Mathematics

Research article

Generalized fractal measures on Cartesian products: Critical cases and multidimensional Cantor sets

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Abstract: This paper investigated the interplay of generalized fractal measures, specifically packing and Hewitt-Stromberg measures, on Cartesian products of symmetric generalized Cantor sets. By developing a unified framework for analyzing product sets in separable metric spaces, we uncovered how these measures interact in both typical and critical scenarios. Through explicit construction of multidimensional Cantor sets, we demonstrated instances where classical inequalities break down, particularly in cases involving vanishing and divergent measures. Our results enhance the theoretical foundation for multifractal analysis, offering new tools to address complexity in product geometries.

1. Introduction

Multifractal analysis has become an indispensable framework for characterizing the non-uniform scaling behaviors observed in many natural and mathematical phenomena. A crucial aspect of this analysis lies in describing the distribution of local singularities of geometric measures. In this context, generalized fractal measures, such as packing measures and Hewitt–Stromberg measures, play a central role [1]. Recent work (see [2]) has demonstrated that these generalized approaches are particularly effective for understanding the complexity of measures in separable metric spaces. In this context, the study of Cartesian products of fractal sets naturally arises in dynamic settings (e.g., product attractors [3]) as well as in probabilistic frameworks (e.g., multi-scale measures [4, 5]). Despite these advances, the properties of generalized measures on products of sets remain poorly understood,

especially in the critical cases where interactions between divergent and negligible singularities violate classical inequalities [6–8]. In this paper, we present a detailed investigation of generalized fractal measures on Cartesian products of fractal sets. Our results contribute to a deeper understanding of the multifractal formalism and its applications in analyzing complex systems, particularly in critical cases where standard methods fall short.

We begin by considering two separable metric spaces, $(\mathbb{X}, d_{\mathbb{X}})$ and $(\mathbb{Y}, d_{\mathbb{Y}})$, and their Cartesian product $\mathbb{X} \times \mathbb{Y} := \{(x, y); \ x \in \mathbb{X}, \ y \in \mathbb{Y}\}$ equipped with the maximum metric defined as follows:

$$d\big((x, y), (x', y')\big) = \max\big\{d_{\mathbb{X}}(x, x'), d_{\mathbb{Y}}(y, y')\big\}.$$

Let $\mu \in \mathcal{P}(\mathbb{X})$, the family of Borel probability measures on $\mathbb{X}$. Recall that a Borel probability measure μ is said to be doubling on $E \subseteq \mathbb{X}$ if there are constants $C > 0$ and $r_0 > 0$ such that

$$\mu(B(x, \ 2r)) \le C \, \mu(B(x, \ r)),$$

for all $x \in E$ and $0 < r < r_0$. We will write $\mathcal{P}_D(\mathbb{X})$ for the family of Borel probability measures on $\mathbb{X}$ which satisfy the doubling condition. Doubling measures naturally arise in the study of the gradients of convex functions on the Euclidean d-space. One can, cite for example, [9, 10]. A dimension function is a continuous increasing function defined for $r \ge 0$ with $h(0) = 0$ and $h(r) > 0$. We denote the set of dimension functions by $\mathcal{F}$. In addition, we say that $h \in \mathcal{F}$ satisfies the doubling condition if

$$h(2r) \le m_h \, h(r), \quad \text{for } r > 0,$$

for some positive constant m_h. We denote the set of dimension functions satisfying the doubling condition by $\mathcal{F}_0$.

A natural question in geometric measure theory is whether the measure of a set E defined in a product space can be determined from the measures of its components. In classical settings, this question is often addressed through product formulas or dimensional heuristics, which suggest that the behavior of the product should be trivial. However, for general measures such principles fail, and one can usually derive only upper and lower bounds. In particular, in the critical case, the corresponding product set may exhibit distinct regimes, including zero, finite positive, or infinite measure. This shows that a refined analysis is necessary to control the measure of sets in product spaces beyond classical situations.

Observe first that when the space $\mathbb{X}$ is $\mathbb{R}$, the 1–dimensional Hausdorff measure $\mathcal{H}^1$ coincides with the Lebesgue measure $\mathcal{L}^1$. Let Γ be a Lebesgue–measurable subset of $\mathbb{R}^2$. For each $x \in \mathbb{R}$, define the vertical section

$$\Gamma_x := \{y \in \mathbb{R} : (x, y) \in \Gamma\}.$$

By Fubini's theorem, Γ_x is $\mathcal{L}^1$–measurable for almost every x, and the two-dimensional Lebesgue measure $\mathcal{L}^2$ of Γ can be expressed as

$$\mathcal{L}^2(\Gamma) = \int_{\mathbb{R}} \mathcal{H}^1(\Gamma_x) \, d\mathcal{H}^1(x).$$

In the particular case where Γ is a Cartesian product $\Gamma = E \times F$, this formula reduces to

$$\mathcal{L}^2(\Gamma) = \mathcal{H}^1(E) \, \mathcal{H}^1(F).$$

These identities show, in particular, that such a factorization does not hold in general for the Hausdorff measure of dimension 2, namely

$$\mathcal{H}^2(E \times F) \neq \mathcal{H}^1(E)\,\mathcal{H}^1(F).$$

The problem of determining whether equality may occur, as well as quantifying the inevitable inequalities has been extensively studied in the literature, especially in the setting of Hausdorff and packing measures for fractal sets. These investigations rely on methods from fractal geometry, geometric measure theory, and real analysis, and aim to clarify the structural conditions under which equality can be recovered. A natural initial approach, when $\Gamma = E \times F$, consists in covering the sets E and F by collections of intervals with diameters smaller than a given $\delta > 0$. This construction induces a covering of Γ by squares of side length less than δ, from which one may attempt to derive estimates for the corresponding Hausdorff measures.

Such results have been studied and discussed for different fractal measures, and the interested readers may consult [8] for the upper and lower Hewitt-Stromberg measure and also [11] for Hausdorff and packing measures. In particular, for $E \subseteq \mathbb{X}$, $F \subseteq \mathbb{Y}$, and $h,\, g \in \mathcal{F}$, one has [12]

$$\mathcal{H}^h(E)\mathcal{P}^g(F) \leq \mathcal{P}^{hg}(E \times F) \leq \mathcal{P}^h(E)\mathcal{P}^g(F),$$

except in the $0 - \infty$ cases; that is, the products on the left-hand and right-hand sides are not of the form $0 \times \infty$ or $\infty \times 0$. In this paper, we explore the generalized packing (resp., Hewitt-Stromberg) h-measure $\mathcal{P}_\mu^{q,h}$ (resp., $\mathcal{V}_\mu^{q,h}$) to establish product inequalities similar to the one mentioned above in metric spaces. By employing the generalized Hewitt-Stromberg h-measure $\mathcal{V}_\mu^{q,h}$, which is more extensive than the generalized Hausdorff h-measure $\mathcal{H}_\mu^{q,h}$, we provide a broader framework for these inequalities. More precisely, we have the following result.

Theorem 1.1. *Let $\mu \in \mathcal{P}(\mathbb{X})$, $\nu \in \mathcal{P}(\mathbb{Y})$, $q \in \mathbb{R}$, and $h, g \in \mathcal{F}$. For any set $E \subseteq \mathbb{X} \times \mathbb{Y}$, we have*

$$\mathcal{P}_{\mu \times \nu}^{q,hg}(E) \geq \int^* \underrightarrow{\mathcal{V}}_\nu^{q,g}(E^x)\, d\mathcal{P}_{\Delta,\mu}^{q,h}(x). \tag{1.1}$$

Letting $E = A \times B$, we get the following estimates for Cartesian rectangles,

$$\mathcal{V}_\mu^{q,h}(A)\mathcal{P}_\nu^{q,g}(B) \leq \mathcal{P}_{\mu \times \nu}^{q,hg}(A \times B) \leq \mathcal{P}_\mu^{q,h}(A)\mathcal{P}_\nu^{q,g}(B),$$

except in the $0 - \infty$ cases. It is natural to ask what happens in the $0 - \infty$ cases. Is $\mathcal{P}_{\mu \times \nu}^{q,hg}(E \times F)$ always equal to ∞? This question is important for understanding the behavior of product measures in different contexts. Our study aims to explore this issue by examining specific cases and conditions. By doing so, we seek to determine whether $\mathcal{P}_{\mu \times \nu}^{q,hg}(E \times F)$ is always equal to ∞ when $\mathcal{P}_\mu^{q,h}(E)$ is infinite and $\mathcal{V}_\nu^{q,g}(F)$ is zero. This investigation will enhance our understanding of generalized measures and their applications. In Section 4, we construct the generalized symmetric Cantor set in d-dimensional space created based on the system $\left\{l, \{n_k\}_{k \geq 1}, \{\lambda_k\}_{k \geq 1}\right\}$ denoted by $\mathcal{K}^d$, and we prove the following result.

