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Research article

Normalized solutions for a kind of mass–supercritical Schrödinger–Choquard equations

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Abstract: We investigate the normalized solutions for the following Schrödinger–Choquard equations constrained to L^2-sphere with general potential:

$$\begin{cases} -\Delta v - V(x)v - \lambda v = (I_\alpha * F(v))f(v), & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v|^2 \mathrm{d}x = a, & \text{in } \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $a > 0$ is a prescribed mass, $V \in C^1(\mathbb{R}^N, \mathbb{R})$ is an external potential, $f \in C(\mathbb{R}, \mathbb{R})$, $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ denotes the Riesz potential with order $\alpha \in (0, N)$, and $\lambda \in \mathbb{R}$ is not prescribed in advance but appears as a Lagrange multiplier. By using critical point theory, we develop reliable arguments to construct the existence results of normalized solutions to this kind of Schrödinger–Choquard equations. The obtained results generalize and improve existing results in the literature.

Keywords: mass–supercritical; normalized solution; existence; Schrödinger–Choquard equation
Mathematics Subject Classification: 35B09, 35B38, 35J20, 35Q55

1. Introduction and main results

In this paper, the existence of solutions for the following nonlinear Schrödinger–Choquard equation with general potential and prescribed mass is studied:

$$\begin{cases} -\Delta v - V(x)v - \lambda v = (I_\alpha * F(v))f(v), & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v|^2 \mathrm{d}x = a, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 3$, $a > 0$ is a prescribed mass, $V \in C^1(\mathbb{R}^N, \mathbb{R})$ is an external potential, $f \in C(\mathbb{R}, \mathbb{R})$, $F(s) := \int_0^s f(t)\mathrm{d}t$, $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier depending on the solution $v \in H^1(\mathbb{R}^N)$ but is not prescribed in advance, and $*$ denotes the convolution on $\mathbb{R}^N$. I_α is called the Riesz potential with

$\alpha \in (0, N)$ and is defined by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\pi^{\frac{N}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})|x|^{N-\alpha}}, \quad \forall\, x \in \mathbb{R}^N \backslash \{0\},$$

where Γ denotes the Gamma function [1, 2]. Let $A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\pi^{\frac{N}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}$.

Equation (1.1) is closely related to the following focusing time-dependent Hartree equation:

$$i\partial_t \phi = -\Delta\phi - V(x)v - (I_\alpha * F(v))f(v), \quad \forall\, (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N. \tag{1.2}$$

It is known that if v is a solution of the first equation of (1.1), then $\phi(t, x) = e^{it}v(x)$ is a solitary wave solution of (1.2). Solitary wave solutions are very important in the dynamical behaviors of complex systems such as the nonlinear Schrödinger equation or Choquard equation. For the physical meanings of a solitary wave solution, one can refer to [3, 4], in which the authors gave meaningful physical perspective for a deeper understanding of wave phenomena in complex systems. v_a is often called a normalized solution of problem (1.1) if a pair (λ_a, v_a) satisfies problem (1.1). In this case, λ is not fixed. If λ is fixed, then the problem becomes unconstrained. There are many works on unconstrained problems, see [5–7] for the existence of solutions to the Choquard equation and [8, 9] for solutions to the N-Laplacian Schrödinger equation.

If $N = 3$, $V(x) \equiv 0$, $\lambda = -1$, $F(v) = v^2$, and $I_\alpha = \frac{1}{2|x|}$, then the first equation of (1.1) reduces to the following form:

$$-\Delta v + v = \left(\frac{1}{|x|} * v^2\right)v, \quad \forall\, x \in \mathbb{R}^3. \tag{1.3}$$

In [10], Pekar proposed (1.3) to described the quantum mechanics of a polaron. (1.3) was used to describe an electron trapped in its hole in the work of Choquard [11] in 1976, and that is why (1.3) is called the Choquard equation. From then on, if the nonlinearity of an equation has a convolution form, then this equation is also called a Choquard equation. In fact, since (1.3) modeled self-gravitating matter, it is also called the Newton-Schrödinger equation or the Hartree equation [12]. When (1.3) models self-gravitating matter, the quantum state reduction is known as a gravitational phenomenon. In fact, if the nonlinearity of an equation doesn't have a convolution form, the equation has the form $-\Delta v + V(x)v = f(v)$, and one calls such an equation a Schrödinger equation. The Schrödinger equation is the fundamental equation of nonrelativistic quantum mechanics [13]. Its physical significance is profound and multifaceted. For example, it defines the wave function and its evolution, embodies wave–particle duality, provides a probabilistic interpretation (born rule), quantizes physical quantities, and unifies the description of quantum systems. So, if an equation has the form of a Schrödinger equation with convolution nonlinearity, then one usually calls it a Schrödinger–Choquard equation.

When $V(x) \equiv 0$, $f(v) = |v|^{p-2}v$, problem (1.1) has the following form:

$$\begin{cases} -\Delta v - \lambda v = (I_\alpha * |v|^p)|v|^{p-2}v, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v|^2 \mathrm{d}x = a, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $N \geq 3$, $p \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}]$, $\alpha \in (0, N)$, and λ is a Lagrange multiplier.

Xing and Lei [14] studied problem (1.4) with combined nonlinearity and prescribed mass, where the nonlinearity is $(I_\alpha * |v|^{\frac{N+\alpha}{N-2}})|v|^{\frac{4+\alpha-N}{N-2}}v + |v|^{p_1-2}v$ with $p_1 \in (2+\frac{4}{N}, \frac{N+\alpha}{N-2})$. By using concentration compactness and the Ekeland variational principle, they established some results on multiple radially normalized solutions. Feng and Li [15] investigated problem (1.4) with mass prescribed where the nonlinearity is $(I_\alpha * |v|^{p_2})|v|^{p_2-2}v + |v|^4 v$ with $p_2 \in (\frac{5}{3}, 3)$. They proved that the positive ground states converge to a least energy solution of the limited critical local problem of (1.4) when $\alpha \to 0^+$. There are other results of normalized solutions to Choquard equations or Schrödinger equations; one can refer to [16–26]. Specially, in [16], the authors studied normalized solutions to a critical Choquard equation with nonlocal perturbation and weakly attractive potential. In [17], the authors obtained sufficient and necessary conditions for a normalized solution of a Choquard equation where the ground state solutions and the normalized solutions are equivalent. In [18], normalized solutions for a Choquard equation with a logarithmic perturbation and critical exponents was studied. In [19], concentration and multiplicity of normalized solutions for the Choquard equation with potential were established. In [20], the authors obtained concentration and multiplicity of semiclassical states for the double phase problems with a Choquard term. In [21], the existence of a local minimum and a mountain pass-type solution for a class of Choquard–Schrodinger–Poisson systems involving p-Laplacian were obtained. [22, 23] concern the planar Choquard equation, and [24–26] deal with other kinds of Choquard equations and Schrödinger equations. From the above works, we can see that normalized solutions of Choquard equations and Schrödinger equations play an important role in physics. To the best of our knowledge, it seems that there is little work on (1.1) since it involves the potential term $V(x)$ and the convolution term $[I_\alpha * F(v)]F(v)$–that is why we investigate this equation in this paper.

Namely, the critical points of the following functional

$$I(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} V(x)v^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} [I_\alpha * F(v)]F(v)\mathrm{d}x \tag{1.5}$$

under the constrained

$$S_a := \{v \in H^1(\mathbb{R}^N) : \|v\|_2^2 := \int_{\mathbb{R}^N} |v|^2 \mathrm{d}x = a\} \tag{1.6}$$

are normalized solutions of problem (1.1). We say that a solution $v \in S_a$ is called a ground state solution if the energy functional I achieves its minimal energy at v among all the nontrivial solutions that is

$$I(v) = \inf\{I(v) : v \in S_a, I|'_{S_a}(v) = 0\}. \tag{1.7}$$

When $V(x) \equiv 0$, problem (1.1) reduces to the following form:

$$\begin{cases} -\Delta v - \lambda v = (I_\alpha * F(v))f(v), & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v|^2 \mathrm{d}x = a, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.8}$$

where $N \geq 3$. Li and Ye [27] studied problem (1.8) and established existence results of a ground state (v_a, λ_a) under the following conditions on f:
(f1) $f \equiv 0$ for $t \leq 0$, and there exists $p \in \left(\frac{N+\alpha+2}{N}, \frac{N+\alpha}{N-2}\right)$ such that

$$\lim_{|t|\to 0} \frac{f(t)}{|t|^{p-2}t} = 0, \lim_{|t|\to+\infty} \frac{F(t)}{|t|^p} = +\infty, \lim_{|t|\to+\infty} \frac{F(t)}{|t|^{\frac{N+\alpha}{N-2}}} = 0;$$

(f2) let $\tilde{F}(t) := f(t)t - \frac{N+\alpha+2}{2}F(t)$, and let there exist $\mu_1 \geq 1$ such that

$$\tilde{F}(s_1 t) \leq \mu_1 \tilde{F}(t), \quad \text{for } t \in \mathbb{R} \text{ and } s_1 \in [0,1];$$

(f3) let $\bar{F}(t) := f(t)t - \frac{N+\alpha}{N}F(t)$, $\bar{F}'(t)$ exist, and $\frac{N+2+\alpha}{N}\bar{F}(t) < \bar{F}(t)t$;
(f4) there exist $\mu_2 \in (0,1)$ and $s_2 > 1$ such that, for all $t \in \mathbb{R}$ and $|s| \leq s_2$, $F(st) \leq \mu_2 |s|^{\frac{N+\alpha+2}{N}} F(t)$;
(f5) $f(t)t < \frac{N+\alpha}{N-2}F(t)$, and $\forall\ t > 0$.

It is known that $\frac{N+\alpha+2}{N}$ is called a mass–critical exponent or L^2–critical exponent when we investigate Choquard equations with mass constrained. The critical exponent is very important since it can ensure the boundedness of the energy functional from below. From (f1), we know that $F(t)$ is larger than $|t|^{\frac{N+\alpha+2}{N}}$ when $t \to +\infty$; then, the functional of (1.8) is unbounded from below on the constrained set S_a, and problem (1.8) is called mass–supercritical. If $F(t)$ is smaller than $|t|^{\frac{N+\alpha+2}{N}}$ when $t \to +\infty$, then the functional of the related Choquard equation on the constrained set S_a is called mass–subcritical. Since the structure of an unbounded functional is more complicated, dealing with mass–supercritical problems is more difficult than dealing with mass–subcritical problems.

In 2020, Bartsch et al. [28] investigated problem (1.8) with mass–supercritical and established existence results of infinitely many radial normalized solutions and normalized ground state solutions under the following assumptions on f:
(A1) $f \in C(\mathbb{R},\mathbb{R})$, and there exist $\theta_1, \theta_2 > 0$ with $\frac{N+\alpha+2}{N} < \theta_1 \leq \theta_2 < \frac{N+\alpha}{N-2}$ such that

$$0 < \theta_1 F(t) \leq f(t)t < \theta_2 F(t), \quad \forall\ t \in \mathbb{R}\backslash\{0\}; \tag{1.9}$$

(A2) the function $\frac{\bar{F}(t)}{|t|^{\frac{N+\alpha+2}{N}}}$ is nondecreasing in $(0,+\infty)$ and nonincreasing in $(-\infty,0)$, where $\bar{F}(t)$ is defined as that in (f3).

Subsequently, Xia and Zhang [29] obtained some results under the following weaker condition than (A2):
(A2') $\frac{\bar{F}(t)}{|t|^{\frac{N+\alpha+2}{N}}}$ is nondecreasing in $(0,+\infty)$.

