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Research article

Existence and uniqueness theorems for solutions to Caputo fractional differential equations with nonlocal and irregular boundary conditions

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Abstract: In this work, we introduce a analytical framework for proving the existence and uniqueness of solutions to nonlinear fractional differential equations (NFDEs) of order $\nu \in (1, 2]$, subject to irregular boundary conditions, in a Banach space. Several fundamental definitions, theorems, and auxiliary lemmas are presented for the analysis of the proposed problem. We establish sufficient conditions for existence and uniqueness for the proposed system by applying Krasnosel'skii's fixed-point theorem and the Banach contraction principle. Finally, an illustrative example is constructed to validate the main theoretical results and demonstrate the effectiveness and applicability of the proposed approach, showing how the generalized framework can be systematically applied to analyze fractional boundary value problems.

Keywords: differential equations; fractional derivatives; existence; contraction principle
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1. Introduction

This article investigates the existence and uniqueness of solutions for

$$\begin{cases} {}^{c}D^{\nu}\phi(\rho) = \Phi(\rho, \phi(\rho)), \rho \in [0, \xi], 1 < \nu \leq 2, \ 0 \leq \kappa < \xi, \\ \phi'(\kappa) + (-1)^{\omega}\phi'(\xi) + b\phi(\xi) = 0, \phi(\kappa) + (-1)^{\omega+1}\phi(\xi) = 0, \ \ b \neq 0, \ \ \omega = 0, 1, \end{cases} \tag{1.1}$$

where the Caputo fractional derivative of order ν is denoted by ${}^{c}D^{\nu}$ and $\Phi : [0, \xi] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, we generalize and extend previous results by applying Krasnoselskii's fixed point theorem and the Banach contraction principle.

Fractional-order differential equations extend classical differential equations by allowing derivatives of non-integer order. This generalization enables the modeling of systems with memory and heredity

effects, or complex nonlinear behavior, that cannot be adequately captured by integer-order models. In the late seventeenth century, Leibniz posed his famous question concerning the interpretation of a derivative of half order; subsequent contributions from Lagrange, Fourier, Riemann, and Liouville gradually shaped the foundational theory. Over time, several key formal definitions emerged—most notably the Riemann–Liouville, Caputo, and Grünwald–Letnikov formulations—and, with the advent of numerical computation in the twentieth century, fractional-order differential models gained widespread application in engineering and the physical sciences [1]. These equations may appear as fractional ordinary or fractional partial differential equations, and their formulations vary according to the specific definition of the fractional derivative employed. The Caputo derivative is especially attractive in applications due to its natural compatibility with classical initial conditions, whereas the Grünwald–Letnikov formulation is widely used in numerical simulation. Fractional-order differential equations provide a powerful mathematical framework for describing complex dynamical systems and have become central topic of research in the physical sciences, engineering, and applied mathematics [2–4].

Recent progress in fractional differential equations has really pushed forward our understanding and practical uses of irregular and regular boundary conditions. A central research focus has been on proving when solutions exist and are unique, exploring different fractional operators, and a wide class of boundary conditions. In [5], the author provided a solid groundwork by proving that solutions exist for FDEs with irregular boundaries, $\phi^{'}(0) + (-1)^{\omega}\phi^{'}(\xi) + b\phi(\xi) = 0, \phi(0) + (-1)^{\omega+1}\phi(\xi) = 0$ by fixed-point theorems. On the other hand, the authors in [6] investigated fractional differential equations (FDEs) subject to anti-periodic boundary conditions by employing the Leray–Schauder degree theory. Furthermore, numerous studies on FDEs with nonlocal and integral boundary conditions highlight the flexibility of fractional calculus in modeling memory-dependent and hereditary phenomena and reinforce the fundamental role of fixed-point techniques and topological methods in establishing the solvability of such problems. Collectively, these contributions form a strong theoretical foundation for the modern analysis of fractional boundary value problems involving complex boundary structures [7–10]. Recent advances have also focused on the development of high-accuracy numerical schemes for time–space fractional convection–diffusion equations. For instance, Chen et al. [11] proposed a high-order interpolation-based collocation method combined with Gauss–Legendre quadrature, providing rigorous error estimates and improved computational efficiency. These results demonstrate the growing effectiveness of structure-preserving and high-order numerical techniques for solving challenging fractional transport problems.

