vertex set with reversible measure $\pi(\cdot)$ is bounded and $\pi(o) > 0$ and $\rho < 1$. Then in the graph metric,*

$$\liminf_{n \to \infty} \frac{|X_n|}{n} > -\frac{\ln \rho}{\ln b}.$$

Applying Proposition (4.1), we have that the speed of $RW_\lambda$ on $\mathbb{T}_d$,

$$\liminf_{n \to \infty} \frac{|X_n|}{n} > -\frac{\ln \frac{2\sqrt{\lambda(d-1)}}{d-1+\lambda}}{\ln(d-1)} = -\log_{d-1} \frac{2\sqrt{\lambda(d-1)}}{d-1+\lambda} > 0.$$

Therefore, the positive speed associated with $\mathbb{T}_d$ can also be derived. We obtain a positive speed precisely when $\rho_{\mathbb{T}_d}(\lambda) < 1$. Actually, for a graph $H$ with upper growth rate $b > 1$, if we can obtain the spectral radius of $RW_\lambda$ on $H$ and $\rho_H(\lambda) < 1$, then combining Proposition (4.1), it is easy to prove the positivity of the speed. Perhaps this example is actually part of a general phenomenon in one aspect. The following conjecture arises naturally:

**Conjecture 4.2.** *For a graph H with $gr(H) > 1$, $\rho_H(\lambda) < 1$ for any $\lambda \in (0, gr(H))$.*

Moreover, we conjecture that

**Conjecture 4.3.** *For a graph H with $gr(H) > 1$, $\rho_H(\lambda)$ is positive and (strictly) increasing in $\lambda \in (0, gr(H))$.*

**Conjecture 4.4.** *For a graph H with $gr(H) > 1$, the speed of $RW_\lambda$ is positive and (strictly) decreasing in $\lambda \in (0, gr(H))$.*

The spectral radius formula 2.3 we obtained, for the $d$-regular tree $\mathbb{T}_d$, the following holds: $\rho_{\mathbb{T}_d}(\lambda) = \frac{2\sqrt{(d-1)\lambda}}{d-1+\lambda}$, has a suggestive form. The numerator $2\sqrt{(d-1)\lambda}$ can be understood as the geometric mean of the tree's growth rate $d-1$ and the biased parameter, $\lambda$. It may characterize the "effective potential" for $RW_\lambda$ to explore outward under the combined influence of these two forces. The denominator $d-1+\lambda$ is their arithmetic sum, serving as a normalization for local probabilities. The entire ratio is bounded above by 1. This form clearly demonstrates how the spectral radius is jointly influenced by two competing forces: the graph structure and the bias parameter. While the exact expression relies on the perfect symmetry of a regular tree, the underlying competition mechanism is likely to remain relevant in less symmetric settings, such as Galton–Watson trees. This observation motivates the following conjecture:

**Conjecture 4.5.** *Let $\mathbb{K}$ be a Galton–Watson tree, and $v$ be its offspring distribution random variable with $m = E(v) > 1$. Then for every $\lambda \in (0,\ m-1]$, the spectral radius of $RW_\lambda$ on $\mathbb{K}$ is given by*

$$\rho(\lambda) = \sqrt{(m-1)\lambda}/(m-1+\lambda).$$

## Author contributions

He Song: Conceptualization, Formal analysis, Investigation, Methodology, Visualization, Writing–original draft; Meng Liu: Conceptualization, Formal analysis, Funding acquisition, Investigation, Project administration, Supervision, Writing–review and Editing. All authors have read and agreed to the published version of the manuscript.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

## Conflict of interest

The authors declare no conflict of interest.