Very recently, Jin et al. [30] used the following conditions to study normalized solutions for problem (1.8):
(S1) $f \in C(\mathbb{R},\mathbb{R})$, $\lim_{s\to 0}\frac{f(s)}{|s|^{\frac{2+\alpha}{N}}} = 0$, $\lim_{|s|\to+\infty}\frac{F(s)}{|s|^{\frac{N+2+\alpha}{N}}} = +\infty$, $\lim_{|s|\to+\infty}\frac{f(s)}{|s|^{\frac{N+\alpha}{N-2}-1}} = 0$;
(S2) $0 < \frac{N+2+\alpha}{N}F(s) \leq f(s)s < \frac{N+\alpha}{N-2}F(s)$, $\forall\ s \in \mathbb{R}\backslash\{0\}$;
(S3) $\lim_{|s|\to+\infty}\sup\frac{F(s)}{Nf(s)s-(N+2+\alpha)F(s)} < +\infty$;
(S3') there exist $k \in (\frac{N+\alpha}{2+\alpha}, 1+\frac{N}{\alpha})$ and $C_0 > 0$ such that

$$\left|\frac{f(s)s - \frac{N+\alpha}{N}F(s)}{s}\right|^k \leq C_0[Nf(s)s - (N+2+\alpha)F(s)],\ \forall\ s \in \mathbb{R}\backslash\{0\}.$$

It must be pointed out that (1.9) acts as the classical Ambrosetti–Rabinowtz (AR) condition in the study of the Choquad equation in the unconstrained case. It is also very important and necessary in proving the boundedness of Palais–Smale (PS) sequences when studying the Choquad equation with constrained mass. Li and Ye in [27] used more strict conditions (f2) and (f4) than (A1) to obtain the boundedness of the PS sequences. The method used by them seems more intricate. We can see that (S1)–(S3) (or (S3')) are weaker than (A1)–(A2) (or (A2')). Jin and Tang [26] and Jin et al. [30]

proposed a very new method to prove the boundedness of the PS sequences, which seems very different from existing methods in the literature. To the best of our knowledge, when $V(x) \neq 0$, there seem to be few works on normalized solutions for (1.1). Since the additional terms $\int_{\mathbb{R}^N} V(x)v^2\mathrm{d}x$ and $\int_{\mathbb{R}^N}(\nabla V(x), x)v^2\mathrm{d}x$ in the related functional bring difficulties, it is also more difficult to prove the boundedness of PS sequences. There is one question: **Do the results of [30] hold when the Schrödinger–Choquard equation involves a general potential?**

Motivated by the above mentioned works, especially by [26, 30], we will consider normalized solutions of (1.1). The aim of this paper is to give an affirmate answer to the above question. The following conditions on V will be needed:

(V1) $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and $2V(x) + (\nabla V(x), x) = 0$;

(V2) there is $\vartheta \in (0, 1)$ such that $\left|\int_{\mathbb{R}^N}(\nabla V(x), x)v^2\mathrm{d}x\right| \leq \vartheta\|\nabla v\|_2^2$, $v \in H^1(\mathbb{R}^N)$.

The main results of this paper are presented in the following:

Theorem 1.1. *Assume that* (V1) *and* (V2) *hold. If f satisfies* (S1)–(S3), *then problem (1.1) possesses a solution pair $(\lambda_a, v_a) \in (-\infty, 0) \times H^1_{rad}(\mathbb{R}^N)$.*

Theorem 1.2. *Assume that* (V1) *and* (V2) *hold. If f satisfies* (S1), (S2), and (S3'), *then problem (1.1) has a solution pair $(\lambda_a, v_a) \in (-\infty, 0) \times H^1_{rad}(\mathbb{R}^N)$ such that*

$$I(v_a) = \inf\{I : v \in S_a \cap H^1_{rad}(\mathbb{R}^N) : I|'_{S_a}(v) = 0\}.$$

Remark 1.3. It is worth pointing out that the conditions on V are much simpler. It seems that there is no result on (1.1) with general potential. From this point, the results of this paper are new for the Schrödinger–Choquard equation with potential to some extent. The results obtained in this paper generalize and extend the existing results for the mass–constrained Schrödinger–Choquard equation.

The organization of this paper is as follows. In Section 2, some preliminary lemmas and the variational structure are given. In Section 3, we prove Theorems 1.1 and 1.2. In Section 4, we give some conclusions on this kind of Schrödinger–Choquard equation. The following notations are used in the this paper:

♣ $L^s(\mathbb{R}^N)(s \in [1 + \infty))$ appears as the Lebesgue space with norm $\|v\|_s^s = \int_{\mathbb{R}^N} |v|^s\mathrm{d}x$;

♣ $H^1(\mathbb{R}^N)$ is the usual Sobolev space equipped with the inner product and norm

$$(v, u) = \int_{\mathbb{R}^N} [\nabla v\nabla u + uv]\mathrm{d}x, \quad \|v\| = (v, v)^{1/2}, \quad \forall\, u, v \in H^1(\mathbb{R}^N);$$

♣ $H^1_{rad}(\mathbb{R}^N) = \{v \in H^1(\mathbb{R}^N) : v(x) = v(|x|)\}$;

♣ $v_t(x) := v(tx)$ for $t > 0$ and $v \in H^1(\mathbb{R}^N)$;

♣ $B_r(x) := \{y \in \mathbb{R}^N : |y - x| < r\}$ for any $x \in \mathbb{R}^N$ and $r > 0$, and $B_r = B_r(0)$.

♣ $\mathcal{M} := \{v \in H^1_{rad}(\mathbb{R}^N) : \|v\|_2^2 = a\}$.

♣ $C_1, C_2, \cdots$ are positive constants depending on the mass $a > 0$, which may vary in different places.

2. Preliminaries and variational structure

Under conditions (V1), (V2), and (S1), it is well known that the energy functional $I(v)$ is a well defined C^1 functional,

$$\langle I'(v), u\rangle = \int_{\mathbb{R}^N} \nabla v\nabla u\mathrm{d}x - \int_{\mathbb{R}^N} V(x)uv\mathrm{d}x - \int_{\mathbb{R}^N} [I_\alpha * F(v)]f(v)u\mathrm{d}x, \quad \forall\, u, v \in H^1(\mathbb{R}^N), \tag{2.1}$$

and $I(v)$ is unbounded from below on S_a. Therefore, the minimizing argument constrained on S_a is not valid. Following the ideas from [25, 30], we will use a subset $S_a \cap H^1_{rad}(\mathbb{R}^N)$ of S_a to investigate the minimization of $I(v)$. If v is a solution of problem (1.1), then v satisfies the following Nehari identity:

$$\int_{\mathbb{R}^N} [|\nabla v|^2 - V(x)v^2]\mathrm{d}x - \lambda \int_{\mathbb{R}^N} v^2\mathrm{d}x - \int_{\mathbb{R}^N} [I_\alpha * F(v)]f(v)v\mathrm{d}x = 0, \tag{2.2}$$

and the Pohozăev identity:

$$\begin{aligned} &\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} [NV(x) + (\nabla V(x), x)]v^2\mathrm{d}x \\ &- \frac{N\lambda}{2}\int_{\mathbb{R}^N} v^2\mathrm{d}x - \frac{N+\alpha}{2}\int_{\mathbb{R}^N} [I_\alpha * F(v)]F(v)\mathrm{d}x = 0. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), v satisfies

$$\int_{\mathbb{R}^N} |\nabla v|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} (\nabla V(x), x)v^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} [I_\alpha * F(v)][Nf(v)v - (N+\alpha)F(v)]\mathrm{d}x = 0. \tag{2.4}$$

Let

$$P(v) := \int_{\mathbb{R}^N} |\nabla v|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} (\nabla V(x), x)v^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} [I_\alpha * F(v)][Nf(v)v - (N+\alpha)F(v)]\mathrm{d}x, \tag{2.5}$$

then $P(v)$ is a Pohozăev functional of (1.1), and $P(v) = 0$ is also called a Pohozăev identity of (1.1), which is commonly used to study prescribed–mass problems. The following preliminary results are often used in the study of existence of solutions for elliptic equations:

Lemma 2.1 [31]. *For $N \geq 3$, there is an optimal positive constant S depending only on N such that*

$$\|v\|^2_{2^*} \leq S^{-1}\|\nabla v\|^2_2, \quad \forall\, v \in D^{1,2}(\mathbb{R}^N), \tag{2.6}$$

where $D^{1,2}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|v\|_{D^{1,2}} := \|\nabla v\|_2$.

Lemma 2.2 [32]. *Let $N \geq 3$, $p, q > 1$, and $\alpha \in (0, N)$ with $\frac{N-\alpha}{N} + \frac{1}{p} + \frac{1}{q} = 2$, $g \in L^p(\mathbb{R}^N)$, and $h \in L^q(\mathbb{R}^N)$. There exists a sharp constant $C_{\alpha,p,q}$ independent of g and h, such that*

$$\int_{\mathbb{R}^N} (I_\alpha * g)h\mathrm{d}x \leq C_{\alpha,p,q}\|g\|_p\|h\|_q. \tag{2.7}$$

Particularly,

$$\int_{\mathbb{R}^N} (I_\alpha * g)h\mathrm{d}x \leq C_{\alpha,2N/(N+\alpha),2N/(N+\alpha)}\|g\|_{2N/(N+\alpha)}\|h\|_{2N/(N+\alpha)}. \tag{2.8}$$

Lemma 2.3 [33]. *For $N \geq 3$, let $s \in (2, 2^*)$. Then, there is a sharp constant $C_{s,N} > 0$ such that*

$$\|v\|_s \leq C_{s,N}\|\nabla v\|_2^q\|v\|_2^{1-q}, \tag{2.9}$$

where $q = \frac{N(s-2)}{2s}$.

It is well known that (2.6) is called a Sobolev inequality, (2.7) and (2.8) are Hardy–Littlewood–Sobolev inequalities, and (2.9) is called a Gagliardo–Nirenberg inequality. Now, we present some new ideas proposed by Chen and Tang [25], in which normalized solutions of Schrödinger euqations are obtained under some new critical point theorems on a manifold. Let H and E denote real Hilbert space and real infinite dimensional Banach space, respectively. The scalar product and norm of H are denoted by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$. The norm of E is denoted by $\|\cdot\|_E$. We assume that the following continuous injections hold:

$$E \hookrightarrow H \hookrightarrow E^*, \tag{2.10}$$

where E^* denotes the dual space of E. From (2.9), we know that H is equivalent to its dual space. Define the trace of the unit sphere of H in E as the following:

$$\mathcal{M} := \{v \in E : \|v\|_H = 1\}. \tag{2.11}$$

It is known that $\mathcal{M}$ is unbounded and inherits the topology from E. It is easy to see that $\mathcal{M}$ is submanifold of E whose codimension is 1. For a given point $v \in \mathcal{M}$, the tangent space at this point is given as the following:

$$T_v\mathcal{M} := \{u \in E : (v,u)_H = 0\}, \tag{2.12}$$

which is a closed subspace of E and the codimension is 1.