In addition to the aforementioned contributions, the principal novelty of the present work lies in the formulation of a more general class of three-point nonlocal fractional boundary conditions involving both the solution and its first derivative. Specifically, the proposed model incorporates weighted linear combinations of endpoint and interior values, thereby encompassing shifted, and irregular boundary conditions as special cases. This unified framework allows greater flexibility in modeling realistic physical constraints where measurements or controls may occur at multiple locations. Moreover, the use of fixed-point techniques enables the derivation of existence and uniqueness criteria under weaker and more general assumptions on the nonlinear term. Consequently, the proposed approach not only generalizes and unifies several existing results in the literature, but also broadens the applicability of fractional boundary value problems to more complex and practically relevant dynamical systems.

Motivated by these developments, this paper proposes a new class of fractional boundary value

problems that unifies irregular and nonlocal boundary conditions within a single framework. This approach generalizes previous models and makes it possible to handle a much wider variety of real-world physical constraints. Unlike earlier studies, the results obtained here provide stronger and more precise conditions for the existence of solutions, while also accommodating a larger class of fractional differential operators. These improvements greatly expand the practical usefulness of the theory.

The article is organized into five sections to ensure a clear and closure. This section presents the introduction, outlining the motivation, objectives, and significance of the study. Section 2 provides the necessary preliminaries, including key definitions, notations, and relevant theorems from fractional calculus and fixed-point theory. Section 3 contains the main results, where a generalized framework is established for proving the existence and uniqueness of solutions to the considered NFDEs. Section 4 provides an illustrative example that demonstrates the applicability and effectiveness of the theoretical findings. Finally, Section 5 concludes the article by summarizing the main contributions, highlighting the advancements over existing literature, and suggesting potential directions for future research.

2. Preliminary

Definition 2.1. [13] The Caputo fractional derivative of order $\nu > 0$, denoted ${}^cD^\nu$, for a given function $\phi \in C^n([0,\xi])$ is defined by

$$ {}^cD^\nu\phi(t) = \frac{1}{\Gamma(n-\nu)} \int_0^\rho (\rho-\tau)^{n-\nu-1}\phi^{(n)}(\tau)d\tau. $$

Definition 2.2. [13] The Riemann-Liouville fractional integral I^ν of a function $\phi(\rho)$ is defined by

$$ I^\nu\phi(\rho) = \frac{1}{\Gamma(\nu)} \int_0^\rho (\rho-\tau)^{\nu-1}\phi(\tau)d\tau. $$

Theorem 2.1. *[12] Let M be a non-empty, closed, convex subset of a Banach space X. Suppose that the operators F and G are defined on M and satisfy the following conditions*

i. for all $u, v \in M$, the combination $F(u) + G(v) \in M$,
ii. F is continuous and maps bounded subsets of M into relatively compact subsets of X,
iii. the operator G is a contraction.

Then, there exists $z \in M$ such that $z = F(z) + G(z)$.

Lemma 2.2. *[13] For $\nu > 0$, the general solution of the FDE ${}^cD^\nu\phi(\rho) = 0$, is given by; $\phi(\rho) = \tau_0 + \tau_1\rho + \tau_2\rho^2 + \cdots + \tau_{n-1}\rho^{n-1}$, where $\tau_i \in \mathbb{R}, i = 0, 1, \cdots, n-1$.*

Lemma 2.3. *For a given $g \in C[0,\xi]$, the unique solution of*

$$ \begin{cases} {}^cD^\nu\phi(\rho) = g(\rho), \rho \in [0,\xi],\ 1 < \nu \le 2,\ 0 \le \kappa < \xi, \\ \phi'(\kappa) + (-1)^\omega\phi'(\xi) + b\phi(\xi) = 0,\ \phi(\kappa) + (-1)^{\omega+1}\phi(\xi)) = 0,\ b \ne 0,\ \omega = 0, 1, \end{cases} \tag{2.1} $$

is given by

$$ \phi(\rho) = \int_0^\rho \frac{(\rho-s)^{\nu-1}}{\Gamma(\nu)} g(s)\,ds + \frac{1}{(\xi-\kappa)b}\Big[\Big[\kappa b + \Big(1-(-1)^{\omega+1}\Big)\Big] \int_0^\xi \frac{(\xi-s)^{\nu-1}}{\Gamma(\nu)} g(s)\,ds \tag{2.2} $$