Theorem 1.2. *Let $\mathcal{K}^d$ be the generalized symmetric Cantor set in d-dimensional space created based the system $\left\{l, \{n_k\}_{k \geq 1}, \{\lambda_k\}_{k \geq 1}\right\}$, which satisfies (4.1). Let $\mathcal{N} = \{k : n_k > l\}$. Assume that there exists a positive constant C such that $\delta_k \leq C\mu(I_k)$ for all $k \in \mathcal{N}$. Then, there exists a constant M such that*

$$\limsup_{k \to \infty} (n_1 n_2 \ldots n_k)^d \mu(I_k)^q h(\lambda_k) \leq \mathcal{P}_{\mu,0}^{q,h}(\mathcal{K}^d) \leq M \limsup_{k \to \infty} (n_1 n_2 \ldots n_k)^d \mu(I_k)^q h(\lambda_k).$$

To prove Theorem 1.2, we will construct a new pre-packing measure $\widetilde{\mathcal{P}}_{\mu,0}^{q,h}$ using closed cubes rather than closed balls, which is much easier to investigate and more appropriate in our study. To answer the question asked above, we will construct four sets $\mathcal{K}_1$ and $\mathcal{K}_2^j$, $j = 1, 2, 3$ such, that $\mathcal{P}_\mu^{q,h}(\mathcal{K}_1) = \infty$, and $\mathcal{V}_\nu^{q,g}(\mathcal{K}_2^j) = 0$, while $\widetilde{\mathcal{P}}_{\mu \times \nu,0}^{q,hg}(\mathcal{K}_1 \times \mathcal{K}_2^j)$ is infinite, positive finite, or zero accordingly as $j = 1, 2, 3$, respectively. Therefore, we give a negative answer to this question. To this end, we must estimate the generalized packing measure of the product set $\mathcal{K}_1 \times \mathcal{K}_2$, where $\mathcal{K}_1$ (resp., $\mathcal{K}_2$) is the one-dimensional generalized Cantor set (see Theorem 5.1).

The present paper is structured as follows. The next section is dedicated to constructing fractal measures and preliminary results. We define the generalized packing and Hewitt-Stromberg measures, introducing necessary notation and establishing fundamental properties. In Section 3, we present and prove integral inequalities involving these measures, with a particular focus on Cartesian rectangles. Section 4 focuses on the proof of Theorem 1.2, where we construct a new packing premeasure using closed cubes. Finally, Section 5 discusses the implications of our results and explores potential applications in multifractal analysis.

2. Construction of fractal measures and preliminary results

2.1. Generalized packing and Hewitt-Stromberg h-measures

Throughout this work, we consider separable metric spaces. This assumption is natural in the present setting, since the classical definitions of Hausdorff and packing measures are based on countable coverings or packings. Separability guarantees the availability of such countable families and ensures that the standard measure constructions used below are well defined.

$$\xi(x,r) = \mu(B(x,r))^q h(2r) \quad \text{and} \quad \zeta(x,r) = \nu(B(x,r))^q g(2r), \tag{2.1}$$

where $x \in \mathbb{X}$, $q \in \mathbb{R}$, and $r > 0$. We say that ξ satisfies the doubling condition if and only if $\mu \in \mathcal{P}_D(\mathbb{X})$ and $h \in \mathcal{F}_0$. We start this section by recalling the construction of various fractal measures that we will use throughout this paper. To this end, consider a packing of the set E, that is, a sequence $(x_i, r_i)_{i \in I}$, $x_i \in E$ and $r_i > 0$, such that $x_i \notin B(x_j, r_j)$ for all $i \neq j$ in I. Equivalently, for all i, j in I, we have

$$i \neq j \implies d(x_i, x_j) > \max(r_i, r_j).$$

It is said to be δ-fine if $r_i \leq \delta$ for every i. Now, we write

$$\mathcal{P}_\delta^\xi(E) = \sup \sum \xi(x_i, r_i),$$

where the supremum is evaluated over all δ-fine packings of E. The generalized packing premeasure $\mathcal{P}_0^\xi(E)$ and measure $\mathcal{P}^\xi(E)$ of E with respect to the gauge $\xi(x,r)$ are respectively given by the following definitions:

$$\mathcal{P}_0^\xi(E) = \inf_{\delta > 0} \mathcal{P}_\delta^\xi(E) = \lim_{\delta \to 0} \mathcal{P}_\delta^\xi(E)$$

and

$$\mathcal{P}^\xi(E) = \inf \left\{ \sum_i \mathcal{P}_0^\xi(E_i); \; E \subseteq \bigcup_i E_i \right\}.$$

The packings we utilize are occasionally referred to as weak packings or pseudo-packings. We refer the reader to [13–15] for more information about these types of packings. Consider the case, for $t \in \mathbb{R}$, $h(r) = (2r)^t$ then ξ will be denoted by ξ_t. Therefore, $\mathcal{P}^\xi$ reduces to generalized packing measure $\mathcal{W}_\mu^{q,t}$ [2] and the standard packing measure $\mathcal{R}^h$ [12] by taking $q = 0$. The packing premeasure and measure are fundamental in the study of multifractal analysis [4, 16]. Therefore, we should analyze the link between the packing premeasure and the measure. In [17], the authors discussed the case of the s-dimensional packing premeasure and measure on $\mathbb{R}^n$, and it has been shown that for every compact set $K \subset \mathbb{R}^n$ of finite $\mathcal{P}_0^s$-premeasure and every $0 \leq s \leq n$, $\mathcal{P}_0^s(K) = \mathcal{P}^s(K)$. This result has been extended in [18] in a general metric space, using the multifractal premeasure and measure instead of the s-dimensional premeasure and measure. That is to say, the authors proved that for $E \subset \mathrm{supp}(\mu)$, a compact subset with finite $\mathcal{P}_{\mu,0}^{q,t}$-premeasure, if $q \leq 0$, then $\mathcal{P}_{\mu,0}^{q,t}(E) = \mathcal{P}_\mu^{q,t}(E)$; and if $q > 0$ and μ is doubling on E, then $\mathcal{P}_{\mu,0}^{q,t}(E)$ and $\mathcal{P}_\mu^{q,t}(E)$ are either zero or neither. This result can be extended in our case. More precisely, we have the following result:

Lemma 2.1. *Let* $\mathbb{X}$ *be a metric space and* ξ *the gauge function defined in* (2.1). *Suppose K is a compact subset of* $\mathrm{supp}(\mu)$ *such that* $\mathcal{P}_0^\xi(K) < \infty$. *Then:*

(1) If $q \leq 0$, then $\mathcal{P}^\xi(K) = \mathcal{P}_0^\xi(K)$.
(2) If $q > 0$ and ξ satisfies the doubling condition, then there exists a constant C such that

$$C \, \mathcal{P}_0^\xi(K) \leq \mathcal{P}^\xi(K) \leq \mathcal{P}_0^\xi(K).$$

Proof. The proof is straightforward and mimics that in Theorem 1 in [18]. □

In the following, we will set up the generalized Hewitt-Stromberg measure $\mathcal{V}^\xi$. We need first to define the uniform δ-packing, that is, a δ-packing such that $r_i = r_j$ for all i, j in I. For $E \subseteq \mathbb{X}$, we define the packing number $\mathcal{M}_{\mu,\delta}^q(E)$ by

$$\mathcal{M}_{\mu,\delta}^q(E) = \sup \left\{ \sum_i \mu(B(x_i, r_i))^q \, \middle| \, (x_i, r_i) \text{ is a uniform } \delta\text{-packing of } E \right\}$$

and let

$$\mathcal{V}_0^\xi(E) = \liminf_{\delta \to 0} \mathcal{M}_{\mu,\delta}^q(E) h(2\delta).$$

It is clear that $\mathcal{V}_0^\xi$ is increasing and $\mathcal{V}_0^\xi(\emptyset) = 0$. However, it is not σ-additive. For this, we introduce the $\mathcal{V}^\xi$ measure defined by

$$\mathcal{V}^\xi(E) = \inf \left\{ \sum_i \mathcal{V}_0^\xi(E_i) \, \middle| \, E \subseteq \bigcup_i E_i \right\}.$$