Consider a C^1 functional $\psi : E \to \mathbb{R}$. Denote the trace of ψ on $\mathcal{M}$ by $\psi|_{\mathcal{M}}$. Then $\psi|_{\mathcal{M}}$ is of C^1, and the derivative of ψ is

$$\langle \psi|'_{\mathcal{M}}(v), u\rangle = \langle \psi'(v), u\rangle, \quad \forall\ v \in M,\ u \in T_vM. \tag{2.13}$$

The norm of $\{\psi|'_M(v)\}$ is defined as

$$\|\psi|'_{\mathcal{M}}(v)\| = \sup_{u\in T_v\mathcal{M}, \|u\|_E=1} |\langle \psi'(v), u\rangle|, \quad \forall\ v \in M. \tag{2.14}$$

For any $\delta > 0$ and any set $A \subset E$, denote

$$A_\delta := \{u \in E : \|v - u\| < \delta,\ \forall\ v \in A\}.$$

For any $b \in \mathbb{R}$ and any $\psi : E \to \mathbb{R}$, denote $\psi^b := \{v \in E : \psi(v) \le b\}$. The scalar product of $E \times \mathbb{R}$ is given as

$$\langle (v,s), (u,t)\rangle_{E\times\mathbb{R}} := (v,u)_H + st, \quad \forall\ (v,s), (u,t) \in E \times \mathbb{R}, \tag{2.15}$$

and the norm of $E \times \mathbb{R}$ is

$$\|(v,s)\|_{E\times\mathbb{R}} := \sqrt{\|v\|_H + s^2}, \quad \forall\ (v,s) \in E \times \mathbb{R}. \tag{2.16}$$

Let $\tilde{\psi} : E \times \mathbb{R} \to \mathbb{R}$ be a C^1 functional. Denote the trace of $\tilde{\psi}$ on $M \times \mathbb{R}$ by $\tilde{\psi}_{\mathcal{M}\times\mathbb{R}}$. Then, on $\mathcal{M} \times \mathbb{R}$, $\tilde{\psi}_{\mathcal{M}\times\mathbb{R}}$ is of class C^1 functional, and

$$\langle \tilde{\psi}'_{M\times\mathbb{R}}(v,s), (u,t)\rangle := \langle \tilde{\psi}(v,s), (u,t)\rangle, \quad \forall\ (v,s) \in \mathcal{M} \times \mathbb{R},\ (u,t) \in \tilde{T}_{(v,s)}(\mathcal{M} \times \mathbb{R}), \tag{2.17}$$

where

$$\widetilde{T}_{(v,s)}(\mathcal{M}\times\mathbb{R}) := \{(u,t)\in E\times\mathbb{R} : (v,u)_H = 0\}. \tag{2.18}$$

Define the norm of $\{\tilde{\psi}'_{\mathcal{M}\times\mathbb{R}}(v,s)\}$ by

$$\|\tilde{\psi}'_{\mathcal{M}\times\mathbb{R}}(v,s)\| = \sup_{(u,t)\in\tilde{T}_{(v,s)}(\mathcal{M}\times\mathbb{R}),\|(u,t)\|_{E\times\mathbb{R}}=1} |\langle\tilde{\psi}'_{\mathcal{M}\times\mathbb{R}}(v,s),(u,t)\rangle|, \quad \forall (v,s)\in M\times\mathbb{R}. \tag{2.19}$$

Lemma 2.4 [34]. *Let $\psi\in C^1(H^1_{rad}(\mathbb{R}^N),\mathbb{R})$ and $\{v_n\}\subset\mathcal{M}$ be a bounded sequence in $H^1_{rad}(\mathbb{R}^N)$. Then the following are equivalent:*
(i) $\|\psi|'_{\mathcal{M}}(v_n)\|\to 0$ *as* $n\to\infty$;
(ii) $\psi'(v_n)-\langle\psi'(v_n),v_n\rangle v_n$ *in* E^* *as* $n\to\infty$, *where* E^* *is the dual space of* $H^1_{rad}(\mathbb{R}^N)$.

Lemma 2.5 [30]. *Assume that $c\in\mathbb{R}$, $\bar{\Upsilon}\subset M\times\mathbb{R}$ is a closed set, and $\tilde{\psi}\in C^1(E\times\mathbb{R},\mathbb{R})$. Let*

$$\bar{\Gamma} := \{\bar{\gamma}\in C([0,1],M\times\mathbb{R}) : \bar{\gamma}(0)\in\bar{\Upsilon}, \tilde{\psi}(\bar{\gamma}(1))<c\}. \tag{2.20}$$

Suppose that $\tilde{\psi}$ satisfies

$$\sigma := \inf_{\bar{\gamma}\in\bar{\Gamma}}\max_{t\in[0,1]}\tilde{\psi}(\bar{\gamma}(t))\geq\varrho := \sup_{\bar{\gamma}\in\bar{\Gamma}}\{\tilde{\psi}(\bar{\gamma}(0)),\tilde{\psi}(\bar{\gamma}(1))\}. \tag{2.21}$$

Let $\{\bar{\gamma}_n\}\subset\bar{\Gamma}$ be such that

$$\sup_{t\in[0,1]}\tilde{\psi}(\bar{\gamma}(t))\leq\sigma+\frac{1}{n}, \quad \forall\, n\in\mathbb{N}. \tag{2.22}$$

Then there exists a sequence $\{(v_n,t_n)\}\subset M\times\mathbb{R}$ satisfying
(i) $\sigma-\frac{2}{n}\leq\tilde{\psi}(v_n,t_n)\leq\sigma+\frac{2}{n}$;
(ii) $\min_{t\in[0,1]}\|(v_n,t_n)-\bar{\gamma}_n(t)\|_{E\times\mathbb{R}}\leq\frac{2}{\sqrt{n}}$;
(iii) $\|\tilde{\psi}|'_{M\times\mathbb{R}}(v_n,t_n)\|\leq\frac{8}{\sqrt{n}}$.

In order to use Lemma 2.5, let $H=L^2(\mathbb{R}^N)$ and $E=H^1(\mathbb{R}^N)(H^1_{rad}(\mathbb{R}^N))$. The inner product of H and E are given as

$$(v,u)_H := \frac{1}{a}\int_{\mathbb{R}^N} uv\mathrm{d}x, \ (v,u)_E := \int_{\mathbb{R}^N}(\nabla v\cdot\nabla u+vu)\mathrm{d}x, \quad \forall\, v,u\in E, \tag{2.23}$$

and the norms of H and E are denoted as the following, respectively:

$$\|v\|_H := \frac{1}{\sqrt{a}}\left(\int_{\mathbb{R}^N} v^2\mathrm{d}x\right)^{\frac{1}{2}}, \ \|v\|_E := \left[\int_{\mathbb{R}^N}(|\nabla v|^2+|v|^2)\mathrm{d}x\right]^{\frac{1}{2}}, \quad \forall\, v\in E. \tag{2.24}$$

When identifying H with its dual space, the injections $E\hookrightarrow H\hookrightarrow E^*$ are continuous. Define a continuous map $\theta: H^1(\mathbb{R}^N)\times\mathbb{R}\to H^1(\mathbb{R}^N)$ by

$$\theta(u,s)(x) := e^{\frac{Nt}{2}}u(e^t x), \quad \forall\, u\in H^1(\mathbb{R}^N), \ x\in\mathbb{R}^N, \ t\in\mathbb{R}. \tag{2.25}$$

The following auxiliary functional is needed:

$$\begin{aligned}\tilde{I}(u,s)(x) : &= I(\theta(u,s))(x)\\ &= \frac{e^{2s}}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N} V(e^{-s}x)u^2\mathrm{d}x - \frac{1}{2e^{(N+\alpha)s}}\int_{\mathbb{R}^N}[I_\alpha * F(e^{\frac{Ns}{2}}u)]F(e^{\frac{Ns}{2}}u)\mathrm{d}x. \end{aligned} \tag{2.26}$$

We can see that $\tilde{I}$ is of class C^1, and

$$\begin{aligned}\langle \tilde{I}'(u,s),(w,t)\rangle &= \langle \tilde{I}'(u,s),(w,0)\rangle + \langle \tilde{I}'(u,s),(0,t)\rangle\\ &= e^{2s}\int_{\mathbb{R}^N}\nabla u\cdot\nabla w\mathrm{d}x + te^{2s}\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} V(e^{-s}x)uw\mathrm{d}x\\ &\quad + \frac{t}{2}\int_{\mathbb{R}^N}(\nabla V(e^{-s}x),e^{-s}x)u^2\mathrm{d}x - \frac{1}{e^{(N+\alpha)s}}\int_{\mathbb{R}^N}[I_\alpha * F(e^{\frac{Ns}{2}}u)]f(e^{\frac{Ns}{2}}u)e^{\frac{Ns}{2}}w\mathrm{d}x\\ &\quad - \frac{t}{2e^{(N+\alpha)s}}\int_{\mathbb{R}^N}[I_\alpha * F(e^{\frac{Ns}{2}}u)][(N+\alpha)F(e^{\frac{Ns}{2}}u) - Nue^{\frac{Ns}{2}}f(e^{\frac{Ns}{2}}u)]\mathrm{d}x\\ &= \langle \tilde{I}'(u,s),\theta(w,s)\rangle + tP(\theta(w,s)). \end{aligned} \tag{2.27}$$

Let

$$v(x) := \theta(u,s) = e^{\frac{Ns}{2}}u(e^s x),\quad \phi(x) := \theta(w,s) = e^{\frac{Ns}{2}}w(e^s x), \tag{2.28}$$

then

$$\begin{aligned}(v,\phi)_H &= \frac{1}{a}\int_{\mathbb{R}^N}\phi(x)v(x)\mathrm{d}x\\ &= \frac{1}{a}\int_{\mathbb{R}^N} e^{Ns}u(e^s x)w(e^s x)\mathrm{d}x\\ &= \frac{1}{a}\int_{\mathbb{R}^N} u(y)w(y)\mathrm{d}y\\ &= (u,w)_H. \end{aligned} \tag{2.29}$$

From (2.29), we know that

$$\phi\in T_v(S_a) \Leftrightarrow (w,s)\in \widetilde{T}_{u,s}(S_a\times\mathbb{R}),\quad \forall\, t,s\in\mathbb{R}. \tag{2.30}$$

From (2.27), (2.28), and (2.30), we have

$$\|P(v)\| = |\langle \tilde{I}'(u,s),(0,1)\rangle| \le \|\tilde{I}|'_{S_a\times\mathbb{R}}(u,s)\|, \tag{2.31}$$

and

$$\begin{aligned}\|\tilde{I}|'_{S_a}(v)\| &= \sup_{\phi\in T_v(S_a)}\frac{1}{\sqrt{\|\nabla\phi\|_2^2+\|\phi\|_2^2}}|\langle \tilde{I}'(u,s),(w,0)\rangle|\\ &= \sup_{(w,0)\in\widetilde{T}_{(u,s)}(S_a\times\mathbb{R})}\frac{1}{\sqrt{e^{2s}\|\nabla w\|_2^2+\|w\|_2^2}}|\langle \tilde{I}'(u,s),(w,0)\rangle| \end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{(w,0)\in\widetilde{T}_{(u,s)}(S_a\times\mathbb{R})} \frac{1}{\sqrt{e^{-2s}(\|\nabla w\|_2^2+\|w\|_2^2)}}|\langle\tilde{I}'(u,s),(w,0)\rangle| \\
&\leq \sup_{(w,0)\in\widetilde{T}_{(u,s)}(S_a\times\mathbb{R})} \frac{1}{\sqrt{e^{-2s}\|w\|_2^2}}|\langle\tilde{I}'(u,s),(w,0)\rangle| \\
&\leq \sup_{(w,0)\in\widetilde{T}_{(u,s)}(S_a\times\mathbb{R})} \frac{e^{|s|}}{\|w\|_2}|\langle\tilde{I}'(u,s),(w,0)\rangle| \\
&= \sup_{(w,0)\in\widetilde{T}_{(u,s)}(S_a\times\mathbb{R})} \frac{e^{|s|}}{\sqrt{a}\|(w,0)\|_{E\times\mathbb{R}}}|\langle\tilde{I}'(u,s),(w,0)\rangle| \\
&\leq \frac{e^{|s|}}{\sqrt{a}}\|\tilde{I}|'_{S_a\times\mathbb{R}}(u,s)\|. \qquad (2.32)
\end{aligned}
$$

Lemma 2.6. *Suppose that* (V1), (V2), *and* (S2) *hold. Then,*
(i) *there exists small enough* $\rho(a)>0$ *such that* $I(v)>0$ *if* $v\in A_{4\rho}$ *and*

$$0<\sup_{v\in A_\rho} I(v)<\varrho_a:=\inf\{I(v):v\in S_a,\|\nabla v\|_2^2=4\rho(a)\}, \qquad (2.33)$$

where

$$A_\rho=\{v\in S_a:\|\nabla v\|_2^2\leq\rho(a)\},\ \ A_{4\rho}=\{v\in S_a:\|\nabla v\|_2^2\leq 4\rho(a)\}; \qquad (2.34)$$