$$
\begin{aligned}
&- \left[1 + (-1)^{\omega} + b\xi\right] \int_0^{\kappa} \frac{(\kappa - s)^{\nu-1}}{\Gamma(\nu)} g(s)\, ds \\
&+ \left[\kappa + (-1)^{\omega}\xi\right] \left((-1)^{\omega} \int_0^{\xi} \frac{(\xi - s)^{\nu-2}}{\Gamma(\nu-1)} g(s)\, ds + \int_0^{\kappa} \frac{(\kappa - s)^{\nu-2}}{\Gamma(\nu-1)} g(s)\, ds \right) \Bigg] \\
+ \frac{\rho}{(\xi - \kappa)b} \Bigg[&- b \int_0^{\xi} \frac{(\xi - s)^{\nu-1}}{\Gamma(\nu)} g(s)\, ds + b \int_0^{\kappa} \frac{(\kappa - s)^{\nu-1}}{\Gamma(\nu)} g(s)\, ds \\
&- \left[1 + (-1)^{\omega+1}\right] \left((-1)^{\omega} \int_0^{\xi} \frac{(\xi - s)^{\nu-2}}{\Gamma(\nu-1)} g(s)\, ds + \int_0^{\kappa} \frac{(\kappa - s)^{\nu-2}}{\Gamma(\nu-1)} g(s)\, ds \right) \Bigg].
\end{aligned}
$$

Proof. Using Lemma 2.2, it follows that

$$
\phi(\rho) = I^{\nu} g(\rho) - \sum_{i=0}^{n} \tau_i \rho^i, \quad \tau_i \in \mathbb{R}, \text{and } n = [\nu] + 1, \text{where } \nu \text{ is the integer part of the real number } \nu.
$$

Here I^{ν} denotes the Riemann–Liouville fractional integral of order ν and since the order $\nu \in (1, 2]$ the solution is given by

$$
\phi(\rho) = I^{\nu} g(\rho) - \tau_0 - \tau_1 \rho. \tag{2.3}
$$

By using the irregular boundaries of (2.1) we see that

$$
\begin{aligned}
\tau_0 = \frac{-1}{(\xi - \kappa)b} \Bigg[&\left[\kappa b + (1 - (-1)^{\omega+1}\right] \int_0^{\xi} \frac{(\xi - s)^{\nu-1}}{\Gamma(\nu)} g(s) ds - \left[1 + (-1)^{\omega} + b\xi\right] \int_0^{\kappa} \frac{(\kappa - s)^{\nu-1}}{\Gamma(\nu)} g(s) ds \\
&+ \left[\kappa + (-1)^{\omega+1}\xi\right] \left((-1)^{\omega} \int_0^{\xi} \frac{(\xi - s)^{\nu-2}}{\Gamma(\nu-1)} g(s) ds + \int_0^{\kappa} \frac{(\kappa - s)^{\nu-2}}{\Gamma(\nu-1)} g(s) ds \right) \Bigg], \\
\tau_1 = \frac{-1}{(\xi - \kappa)b} \Bigg[&- b \int_0^{\xi} \frac{(\xi - s)^{\nu-1}}{\Gamma(\nu)} g(s) ds + b \int_0^{\kappa} \frac{(\kappa - s)^{\nu-1}}{\Gamma(\nu)} g(s) ds \\
&- \left[1 + (-1)^{\omega+1}\right] \left((-1)^{\omega+1} \int_0^{\xi} \frac{(\xi - s)^{\nu-2}}{\Gamma(\nu-1)} g(s) ds + \int_0^{\kappa} \frac{(\kappa - s)^{\nu-2}}{\Gamma(\nu-1)} g(s) ds \right) \Bigg].
\end{aligned}
$$

Substituting τ_0 and τ_1 in (2.3) we get the solution (2.2). □

Remark. When setting $\kappa = 0$, $\xi = \pi$ we see that the solution (2.2) in Lemma 2.3 reduces to the solution in [5, lemma 2.2].