One can cite [5, 19, 20] for more details on the Hewitt-Stromberg measures and their importance in analyzing the local properties of fractals and products of fractals. Now, we introduce our definitions of scaled measures. A set $\Delta \subseteq (0, \infty)$ such that $0 \in \overline{\Delta}$ is termed a scale, where $\overline{\Delta}$ is the closure of Δ. A packing (x_i, r_i) is Δ-valued if $r_i \in \Delta$ for every i. A (Δ, δ)-packing is a packing that is $\Delta \cap (0, \delta]$-valued, i.e., Δ-valued and δ-fine. Consider the gauge function defined in (2.1) and $E \subseteq \mathbb{X}$. We write:

$$\mathcal{P}_{\Delta,\delta}^\xi(E) = \sup \sum \xi(x_i, r_i),$$

where the supremum is evaluated for every (Δ, δ)-packing of the set E. The generalized Δ-scaled packing premeasure $\mathcal{P}^{\xi}_{\Delta,0}(E)$ and measure $\mathcal{P}^{\xi}_{\Delta}(E)$ of E with respect to the gauge $\xi(x, r)$ are respectively given by the following definitions:

$$\mathcal{P}^{\xi}_{\Delta,0}(E) = \inf_{\delta>0} \mathcal{P}^{\xi}_{\Delta,\delta}(E)$$

and

$$\mathcal{P}^{\xi}_{\Delta}(E) = \inf\Big\{ \sum_i \mathcal{P}^{\xi}_{\Delta,0}(E_i); \ E \subseteq \bigcup_i E_i \Big\}.$$

The generalized Δ-scaled packing measure $\mathcal{P}^{\xi}_{\Delta}$ represents a direct generalization of the generalized packing h-measure $\mathcal{P}^{\xi}$. Now, we define the Δ-scaled generalized Hewitt-Stromberg measure. We write

$$\mathcal{V}^{\xi}_{\Delta,\delta}(E) = \sup \sum_i \xi(x_i, r_i),$$

where the supremum is taken over all (Δ, δ)-valued uniform packings of the set E. The generalized Δ-scaled Hewitt-Stromberg premeasure $\mathcal{V}^{\xi}_{\Delta,0}(E)$ and measure $\mathcal{V}^{\xi}_{\Delta}(E)$ of E with respect to the gauge ξ are respectively defined as follows:

$$\mathcal{V}^{\xi}_{\Delta,0}(E) = \limsup_{\substack{\delta\in\Delta \\ \delta\to 0}} \mathcal{M}^{q}_{\mu,\delta}(E)h(2\delta) \tag{2.2}$$

and

$$\mathcal{V}^{\xi}_{\Delta}(E) = \inf\Big\{ \sum_i \mathcal{V}^{\xi}_{\Delta,0}(E_i); \ E \subseteq \bigcup_i E_i \Big\}.$$

Remark 2.1. *(1) The introduction of Δ–scaled packing and Hewitt–Stromberg measures is motivated by the need for finer control on the set of admissible radii in packings. Restricting radii to a prescribed scale Δ allows us to localize measure–theoretic contributions at specific resolutions, which is essential for establishing the integral inequalities in Section 3 and for constructing examples exhibiting non-classical behavior of product measures.*

(2) The Δ-scaled packing (resp., Hewitt-Stromberg) measure is a straight generalization of the packing (resp., Hewitt-Stromberg) measure. The only distinction is that the radii permitted in the packings are restricted to Δ.

(3) Clearly, we have

$$\mathcal{V}^{\xi}_{\Delta,0} \leq \mathcal{P}^{\xi}_{\Delta,0} \quad and \quad \mathcal{V}^{\xi}_{\Delta} \leq \mathcal{P}^{\xi}_{\Delta}.$$

(4) It is easy to check that the set function $\mathcal{V}^{\xi}_{\Delta,0}$ can be equivalently defined as

$$\mathcal{V}^{\xi}_{\Delta,0}(E) = \inf_{\delta>0} \mathcal{V}^{\xi}_{\Delta,\delta}(E). \tag{2.3}$$

The study of generalized packing and Hewitt-Stromberg measures has been the subject of several previous works. In particular, the regularity properties of Hewitt-Stromberg h-measures were investigated in [21, 22], providing important insights into their structural behavior in various metric spaces. Moreover, [23] explored Hausdorff measures on Cartesian product sets, offering results that complement the framework considered in this paper. By extending these approaches, we establish new inequalities and estimation techniques that apply to a broader class of fractal measures in metric spaces.

2.2. The upper and lower generalized measures

We start by defining upper generalized packing and Hewitt-Stromberg measures as special cases of their associated scaled measures. Let ξ be the gauge function defined in (2.1) and $E \subseteq \mathbb{X}$. We write:

$$\overline{\mathcal{P}}_0^\xi(E) = \sup_\Delta \mathcal{P}_{\Delta,0}^\xi(E) = \mathcal{P}_{(0,\infty)}^\xi(E)$$

and

$$\overline{\mathcal{V}}_0^\xi(E) = \sup_\Delta \mathcal{V}_{\Delta,0}^\xi(E) = \mathcal{V}_{(0,\infty)}^\xi(E).$$

Then, the upper generalized packing and Hewitt-Stromberg measure are defined, respectively, by

$$\overline{\mathcal{P}}^\xi(E) = \inf\Big\{\sum_i \overline{\mathcal{P}}_0^\xi(E_i);\ E \subseteq \bigcup_i E_i\Big\}$$

and

$$\overline{\mathcal{V}}^\xi(E) = \inf\Big\{\sum_i \overline{\mathcal{V}}_0^\xi(E_i);\ E \subseteq \bigcup_i E_i\Big\}.$$

Note that the upper generalized packing measure $\overline{\mathcal{P}}^\xi$ is essentially identical to the generalized packing measure $\mathcal{P}^\xi$ mentioned at the beginning of this section. We refer the reader to [24, 25] for more information about the upper Hewitt-Stromberg measure, also called box measures. It follows from (2.2) that

$$\overline{\mathcal{V}}_0^\xi(E) = \limsup_{\delta\to 0} \mathcal{M}_{\mu,\delta}^q(E)h(2\delta).$$

Similarly, we define the lower generalized packing and Hewitt-Stromberg measure as the lower extreme cases of their associated scaled measures. Let ξ be the gauge function defined in (2.1) and $E \subseteq \mathbb{X}$. We write:

$$\underline{\mathcal{P}}_0^\xi(E) = \inf_\Delta \mathcal{P}_{\Delta,0}^\xi(E) \qquad \text{and} \qquad \underline{\mathcal{V}}_0^\xi(E) = \inf_\Delta \mathcal{V}_{\Delta,0}^\xi(E),$$

where the infima is taken for every scale Δ. Then, the lower generalized packing and Hewitt-Stromberg measure are defined, respectively, by

$$\underline{\mathcal{P}}^\xi(E) = \inf\Big\{\sum_i \underline{\mathcal{P}}_0^\xi(E_i);\ E \subseteq \bigcup_i E_i\Big\}$$

and

$$\underline{\mathcal{V}}^\xi(E) = \inf\Big\{\sum_i \underline{\mathcal{V}}_0^\xi(E_i);\ E \subseteq \bigcup_i E_i\Big\}.$$

As above, it follows from (2.2) that

$$\underline{\mathcal{V}}_0^\xi(E) = \liminf_{\delta\to 0} \mathcal{M}_{\mu,\delta}^q(E)h(2\delta).$$

In other words, the lower generalized Hewitt-Stromberg measure is exactly the Hewitt-Stromberg measure $\mathcal{V}^\xi$ defined at the beginning of this section. The lower premeasures $\underline{\mathcal{P}}_0^\xi$ and $\underline{\mathcal{V}}_0^\xi$ are not sub-additive, so for this we introduce

$$\underrightarrow{\mathcal{P}}^\xi(E) = \liminf_{E_n \nearrow E} \underline{\mathcal{P}}_0^\xi(E_n) = \inf\Big\{\sup_n \underline{\mathcal{P}}_0^\xi(E_n):\ E_n \nearrow E\Big\}$$

and

$$\underrightarrow{\mathcal{V}}^{\xi}(E) = \liminf_{E_n \nearrow E} \underline{\mathcal{V}}^{\xi}_0(E_n) = \inf\left\{\sup_n \underline{\mathcal{V}}^{\xi}_0(E_n) : E_n \nearrow E\right\}.$$

Clearly

$$\underline{\mathcal{P}}^{\xi} \leq \underrightarrow{\mathcal{P}}^{\xi} \leq \underline{\mathcal{P}}^{\xi}_0 \quad \text{and} \quad \underline{\mathcal{V}}^{\xi} \leq \underrightarrow{\mathcal{V}}^{\xi} \leq \underline{\mathcal{V}}^{\xi}_0.$$

3. Integral inequalities

Throughout this section, E^x denotes the vertical section defined by $\{y \in Y, (x, y) \in E\}$ and $\int^*$ denotes the upper Lebesgue integral defined as follows:

$$\int^* f d\mu = \inf\left\{\int \phi \, d\mu : \phi \geq f \text{ Borel measurable}\right\}.$$

This section aims to prove Theorem 1.1.