(ii) *there holds*

$$m(a):=\inf_{\gamma\in\Gamma_a}\max_{t\in[0,1]} I(\gamma(t))\geq\varrho_a>\sup_{\gamma\in\Gamma_a}\{I(\gamma(0)),I(\gamma(1))\}, \qquad (2.35)$$

where

$$\Gamma_a:=\{\gamma\in C([0,1],S_a):\|\nabla\gamma(0)\|_2^2\leq\rho(a),I(\gamma(1))<0\}. \qquad (2.36)$$

Proof. (i) By (S1), for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|f(t)|\leq\varepsilon|u|^{\frac{2+\alpha}{N}}+C_\varepsilon|u|^{\frac{\alpha+N}{N-2}-1},\ \ \forall\, t\in\mathbb{R}. \qquad (2.37)$$

From (2.37), we obtain that for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|F(t)|\leq\varepsilon|t|^{\frac{2+N+\alpha}{N}}+C_\varepsilon|t|^{\frac{\alpha+N}{N-2}},\ \ \forall\, t\in\mathbb{R}. \qquad (2.38)$$

By (2.38), Lemmas 2.1 and 2.3, one has

$$
\begin{aligned}
\left(\int_{\mathbb{R}^N}|F(v)|^{\frac{2N}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{N}} &\leq \left[\int_{\mathbb{R}^N}(\varepsilon|v|^{\frac{2+N+\alpha}{N}}+C_\varepsilon|v|^{\frac{\alpha+N}{N-2}})^{\frac{2N}{N+\alpha}}\mathrm{d}x\right]^{\frac{N+\alpha}{N}} \\
&\leq 4\varepsilon^2\left(\int_{\mathbb{R}^N}|v|^{\frac{2(2+N+\alpha)}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{N}}+4C_\varepsilon^2\left(\int_{\mathbb{R}^N}|v|^{\frac{2N}{N-2}}\mathrm{d}x\right)^{\frac{N+\alpha}{N}}
\end{aligned}
$$

$$
\begin{aligned}
&= 4\varepsilon^2\|v\|_{2(2+N+\alpha)/(N+\alpha)}^{\frac{2(2+N+\alpha)}{N}} + 4C_\varepsilon^2\|v\|_{2^*}^{\frac{2(N+\alpha)}{N-2}} \\
&\le 4\varepsilon^2 C_{N,2(2+\alpha+N)/(N+\alpha)}^{\frac{2(2+\alpha+N)}{N}} a^{\frac{2+\alpha}{N}}\|\nabla v\|_2^2 + 4C_\varepsilon^2 S^{-\frac{N+\alpha}{N-2}}\|\nabla v\|_2^{\frac{2(N+\alpha)}{N-2}}. \qquad (2.39)
\end{aligned}
$$

From (2.39), Lemmas 2.1, 2.2 and 2.3, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N}[I_\alpha * F(v)]F(v)\mathrm{d}x &= \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha\pi^{\frac{N}{2}}\Gamma(\frac{\alpha}{2})}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(v(x))F(v(y))}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \\
&\le C_0\|F(v)\|_{2N/(N+\alpha)}^2 \\
&\le 4C_0\varepsilon^2 C_{N,2(2+\alpha+N)/(N+\alpha)}^{\frac{2(2+\alpha+N)}{N}} a^{\frac{2+\alpha}{N}}\|\nabla v\|_2^2 + 4C_0C_\varepsilon^2 S^{-\frac{N+\alpha}{N-2}}\|\nabla v\|_2^{\frac{2(N+\alpha)}{N-2}}. \qquad (2.40)
\end{aligned}
$$

Let $\rho > 0$ be fixed and be arbitrary, and let $v, v_0, u \in S_a$ be such that

$$
\|\nabla v\|_2^2 \le \rho, \ \ \|\nabla v_0\|_2^2 = 4\rho, \ \ \|\nabla u\|_2^2 \le 4\rho. \qquad (2.41)
$$

Let $\varepsilon > 0$ be such that $4C_0\varepsilon^2 C_{N,2(2+\alpha+N)/(N+\alpha)}^{\frac{2(2+\alpha+N)}{N}} a^{\frac{2+\alpha}{N}} < \frac{1}{20}$ in (2.40). By (V1), (V2), and (2.40), we have

$$
\begin{aligned}
I(v) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}V(x)v^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v)]F(v)\mathrm{d}x \\
&\ge \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{4}\|\nabla v\|_2^2 - 2C_0\varepsilon^2 C_{N,2(2+\alpha+N)/(N+\alpha)}^{\frac{2(2+\alpha+N)}{N}} a^{\frac{2+\alpha}{N}}\|\nabla v\|_2^2 - 2C_0C_\varepsilon^2 S^{-\frac{N+\alpha}{N-2}}\|\nabla v\|_2^{\frac{2(N+\alpha)}{N-2}} \\
&\ge \frac{9}{40}\|\nabla v\|_2^2 - C_1\|\nabla v\|_2^{\frac{2(N+\alpha)}{N-2}}, \qquad (2.42)
\end{aligned}
$$

$$
\begin{aligned}
I(v) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}V(x)v^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v)]F(v)\mathrm{d}x \\
&\le \frac{1}{2}\|\nabla v\|_2^2 + \frac{\vartheta}{4}\|\nabla v\|_2^2 \le \frac{3}{4}\rho, \qquad (2.43)
\end{aligned}
$$

$$
\begin{aligned}
I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}[I_\alpha * F(u)]F(u)\mathrm{d}x \\
&\ge \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{4}\|\nabla u\|_2^2 - 2C_0\varepsilon^2 C_{N,2(2+\alpha+N)/(N+\alpha)}^{\frac{2(2+\alpha+N)}{N}} a^{\frac{2+\alpha}{N}}\|\nabla u\|_2^2 - 2C_0C_\varepsilon^2 S^{-\frac{N+\alpha}{N-2}}\|\nabla u\|_2^{\frac{2(N+\alpha)}{N-2}} \\
&\ge \frac{9}{40}\|\nabla u\|_2^2 - C_1\|\nabla u\|_2^{\frac{2(N+\alpha)}{N-2}}, \qquad (2.44)
\end{aligned}
$$

$$
\begin{aligned}
I(v_0) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_0|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}V(x)v_0^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v_0)]F(v_0)\mathrm{d}x \\
&\ge \frac{1}{2}\|\nabla v_0\|_2^2 - \frac{1}{4}\|\nabla v_0\|_2^2 - \int_{\mathbb{R}^N}[I_\alpha * F(v_0)]F(v_0)\mathrm{d}x \\
&\ge \frac{9}{40}\|\nabla v_0\|_2^2 - C_1\|\nabla v_0\|_2^{\frac{2(N+\alpha)}{N-2}} \\
&= \frac{9}{10}\rho - C_1(4\rho)^{\frac{\alpha+N}{N-2}} \ge \frac{33}{40}\rho. \qquad (2.45)
\end{aligned}
$$

From (2.42)–(2.45), there exists sufficiently small $\rho = \rho(a) > 0$ such that $I(v) > 0$ if $v \in A_{4\rho}$. Thus, (i) of Lemma 2.4 holds.

(ii) We need to prove that Γ_a is not empty. For $t > 0$ and any $v \in S_a$, we have

$$\|t^{\frac{N}{2}} v_t\|_2^2 = \|v\|_2^2, \ \ \|\nabla(t^{\frac{N}{2}} v_t)\|_2^2 = t^2 \|\nabla v\|_2^2. \tag{2.46}$$

From (2.46), we know that $t^{\frac{N}{2}} v_t \in S_a$. From (S1), one has

$$\begin{aligned}
&\liminf_{t\to+\infty} \frac{1}{t^{N+2+\alpha}} \int_{\mathbb{R}^N} [I_\alpha * F(t^{\frac{N}{2}} v)] F(t^{\frac{N}{2}} v) \mathrm{d}x \\
\geq\ & \frac{1}{t^{N+2+\alpha}} \int_{\mathbb{R}^N} \liminf_{t\to+\infty} [I_\alpha * F(t^{\frac{N}{2}} v)] F(t^{\frac{N}{2}} v) \mathrm{d}x \\
=\ & A_\alpha \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(y)|^{\frac{N+2+\alpha}{N}} |v(x)|^{\frac{N+2+\alpha}{N}}}{|x-y|^{N-\alpha}} \liminf_{t\to+\infty} \left[\frac{F(t^{\frac{N}{2}} v(y))}{|t^{\frac{N}{2}} v(y)|^{\frac{N+2+\alpha}{N}}} \times \frac{F(t^{\frac{N}{2}} v(x))}{|t^{\frac{N}{2}} v(x)|^{\frac{N+2+\alpha}{N}}} \right] \mathrm{d}y\mathrm{d}x \\
\geq\ & A_\alpha \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(y)|^{\frac{N+2+\alpha}{N}} |v(x)|^{\frac{N+2+\alpha}{N}}}{|x-y|^{N-\alpha}} \liminf_{t\to+\infty} \left[\frac{F(t^{\frac{N}{2}} v(y))}{|t^{\frac{N}{2}} v(y)|^{\frac{N+2+\alpha}{N}}} \right] \times \liminf_{t\to+\infty} \left[\frac{F(t^{\frac{N}{2}} v(x))}{|t^{\frac{N}{2}} v(x)|^{\frac{N+2+\alpha}{N}}} \right] \mathrm{d}y\mathrm{d}x \\
\geq\ & A_\alpha \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(y)|^{\frac{N+2+\alpha}{N}} |v(x)|^{\frac{N+2+\alpha}{N}}}{|x-y|^{N-\alpha}} (+\infty) \mathrm{d}y\mathrm{d}x \\
\geq\ & (+\infty) \int_{\mathbb{R}^N} [I_\alpha * |v|^{\frac{N+2+\alpha}{N}}] |v|^{\frac{N+2+\alpha}{N}} \mathrm{d}x \geq +\infty.
\end{aligned} \tag{2.47}$$

From (2.47), (V1), and (V2), we have

$$\begin{aligned}
I(t^{\frac{N}{2}} v_t) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla(t^{\frac{N}{2}} v_t)| \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^N} V(x) |t^{\frac{N}{2}} v_t|^2 \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^N} [I_\alpha * F(t^{\frac{N}{2}} v_t)] F(t^{\frac{N}{2}} v_t) \mathrm{d}x \\
&\leq \frac{t^2}{2} \|\nabla v\|_2^2 + \frac{\vartheta}{4} \|\nabla(t^{\frac{N}{2}} v_t)\|^2 \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^N} [I_\alpha * F(t^{\frac{N}{2}} v_t)] F(t^{\frac{N}{2}} v_t) \mathrm{d}x \\
&= \frac{t^2}{2} \|\nabla v\|_2^2 + \frac{\vartheta t^2}{4} \|\nabla v\|_2^2 - \frac{1}{2t^{N+\alpha}} \int_{\mathbb{R}^N} [I_\alpha * F(t^{\frac{N}{2}} v)] F(t^{\frac{N}{2}} v) \mathrm{d}x \\
&= \frac{t^2}{2} \left[(1 + \frac{\vartheta}{2}) \|\nabla v\|_2^2 - \frac{1}{t^{N+\alpha+2}} \int_{\mathbb{R}^N} [I_\alpha * F(t^{\frac{N}{2}} v)] F(t^{\frac{N}{2}} v) \mathrm{d}x \right] \\
&\to -\infty, \ \ t \to +\infty.
\end{aligned} \tag{2.48}$$