3. Results

Define $C = C([0, \xi], \mathbb{R})$, the Banach space of all continuous functions from $[0, \xi] \longrightarrow \mathbb{R}$

Theorem 3.1. *Let $\Phi : [0, \xi] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying the condition $\|\Phi(\rho, \phi_2) - \Phi(\rho, \phi_1)\| \leq L\|\phi_2 - \phi_1\|, \forall \rho \in [0, \xi]$, and $\phi_1, \phi_2 \in \mathbb{R}$ then the problem (1.1) has a unique solution provided $L\,\Delta < 1$, where*

$$
\Delta = \frac{1}{(\xi - \kappa)|b|\Gamma(\nu + 1)} \left[(2\xi|b| + 1 - (-1)^{\omega+1})(\xi^{\nu} + \kappa^{\nu}) + \nu(\kappa + 2\xi + \xi(-1)^{\omega+1})(\xi^{\nu-1} + \kappa^{\nu-1}) \right].
$$

Proof. Define $\mathcal{T} : C \longrightarrow C$ by

$$(\mathcal{T}\phi)(\rho) = \int_0^\rho \frac{(\rho-s)^{\nu-1}}{\Gamma(\nu)}\Phi(s,\phi(s))\,ds + \frac{1}{(\xi-\kappa)b}\Big[\Big[\kappa b + \Big(1-(-1)^{\omega+1}\Big)\Big]\int_0^\xi \frac{(\xi-s)^{\nu-1}}{\Gamma(\nu)}\Phi(s,\phi(s))\,ds \tag{3.1}$$

$$- \big[1+(-1)^\omega + b\xi\big]\int_0^\kappa \frac{(\kappa-s)^{\nu-1}}{\Gamma(\nu)}\Phi(s,\phi(s))\,ds$$

$$+ \big[\kappa + (-1)^\omega\xi\big]\Big((-1)^\omega\int_0^\xi \frac{(\xi-s)^{\nu-2}}{\Gamma(\nu-1)}\Phi(s,\phi(s))\,ds + \int_0^\kappa \frac{(\kappa-s)^{\nu-2}}{\Gamma(\nu-1)}\Phi(s,\phi(s))\,ds\Big)\Big]$$

$$+ \frac{\rho}{(\xi-\kappa)b}\Big[-b\int_0^\xi \frac{(\xi-s)^{\nu-1}}{\Gamma(\nu)}\Phi(s,\phi(s))\,ds + b\int_0^\kappa \frac{(\kappa-s)^{\nu-1}}{\Gamma(\nu)}\Phi(s,\phi(s))\,ds$$

$$- \Big[1+(-1)^{\omega+1}\Big]\Big((-1)^\omega\int_0^\xi \frac{(\xi-s)^{\nu-2}}{\Gamma(\nu-1)}\Phi(s,\phi(s))\,ds + \int_0^\kappa \frac{(\kappa-s)^{\nu-2}}{\Gamma(\nu-1)}\Phi(s,\phi(s))\,ds\Big)\Big]$$

Let $\sup_{\rho\in[0,T]}\|\Phi(\rho,0)\| = M$, where $M > 0$ and $r \geq \frac{M\,\Delta}{1-L\,\Delta}$.
We will show that $\mathcal{T}B_r \subset B_r$, where $B_r = \{\phi \in C : \|\phi\| \leq r\}$, that is,

$$\|(\mathcal{T}\phi)(\rho)\| \leq \int_0^\rho \frac{(\rho-s)^{\nu-1}}{\Gamma(\nu)}\|\Phi(s,\phi(s))\|\,ds$$

$$+ \frac{1}{(\xi-\kappa)|b|}\Big[\Big[\kappa|b| + \Big(1-(-1)^{\omega+1}\Big)\Big]\int_0^\xi \frac{(\xi-s)^{\nu-1}}{\Gamma(\nu)}\|\Phi(s,\phi(s))\|\,ds$$

$$+ \big[1+(-1)^\omega + |b|\xi\big]\int_0^\kappa \frac{(\kappa-s)^{\nu-1}}{\Gamma(\nu)}\|\Phi(s,\phi(s))\|\,ds$$

$$+ \big[|\kappa + (-1)^\omega\xi|\big]\Big(\int_0^\xi \frac{(\xi-s)^{\nu-2}}{\Gamma(\nu-1)}\|\Phi(s,\phi(s))\|\,ds + \int_0^\kappa \frac{(\kappa-s)^{\nu-2}}{\Gamma(\nu-1)}\|\Phi(s,\phi(s))\|\,ds\Big)\Big]$$