Lemma 3.1. *Let ξ and ζ be the gauge functions defined in* (2.1). *For any set $E \subseteq \mathbb{X} \times \mathbb{Y}$, we have*

$$\mathcal{P}^{\xi\zeta}_{\Delta,0}(E) \geq \mathcal{P}^{\xi}_{\Delta,0}(\mathbb{X}) \inf_{x \in \mathbb{X}} \underline{\mathcal{V}}^{\zeta}_0(E^x). \tag{3.1}$$

Proof. Let $m < \inf_{x \in \mathbb{X}} \underline{\mathcal{V}}^{\zeta}_0(E^x)$. For each x, there is a number $n \in \mathbb{N}$ such that $\underline{\mathcal{V}}^{\zeta}_\delta(E^x) > m$ for all $\delta < \frac{1}{n}$. We set

$$B_n = \left\{x : \underline{\mathcal{V}}^{\zeta}_\delta(E^x) > m \text{ for all } \delta < \frac{1}{n}\right\},$$

and clearly, we have $B_n \nearrow \mathbb{X}$. Let $m' < \mathcal{P}^{\xi}_{\Delta,0}(\mathbb{X})$. Since $B_n \nearrow \mathbb{X}$, there exists n such that

$$m' < \mathcal{P}^{\xi}_{\Delta,0}(B_n).$$

Consequently, there exists $\delta_0 > 0$ such that for every $\delta < \delta_0$, there is a (Δ, δ)-packing $\pi = \{(x_i, r_i)\}_{i \in I}$ of B_n satisfying

$$\sum_i \xi(x_i, r_i) > m'.$$

We can take $\delta_0 < \frac{1}{n}$. Therefore, for all $i \in I$, there is a uniform δ-fine packing $\pi_i = \{(y_{ij}, r_i)\}_{i \in I, j \in K_i}$ of E_{x_i} such that

$$\sum_{j \in K_i} \zeta(y_{ij}, r_i) > m.$$

The set $\omega = \left\{(x_i, y_{ij}), r_i\right\}_{i \in I, j \in K_i}$ is thus a (Δ, δ)-packing of E, which implies that

$$\sum_{i \in I} \sum_{j \in K_i} \xi(x_i, r_i) \zeta(y_{ij}, r_i) \geq m \sum_{i \in I} \xi(x_i, r_i) \geq m \, m'.$$

Therefore $\mathcal{P}^{\xi\zeta}_\Delta(E) > m \, m'$. Since this remains valid for any $\delta < \delta_0$, all $m' < \mathcal{P}^{\xi}_\delta(\mathbb{X})$, and all $m < \inf_{x \in \mathbb{X}} \underline{\mathcal{V}}^{\zeta}_0(E_x)$, we get the desired result. $\qquad\square$

Lemma 3.2. *Let ξ and ζ be the gauge functions defined in (2.1). If $E \subseteq \mathbb{X} \times \mathbb{Y}$ is compact, then the mapping $x \longmapsto \underline{\mathcal{V}}_0^\zeta(E^x)$ is Borel measurable and*

$$\mathcal{P}_{\Delta,0}^{\xi\zeta}(E) \geq \int \underline{\mathcal{V}}_0^\zeta(E^x)\, d\mathcal{P}_\Delta^\xi(x). \tag{3.2}$$

Proof. We start the proof by proving that the function $x \mapsto \underline{\mathcal{V}}_0^\zeta(E^x)$ is Borel measurable whenever E is compact. We have

$$\underline{\mathcal{V}}_0^\zeta(E^x) = \liminf_{\delta \to 0} \mathcal{M}_{\nu,\delta}^q(E^x)\, g(2\delta),$$

so to show the Borel measurability of $x \mapsto \underline{\mathcal{V}}_0^\zeta(E^x)$, it suffices to show that for every $\delta > 0$, the mapping

$$x \longmapsto \mathcal{M}_{\nu,\delta}^q(E^x)$$

is Borel measurable. Fix $\delta > 0$. For a compact set $K \subseteq \mathbb{Y}$, define

$$\Phi_\delta(K) := \mathcal{M}_{\nu,\delta}^q(K).$$

By definition, Φ_δ is the supremum over all finite families $\{y_1, \ldots, y_m\} \subset K$ satisfying $d(y_i, y_j) > \delta$ of the quantities $\sum_{i=1}^m \nu(B(y_i, \delta))^q$. Each such quantity depends continuously on the points $(y_1, \ldots, y_m)$, and therefore Φ_δ is upper semicontinuous on the space of compact subsets of $\mathbb{Y}$ endowed with the Hausdorff topology. Since E is compact, the section map $x \longmapsto E^x$ is Borel measurable as a map from $\mathbb{X}$ into the space of compact subsets of $\mathbb{Y}$. Consequently, the composition

$$x \longmapsto \mathcal{M}_{\nu,\delta}^q(E^x) = \Phi_\delta(E^x)$$

is Borel measurable. Finally, since $\underline{\mathcal{V}}_0^\zeta(E^x)$ is obtained as a $\liminf$ of the Borel measurable functions $x \mapsto \mathcal{M}_{\nu,\delta}^q(E^x)\, g(2\delta)$, the mapping $x \mapsto \underline{\mathcal{V}}_0^\zeta(E^x)$ is Borel measurable.

Next, we use this result to show that $\mathcal{P}_{\Delta,0}^{\xi\zeta}(E)$ can be bounded below by the integral of $\mathcal{V}_0^\zeta(E^x)$. In other words, we must prove that

$$\mathcal{P}_{\Delta,0}^{\xi\zeta}(E) \geq \int s\, d\mathcal{P}_\Delta^\xi(x),$$

for each function $s \leq \underline{\mathcal{V}}_0^\zeta(E^x)$. Let $s = \sum_{i=1}^m a_i \chi_{F_i}$ be such a function with F_i disjoint Borel sets and a_i positive. If there is i such that $\mathcal{P}_\Delta^\xi(F_i) = \infty$, then by Lemma 3.2, we have $\mathcal{P}_{\Delta,0}^{\xi\zeta}(E) \geq a_i \mathcal{P}_\Delta^\xi(F_i) = \infty$. Otherwise $\mathcal{P}_\Delta^\xi(F_i) < \infty$ for all i and it follows that each F_i is approached from within by compact sets since $\mathbb{X}$ is compact. Then for every $\epsilon > 0$ and each i, we can find a compact set $C_i \subseteq F_i$ satisfying $\mathcal{P}_\Delta^\xi(C_i) \geq \mathcal{P}_\Delta^\xi(F_i) - \frac{\epsilon}{ma_i}$. Therefore,

$$
\begin{aligned}
\int s\, d\mathcal{P}_\Delta^\xi(x) &= \sum_{i=1}^m a_i \mathcal{P}_\Delta^\xi(F_i) \leq \sum_{i=1}^m a_i \left(\mathcal{P}_\Delta^\xi(C_i) + \frac{\epsilon}{ma_i} \right) \\
&= \epsilon + \sum_{i=1}^m a_i \mathcal{P}_\Delta^\xi(C_i).
\end{aligned}
$$

For every i, define $E_i = E \cap (C_i \times \mathbb{Y})$. Then, using Lemma 3.2, we find

$$a_i\, \mathcal{P}_\Delta^\xi(C_i) \leq \mathcal{P}_{\Delta,0}^{\xi\zeta}(E_i), \qquad i = 1, 2, \ldots, m.$$

Hence, $\int s\, \mathcal{P}^{\xi}_{\Delta}(x) \le \epsilon + \sum_{i=1}^{m} \mathcal{P}^{\xi\zeta}_{\Delta,0}(E_i)$. Given that C_i are disjoint compact sets, E_i are likewise disjoint and compact. Therefore, we have

$$\sum_{i=1}^{m} \mathcal{P}^{\xi\zeta}_{\Delta,0}(E_i) = \mathcal{P}^{\xi\zeta}_{\Delta,0}\left(\bigcup_{i=1}^{m} E_i\right) \le \mathcal{P}^{\xi\zeta}_{\Delta,0}(E).$$

Then, $\int s\, d\mathcal{P}^{\xi}_{\Delta}(x) \le \epsilon + \mathcal{P}^{\xi\zeta}_{\Delta,0}(E)$. Since $\epsilon > 0$ and $s \le \underline{\mathcal{V}}^{\zeta}_0(E^x)$ were chosen arbitrarily, we obtain the desired result. $\qquad\square$

Lemma 3.3. *Let ξ and ζ be the gauge functions defined in (2.1). For every $E \subseteq \mathbb{X} \times \mathbb{Y}$, we have*

$$\mathcal{P}^{\xi\zeta}_{\Delta,0}(E) \ge \int^{*} \underline{\mathcal{V}}^{\zeta}_0(E^x)\, d\mathcal{P}^{\xi}_{\Delta}(x).$$