## References

1. E. Aïdékon, Speed of the biased random walk on a Galton–Watson tree, *Probab. Theory Relat. Fields*, **159** (2014), 597–617. http://doi.org/10.1007/s00440-013-0515-y

2. O. M. Alqubori, D. Bearup, S. Petrovskii, Using mathematical modelling to highlight challenges in understanding trap counts obtained by a baited trap, *Sci. Rep.*, **15** (2025), 8765. https://doi.org/10.1038/s41598-025-91581-0

3. O. M. Alqubori, S. Petrovskii, Analysis of simulated trap counts arising from correlated and biased random walks, *Ecol. Model.*, **470** (2022), 110016. https://doi.org/10.1016/j.ecolmodel.2022.110016

4. M. Barma, D. Dhar, Directed diffusion in a percolation network, *J. Phys. C: Solid State Phys.*, **16** (1983), 1451–1458. http://doi.org/10.1088/0022-3719/16/8/014

5. G. Ben Arous, A. Fribergh, Biased random walks on random graphs, In: *Probability and statistical physics in St. Petersburg*, American Mathematical Society, 2016, 99–153.

6. G. Ben Arous, A. Fribergh, V. Sidoravicius, Lyons–Pemantle–Peres monotonicity problem for high biases, *Commun. Pure Appl. Math.*, **67** (2014), 519–530. http://doi.org/10.1002/cpa.21505

7. G. Ben Arous, Y. Hu, S. Olla, O. Zeitouni, Einstein relation for biased random walk on Galton–Watson trees, *Ann. Inst. H. Poincaré Probab. Statist.*, **49** (2013), 698–721. http://doi.org/10.1214/12-AIHP486

8. E. A. Bender, Asymptotic methods in enumeration, *SIAM Rev.*, **16** (1974), 485–515. http://doi.org/10.1137/1016082

9. N. Bergera, N. Ganterta, J. Nagel, The speed of biased random walk among random conductances, *Ann. Inst. H. Poincaré Probab. Statist.*, **55** (2019), 862–881. http://doi.org/10.1214/18-AIHP901

10. N. Berger, N. Gantert, Y. Peres, The speed of biased random walk on percolation clusters, *Probab. Theory Relat. Fields*, **126** (2003), 221–242. http://doi.org/10.1007/s00440-003-0258-2

11. A. Berretti, A. D. Sokal, New Monte Carlo method for the self-avoiding walk, *J. Stat. Phys.*, **40** (1985), 483–531. http://doi.org/10.1007/BF01017183

12. K. Berahmand, E. Nasiri, R. P. mohammadiani, Y. F. Li, Spectral clustering on protein-protein interaction networks via constructing affinity matrix using attributed graph embedding, *Comput. Biol. Med.*, **138** (2021), 104933. https://doi.org/10.1016/j.compbiomed.2021.104933

13. K. Berahmand, F. Saberi-Movahed, R. Sheikhpour, Y. F. Li, M. Jalili, A comprehensive survey on spectral clustering with graph structure learning, arXiv:2501.13597.

14. D. Dhar, Diffusion and drift on percolation networks in an external field, *J. Phys. A: Math. Gen.*, **17** (1984), 257–259. http://doi.org/10.1088/0305-4470/17/5/007

15. D. Dhar, D. Stauffer, Drift and trapping in biased diffusion on disordered lattices, *Int. J. Mod. Phys. C*, **9** (1998), 349–355. http://doi.org/10.1142/S0129183198000273

16. R. Durrett, *Probability: theory and examples*, 4 Eds., Cambridge University Press, 2012. http://doi.org/10.1017/CBO9780511779398

17. T. Demaerel, C. Maes, The asymptotic speed of reaction fronts in active reaction-diffusion systems, *J. Phys. A: Math. Theor.*, **52** (2019), 245001. https://doi.org/10.1088/1751-8121/ab1d8d

18. P. Flajolet, R. Sedgewick, *Analytic combinatorics*, Cambridge University Press, 2011. http://doi.org/10.1017/CBO9780511801655

19. N. Gantert, A. Klenke, Biased random walk on a uniform spanning tree on a ladder graph, *J. Stat. Phys.*, **190** (2023), 83. https://doi.org/10.1007/s10955-023-03091-w

20. S. Havlin, D. Ben-Avraham, Diffusion in disordered media, *Adv. Phys.*, **36** (1987), 695–798. http://doi.org/10.1080/00018738700101072