$$+ \frac{\rho}{(\xi-\kappa)|b|}\Big[|b|\int_0^\xi \frac{(\xi-s)^{\nu-1}}{\Gamma(\nu)}\|\Phi(s,\phi(s))\|\,ds + |b|\int_0^\kappa \frac{(\kappa-s)^{\nu-1}}{\Gamma(\nu)}\|\Phi(s,\phi(s))\|\,ds$$

$$+ \Big[1+(-1)^{\omega+1}\Big]\Big(\int_0^\xi \frac{(\xi-s)^{\nu-2}}{\Gamma(\nu-1)}\|\Phi(s,\phi(s))\|\,ds + \int_0^\kappa \frac{(\kappa-s)^{\nu-2}}{\Gamma(\nu-1)}\|\Phi(s,\phi(s))\|\,ds\Big)\Big]$$

$$\leq \int_0^\rho \frac{(\rho-s)^{\nu-1}}{\Gamma(\nu)}\left(\|\Phi(s,\phi(s)) - \Phi(s,0)\| + \|\Phi(s,0)\|\right)ds$$

$$+ \frac{1}{(\xi-\kappa)|b|}\Big[\Big(\kappa|b| + 1 - (-1)^{\omega+1}\Big)\int_0^\xi \frac{(\xi-s)^{\nu-1}}{\Gamma(\nu)}\left(\|\Phi(s,\phi(s)) - \Phi(s,0)\| + \|\Phi(s,0)\|\right)ds$$

$$+ (1+(-1)^\omega + |b|\xi)\int_0^\kappa \frac{(\kappa-s)^{\nu-1}}{\Gamma(\nu)}\left(\|\Phi(s,\phi(s)) - \Phi(s,0)\| + \|\Phi(s,0)\|\right)ds$$

$$+ |\kappa + (-1)^\omega\xi|\Big(\int_0^\xi \frac{(\xi-s)^{\nu-2}}{\Gamma(\nu-1)}\left(\|\Phi(s,\phi(s)) - \Phi(s,0)\| + \|\Phi(s,0)\|\right)ds$$

$$+ \int_0^\kappa \frac{(\kappa-s)^{\nu-2}}{\Gamma(\nu-1)}\left(\|\Phi(s,\phi(s)) - \Phi(s,0)\| + \|\Phi(s,0)\|\right)ds\Big)\Big]$$

$$
\begin{aligned}
&+\frac{\rho}{(\xi-\kappa)|b|}\Big[|b|\int_0^{\xi}\frac{(\xi-s)^{\nu-1}}{\Gamma(\nu)}\left(\|\Phi(s,\phi(s))-\Phi(s,0)\|+\|\Phi(s,0)\|\right)ds\\
&\quad+|b|\int_0^{\kappa}\frac{(\kappa-s)^{\nu-1}}{\Gamma(\nu)}\left(\|\Phi(s,\phi(s))-\Phi(s,0)\|+\|\Phi(s,0)\|\right)ds\\
&\quad+\left(1+(-1)^{\omega+1}\right)\Big(\int_0^{\xi}\frac{(\xi-s)^{\nu-2}}{\Gamma(\nu-1)}\left(\|\Phi(s,\phi(s))-\Phi(s,0)\|+\|\Phi(s,0)\|\right)ds\\
&\quad+\int_0^{\kappa}\frac{(\kappa-s)^{\nu-2}}{\Gamma(\nu-1)}\left(\|\Phi(s,\phi(s))-\Phi(s,0)\|+\|\Phi(s,0)\|\right)ds\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\leq\frac{Lr+M}{(\xi-\kappa)|b|\,\Gamma(\nu+1)}\Big[&((\xi-\kappa)|b|+\kappa|b|+\xi|b|+1-(-1)^{\omega+1})\xi^{\nu}+(2\xi|b|+1-(-1)^{\omega+1})\kappa^{\nu}\\
&+(\nu(\kappa+\xi)+\nu\xi(1+(-1)^{\omega+1}))(\xi^{\nu-1}+\kappa^{\nu-1})\Big]=L\,\Delta\,r+\Delta M\leq r.
\end{aligned}
$$