Proof. We can assume without loss of generality that $\mathbb{X}$ and $\mathbb{Y}$ are complete metric spaces. If $\mathcal{P}^{\xi\zeta}_{\Delta,0}(E) = \infty$, the statement is immediate. Now, assume that $\mathcal{P}^{\xi\zeta}_{\Delta,0}(E) < \infty$ and then E is totally bounded. It follows that $\overline{E}$ is compact. Hence,

$$\mathcal{P}^{\xi\zeta}_{\Delta,0}(E) = \mathcal{P}^{\xi\zeta}_{\Delta,0}(\overline{E}) \underset{\text{by Lemma 3.2}}{\ge} \int \underline{\mathcal{V}}^{\zeta}_0((\overline{E})^x)\, d\mathcal{P}^{\xi}_{\Delta}(x) \ge \int^{*} \underline{\mathcal{V}}^{\zeta}_0(E^x)\, d\mathcal{P}^{\xi}_{\Delta}(x).$$

$\qquad\square$

3.1. Proof of Theorem 1.1

Let $E_n \nearrow E$. Then for each n, by Lemma 3.3, we have $\mathcal{P}^{\xi\zeta}_{\Delta,0}(E_n) \ge \int^{*} \underline{\mathcal{V}}^{\zeta}_0(E^x_n)\, d\mathcal{P}^{\xi}_{\Delta}(x)$. Therefore,

$$\sup_n \mathcal{P}^{\xi\zeta}_{\Delta,0}(E_n) \ge \int^{*} \sup \underline{\mathcal{V}}^{\zeta}_0(E^x_n)\, d\mathcal{P}^{\xi}_{\Delta}(x) \ge \int^{*} \overrightarrow{\mathcal{V}}^{\zeta}(E^x)\, d\mathcal{P}^{\xi}_{\Delta}(x),$$

since $E^x_n \nearrow E^x$ for every $x \in \mathbb{X}$. Consider the infimum over all sequences $E_n \nearrow E$ to obtain $\overrightarrow{\mathcal{P}}^{\xi\zeta}_{\Delta,0}(E) \ge \int^{*} \overrightarrow{\mathcal{V}}^{\zeta}(E^x)\, d\mathcal{P}^{\xi}_{\Delta}(x)$. Since $\overrightarrow{\mathcal{P}}^{\xi\zeta}_{\Delta,0}(E) = \mathcal{P}^{\xi\zeta}(E)$, we get the desired result.

Remark 3.1. *Lemma 3.3 is the key technical tool in the proof of Theorem 1.1, as it provides an integral lower bound for the Δ–scaled packing premeasure on the product space $\mathbb{X} \times \mathbb{Y}$ in terms of the generalized Hewitt–Stromberg premeasure of the vertical sections. As a simple illustration, consider a Cartesian rectangle $E = A \times B$. In this case, the vertical sections satisfy $E^x = B$ for $x \in A$ and $E^x = \emptyset$ otherwise. Applying Lemma 3.3 yields*

$$\mathcal{P}^{\xi\zeta}_{\Delta,0}(A \times B) \ge \mathcal{P}^{\xi}_{\Delta}(A)\, \underline{\mathcal{V}}^{\zeta}_0(B),$$

which already reflects the product-type behavior underlying the main result.

4. Estimation of the generalized packing premeasure

This section discusses estimation methods for generalized packing measures applied to generalized Cantor sets. We will introduce a generalized packing premeasure $\widetilde{\mathcal{P}}_0^\xi$, equivalent to the measure $\mathcal{P}_0^\xi$ defined in Section 2, by using closed cubes with side lengths less than δ instead of closed balls. This approach will enable precise estimates of the measure of the d-dimensional symmetric generalized Cantor set $\mathcal{K}^d$. We will then demonstrate that this premeasure is suitable for estimating the measure of generalized Cantor sets (see Theorem 1.2). Recall that a cube $I(x, r)$ in $\mathbb{R}^d$ is a subset of the form $I(x, r) = \prod_{i=1}^n [x_i - r, \ x_i + r]$. For a cube I, we denote its side length by $l(I)$. Let ξ be a gauge function that satisfies the doubling condition. We define

$$\widetilde{\mathcal{P}}_0^\xi(E) = \lim_{\delta \to 0} \widetilde{\mathcal{P}}_\delta^\xi(E),$$

where $\widetilde{\mathcal{P}}_\delta^\xi(E) = \sup \sum \xi(x_i, r_i)$ (of course the gauge function $\xi(x_i, r_i)$ in this case is defined by $\mu(I(x_i, \ r_i))^q h(l(I_i)))$ with the supremum being over all packings of E by cubes of side length $\leq \delta$ centered in E. Then there exists a constant C [26] such that

$$C^{-1} \widetilde{\mathcal{P}}_0^\xi(E) \leq \mathcal{P}_0^\xi(E) \leq C \widetilde{\mathcal{P}}_0^\xi(E).$$

Taking $l(I) = 2r$, we will compute the estimation of the generalized packing measure of $\mathcal{K}^d$. Let $\mathcal{G}$ be the set of all continuous and increasing functions h on $[0, r_0)$ for some $r_0 > 0$ with $h(0) = 0$. $r \longmapsto h(r)/r^d$ is decreasing and assume in this section that $h \in \mathcal{G}$. Since $t \longmapsto h(r)/r^d$ is decreasing, we can see that h satisfies the doubling condition, that is,

$$h(2r) \leq 2^d h(r), \quad \text{for } \ 0 < r < r_0/2.$$

For the remainder of this article, we assume that $h \in \mathcal{G}$. Let us start by recalling the construction of the generalized symmetric Cantor set in d-dimensional space $\mathcal{K}^d$.

4.1. Construction of generalized Cantor set

First, we recall the construction of the one-dimensional generalized Cantor set $\mathcal{K}$. Let l be a positive number, $\{n_k\}_{k \geq 1}$ be a sequence of integers, and $\{\lambda_k\}_{k \geq 1}$ be a sequence of positive numbers such that

$$n_k > 1, \qquad n_1 \lambda_1 < l, \qquad \text{and} \qquad \lambda_{k+1} \, n_{k+1} < \lambda_k \tag{4.1}$$

for every $k \geq 1$. The construction of the generalized Cantor set $\left\{l, \{n_k\}_{k \geq 1}, \{\lambda_k\}_{k \geq 1}\right\}$ is as follows. First, starting from a closed interval of length l, remove $(n_1 - 1)$ open intervals, resulting in n_1 closed intervals of length λ_1, denoted by $I_1, \dots, I_{n_1}$. Let

$$J_1 = \bigcup_{j_1=1}^{n_1} I_{j_1}.$$

Second, from every remaining closed interval of length λ_1, remove $(n_2 - 1)$ open intervals, resulting in n_2 closed intervals of length λ_2. These are denoted by I_{j_1, j_2}. We now have

$$J_2 = \bigcup_{j_1=1}^{n_1} \bigcup_{j_2=1}^{n_2} I_{j_1, j_2}.$$

We continue this process and in the k-th step obtain $n_1 n_2 \cdots n_k$ closed intervals of length λ_k, denoted by $I_{j_1, j_2, \ldots, j_k}$, and represent their union by J_k. Thus let $\mathcal{K} = \bigcap_{k=0}^{\infty} J_k$. Let μ be a geometric length-based content associated with the construction, defined on each k-th level cylinder Q^k by $\mu(Q^k) = \lambda_k$. Define

$$S_k = \frac{\mu(Q^{k+1})}{\mu(Q^k)} = \frac{\lambda_{k+1}}{\lambda_k}. \tag{4.2}$$

This construction can be generalized to $\mathbb{R}^d$ as follows, and the resulting generalized Cantor set is denoted by $\mathcal{K}^d$. At each step k, the set J_k consists of $n_1 \cdots n_k$ disjoint cubes of side length λ_k, obtained by uniformly removing gaps from the previous generation. Then, the set $\mathcal{K}^d$ is defined as the Cartesian product of $\mathcal{K}$, resulting in a family of d–dimensional cubes. Let F_k be the product set of d copies J_k. Then it is clear that F_k is the union of $(n_1 n_2 \ldots n_k)^d$ closed cubes with the side λ_k, represented by $Q^{(k)}$, and $\mathcal{K}^d = \bigcap_{k=0}^{\infty} F_k$.