From (2.48), we can choose sufficiently small $t_1 > 0$ and sufficiently large $t_2 > 0$ such that

$$\|\nabla(t_1^{\frac{N}{2}} v_{t_1})\|_2^2 = t_1^2 \|\nabla v\|_2^2 \leq \rho(a), \ \ \|\nabla(t_2^{\frac{N}{2}} v_{t_2})\|_2^2 = t_2^2 \|\nabla v\|_2^2 \geq 4\rho(a), \ \ \text{and } I(t_2^{\frac{N}{2}} v_{t_2}) < 0.$$

Let $\gamma_0 := [t_1 + (t_2 - t_1)t]^{\frac{N}{2}} v_{t_1 + (t_2 - t_1)t}$. It is easy to see that $\gamma_0 \in \Gamma_a$, which shows that Γ_a is not empty. For any $\gamma \in \Gamma_a$, it follows from the intermediate value theorem that there exists $t_0 \in (0, 1)$, depending on γ such that $\|\nabla \gamma(t_0)\|_2^2 = 4\rho(a)$ and

$$\max_{t\in[0,1]} I(\gamma(t)) \geq I(\gamma(t_0)) \geq \inf\{I(v) : v \in S_a, \|\nabla v\|_2^2 = 4\rho(a)\}. \tag{2.49}$$

From (2.49) and the arbitrariness of $\gamma \in \Gamma_a$, one has

$$m(a) = \inf_{\gamma\in\Gamma_a} \max_{t\in[0,1]} I(\gamma(t)) \geq \inf\{I(v) : v \in S_a, \|\nabla v\|_2^2 = 4\rho(a)\}. \tag{2.50}$$

Therefore, it follows from the proof of (i) and (2.50) that (ii) of Lemma 2.6 holds. The proof is complete. □

Similar to [30], we have the following two lemmas:

Lemma 2.7. *Suppose that (S1), (V1), and (V2) hold, and there exist $\rho(a) > 0$ and $\varrho_a > 0$ such that (2.33) and (2.35) hold. Then, there exists a sequence $\{v_n\}$ of S_a satisfying*

$$I(v_n) \to m(a), \quad I|'_{S_a}(v_n) \to 0, \quad \text{and } P(v_n) \to 0. \tag{2.51}$$

Lemma 2.8. *Suppose that* (S1), (V1), and (V2) *hold, and there exists $\rho(a) > 0$ and $\varrho_a > 0$ such that (2.33) and (2.35) hold. Then, there exists a sequence $\{v_n\}$ of $S_a \cap H^1_{rad}(\mathbb{R}^N)$ satisfying*

$$I(v_n) \to m(a), \quad I|'_{S_a}(v_n) \to 0, \quad \text{and } P(v_n) \to 0. \tag{2.52}$$

Similar to [27], one has the following lemma:

Lemma 2.9 [27]. *Suppose that* (S1), (V1), and (V2) *hold. If there exists $v \in H^1(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$ such that*

$$-\Delta v - V(x)v - \lambda v = [I_\alpha * F(v)]f(v), \quad \text{in } \mathbb{R}^N, \tag{2.53}$$

then

$$P(v) = \|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} (\nabla V(x), x)v^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} [I_\alpha * F(v)][Nf(v)v - (N+\alpha)F(v)]\mathrm{d}x = 0. \tag{2.54}$$

Lemma 2.10. *Suppose that* (V1), (V2), and (S1)–(S3) *hold. Let $\{v_n\} \subset S_a \cap H^1_{rad}(\mathbb{R}^N)$ and satisfy (2.52), then $\{v_n\}$ is bounded in $H^1_{rad}(\mathbb{R}^N)$.*

Proof. We have $\|v_n\|_2^2 = a$ since $\{v_n\} \subset S_a \cap H^1_{rad}(\mathbb{R}^N)$. Then, to prove $\{v_n\}$ is bounded in $H^1_{rad}(\mathbb{R}^N)$, we just need to prove that $\{\|\nabla v_n\|_2\}$ is bounded. Suppose by contradiction that $\|\nabla v_n\|_2 \to +\infty$. By (1.5), (2.5), and (2.52), we have

$$m(a) + o_n(1) = \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N} V(x)v_n^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} [I_\alpha * F(v_n)]F(v_n)\mathrm{d}x, \tag{2.55}$$

and

$$\begin{aligned} o_n(1) = P(v_n) &= \|\nabla v_n\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} (\nabla V(x), x)v_n^2 \mathrm{d}x \\ &\quad - \frac{N}{2}\int_{\mathbb{R}^N} [I_\alpha * F(v_n)]\left[f(v_n)v_n - \frac{N+\alpha}{N}F(v_n)\right]\mathrm{d}x. \end{aligned} \tag{2.56}$$

By (V1), (2.55), and (2.56), we have

$$\begin{aligned} 2m(a) + o_n(1) &= 2I(v_n) - P(v_n) \\ &= -\frac{1}{2}\int_{\mathbb{R}^N} [2V(x) + (\nabla V(x), x)]v_n^2 \mathrm{d}x \\ &\quad + \frac{N}{2}\int_{\mathbb{R}^N} [I_\alpha * F(v_n)]\left[f(v_n)v_n - \frac{N+2+\alpha}{N}F(v_n)\right]\mathrm{d}x. \end{aligned} \tag{2.57}$$

From (S1), one has

$$\limsup_{|s|\to\infty} \frac{|f(s)s - \frac{N+\alpha}{N}F(s)|}{|s|^{\frac{N+\alpha}{N-2}}} = 0. \tag{2.58}$$

From (2.58), we have

$$\limsup_{|s|\to\infty} \frac{|f(s)s - \frac{N+\alpha}{N}F(s)|^{\frac{N-2}{2+\alpha}}}{|s|^{\frac{N+\alpha}{2+\alpha}}} = \limsup_{|s|\to\infty} \left(\frac{|f(s)s - \frac{N+\alpha}{N}F(s)|}{|s|^{\frac{N+\alpha}{N-2}}} \right)^{\frac{N-2}{2+\alpha}} = 0. \tag{2.59}$$

From (S1) and (S2), we have

$$\limsup_{|s|\to\infty} \frac{|f(s)s - \frac{N+\alpha}{N}F(s)|}{Nf(s)s - (2+N+\alpha)F(s)} = \frac{1}{N} \limsup_{|s|\to\infty} \left[\frac{2F(s)}{Nf(s)s - (2+N+\alpha)F(s)} + 1 \right] < +\infty. \tag{2.60}$$

It follows from (2.59) and (2.60) that

$$\begin{aligned} &\limsup_{|s|\to\infty} \frac{|f(s)s - \frac{N+\alpha}{N}F(s)|^{\frac{\alpha+N}{2+\alpha}}}{|s|^{\frac{\alpha+N}{2+\alpha}}[Nf(s)s - (2+N+\alpha)F(s)]} \\ = &\limsup_{|s|\to\infty} \left(\frac{|f(s)s - \frac{N+\alpha}{N}F(s)|^{\frac{N-2}{2+\alpha}}}{|s|^{\frac{\alpha+N}{2+\alpha}}} \cdot \frac{|f(s)s - \frac{N+\alpha}{N}F(s)|}{[Nf(s)s - (2+N+\alpha)F(s)]} \right) = 0. \end{aligned} \tag{2.61}$$

From (S1), (S2), and (2.61), for some positive constants $C_2 > 0$ and $R > 0$, we get

$$\left| \frac{f(s)s - \frac{N+\alpha}{N}F(s)}{s} \right|^{\frac{\alpha+N}{\alpha+2}} \le C_2[Nf(s)s - (2+N+\alpha)F(s)], \quad \forall\, |s| \ge R, \tag{2.62}$$

and

$$|F(s)| \le C_2[Nf(s)s - (2+N+\alpha)F(s)], \quad \forall\, |s| \ge R. \tag{2.63}$$

Let $\Lambda_n := \{x \in \mathbb{R}^N : |v_n(x)| \le R\}$. From (S1) and the Hardy–Little–Hood inequality, we have

$$\begin{aligned} & A_\alpha \int_{\Lambda_n}\int_{\Lambda_n} \frac{F(v_n(x))}{|x-y|^{N-\alpha}} F(v_n(y))\mathrm{d}x\mathrm{d}y \\ = & \int_{\Lambda_n} [I_\alpha * F(v_n)]F(v_n)\mathrm{d}x \\ \le & C_3 \left(\int_{\Lambda_n} |F(v_n)|^{\frac{2N}{N+\alpha}}\mathrm{d}x \right)^{\frac{N+\alpha}{N}} \\ \le & \left[\int_{\Lambda_n} (C_4|v_n|^{\frac{N+2+\alpha}{N}} + C_5|v_n|^{\frac{N+\alpha}{N-2}})^{\frac{2N}{N+\alpha}}\mathrm{d}x \right]^{\frac{N+\alpha}{N}} \\ \le & C_6 \left(\int_{\Lambda_n} |v_n|^2 R^{\frac{4}{N+\alpha}}\mathrm{d}x + \int_{\Lambda_n} |v_n|^2 R^{\frac{4}{N-2}}\mathrm{d}x \right)^{\frac{N+\alpha}{N}} \end{aligned}$$

$$\leq C_7\left(\int_{\Lambda_n}|v_n|^2\mathrm{d}x\right)^{\frac{N+\alpha}{N}}$$
$$\leq C_7 a^{\frac{N+\alpha}{N}}. \tag{2.64}$$

By (S1), (2.57), (2.62), and (2.64), there exists $C_8 > 0$ such that

$$\begin{aligned}
&\frac{1}{\|\nabla v_n\|_2^2}\int_{\Lambda_n}[I_\alpha * F(v_n)]\left[f(v_n)v_n - \frac{N+\alpha}{N}F(v_n)\right]\mathrm{d}x\\
\leq\ & \frac{1}{\|\nabla v_n\|_2^2}\int_{\Lambda_n}[I_\alpha * F(v_n)]f(v_n)v_n\mathrm{d}x\\
=\ & \frac{A_\alpha}{\|\nabla v_n\|_2^2}\int_{\Lambda_n}\int_{\mathbb{R}^N}\frac{F(v_n(y))}{|x-y|^{N-\alpha}}f(v_n(x))v_n(x)\mathrm{d}y\mathrm{d}x\\
<\ & \frac{(N+\alpha)A_\alpha}{(N-2)\|\nabla v_n\|_2^2}\int_{\Lambda_n}\int_{\mathbb{R}^N}\frac{F(v_n(y))}{|x-y|^{N-\alpha}}F(v_n(x))\mathrm{d}y\mathrm{d}x\\
=\ & \frac{(N+\alpha)A_\alpha}{(N-2)\|\nabla v_n\|_2^2}\int_{\mathbb{R}^N}\int_{\Lambda_n}\frac{F(v_n(y))}{|x-y|^{N-\alpha}}F(v_n(x))\mathrm{d}x\mathrm{d}y\\
=\ & \frac{(N+\alpha)A_\alpha}{(N-2)\|\nabla v_n\|_2^2}\int_{\Lambda_n}\int_{\Lambda_n}\frac{F(v_n(y))}{|x-y|^{N-\alpha}}F(v_n(x))\mathrm{d}x\mathrm{d}y\\
&+\frac{(N+\alpha)A_\alpha}{(N-2)\|\nabla v_n\|_2^2}\int_{\mathbb{R}^N\setminus\Lambda_n}\int_{\Lambda_n}\frac{F(v_n(y))}{|x-y|^{N-\alpha}}F(v_n(x))\mathrm{d}x\mathrm{d}y\\
\leq\ & \frac{C_2(N+\alpha)A_\alpha}{(N-2)\|\nabla v_n\|_2^2}\int_{\mathbb{R}^N\setminus\Lambda_n}\int_{\Lambda_n}\frac{F(v_n(y))}{|x-y|^{N-\alpha}}[Nf(v_n(x))v_n(x)-(N+2+\alpha)F(v_n(x))]\mathrm{d}x\mathrm{d}y\\
&+\frac{(N+\alpha)C_8 a^{\frac{N+\alpha}{N}}}{(N-2)\|\nabla v_n\|_2^2}\\
\leq\ & \frac{C_2(N+\alpha)}{(N-2)\|\nabla v_n\|_2^2}\int_{\mathbb{R}^N\setminus\Lambda_n}[I_\alpha * F(v_n)][Nf(v_n)v_n-(N+2+\alpha)F(v_n)]\mathrm{d}y + o_n(1)\\
\leq\ & \frac{C_2(N+\alpha)(2m(a)+o_n(1))}{(N-2)\|\nabla v_n\|_2^2} + o_n(1) = o_n(1).
\end{aligned} \tag{2.65}$$