Now, for $\phi,\psi\in C$ and each $\rho\in[0,\xi]$, we obtain

$$
\begin{aligned}
\|(\mathcal{T}\phi)(\rho)-(\mathcal{T}\psi)(\rho)\|\leq&\int_0^{\rho}\frac{(\rho-s)^{\nu-1}}{\Gamma(\nu)}(\|\Phi(s,\phi(s))-\Phi(s,\psi(s))\|)\,ds\\
&+\frac{1}{(\xi-\kappa)|b|}\Big[[\kappa|b|+1-(-1)^{\omega+1}]\int_0^{\xi}\frac{(\xi-s)^{\nu-1}}{\Gamma(\nu)}(\|\Phi(s,\phi(s))-\Phi(s,\psi(s))\|)\,ds\\
&+[1+(-1)^{\omega}+|b|\xi]\int_0^{\kappa}\frac{(\kappa-s)^{\nu-1}}{\Gamma(\nu)}(\|\Phi(s,\phi(s))-\Phi(s,\psi(s))\|)\,ds\\
&+\left|\kappa+(-1)^{\omega}\xi\right|\Big(\int_0^{\xi}\frac{(\xi-s)^{\nu-2}}{\Gamma(\nu-1)}(\|\Phi(s,\phi(s))-\Phi(s,\psi(s))\|)\,ds\\
&\qquad+\int_0^{\kappa}\frac{(\kappa-s)^{\nu-2}}{\Gamma(\nu-1)}(\|\Phi(s,\phi(s))-\Phi(s,\psi(s))\|)\,ds\Big)\Big]\\
&+\frac{\rho}{(\xi-\kappa)|b|}\Big[|b|\int_0^{\xi}\frac{(\xi-s)^{\nu-1}}{\Gamma(\nu)}(\|\Phi(s,\phi(s))-\Phi(s,\psi(s))\|)\,ds\\
&\quad+|b|\int_0^{\kappa}\frac{(\kappa-s)^{\nu-1}}{\Gamma(\nu)}(\|\Phi(s,\phi(s))-\Phi(s,\psi(s))\|)\,ds\\
&\quad+(1+(-1)^{\omega+1})\Big(\int_0^{\xi}\frac{(\xi-s)^{\nu-2}}{\Gamma(\nu-1)}(\|\Phi(s,\phi(s))-\Phi(s,\psi(s))\|)\,ds\\
&\qquad+\int_0^{\kappa}\frac{(\kappa-s)^{\nu-2}}{\Gamma(\nu-1)}(\|\Phi(s,\phi(s))-\Phi(s,\psi(s))\|)\,ds\Big)\Big]
\end{aligned}
$$

$$
\leq\frac{L\|\phi-\psi\|}{(\xi-\kappa)|b|\Gamma(\nu+1)}\Big[(2\xi|b|+1-(-1)^{\omega+1})(\xi^{\nu}+\kappa^{\nu})+\nu(\kappa+2\xi+\xi(-1)^{\omega+1})(\xi^{\nu-1}+\kappa^{\nu-1})\Big]=L\Delta\|\phi-\psi\|.
$$

As $L\Delta<1$, we see that $\mathcal{T}$ is a contraction. Therefore, by the contraction principle we proved the conclusion. □

Theorem 3.2. *Assume that $\Phi:[0,\xi]\times\mathbb{R}\to\mathbb{R}$ is a continuous function mapping bounded sets into relatively compact subsets of $\mathbb{R}$. Suppose that*

(A_1) $\|\Phi(\rho,\phi)-\Phi(\rho,\psi)\| \le L\|\phi-\psi\|$ *for all* $\rho\in[0,\xi]$ *and* $\phi,\psi\in\mathbb{R}$*;*
(A_2) $\|\Phi(\rho,\phi)\| \le \mu(\rho)$ *for all* $(\rho,\phi)\in[0,\xi]\times\mathbb{R}$*, with* $\mu\in L^1([0,\xi],\mathbb{R}^+)$.

If $L\Lambda<1$*, then (1.1) has at least one solution on* $[0,\xi]$*. where,*

$$\Lambda=\frac{1}{(\xi-\kappa)|b|\Gamma(\nu+1)}\Big[\xi^{\nu-1}\Big[\xi(|b|(\kappa+\xi)+1-(-1)^{\omega+1})+\nu(\kappa+2\xi+\xi(-1)^{\omega+1})\Big]$$
$$+\kappa^{\nu-1}\Big[\kappa(2\xi|b|+1-(-1)^{\omega+1})+\nu(\kappa+2\xi+\xi(-1)^{\omega+1})\Big]\Big].$$