4.2. Proof of Theorem 1.2

Before commencing the proof, it is necessary to establish certain preliminary results: a technical lemma, and a new function Ψ that satisfies certain conditions. For each $k \geq 1$, denote by δ_k the length of the gaps removed at step k of the construction, so that

$$\lambda_{k-1} = n_k \lambda_k + (n_k - 1)\delta_k, \qquad \delta_k = \frac{\lambda_{k-1} - n_k \lambda_k}{n_k - 1}.$$

Lemma 4.1. *[27] Let I be an open interval of length l such that the center of I belongs to $\mathcal{K}$. Define k such that $2\lambda_k < l \leq 2\lambda_{k-1}$. Let $N_k(I)$ be the number of $I_{j_1, \ldots, j_k}$ that are contained in I.*

(1) Assume that $l \leq \lambda_{k-1}$, and then

$$(N_k(I) - 1)(\lambda_k + \delta_k) < l \leq 2\lambda_k + 2(N_k(I) + 1)(\lambda_k + \delta_k). \tag{4.3}$$

(2) Assume that $l > \lambda_{k-1}$, and then

$$N_k(I) \geq \max\{(n_k - 4)/2, 1\}. \tag{4.4}$$

Proof. Let $x \in \mathcal{K}$ and let $I_{j_1, \ldots, j_k}(x)$ be the unique interval in generation k containing x. Since $2\lambda_k < l$ then $N_k(I) \geq 1$. Now, assume that $l \leq \lambda_{k-1}$. Let y be one of the endpoints of $I_{j_1, \ldots, j_{k-1}}(x)$, which is at a distance from point x that is either longer or equal. It follows that

$$|y - x| \geq \frac{1}{2}\lambda_{k-1} \geq l/2$$

and then the interval $J = (x, x + l/2)$ or $J = (x - l/2, x)$ contained in $I_{j_1, \ldots, I_{j_{k-1}}}(x)$. Since the distance between two consecutive intervals at level k is $\lambda_k + \delta_k$, we obtain

$$N_k(J)(\lambda_k + \delta_k) < l/2 \leq \lambda_k + (N_k(J) + 1)(\lambda_k + \delta_k)$$

and

$$N_k(J) \leq N_k(I) \leq 2N_k(J) + 1.$$

To prove the second assertion, we will consider the special case of the first one with $l = \lambda_{k-1}$. We obtain

$$2(N_k(I) + 1)(\lambda_k + \delta_k) \geq \lambda_{k-1} - 2\lambda_k \geq (n_k - 2)(\lambda_k + \delta_k),$$

which implies that $N_k(I) \geq (n_k - 4)/2$ as required since $N_k(I) \geq 1$. $\qquad \square$

Proposition 4.1. *Let $\mathcal{K}^d$ be the generalized symmetric Cantor set in d-dimensional space. There exists a nonnegative set function Ψ defined on every closed subset Ω that satisfies the conditions given below:*

(1) If $\Omega_1 \cap \Omega_2 = \emptyset$, then $\Psi(\Omega_1 \cup \Omega_2) = \Psi(\Omega_1) + \Psi(\Omega_2)$.
(2) There is a positive constant B such that $\Psi(\Omega) \le B$ for all Ω.
(3) There are positive constants a and r_0 such that

$$\xi(x, r) \le a\Psi(I(x, r)),$$

for all $x \in \mathcal{K}^d$ and $0 < r < r_0$.

Proof. Recall that μ is the scale-dependent weight defined by $\mu(Q^{(k)}) = \lambda_k$. Let $B > \limsup_{k\to\infty}(n_1 n_2 \dots, n_k)^d \lambda_k^q h(\lambda_k)$. Hence, one can find an integer k_0 such that

$$(n_1 n_2 \dots n_k)^d \lambda_k^q h(\lambda_k) < B < (n_1 n_2 \dots n_{k_0})^d \lambda_{k_0}^q h(\lambda_{k_0})$$

for all $k \ge k_0$. Now choose $\widetilde{\lambda}_k$ such that $B = (n_1 n_2 \dots n_k)^d \lambda_k^q h(\widetilde{\lambda}_k)$ for all $k \ge k_0$. We have $\lambda_k < \widetilde{\lambda}_k < \lambda_0$ and

$$
\begin{aligned}
h(\widetilde{\lambda}_{k+1}) &= \frac{B}{(n_1 n_2 \dots n_{k+1})^d \lambda_{k+1}^q} \\
&= \frac{B}{(n_1 n_2 \dots n_{k+1})^d S_k^q \lambda_k^q} \\
&= \frac{h(\widetilde{\lambda}_k)}{n_{k+1}^d S_k^q},
\end{aligned}
$$

where $S_k = \frac{\lambda_{k+1}}{\lambda_k}$. Let A be a set and define

$$M_{\mu,k}^q(A) := \Big\{ \sum \mu(Q^{(k)})^q, \quad Q^{(k)} \text{ are entirely contained in } A\Big\}.$$

Next, we obtain

$$
\begin{aligned}
M_{\mu,k+1}^q(A)h(\widetilde{\lambda}_{k+1}) &= \Big\{ \sum \mu(Q^{(k+1)})^q, \quad Q^{(k+1)} \text{ are entirely contained in } A\Big\}h(\widetilde{\lambda}_{k+1}) \\
&\ge n_{k+1}^d S_k^q h(\widetilde{\lambda}_{k+1})\Big\{ \sum \mu(Q^{(k)})^q, \quad Q^{(k)} \text{ are entirely contained in } A\Big\} \\
&\ge M_{\mu,k}^q(A)h(\widetilde{\lambda}_k)
\end{aligned}
$$

It follows that the sequence $\{M_{\mu,k}^q(A)h(\widetilde{\lambda}_k)\}$ is increasing, and then we may define the function of sets

$$\Psi(A) = \lim_{k\to\infty} M_{\mu,k}^q(A)h(\widetilde{\lambda}_k).$$

It is not hard to check that conditions (1) and (2) of Proposition 4.1 are verified for this set function using the number B defined earlier. More precisely, the additivity in (1) holds for disjoint closed sets Ω_1, Ω_2 whose intersections with each generation consist of disjoint families of k-th-level cubes, so that the corresponding packing sums $M_{\mu,k}^q$ are additive. It still needs to be proven that condition (3) of Proposition 4.1 is satisfied. Let $r_0 = \lambda_{k_0}$ and $I(x, r)$ be a closed cube centered at $x \in \mathcal{K}^d$ with side $2r \le r_0$. Hence, one can find an integer $k > k_0$ for which $\lambda_k < r \le \lambda_{k-1}$. Clearly, x belongs to some cube in generation k denoted by $Q^{(k)}$ and then $Q^{(k)} \subset I(x, r)$. This implies that

$$M_{\mu,k}^q(I(x, r)) \ge \lambda_k^q.$$

(1) If $k \notin \mathcal{N}$ and then $n_k \le l$, we get

$$
\begin{aligned}
h(2r)\,\mu(I(x,r))^q &\le\ \lambda_k^q h(2\lambda_{k-1}) \le m_h h(\lambda_{k-1})\lambda_k^q \\
&\le\ m_h \frac{B}{(n_1 n_2 \ldots n_{k-1})^d \mu(Q^{(k-1)})^q} \lambda_k^q \\
&\le\ m_h l^d S_{k-1}^q h(\widetilde{\lambda}_k)\lambda_k^q \\
&\le\ a\, M_{\mu,k}^q(Q(x,r)) h(\widetilde{\lambda}_k) \le a\Psi((Q(x,r))).
\end{aligned}
$$

(2) If $k \in \mathcal{N}$. Let $\mathcal{K}_i$ be the generalized symmetric Cantor set in one-dimensional space constructed by the projection of $\mathcal{K}^d$ and I on every x_i-axis. We also denote by I_i the projection of I. We denote $\mathsf{N}_k(I)$ in Lemma 4.1 with respect to $\mathcal{K}_i$ as $\mathsf{N}_{k,i}(I)$. Here, if $2r \le \lambda_{k-1}$, then

$$
\begin{aligned}
2r\ &\le^{(4.3)}\ 2\lambda_k + 2(\mathsf{N}_{k,i}(I_i) + 1)(\lambda_k + \delta_k) \qquad (l = 2r) \\
&\le\ 6(1 + C)\mathsf{N}_{k,i}(I_i)\lambda_k := \beta_i,
\end{aligned}
$$

since $\delta_k \le C\lambda_k$ and $\mathsf{N}_{k,i}(I_i) \ge 1$. It follows that

$$
\begin{aligned}
h(2r)\mu(I)^q &=\ \psi(2r)^d\,\lambda_k^q \le \psi(\beta_1)\ldots\psi(\beta_d)\,\lambda_k^q \\
&\le\ m\mathsf{N}_{k,1}(I_1)\psi(\lambda_k)\ldots\mathsf{N}_{k,d}(I_d)\psi(\lambda_k)\,\lambda_k^q \\
&\le\ m\,M_{\mu,k}^q(I)h(2r) \\
&\le\ m\Psi(I),
\end{aligned}
$$

where $\psi^d = h$ and $m = (6(1 + C))^d$. Consider the case where $2r > \lambda_{k-1}$. Then using Lemma 4.1, we have

$$
n_k^d \lambda_k^q \le 6^d M_{\mu,k}^q(I) \quad \text{and} \quad 2r \le 2\lambda_{k-1} \le 2(1 + C)n_k\lambda_k
$$

and

$$
\begin{aligned}
h(2r)\mu(I)^q &\le\ h(2\lambda_{k-1})\lambda_k^q \le (2(1 + C)n_k))^d h(\lambda_k)\lambda_k^q \\
&\le\ M' M_{\mu,k}^q(I)h(\lambda_k) \le M\Psi(I).
\end{aligned}
$$