From (S1), for any $\varepsilon > 0$, there exists $C'_\varepsilon > 0$ such that

$$|F(s)| \leq C'_\varepsilon|s|^{\frac{N+2+\alpha}{N}} + \varepsilon|s|^{\frac{N+\alpha}{N-2}},\ \ \forall\ s\in\mathbb{R}. \tag{2.66}$$

From the Hölder inequlity, (2.57), (2.62), and (2.66), we have

$$\begin{aligned}
&\frac{1}{\|\nabla v_n\|_2^2}\int_{\mathbb{R}^N\setminus\Lambda_n}[I_\alpha * F(v_n)]\left[f(v_n)v_n - \frac{N+\alpha}{N}F(v_n)\right]\mathrm{d}x\\
\leq\ & \frac{1}{\|\nabla v_n\|_2^2}\left(\int_{\mathbb{R}^N\setminus\Lambda_n}|[I_\alpha * F(v_n)]\|v_n|^{\frac{\alpha+N}{N-2}}\mathrm{d}x\right)^{\frac{N-2}{\alpha+N}}\left(\int_{\mathbb{R}^N\setminus\Lambda_n}\left|\frac{f(v_n)v_n-\frac{N+\alpha}{N}F(v_n)}{v_n}\right|^{\frac{N+\alpha}{2+\alpha}}\mathrm{d}x\right)^{\frac{\alpha+2}{\alpha+N}}\\
\leq\ & \frac{1}{\|\nabla v_n\|_2^2}\left(\int_{\mathbb{R}^N\setminus\Lambda_n}C_2[Nf(v_n)v_n-(2+N+\alpha)F(v_n)]\mathrm{d}x\right)^{\frac{\alpha+2}{N+\alpha}}\left(\int_{\mathbb{R}^N\setminus\Lambda_n}|I_\alpha * F(v_n)\|v_n|^{\frac{N+\alpha}{N-2}}\mathrm{d}x\right)^{\frac{N-2}{N+\alpha}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{[2C_2(2m(a)+1)]^{\frac{\alpha+2}{\alpha+N}}}{\|\nabla v_n\|_2^2}\left[\int_{\mathbb{R}^N}(C'_\varepsilon|v_n|^{\frac{N+2+\alpha}{N}}+\varepsilon|v_n|^{\frac{N+\alpha}{N-2}})^{\frac{2N}{N+\alpha}}\mathrm{d}x\right]^{\frac{N-2}{2N}}\left(\int_{\mathbb{R}^N}|v_n|^{\frac{2N}{N-2}}\mathrm{d}x\right)^{\frac{N-2}{2N}}\\
&\leq \frac{C_9[2C_2(2m(a)+1)]^{\frac{\alpha+2}{\alpha+N}}}{\|\nabla v_n\|_2^2}((C'_\varepsilon)^{\frac{N-2}{N+\alpha}}\|\nabla v_n\|_2^{\frac{N-2}{N+\alpha}}+\varepsilon^{\frac{N-2}{N+\alpha}}\|\nabla v_n\|_2)\|\nabla v_n\|_2\\
&= C_9(C'_\varepsilon)^{\frac{N-2}{N+\alpha}}[2C_2(2m(a)+1)]^{\frac{\alpha+2}{\alpha+N}}\|\nabla v_n\|_2^{-\frac{2+\alpha}{N+\alpha}}+C_9[2C_2(2m(a)+1)]^{\frac{\alpha+2}{\alpha+N}}\varepsilon^{\frac{N-2}{N+\alpha}}\\
&= o_n(1)+C_9[2C_2(2m(a)+1)]^{\frac{\alpha+2}{\alpha+N}}\varepsilon^{\frac{N-2}{N+\alpha}}. \qquad (2.67)
\end{aligned}
$$

From (V1), (V2), (2.56), (2.65), and (2.67), we have

$$
\begin{aligned}
o_n(1)+\frac{2}{N} &= \frac{1}{\|\nabla v_n\|_2^2}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]\left[f(v_n)v_n-\frac{N+\alpha}{N}F(v_n)\right]\mathrm{d}x\\
&\quad -\frac{1}{N\|\nabla v_n\|_2^2}\int_{\mathbb{R}^N}(\nabla V(x),x)|v_n|^2\mathrm{d}x\\
&\leq \frac{\vartheta}{N}+\frac{1}{\|\nabla v_n\|_2^2}\int_{\Lambda_n}[I_\alpha * F(v_n)]\left[f(v_n)v_n-\frac{N+\alpha}{N}F(v_n)\right]\mathrm{d}x\\
&\quad +\frac{1}{\|\nabla v_n\|_2^2}\int_{\mathbb{R}^N\setminus\Lambda_n}[I_\alpha * F(v_n)]\left[f(v_n)v_n-\frac{N+\alpha}{N}F(v_n)\right]\mathrm{d}x\\
&= \frac{\vartheta}{N}+o_n(1)+C_9[2C_2(2m(a)+1)]^{\frac{\alpha+2}{\alpha+N}}\varepsilon^{\frac{N-2}{N+\alpha}}. \qquad (2.68)
\end{aligned}
$$

It is easy to see (2.68) is a contradiction since ε is arbitrary. Therefore, $\{v_n\}$ is bounded in $H^1_{rad}(\mathbb{R}^N)$. □

Lemma 2.11. *Suppose that $N \geq 4$ and* (V1), (V2), (S1), (S2), and (S3') *hold. Let $\{v_n\} \subset S_a \cap H^1_{rad}(\mathbb{R}^N)$ and satisfy (2.52); then, $\{v_n\}$ is bounded in $H^1_{rad}(\mathbb{R}^N)$.*

Proof. Let $\{v_n\} \subset S_a \cap H^1_{rad}(\mathbb{R}^N)$. Then $\|v_n\|_2^2 = a$. So, we only need to prove that $\{\|\nabla v_n\|_2\}$ is bounded. By (1.5), (2.5), and (2.53), we have

$$
m(a)+o_n(1)=\frac{1}{2}\|\nabla v_n\|_2^2-\frac{1}{2}\int_{\mathbb{R}^N}V(x)v_n^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]F(v_n)\mathrm{d}x \qquad (2.69)
$$

and

$$
\begin{aligned}
o_n(1)=P(v_n) &= -\frac{N}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]\left[f(v_n)v_n-\frac{N+\alpha}{N}F(v_n)\right]\mathrm{d}x\\
&\quad +\|\nabla v_n\|_2^2+\frac{1}{2}\int_{\mathbb{R}^N}(\nabla V(x),x)v_n^2\mathrm{d}x. \qquad (2.70)
\end{aligned}
$$

From (V1), (2.69), and (2.70), we have

$$
\begin{aligned}
2m(a)+o_n(1) &= 2I(v_n)-P(v_n)\\
&= \frac{N}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]\left[f(v_n)v_n-\frac{N+2+\alpha}{N}F(v_n)\right]\mathrm{d}x\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^N}[2V(x)+(\nabla V(x),x)]v_n^2\mathrm{d}x
\end{aligned}
$$

$$= \frac{N}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]\left[f(v_n)v_n - \frac{N+2+\alpha}{N}F(v_n)\right]\mathrm{d}x. \tag{2.71}$$

By (V2), (S1), (S2), Lemmas 2.1–2.3, and (2.70), we get

$$\begin{aligned}
\|\nabla v_n\|_2^2 &= -\frac{1}{2}\int_{\mathbb{R}^N}(\nabla V(x),x)v_n^2\mathrm{d}x + \frac{N}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]\left[f(v_n)v_n - \frac{N+\alpha}{N}F(v_n)\right]\mathrm{d}x\\
&\le C_8\left(\int_{\mathbb{R}^N}|F(v_n)|^{\frac{2N}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{2N}}\left(\int_{\mathbb{R}^N}|f(v_n)v_n|^{\frac{2N}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{2N}}\\
&\quad +\frac{\vartheta}{2}\|\nabla v_n\|_2^2 + C_8\left(\int_{\mathbb{R}^N}|F(v_n)|^{\frac{2N}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{N}}\\
&\le \frac{\vartheta}{2}\|\nabla v_n\|_2^2 + C_9\left(\int_{\mathbb{R}^N}|F(v_n)|^{\frac{2N}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{N}}\\
&\le \frac{\vartheta}{2}\|\nabla v_n\|_2^2 + C_9\left[\int_{\mathbb{R}^N}(\varepsilon|v_n|^{\frac{N+2+\alpha}{N}} + C_\varepsilon|v_n|^{\frac{N+\alpha}{N-2}})\mathrm{d}x\right]^{\frac{N+\alpha}{N}}\\
&\le \frac{\vartheta}{2}\|\nabla v_n\|_2^2 + C_{10}\varepsilon^2\left(\int_{\mathbb{R}^N}|v_n|^{\frac{2(N+2+\alpha)}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{N}} + C_{10}C_\varepsilon^2\left(\int_{\mathbb{R}^N}|v_n|^{\frac{2N}{N-2}}\mathrm{d}x\right)^{\frac{N+\alpha}{N}}\\
&= \frac{\vartheta}{2}\|\nabla v_n\|_2^2 + C_{10}\|\nabla v_n\|_{\frac{2(N+2+\alpha)}{N+\alpha}}^{\frac{N+\alpha}{N}} + C_{11}C_\varepsilon^2\|\nabla v_n\|_{2^*}^{22^*}\\
&\le \frac{\vartheta}{2}\|\nabla v_n\|_2^2 + C_{10}\varepsilon^2 C_{\frac{2(N+2+\alpha)}{N+\alpha}}^{\frac{2(N+2+\alpha)}{N}}\|v_n\|_2^{\frac{2(2+\alpha)}{N}}\|\nabla v_n\|_2^2 + C_{11}C_\varepsilon^2 S^{-\frac{N+\alpha}{N-2}}\|\nabla v_n\|_2^{\frac{2(N+\alpha)}{N-2}}\\
&= \frac{\vartheta}{2}\|\nabla v_n\|_2^2 + C_{12}\varepsilon^2\|\nabla v_n\|_2^2 + C_{13}C_\varepsilon^2\|\nabla v_n\|_2^{\frac{2(N+\alpha)}{N-2}}.
\end{aligned} \tag{2.72}$$