Proof. Let $B_r=\{\phi\in C:\|\phi\|\le r\}$, where $r\ge\|\mu\|\Delta$, and define

$$(\mathcal{T}_1\phi)(\rho)=\int_0^{\rho}\frac{(\rho-s)^{\nu-1}}{\Gamma(\nu)}\Phi(s,\phi(s))\,ds.$$

$$\begin{aligned}(\mathcal{T}_2\phi)(\rho)=\frac{1}{(\xi-\kappa)b}\Bigg[&[\kappa b+(1-(-1)^{\omega+1})]\int_0^{\xi}\frac{(\xi-s)^{\nu-1}}{\Gamma(\nu)}\Phi(s,\phi(s))\,ds\\&-[1+(-1)^{\omega}+b\xi]\int_0^{\kappa}\frac{(\kappa-s)^{\nu-1}}{\Gamma(\nu)}\Phi(s,\phi(s))\,ds\\&+[\kappa+(-1)^{\omega}\xi]\Big((-1)^{\omega}\int_0^{\xi}\frac{(\xi-s)^{\nu-2}}{\Gamma(\nu-1)}\Phi(s,\phi(s))\,ds+\int_0^{\kappa}\frac{(\kappa-s)^{\nu-2}}{\Gamma(\nu-1)}\Phi(s,\phi(s))\,ds\Big)\Bigg]\\+\frac{\rho}{(\xi-\kappa)b}\Bigg[&-b\int_0^{\xi}\frac{(\xi-s)^{\nu-1}}{\Gamma(\nu)}\Phi(s,\phi(s))\,ds+b\int_0^{\kappa}\frac{(\kappa-s)^{\nu-1}}{\Gamma(\nu)}\Phi(s,\phi(s))\,ds\\&-[1+(-1)^{\omega+1}]\Big((-1)^{\omega}\int_0^{\xi}\frac{(\xi-s)^{\nu-2}}{\Gamma(\nu-1)}\Phi(s,\phi(s))\,ds+\int_0^{\kappa}\frac{(\kappa-s)^{\nu-2}}{\Gamma(\nu-1)}\Phi(s,\phi(s))\,ds\Big)\Bigg].\end{aligned}$$

Now, for any $\phi_1,\phi_2\in B_r$ we have

$$\|\mathcal{T}_1\phi_1+\mathcal{T}_2\phi_2\|\le\|\mu\|\Bigg[\frac{1}{(\xi-\kappa)|b|\Gamma(\nu+1)}\Big[(2\xi|b|+1-(-1)^{\omega+1})(\xi^{\nu}+\kappa^{\nu})+\nu(\kappa+2\xi+\xi(-1)^{\omega+1})(\xi^{\nu-1}+\kappa^{\nu-1})\Big]\Bigg].$$

Therefore, we have

$$\|\mathcal{T}_1\phi_1+\mathcal{T}_2\phi_2\|\le\|\mu\|\Delta\le r.$$

Implying that $\mathcal{T}_1\phi_1+\mathcal{T}_2\phi_2\in B_r$.

Also, due to the continuity of Φ the operator ϕ is continuous, and as $\mathcal{T}_1\phi\le\frac{\xi^{\nu}\|\mu\|}{\Gamma(\nu+1)}$ we see that $\mathcal{T}_1$ is uniformly bounded.

Now, we show the compactness of $\mathcal{T}_1$. Setting $\mathcal{S}=[0,\xi]\times B_r$ and define $\Phi_{max}=\sup_{(\rho,\phi)\in\mathcal{S}}\|\Phi(\rho,\phi)\|$, we have

$$\|(\mathcal{T}_1\phi)(\rho_1)-(\mathcal{T}_1\phi)(\rho_2)\|=\frac{1}{\Gamma(\nu)}\|\int_0^{\rho_1}\Big[(\rho_1-s)^{\nu-1}-(\rho_2-s)^{\nu-1}\Big]\Phi(s,\phi)ds+\int_{\rho_1}^{\rho_2}(\rho_2-s)^{\nu-1}\Phi(s,\phi)ds\|$$
$$\le\Phi_{max}\frac{|2(\rho_2-\rho_1)^{\nu}+\rho_1^{\nu}-\rho_2^{\nu}|}{\Gamma(\nu+1)}.$$

Independent of ϕ. Therefore, $\mathcal{T}_1$ is equicontinuous and $\mathcal{T}_1(B_r)$ is uniformly bounded. The Arzela-Ascoli theorem assures that $\mathcal{T}_1(B_r)$ is relatively compact and therefore $\mathcal{T}_1$ is compact.