$\square$

Now, we can prove Theorem 1.2. Since the lower left corner of every closed cube $Q^{(k)}$ belongs to $\mathcal{K}^d$, one can see that $(n_1 n_2 \ldots n_k)^d$ closed cubes centered at these points with side λ_k are disjoint. Therefore, by the definition of the generalized packing premeasure, the left-part inequality of Theorem 1.2 follows. Thus, we only need to focus on proving the right part. Without loss of generality, assume that $\limsup_{k\to\infty}(n_1 n_2 \ldots n_k)^d \lambda_k^q h(\lambda_k)$, is finite (otherwise, the desired inequality is evident). Let Ψ be the function defined earlier. Let $\epsilon < k_0$, which will be defined earlier and let $\{I(x_i, r_i)\}$ be a disjoint closed cube with $x_i \in \mathcal{K}^d$. Then

$$
\begin{aligned}
B\ &\ge^{C1}\ \Psi\Big(\bigcup I(x_i, r_i)\Big) \ge \sum_i \Psi(I(x_i, r_i)) \\
&\ge^{C2}\ a \sum_i \mu(I)^q h(2r_i).
\end{aligned}
$$

It follows that $\mathcal{P}^\xi(\mathcal{K}^d) \leq \widetilde{\mathcal{P}}_0^\xi(\mathcal{K}^d) \leq Ba^{-1}$. Given that B is any number for which $B > \lim_{k\to\infty}(n_1 n_2 \ldots n_k)^d \lambda_k^q h(\lambda_k)$, the proof is now complete.

Recall that $\mu(I_k) = \lambda_k$ denotes the scale-dependent weight associated with the k-th generation intervals. Then, we get the following result.

Corollary 4.1. *Let $\mathcal{K}^d$ be the generalized symmetric Cantor set in d-dimensional space ($d \geq 1$) generated using the system $\{l, \{n_k\}_{k\geq 1}, \{\lambda_k\}_{k\geq 1}\}$, which satisfies condition (4.1). Suppose that there is a constant η such that $n_k \leq \eta, \quad k = 1, 2, \ldots$. Then,*

(1) $\mathcal{P}_0^\xi(\mathcal{K}^d) = 0$ if and only if $\limsup_{k\to\infty} (n_1 n_2 \ldots n_k)^d \mu(I_k)^q h(\lambda_k) = 0$.

(2) $0 < \mathcal{P}_0^\xi(\mathcal{K}^d) < \infty$ if and only if $0 < \limsup_{k\to\infty} (n_1 n_2 \ldots n_k)^d \mu(I_k)^q h(\lambda_k) < \infty$.

(3) $\mathcal{P}_0^\xi(\mathcal{K}^d) = \infty$ if and only if $\limsup_{k\to\infty} (n_1 n_2 \ldots n_k)^d \mu(I_k)^q h(\lambda_k) = \infty$.

Example 4.1. *In this instance, we focus on the classical standard Cantor set. This set has been extensively studied. One can cite, for instance, [24, 28, 29]. A direct application of Theorem 1.2 allows us to obtain the following result. We consider the case $d = 1$, $l = 1$, and $n_k = 2$. If $\mu(I_k) = \lambda_k = \left(\frac{1}{3}\right)^k$ and $h(r) = r^\alpha$, then we obtain*

$$
\begin{aligned}
\limsup_{k\to+\infty} (n_1 n_2 \ldots n_k)^d \mu(I_k)^q h(\lambda_k) &= \limsup_{k\to+\infty} 2^k \left(\frac{1}{3}\right)^{qk} \left(\frac{1}{3}\right)^{k\alpha} \\
&= \limsup_{k\to+\infty} 2^k \left(\frac{1}{3}\right)^{k(q+\alpha)} \\
&= \limsup_{k\to+\infty} \left(\frac{1}{3}\right)^{k(q+\alpha-\ln(2)/\ln(3))}.
\end{aligned}
$$

Thus, we find that $\widetilde{\mathcal{P}}_0^\xi(\mathcal{K})$ is infinite, positive, and finite, or zero depending on whether, $q + \alpha - \ln(2)/\ln(3)$, is negative, zero, or positive, respectively.

5. Generalized packing measure for the product sets

In this section, we consider $h(r) = r^\alpha$ and $g(r) = r^\beta$, where $\alpha, \beta \in \mathbb{R}$, and μ and ν are the uniform measures. We will study the $0 - \infty$ cases and then address the question introduced in the introduction. To this end, we must estimate the generalized packing measure of $\mathcal{K}_1 \times \mathcal{K}_2$, where $\mathcal{K}_1$ (resp., $\mathcal{K}_2$) is the generalized symmetric Cantor set in one-dimensional space. More specifically, the following result holds.

Theorem 5.1. *Let $\mathcal{K}_1$ (resp. $\mathcal{K}_2$) be the generalized symmetric Cantor set in one-dimensional space generated, using the system $\{l, \{n_k\}_{k\geq 1}, \{\lambda_k\}_{k\geq 1}\}$ $\left(resp., \{l, \{n_k\}_{k\geq 1}, \{\Lambda_k\}_{k\geq 1}\}\right)$, let $\mu \in \mathcal{P}_D(\mathcal{K}_1)$, $\nu \in \mathcal{P}_D(\mathcal{K}_2)$, $h, g \in \mathcal{F}_0$, and $q \in \mathbb{R}$. We have*

$$
\begin{aligned}
\limsup_{k\to\infty} (n_1 n_2 \ldots n_k) \lambda_k^q \Lambda_k^q h(\lambda_k) g(\Lambda_k) &\leq \widetilde{\mathcal{P}}_{\mu\times\nu,0}^{q,hg}(\mathcal{K}_1 \times \mathcal{K}_2) \\
&\leq C \limsup_{k\to\infty} (n_1 n_2 \ldots n_k) \lambda_k^q \Lambda_k^q h(\lambda_k) g(\Lambda_k).
\end{aligned}
$$

5.1. Construction of the function Ψ

Let $B > \limsup_{k\to\infty}(n_1\ldots,n_k)^d\lambda_k^{q+\alpha}\Lambda_k^{q+\beta}$. As a result, we can find k_0 satisfying

$$(n_1\ldots n_k)^d\lambda_k^{q+\alpha}\Lambda_k^{q+\beta} < B < (n_1\ldots n_{k_0})^d\lambda_k^{q+\alpha}\Lambda_{k_0}^{q+\beta}$$

for all $k \geq k_0$. Now choose $\widetilde{\Lambda}_k$ such that $B = (n_1\ldots n_k)^d\lambda_k^{q+\alpha}\widetilde{\Lambda}_k^{\beta}\Lambda_k^q$ for all $k \geq k_0$. We have $\Lambda_k < \widetilde{\Lambda}_k < \Lambda_{k_0}$ and

$$\lambda_{k+1}^\alpha\widetilde{\Lambda}_{k+1}^\beta = \lambda_k^\alpha\widetilde{\Lambda}_k^\beta\frac{\lambda_k^q\widetilde{\Lambda}_k^q}{n_{k+1}^d\lambda_{k+1}^q\widetilde{\Lambda}_{k+1}^q} = \frac{\lambda_k^\alpha\widetilde{\Lambda}_k^\beta}{n_{k+1}^d S_k^q},$$

where $S_k = \frac{\lambda_{k+1}\Lambda_{k+1}}{\lambda_k\Lambda_k}$. Let $A \subset \mathcal{K}_1 \times \mathcal{K}_2$ and denote by $I_1^{(k)}$ (resp., $I_2^{(k)}$) any of the closed intervals of the generation k of $\mathcal{K}_1$ (resp., $\mathcal{K}_2$). Define

$$M_{\mu\times\nu,k}^q(A) := \Big\{\sum\mu\times\nu\big(I_1^{(k)}\times I_2^{(k)}\big)^q, \quad I_1^{(k)}\times I_2^{(k)} \text{ which are completely included in } A\Big\}.$$

Then, we set

$$
\begin{aligned}
M_{\mu\times\nu,k+1}^q(A)\lambda_{k+1}^\alpha\widetilde{\Lambda}_{k+1}^\beta &= \Big\{\sum\mu\times\nu\big(I_1^{(k+1)}\times I_2^{(k+1)}\big)^q, \quad I_1^{(k+1)}\times I_2^{(k+1)} \text{ which are completely included in } A\Big\} \\
&\quad \cdot\lambda_{k+1}^\alpha\widetilde{\Lambda}_{k+1}^\beta \\
&\geq n_{k+1}^d S_k^q\lambda_{k+1}^\alpha\widetilde{\Lambda}_{k+1}^\beta\Big\{\sum\mu\times\nu\big(I_1^{(k)}\times I_2^{(k)}\big)^q, \\
&\qquad \big(I_1^{(k)}\times I_2^{(k)}\big) \text{ which are completely included in } A\Big\} \\
&\geq M_{\mu\times\nu,k}^q(A)\lambda_k^\alpha\widetilde{\Lambda}_k^\beta.
\end{aligned}
$$

Hence the sequence $\{M_{\mu\times\nu,k}^q(A)\lambda_k^\alpha\widetilde{\Lambda}_k^\beta\}$ is increasing, and then we may define the function of sets

$$\Psi(A) = \lim_{k\to\infty}M_{\mu\times\nu,k}^q(A)\lambda_k^\alpha\widetilde{\Lambda}_k^\beta.$$

Lemma 5.1. *There are positive constants a and r_0 satisfying*

$$\mu\times\nu(I_1\times I_2)^q h(2r)g(2r) \leq a\Psi(I_1\times I_2),$$

for every $I_1 = I(x,r)$ and $I_2 = I(x',r)$, $x \in \mathcal{K}_1$, $x' \in \mathcal{K}_2$, and $0 < r < r_0$.