Since $\vartheta \in (0,1)$, ε is arbitrary, from (2.72), the Hölder inequality, Lemmas 2.1 and 2.3, we get that $\|\nabla v_n\|_2^2 \ge C_{14}$. Set $k' = \frac{k}{k-1}$, then $\frac{N+2}{N} < k' < \frac{N+\alpha}{N-2}$. From (V2), (S1), (S2), (S3'), (2.38), (2.71), (2.70), and the Hölder inequality, we have

$$\begin{aligned}
\frac{2}{N} &= -\frac{1}{N\|\nabla v_n\|_2^2}\int_{\mathbb{R}^N}(\nabla V(x),x)|v_n|^2\mathrm{d}x\\
&\quad +\frac{1}{\|\nabla v_n\|_2^2}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]\left[f(v_n)v_n - \frac{N+\alpha}{N}F(v_n)\right]\mathrm{d}x\\
&\le \frac{\vartheta}{N} + \frac{1}{\|\nabla v_n\|_2^2}\left(\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]\left(\frac{|f(v_n)v_n - \frac{N+\alpha}{N}F(v_n)|}{v_n}\right)^k\mathrm{d}x\right)^{\frac{1}{k}}\\
&\quad \times\left(\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]|v_n|^{k'}\mathrm{d}x\right)^{\frac{1}{k'}}\\
&\le \frac{\vartheta}{N} + \frac{1}{\|\nabla v_n\|_2^2}\left(\int_{\mathbb{R}^N}[C_*[I_\alpha * F(v_n)][Nf(v_n)v_n - (2+N+\alpha)F(v_n)]\mathrm{d}x\right)^{\frac{1}{k}}\\
&\quad \times\left(\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]|v_n|^{k'}\mathrm{d}x\right)^{\frac{1}{k'}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\vartheta}{N}+\frac{C_{15}C_*^{\frac{1}{k}}(4m(a)+o_n(1))^{\frac{1}{k}}}{\|\nabla v_n\|_2^2}\left(\int_{\mathbb{R}^N}|F(v_n)|^{\frac{2N}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{2Nk'}}\left(\int_{\mathbb{R}^N}|v_n|^{\frac{2Nk'}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{2Nk'}}\\
&\leq \frac{\vartheta}{N}+\frac{C_{15}C_*^{\frac{1}{k}}(4m(a)+o_n(1))^{\frac{1}{k}}}{\|\nabla v_n\|_2^2}\|\nabla v_n\|_2^{\frac{Nk'-N-\alpha}{2k'}}\|v_n\|_2^{\frac{N+\alpha-(N-2)k'}{2k'}}\\
&\quad\times\left\{\int_{\mathbb{R}^N}[C_\varepsilon|v_n|^{\frac{2+N+\alpha}{N}}+\varepsilon|v_n|^{\frac{N+\alpha}{N-2}}]^{\frac{2N}{N+\alpha}}\mathrm{d}x\right\}^{\frac{N+\alpha}{2Nk'}}\\
&\leq \frac{\vartheta}{N}+\frac{C_{16}C_*^{\frac{1}{k}}(4m(a)+o_n(1))^{\frac{1}{k}}\|\nabla v_n\|_2^{\frac{Nk'-N-\alpha}{2k'}}}{\|\nabla v_n\|_2^2}\\
&\quad\times\left\{C_\varepsilon^{\frac{2N}{N+\alpha}}\int_{\mathbb{R}^N}|v_n|^{\frac{2(2+N+\alpha)}{N+\alpha}}\mathrm{d}x+\varepsilon^{\frac{2N}{N+\alpha}}\int_{\mathbb{R}^N}|v_n|^{\frac{2N}{N-2}}\mathrm{d}x\right\}^{\frac{N+\alpha}{2Nk'}}\\
&\leq \frac{\vartheta}{N}+\frac{C_{17}C_*^{\frac{1}{k}}(4m(a)+o_n(1))^{\frac{1}{k}}\|\nabla v_n\|_2^{\frac{Nk'-N-\alpha}{2k'}}}{\|\nabla v_n\|_2^2}\\
&\quad\times(C_\varepsilon^{-\frac{2N}{N+\alpha}}\|v_n\|_2^{\frac{2(2+\alpha)}{N+\alpha}}\|\nabla v_n\|_2^{\frac{2N}{N+\alpha}}+\varepsilon^{\frac{2N}{N+\alpha}}\|\nabla v_n\|_2^{\frac{2N}{N-2}})^{\frac{N+\alpha}{2Nk'}}\\
&\leq \frac{\vartheta}{N}+\frac{C_{18}C_*^{\frac{1}{k}}(4m(a)+o_n(1))^{\frac{1}{k}}\|\nabla v_n\|_2^{\frac{Nk'-N-\alpha}{2k'}}}{\|\nabla v_n\|_2^2}\times(C_\varepsilon^{-\frac{1}{k'}}\|\nabla v_n\|_2^{\frac{1}{k'}}+\varepsilon^{\frac{1}{k'}}\|\nabla v_n\|_2^{\frac{N+\alpha}{(N-2)k'}})\\
&= \frac{\vartheta}{N}+C_{18}C_*^{\frac{1}{k}}(4m(a)+o_n(1))^{\frac{1}{k}}(C_\varepsilon^{-\frac{1}{k'}}\|\nabla v_n\|_2^{\frac{1}{k'}+\frac{Nk'-N-\alpha}{2k'}-2}+\varepsilon^{\frac{1}{k'}}\|\nabla v_n\|_2^{\frac{N+\alpha}{(N-2)k'}+\frac{Nk'-N-\alpha}{2k'}-2})\\
&= \frac{\vartheta}{N}+o_n(1). \qquad (2.73)
\end{aligned}
$$

From (2.73), we know that $\{\|\nabla v_n\|_2\}$ is bounded. Hence, $\{v_n\}$ is bounded in $H^1_{rad}(\mathbb{R}^N)$. □

3. Proof of main results

Proof of Theorem 1.1. By Lemmas 2.8–2.10, we know that there exists a sequence $\{v_n\}\subset S_a\cap H^1_{rad}(\mathbb{R}^N)$ satisfying (2.52) and $\|v_n\|\leq C_{19}$ for some constant $C_{19}>0$. From Lemma 2.4, we have

$$
\|v_n\|_2^2=a,\ \ I(v_n)\to m(a),\ \ I'(v_n)-\lambda_n v_n\to 0, \qquad (3.1)
$$

where

$$
\begin{aligned}
\lambda_n=\frac{\langle I'(v_n),v_n\rangle}{\|v_n\|_2^2} &= \frac{1}{a}\left[\|\nabla v_n\|_2^2-\int_{\mathbb{R}^N}V(x)v_n^2\mathrm{d}x-\int_{\mathbb{R}^N}[I_\alpha*F(v_n)]f(v_n)v_n\mathrm{d}x\right]\\
&\leq \frac{1}{a}\left(\|\nabla v_n\|_2^2+\frac{\vartheta}{2}\|\nabla v_n\|_2^2\right). \qquad (3.2)
\end{aligned}
$$

From the boundedness of $\{\|\nabla v_n\|_2\}$, (S1), (S2), (V1), (V2), and (3.2), we have

$$
|\lambda_n| \leq \frac{1}{a}\left[\|\nabla v_n\|_2^2+\vartheta\|\nabla v_n\|_2^2+C_0\left(\int_{\mathbb{R}^N}|F(v_n)|^{\frac{2N}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{2N}}\left(\int_{\mathbb{R}^N}|f(v_n)v_n|^{\frac{2N}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{2N}}\right]
$$

$$
\begin{aligned}
&\leq \frac{1+\vartheta}{a}\|\nabla v_n\|_2^2 + \frac{C_0C_{20}}{a}\left(\int_{\mathbb{R}^N}|F(v_n)|^{\frac{2N}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{N}}\\
&\leq \frac{1+\vartheta}{a}\|\nabla v_n\|_2^2 + \frac{C_0C_{20}}{a}\left(\int_{\mathbb{R}^N}(C_{21}|v_n|^{\frac{2+\alpha+N}{N}} + C_{22}|v_n|^{\frac{\alpha+N}{N+2}})^{\frac{2N}{N-\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{N}}\\
&\leq \frac{1+\vartheta}{a}\|\nabla v_n\|_2^2 + \frac{C_0C_{21}}{a}(C_{23}\|\nabla v_n\|_2^2 + C_{24}\|\nabla v_n\|_2^{\frac{2(\alpha+N)}{N-2}})\\
&\leq C_{25}. \qquad (3.3)
\end{aligned}
$$

Hence, passing to a subsequence if necessary, we can assume that $\lambda_n \to \lambda_a$ on $\mathbb{R}$, $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^p(\mathbb{R}^N)$ for $p \in (2, 2^*)$, and $v_n \to v$ a.e. on $\mathbb{R}^N$. From (V1) and (V2), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)v_n^2\mathrm{d}x = \int_{\mathbb{R}^N} V(x)v^2\mathrm{d}x, \qquad (3.4)$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)v_nu\mathrm{d}x = \int_{\mathbb{R}^N} V(x)uv\mathrm{d}x, \quad \forall\ u \in H^1_{rad}(\mathbb{R}^N), \qquad (3.5)$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} (\nabla V(x), x)v_n^2\mathrm{d}x = \int_{\mathbb{R}^N} (\nabla V(x), x)v^2\mathrm{d}x. \qquad (3.6)$$

From (S1) and (S2), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} [I_\alpha * F(v_n)]f(v_n)u\mathrm{d}x = \int_{\mathbb{R}^N} [I_\alpha * F(v)]f(v)u\mathrm{d}x, \quad \forall\ u \in H^1_{rad}(\mathbb{R}^N), \qquad (3.7)$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} [I_\alpha * F(v_n)]f(v_n)v_n\mathrm{d}x = \int_{\mathbb{R}^N} [I_\alpha * F(v)]f(v)v\mathrm{d}x, \qquad (3.8)$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} [I_\alpha * F(v_n)]F(v_n)\mathrm{d}x = \int_{\mathbb{R}^N} [I_\alpha * F(v)]F(v)\mathrm{d}x. \qquad (3.9)$$

Hence, it follows from (V1), (V2), (2.55), (2.56), (3.8), and (3.9) that

$$
\begin{aligned}
2m(a) &= \lim_{n\to\infty}\left\{-\frac{1}{2}\int_{\mathbb{R}^N}[2V(x) + (\nabla V(x), x)]v_n^2\mathrm{d}x\right.\\
&\quad \left. + \frac{N}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]\left[f(v_n)v_n - \frac{2+\alpha+N}{N}F(v_n)\right]\mathrm{d}x\right\}\\
&= \lim_{n\to\infty}\frac{N}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]\left[f(v_n)v_n - \frac{2+\alpha+N}{N}F(v_n)\right]\mathrm{d}x\\
&= \frac{N}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v)]\left[f(v)v - \frac{2+\alpha+N}{N}F(v)\right]\mathrm{d}x. \qquad (3.10)
\end{aligned}
$$

From (S2) and (3.10), we know that $v \neq 0$. For any $u \in E$, from (3.1), (3.2), $\lambda_n \to \lambda_a$ on $\mathbb{R}$, and $v_n \rightharpoonup v$ in E, we have

$$\begin{aligned}
& |\langle I'(v_n) - \lambda_a v_n, u\rangle - \langle I'(v) - \lambda_a v, u\rangle| \\
\leq\ & |(v_n - v, u)_E| + \int_{\mathbb{R}^N} V(x)|(v_n - v)u|\mathrm{d}x + \left|\int_{\mathbb{R}^N} [I_\alpha * F(v)]f(v)u - \int_{\mathbb{R}^N} [I_\alpha * F(v_n)]f(v_n)u\right| \\
& + \left|(1+\lambda_a)\int_{\mathbb{R}^N} vu\mathrm{d}x - (1+\lambda_n)\int_{\mathbb{R}^N} v_n u\mathrm{d}x\right| \\
\leq\ & \left|(1+\lambda_a)\int_{\mathbb{R}^N} (v - v_n)u\mathrm{d}x\right| + \left|(\lambda_a - \lambda_n)\int_{\mathbb{R}^N} v_n u\mathrm{d}x\right| + o_n(1) \\
=\ & o_n(1). \qquad (3.11)
\end{aligned}$$

It follows from (3.1) and (3.11) that

$$I'(v) - \lambda_a v = 0. \qquad (3.12)$$

Hence, from (3.12), we have

$$\langle I'(v) - \lambda_a v, v\rangle = \|\nabla v\|_2^2 - \int_{\mathbb{R}^N} V(x)v^2\mathrm{d}x - \lambda_a\|v\|_2^2 - \int_{\mathbb{R}^N} [I_\alpha * F(v)]f(v)v\mathrm{d}x = 0, \qquad (3.13)$$

and

$$P(v) = \|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} (\nabla V(x), x)v^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} [I_\alpha * F(v)][Nf(v)v - (N+\alpha)F(v)]\mathrm{d}x = 0. \qquad (3.14)$$