Finally, by (A_1) we see that $\mathcal{T}_2$ is a contraction since $L\Lambda<1$. By Theorem 2.1 problem (1.1) has at least one solution on $[0,\xi]$. □

4. Examples

Example 4.1. [5] Consider

$$\begin{cases} {}^{c}D^{\frac{3}{2}}\phi(\rho) = \dfrac{1}{(\rho+7)^2}\dfrac{|\phi|}{1+|\phi|}, \ \rho \in [0,\pi], \\ \phi'(0) + (-1)^{\omega}\phi'(\pi) + \phi(\pi) = 0, \phi(0) + (-1)^{\omega+1}\phi(\pi) = 0, \omega = 0,1. \end{cases} \tag{4.1}$$

Since $\|\Phi(\rho,\phi_1) - \Phi(\rho,\phi_2)\| \leq \frac{1}{49}\|\phi_1 - \phi_2\|$, Then, $L = \frac{1}{49}$ and $\Delta = \frac{2(4\pi+8+(-1)^{\omega+1})}{3}$.

We see that $L\Delta = \frac{2(4\pi+8+(-1)^{\omega+1})}{147} < 1$, for $\omega = 0,1$. Hence, Theorem 3.1 shows that (4.1) has a unique solution.

Example 4.2. Consider the following Caputo fractional boundary value problem:

$$\begin{cases} {}^{c}D^{\frac{3}{2}}\phi(\rho) = \dfrac{e^{-\rho}}{(\rho+3)^4}\dfrac{\phi(\rho)}{1+\phi(\rho)^2}, \qquad \rho \in [0,2], \\ \phi'(1) + (-1)^{\omega}\phi'(2) + \dfrac{1}{3}\phi(2) = 0, \ \phi(1) + (-1)^{\omega+1}\phi(2) = 0, \qquad \omega = 0,1. \end{cases} \tag{4.2}$$

It follows that $\|\Phi(\rho,\phi_1) - \Phi(\rho,\phi_2)| \leq \frac{1}{81}|\phi_1 - \phi_2|$. Hence, $L = \frac{1}{81}$.

Moreover, with $\kappa = 1$, $\xi = 2$, $\nu = \frac{3}{2}$ and $b = \frac{1}{3}$, we have $\Delta = \frac{2}{3\sqrt{\pi}}\Big(73\sqrt{2} + 59 + 6(\sqrt{2}+2)(-1)^{\omega+1}\Big)$. We see that

$$L\Delta \approx \begin{cases} 0.665 \ , \omega = 0, \\ 0.849 \ , \omega = 1. \end{cases}$$

Hence, with $L\Delta < 1$, Theorem 3.2 shows that problem (4.2) has a unique solution on [0,2].

5. Conclusions

In summary, this paper has successfully developed a generalized framework for investigating the existence and uniqueness of solutions to nonlinear fractional differential equations of order $\nu \in (1,2]$ subject to irregular boundary conditions in a Banach space. By employing classical fixed-point techniques—namely, the Banach contraction mapping principle to ensure uniqueness and Krasnoselskii's fixed-point theorem to establish existence—we derived sufficient conditionsguaranteeing well-posedness of the proposed problem. These results extend and unify earlier studies by allowing more flexible and irregular boundary specifications, which naturally arise inmany applied models where standard separated or periodic conditions are inadequate. The theoretical analysis is complemented by an illustrative example, that validates the applicability of the derived criteria and highlights the practical effectiveness of the proposed framework in verifying solution properties for concrete fractional boundary value problems. Overall, this work contributes to the growing theory of fractional differential equations by providing robust analytical tools for a broader class of nonlinear systems governed by fractional-order derivatives. Future research may explore extensions to impulsive or stochastic settings, or incorporate nonlocal effects, to further enhance the modeling capabilities in interdisciplinary applications such as viscoelasticity, anomalous diffusion, and control theory.

Author contributions

Saleh Fahad Aljurbua: Formal analysis, methodology, writing-original draft, validation, supervision; Salman Alotaibi: investigation, writing-original draft, validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors do not have any conflict of interest.

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