Proof. Let $r_0 = \lambda_{k_0}\Lambda_{k_0}$ and $I_1 \times I_2$ be an open cube centered at $(x,x') \in \mathcal{K}_1 \times K_2$ with side $2r \leq r_0$. Then there exists an integer $k > k_0$ such that $\lambda_k\Lambda_k < r \leq \lambda_{k-1}\Lambda_{k-1}$. Clearly, (x,x') belong to some cube in the generation k denoted by $Q^{(k)}$ and then $Q^{(k)} \subset I_1 \times I_2$. This implies that

$$M_{\mu\times\nu,k}^q(I_1\times I_2) \geq \lambda_k^q\Lambda_k^q.$$

(1) If $k \notin \mathcal{N}$ and then $n_k \leq l$, we obtain

$$
\begin{aligned}
h(2r)g(2r)\,\mu(I_1)^q\,\nu(I_2)^q &\leq \lambda_k^q\Lambda_k^q h(2\lambda_{k-1}\Lambda_{k-1})g(2\lambda_{k-1}\Lambda_{k-1}) \\
&\leq m_h m_g h(\lambda_{k-1})g(\lambda_{k-1})\lambda_k^q\Lambda_k^q \\
&\leq m_h m_g \frac{B}{(n_1 n_2\ldots n_{k-1})^d\lambda_{k-1}^q\Lambda_{k-1}^q}\lambda_k^q\Lambda_k^q \\
&\leq m_h m_g L^d S_{k-1}^q h(\lambda_k)g(\tilde{\Lambda}_k) \\
&\leq a\,M_{\mu\times\nu,k}^q(Q(x,r))h(\lambda_k)g(\widetilde{\Lambda}_k) \leq a\Psi((Q(x,r))).
\end{aligned}
$$

(2) If $k \in \mathcal{N}$. Let $\mathcal{K}_i$ be the generalized symmetric Cantor set in one-dimensional space constructed by the projection of $\mathcal{K}^d$ and I on each x_i-axis. We also denote by I_i the projection of I. We write $\mathsf{N}_k(I)$ in Lemma 4.1, associated to $\mathcal{K}_i$, as $\mathsf{N}_{k,i}(I)$. In this case, if $2r \leq \lambda_{k-1}\Lambda_{k-1}$, then

$$
\begin{aligned}
2r \quad &\leq^{(4.3)} \quad 2\lambda_k\Lambda_k + 2(\mathsf{N}_{k,i}(I_i) + 1)(\lambda_k\Lambda_k + \delta_k) \qquad (l = 2r) \\
&\leq \quad 6(1 + C)\mathsf{N}_{k,i}(I_i)\lambda_k\Lambda_k := \beta_i,
\end{aligned}
$$

since $\delta_k \leq C\lambda_k\Lambda_k$ and $\mathsf{N}_{k,i}(I_i) \geq 1$. It follows that

$$
\begin{aligned}
h(2r)g(2r)\mu(I)^q\nu(I)^q &= \psi(2r)^d \, \lambda_k^q \leq \psi(\beta_1)\ldots\psi(\beta_d) \, \lambda_k^q\Lambda_k^q \\
&\leq m\mathsf{N}_{k,1}(I_1)\psi(\lambda_k)\ldots\mathsf{N}_{k,d}(I_d)\psi(\lambda_k) \, \lambda_k^q\Lambda_k^q \\
&\leq m \, M_{\mu\times\nu,k}^q(I)h(2r)g(2r) \\
&\leq m\Psi(I)
\end{aligned}
$$

where $\psi^d = hg$ and $m = (6(1 + C))^d$. Now consider the case where $2r > \lambda_{k-1}\Lambda_{k-1}$. Then using Lemma 4.1, we have

$$
n_k^d\lambda_k^q\Lambda_k^q \leq 6^d M_{\mu\times\nu,k}^q(I) \quad \text{and} \quad 2r \leq 2\lambda_{k-1}\Lambda_{k-1} \leq 2(1 + C)n_k\lambda_k\Lambda_k
$$

and

$$
\begin{aligned}
h(2r)g(2r)\mu(I)^q\nu(I)^q &\leq h(2\lambda_{k-1}\Lambda_{k-1})g(2\lambda_{k-1}\Lambda_{k-1})\lambda_k^q\Lambda_k^q \\
&\leq (2(1 + C)n_k))^d h(\lambda_k)g(\Lambda_k)\lambda_k^q\Lambda_k^q \\
&\leq M'' M_{\mu\times\nu,k}^q(I)h(\lambda_k)g(\Lambda_k) \leq M\Psi(I).
\end{aligned}
$$

$\square$

The proof of Theorem 5.1 employs a similar approach to that of Theorem 1.2, but replaces Proposition 4.1 with Lemma 5.1.

5.2. *Example*

In this example, we consider the generalized symmetric Cantor set in one-dimensional space $\mathcal{K}_1$ (resp., $\mathcal{K}_2$) constructed from the system $\{l, \{n_k\}_{r\geq 1}, \{\lambda_k\}_{k\geq 1}\}$ $\left(\text{resp.}, \{l, \{n_k\}_{k\geq 1}, \{\Lambda_k\}_{k\geq 1}\}\right)$. We set $n_k = 2$ for all k and define the scaling parameters as follows:

$$
\lambda_k = \left(k^2 2^{-k}\right)^{\frac{1}{\alpha+q}} \qquad \text{and} \qquad \Lambda_k = \left(k^{-j} 2^{-k}\right)^{\frac{1}{\beta+q}},
$$

where $0 < \alpha, \beta < 1$, and $j = 1, 2, 3$. The dimension functions are chosen as

$$
h(r) = r^\alpha \qquad \text{and} \qquad g(r) = r^\beta.
$$

We have $\limsup_{k\to\infty} 2^k\lambda_k^{q+\alpha} = \limsup_{k\to\infty} 2^k(k^2 2^{-k}) = \infty$. Therefore, by Corollary 4.1, we obtain $\widetilde{\mathcal{P}}_{\mu,0}^{q,h}(\mathcal{K}_1) = \infty$. Similarly, we have $\limsup_{k\to\infty} 2^k\Lambda_k^{q+\alpha} = \limsup_{k\to\infty} 2^k(k^{-j} 2^{-k}) = 0$. Then, we obtain $\mathcal{V}_{\nu,0}^{q,g}(\mathcal{K}_2) = 0$. On the other hand, we have

$$
\limsup_{k\to\infty} 2^k\lambda_k^{q+\alpha}2^k\Lambda_k^{q+\beta} = \limsup_{k\to\infty} 2^k\lambda_k^{q+\alpha}2^k\Lambda_k^{q+\beta}
$$

$$
\begin{aligned}
&= \limsup_{k \to \infty} (k^2 2^{-k})(k^{-j} 2^{-k}) 2^k \\
&= \limsup_{k \to \infty} k^{2-j}.
\end{aligned}
$$

Therefore $\widetilde{\mathcal{P}}^{q,hg}_{\mu \times \nu, 0}(\mathcal{K}_1 \times \mathcal{K}_2)$ is infinite, positive finite, or zero accordingly as $j = 1, 2, 3$, respectively.

6. Conclusions

This paper develops a unified framework for generalized packing and Hewitt–Stromberg measures on Cartesian products of fractal sets in separable metric spaces. By introducing Δ-scaled measures, we establish refined integral inequalities that relate the measure of a product set to the generalized Hewitt–Stromberg measures of its sections, extending classical product-type results and clarifying their limitations in critical $0 - \infty$ cases. In particular, by constructing symmetric generalized Cantor sets in arbitrary dimensions, we demonstrate that the measure of a product set can be zero, finite, or infinite, depending on the underlying geometric parameters.

Author contributions

Rihab Guedri: Writing – review & editing; Najmeddine Attia: Writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

Funding

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