By (V1), (S2), (3.13), and (3.14), we get

$$\begin{aligned}
-\lambda_a\|v\|_2^2 &= \frac{1}{2}\int_{\mathbb{R}^N} [2V(x) + (\nabla V(x), x)]|v|^2\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} [I_\alpha * F(v)][(N+\alpha)F(v) - (N-2)f(v)v]\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^N} [I_\alpha * F(v)][(N+\alpha)F(v) - (N-2)f(v)v]\mathrm{d}x. \qquad (3.15)
\end{aligned}$$

From (3.15), we know that $\lambda_a < 0$. By (2.1), (3.1), (3.7), (3.9), and (3.12), we have

$$\begin{aligned}
0 &= \lim_{n\to\infty}[\langle I'(v_n) - \lambda_a v_n, v_n - v\rangle - \langle I'(v) - \lambda_a v, v_n - v\rangle] \\
&= \lim_{n\to\infty}\left\{\int_{\mathbb{R}^N} [|\nabla(v_n - v)|^2 - V(x)|v_n - v|^2 - \lambda_a|v_n - v|^2]\mathrm{d}x\right. \\
&\quad + \int_{\mathbb{R}^N} [I_\alpha * F(v_n)]f(v_n)v\mathrm{d}x - \int_{\mathbb{R}^N} [I_\alpha * F(v_n)]f(v_n)v_n\mathrm{d}x \\
&\quad \left. + \int_{\mathbb{R}^N} [I_\alpha * F(v)]f(v)v_n\mathrm{d}x - \int_{\mathbb{R}^N} [I_\alpha * F(v)]f(v)v\mathrm{d}x\right\} \\
&= \lim_{n\to\infty}\left\{\int_{\mathbb{R}^N} [|\nabla(v_n - v)|^2 - \lambda_a|v_n - v|^2 - V(x)|v_n - v|^2]\mathrm{d}x\right.
\end{aligned}$$

$$+\int_{\mathbb{R}^N}[I_\alpha * F(v)]f(v)(v_n - v)\mathrm{d}x\Big\}$$
$$= \lim_{n\to\infty}\left\{\int_{\mathbb{R}^N}[|\nabla(v_n - v)|^2 - \lambda_a|v_n - v|^2 - V(x)|v_n - v|^2]\mathrm{d}x\right\}. \tag{3.16}$$

From (3.16), we know that $v_n \to v$ in E; hence, $\|v\|_2^2 = a$, $I(v) = m(a)$ and $I'_{S_a}(v) = 0$. □

Proof of Theorem 1.2. We should prove that problem (1.1) has a ground state solution in $S_a \cap H^1_{rad}(\mathbb{R}^N)$. In order to do this, let

$$\Upsilon_a := \{v \in S_a \cap H^1_{rad}(\mathbb{R}^N) : I'_{S_a}(v) = 0\}, \quad \tilde{m}(a) := \inf_{\Upsilon_a} I. \tag{3.17}$$

We first prove that Υ_a is not empty. From Lemmas 2.8 and 2.11, we know that there exists a sequence $\{v_n\} \subset S_a \cap H^1_{rad}(\mathbb{R}^N)$ satisfying (2.52) and $\|v_n\| \le C_{26}$ for some constant $C_{26} > 0$. Similar to the proof of Theorem 1.1, one can prove that there exists $\bar{v} \in E$ such that $v_n \to \bar{v}$ in E, $\|\bar{v}\|_2^2 = a$, and $I'_{S_a}(\bar{v}) = 0$. Thus, $\bar{v} \in \Upsilon_a$, which implies that Υ_a is not empty. By (1.5), (2.5), (V1), (V2), and (S2), we get

$$\begin{aligned} 2I(v) &= 2I(v) - P(v) \\ &= -\frac{1}{2}\int_{\mathbb{R}^N}[2V(x) + (\nabla V(x), x)]v_n^2\mathrm{d}x \\ &\quad + \frac{N}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v)]\left[f(v)v - \frac{N+2+\alpha}{N}F(v)\right]\mathrm{d}x \\ &= \frac{N}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v)]\left[f(v)v - \frac{N+2+\alpha}{N}F(v)\right]\mathrm{d}x \ge 0, \ \forall\, v \in \Upsilon_a. \end{aligned} \tag{3.18}$$

It follows from (3.18) that $\tilde{m}(a) \ge 0$. We need to show that $\tilde{m}(a) > 0$. Let $\{v_n\} \subset \Upsilon_a$ be such that

$$I(v_n) \to \tilde{m}(a), \quad I'_{S_a}(v_n) = 0. \tag{3.19}$$

By (1.6), (2.5), (3.19), (V1), (V2), and Lemma 2.9, we have

$$\tilde{m}(a) + o_n(1) = \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N} V(x)v_n^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]F(v_n)\mathrm{d}x, \tag{3.20}$$

and

$$\begin{aligned} o_n(1) = P(v_n) &= \|\nabla v_n\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}(\nabla V(x), x)v_n^2\mathrm{d}x \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)][Nf(v_n)v_n - (N+\alpha)F(v_n)]\mathrm{d}x. \end{aligned} \tag{3.21}$$

From (V1), (3.20), and (3.21), we have

$$\begin{aligned} \tilde{m}(a) + o_n(1) &= -\frac{1}{4}\int_{\mathbb{R}^N}[2V(x) + (\nabla V(x), x)]v_n^2\mathrm{d}x \\ &\quad + \frac{N}{4}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]\left[f(v_n)v_n - \frac{2+N+\alpha}{N}F(v_n)\right]\mathrm{d}x \end{aligned}$$

$$= \frac{N}{4}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]\left[f(v_n)v_n - \frac{2+N+\alpha}{N}F(v_n)\right]\mathrm{d}x. \tag{3.22}$$

Similar to (2.72), we know that $\|v_n\|_2^2 \geq C_{27}$ for some positive constant C_{27}. Set $k' = \frac{k}{k-1}$, then $k' \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$. From (V1), (V2), (S1), (S2), (S3'), (2.62), (2.63), (3.21), (3.22), Lemmas 2.1 and 2.3 and the Hölder inequality, we get

$$\begin{aligned}
\frac{2}{N} &= -\frac{1}{N\|\nabla v_n\|_2^2}\int_{\mathbb{R}^N}(\nabla V(x), x)v_n^2\mathrm{d}x + \frac{1}{\|\nabla v_n\|_2^2}\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]\left[f(v_n)v_n - \frac{N+\alpha}{N}F(v_n)\right]\mathrm{d}x \\
&\leq \frac{\vartheta}{N} + \frac{1}{\|\nabla v_n\|_2^2}\left(\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]\left|\frac{|f(v_n)v_n - \frac{N+\alpha}{N}F(v_n)|}{|v_n|}\right|^k \mathrm{d}x\right)^{\frac{1}{k}}\left(\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]|v_n|^{k'}\mathrm{d}x\right)^{\frac{1}{k'}} \\
&\leq \frac{\vartheta}{N} + \frac{1}{\|\nabla v_n\|_2^2}\left(\int_{\mathbb{R}^N} C_*[I_\alpha * F(v_n)][Nf(v_n)v_n - (2+N+\alpha)F(v_n)]\mathrm{d}x\right)^{\frac{1}{k}} \\
&\quad \times\left(\int_{\mathbb{R}^N}[I_\alpha * F(v_n)]|v_n|^{k'}\mathrm{d}x\right)^{\frac{1}{k'}} \\
&\leq \frac{\vartheta}{N} + \frac{[C_*(4\tilde{m}(a)+o_n(1))]^{\frac{1}{k}}}{\|\nabla v_n\|_2^2}\left[\left(\int_{\mathbb{R}^N}|F(v_n)|^{\frac{2N}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{2Nk'}}\left(\int_{\mathbb{R}^N}|v_n|^{\frac{2Nk'}{N+\alpha}}\mathrm{d}x\right)^{\frac{N+\alpha}{2Nk'}}\right] \\
&\leq \frac{\vartheta}{N} + \frac{[C_*(4\tilde{m}(a)+o_n(1))]^{\frac{1}{k}}C_{28}\|\nabla v_n\|_2^{\frac{Nk'-N-\alpha}{2k'}}\|v_n\|_2^{\frac{N+\alpha-(N-2)k'}{2k'}}}{\|\nabla v_n\|_2^2} \\
&\quad \times\left(C_\varepsilon^{\frac{2N}{N+\alpha}}\int_{\mathbb{R}^N}|v_n|^{\frac{2(N+2+\alpha)}{N+\alpha}}\mathrm{d}x + \varepsilon^{\frac{2N}{N+\alpha}}\int_{\mathbb{R}^N}|v_n|^{\frac{2N}{N-2}}\mathrm{d}x\right)^{\frac{N+\alpha}{2Nk'}} \\
&\leq \frac{\vartheta}{N} + \frac{[C_*(4\tilde{m}(a)+o_n(1))]^{\frac{1}{k}}C_{29}\|\nabla v_n\|_2^{\frac{Nk'-N-\alpha}{2k'}}}{\|\nabla v_n\|_2^2}(C_\varepsilon^{\frac{1}{k'}}\|\nabla v_n\|_2^{\frac{1}{k'}} + \varepsilon^{\frac{1}{k'}}\|\nabla v_n\|_2^{\frac{N+\alpha}{(N-2)k'}}) \\
&= \frac{\vartheta}{N} + [C_*(4\tilde{m}(a)+o_n(1))]^{\frac{1}{k}}C_{29} \\
&\quad \times(C_\varepsilon^{\frac{1}{k'}}\|\nabla v_n\|_2^{\frac{1}{k'}+\frac{Nk'-N-\alpha}{2k'}-2} + \varepsilon^{\frac{1}{k'}}\|\nabla v_n\|_2^{\frac{N+\alpha}{(N-2)k'}+\frac{Nk'-N-\alpha}{2k'}-2}).
\end{aligned} \tag{3.23}$$

From (3.23), we know that $\{\|\nabla v_n\|_2\}$ is bounded and $\tilde{m}(a) > 0$; thus $\{v_n\}$ is bounded. It follows from a standard argument that there exists $v_0 \in \Upsilon_a$ such that $I(v_0) = \tilde{m}(a)$. □

4. Conclusions

This paper investigates normalized solutions for a kind of mass–supercritical Schrödinger–Choquard equation. By applying a new variational framework and some new analytic techniques, we obtain two main existence results for normalized solutions for this kind of mass–supercritical Schrödinger–Choquard equation. In particular, we first prove that the functional I possesses the properties of mountain pass geometry, and the minimum level $m(a)$ or $\tilde{m}(a)$ is greater than or equal to $\inf I$ in S_a. Second, we prove that there exists a sequence $\{v_n\}$ of S_a satisfying $I(v_n) \to m(a)$ (or $\tilde{m}(a)$), $I|'_{S_a}(v_n) \to 0$, and $P(v_n) \to 0$. Third, we prove that the Lagrange multiplier λ_n

is bounded and $\lambda_n \to \lambda_a$ on $\mathbb{R}$, and we prove that $\{v_n\}$ is bounded in $H^1_{rad}(\mathbb{R}^N)$ by using some new analytical methods. Finally, we prove that the bounded $\{v_n\}$ converges to a v in E, and v is the normalized solution of problem (1.1). Besides, we prove that the normalized solution obtained in Theorem 1.2 is in fact a ground state solution. Our results improve and generalize related results in the literature and hope that the methods used in this paper can be applied to study related elliptic equations.

Author contributions

Xingwen Chen: investigation; methodology; formal analysis; writing-original draft; writing-review and editing. Qiongfen Zhang: investigation; methodology; formal analysis; writing-review and editing; supervision; funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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