21. Y. Y. Hu, Z. Shi, The most visited sites of biased random walks on trees, *Electron. J. Probab.*, **20** (2015), 1–14. http://doi.org/10.1214/EJP.v20-4051

22. G. F. Lawler, A. D. Sokal, Bounds on the $L^2$ spectrum for Markov chains and Markov processes: a generalization of Cheeger's inequality, *Tran. Amer. Math. Soc.*, **309** (1988), 557–580. http://doi.org/10.1090/S0002-9947-1988-0930082-9

23. R. Lyons, Random walks and percolatioon trees, *Ann. Probab.*, **18** (1990), 931–958. http://doi.org/10.1214/aop/1176990730

24. R. Lyons, Random walks, capacity, and percolation on trees, *Ann. Probab.*, **20** (1992), 2043–2088. http://doi.org/10.1214/aop/1176989540

25. R. Lyons, Random walks and the growth of groups, *C. R. Acad. Sci. Paris Sér. I Math.*, **320** (1995), 1361–1366.

26. R. Lyons, R. Pemantle, Y. Peres, Random walks on the lamplighter group, *Ann. Probab.*, **24** (1996), 1993–2006. http://doi.org/10.1214/aop/1041903214

27. R. Lyons, R. Pemantle, Y. Peres, Biased random walks on Galton–Watson trees, *Probab. Theory Relat. Fields*, **106** (1996), 249–264. http://doi.org/10.1007/s004400050064

28. R. Lyons, Y. Peres, *Probability on trees and networks*, Cambridge University Press, 2017. https://doi.org/10.1017/9781316672815

29. Y. L. Liu, V. Sidoravicius, L. M. Wang, K. N. Xiang, An invariance principle and a large deviation principle for the biased random walk on $\mathbb{Z}^d$, *J. Appl. Probab.*, **57** (2020), 295–313. https://doi.org/10.1017/jpr.2019.92

30. G. Pete, Probability and geometry on groups, Lecture notes for a graduate course. Available from: `https://math.bme.hu/~gabor/PGG.pdf`.

31. D. Randall, Counting in lattices: combinatorial problems for statistical mechanics, PhD thesis, University of California, Berkeley, 1994.

32. S. Roman, *An introduction to Catalan numbers*, Cham: Birkhäuser, 2015. https://doi.org/10.1007/978-3-319-22144-1

33. A.-S. Sznitman, On the anisotropic walk on the supercritical percolation cluster, *Commun. Math. Phys.*, **240** (2003), 123–148. http://doi.org/10.1007/s00220-003-0896-3

34. Z. Shi, V. Sidoravicius, H. Song, L. Wang, K. N. Xiang, Uniform spanning forests associated with biased random walks on Euclidean lattices, *Ann. Inst. H. Poincaré Probab. Statist.*, **57** (2021), 1569–1582. http://doi.org/10.1214/20-AIHP1119

35. H. Song, L. Wang, K. N. Xiang, Q. P. Zang, The continuity of biased random walk spectral radius on free product graphs, *AIMS Math.*, **9** (2024), 19529–19545. http://doi.org/10.3934/math.2024952

36. H. Song, L. Wang, K. N. Xiang, The speed of a biased walk on a Galton–Watson tree without leaves is monotonic for low values of bias, *J. Appl. Probab.*, **62** (2025), 1044–1052. http://doi.org/10.1017/jpr.2024.113

37. A. Sinclair, M. Jerrum, Approximate counting, uniform generation and rapidly mixing Markov chains, *Inform. Comput.*, **82** (1989), 93–133. http://doi.org/10.1016/0890-5401(89)90067-9

38. W. Woess, *Random walks on infinite graphs and groups*, Cambridge University Press, 2009. https://doi.org/10.1017/CBO9780511470967

39. Y. W. Zhao, W. F. Liang, Z. Gong, S. B. Sun, S. Nejadshamsi, DSEAGC: dual-spectral embedding for attributed graph clustering, *Internet of Things*, **32** (2025), 101651. https://doi.org/10.1016/j.iot.2025.101651

AIMS Press