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Properties for some Cegrell classes of m-subharmonic function

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Abstract: This paper investigated the properties of maximal subextensions for m-subharmonic (m-sh) functions in the context of complex Hessian operators. We established three main theorems that significantly advance the theory. First, we provided a complete characterization of maximal subextensions in the class $\mathcal{F}_m^a(\Omega)$, showing that any subextension satisfying a minimality condition on its Hessian mass must coincide with the maximal subextension. Second, we proved that for functions in $\mathcal{E}_m(\Omega)$, the maximal subextension relates to the original function through an m-maximal function and preserves Lelong numbers. Third, we demonstrated convergence stability, proving that for sequences with boundary values in $\mathcal{F}_m(\Omega, f)$ converging in L^1_{loc} and uniformly bounded below, their maximal subextensions converged to the maximal subextension of the limit function.

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1. Introduction

Pluripotential theory, which studies the properties of the complex Monge–Ampère operator and plurisubharmonic functions, is a cornerstone of the analysis of several complex variables. Since the foundational work of Bedford and Taylor [1], which enabled the definition of a measure theory for this operator on locally bounded functions, this field has developed continuously. A profound and natural generalization of this theory concerns the Hessian operator in the complex setting

$$H_m(\cdot) = (dd^c\cdot)^m \wedge \omega^{n-m},$$

which arises in crucial geometric problems and has given rise to a so-called "m-pluripotential theory". The fundamental work of Blocki [2] on m-subharmonic (m-sh) functions and that of Chinh [3], who

introduced and studied the generalized Cegrell classes $\mathcal{E}_m^0, \mathcal{F}_m, \mathcal{E}_m$ for this operator, are notable illustrations.

A central problem in this framework, both from a theoretical viewpoint and for its applications, is that of extension. Given a function that is m-sh on a domain Ω and a larger domain $\widehat{\Omega} \supset \Omega$, the problem is the existence of an m-sh function $\widetilde{u}$ on $\widehat{\Omega}$ satisfying $\widetilde{u} \leq u$ on Ω as well as its properties, essentially the relationship between $H_m(\widetilde{u})$ and $H_m(u)$. When $m = n$, this problem was systematically investigated in the seminal work of Cegrell and Zeriahi [4] in which the existence of a subextension $\widetilde{u}$ was established in $\mathcal{F}(\widehat{\Omega})$ with Monge–Ampère mass satisfying

$$\int_{\widehat{\Omega}} (dd^c \widetilde{u})^n \leq \int_{\Omega} (dd^c u)^n.$$

Based on this work, the extension problem for plurisubharmonic functions was extensively studied in [5,6]. In the general case, the problem is also well understood for the class $\mathcal{F}_m(\Omega)$, where it has been established in [7,8] that a subextension always exists and that its mass does not increase:

$$\int_{\widehat{\Omega}} H_m(\widetilde{u}) \leq \int_{\Omega} H_m(u).$$

This problem was recently advanced significantly through the study of energy classes and boundary value problems, as seen in works on maximal subextensions and approximation [9], as well as recent advances on the subsolution theorem for weighted energy classes with prescribed boundary values [10]. However, the construction of the maximal subextension

$$\widehat{u} = \sup\{\psi \in \mathcal{SH}_m^-(\widehat{\Omega}) : \psi|_{\Omega} \leq u\}$$

and a fine analysis of its properties for the broader classes in the Cegrell hierarchy, notably $\mathcal{E}_m(\Omega)$ and $\mathcal{F}_m^a(\Omega)$ (functions whose Hessian does not charge m-polar sets), remained to be explored.

This paper continues the study of this problem by systematically studying the properties of the maximal subextension for m-sh functions. Our main object of study is the maximal subextension $\widetilde{u}$ of a function u belonging to the classes $\mathcal{F}_m$, $\mathcal{F}_m^a$, and $\mathcal{E}_m$ from an m-hyperconvex domain Ω to a larger domain $\widehat{\Omega}$. We examine its behavior with respect to the Hessian operator and the Lelong number, and we establish convergence results for sequences of such functions.

Our main contribution is a set of three theorems that significantly advance the theory. First, we provide a complete characterization of the maximal subextension in the class $\mathcal{F}_m^a(\Omega)$. We show that any subextension $v \in \mathcal{F}_m^a(\widehat{\Omega})$ of $u \in \mathcal{F}_m^a(\Omega)$, which satisfies a certain minimality condition on its Hessian mass outside the set where it perfectly matches u, namely if

$$\int_{\{v<u\}\cap\Omega\cup(\widehat{\Omega}\setminus\Omega)} H_m(v) = 0,$$

then v is necessarily the maximal subextension $\widehat{u}$. This result characterizes the maximal subextension by a unique analytic property moving beyond its purely variational definition.

Second, we establish fundamental properties for the largest Cegrell class, $\mathcal{E}_m(\Omega)$. We demonstrate that if u is any element of $\mathcal{E}_m(\Omega)$, then $\widehat{u}$ is intimately related to u through an m-maximal function. Specifically, we demonstrate the existence of an m-maximal function ($H_m(v) = 0$) $v \in \mathcal{E}_m(\Omega)$ that

satisfies $u \geq \widehat{u} \geq u + v$ in Ω. Furthermore, we show that the Lelong number, a key local invariant of singularity, is preserved under the maximal subextension process: $\forall x \in \Omega,\ \nu_{\widehat{u}}(x) = \nu_u(x)$. This preservation is a profound phenomenon that underscores the rigidity and naturality of the maximal construction.

Third, we prove a robust convergence theorem that ensures the stability of our construction. We show that if $f \in \mathcal{E}_m(\Omega)$, $\{u_j\}$ is a sequence in boundary value class $\mathcal{F}_m(\Omega, f)$, which converges toward u in $L^1_{loc}(\Omega)$ and is uniformly bounded below by a certain function in $\mathcal{F}_m(\Omega, f)$, then the sequence of their maximal subextensions $\{\widehat{u}_j\}$ converges to the maximal subextension $\widehat{u}$ in $L^1_{loc}(\widehat{\Omega})$. More precisely, we show the following result:

Theorem 1. *Suppose that $\Omega \Subset \widehat{\Omega} \subset \mathbb{C}^n$. Take $f \in \mathcal{E}_m(\Omega)$ and $\{u_j\} \subset \mathcal{F}_m(\Omega, f)$ satisying $u_j \geq w \in \mathcal{F}_m(\Omega, f)$ for all $j \geq 1$ and $g \in \mathcal{E}_m(\hat{\Omega})$ be such that $f \geq g$ on Ω and*

$$\int_{\Omega} H_m(w) + \int_{\widehat{\Omega}} H_m(g) < +\infty.$$

If $\{u_j\}$ converges to u in $L^1_{\text{loc}}(\Omega)$, then $\widehat{u}^g_j$ converges to $\widehat{u}^g$ in $L^1_{\text{loc}}(\widehat{\Omega})$.

This result is crucial for applications, as it allows the approximation of complicated functions by sequences of nicer ones while maintaining control over their subextensions.

Collectively, these results substantially extend and refine the existing theory of subextension in the m-sh context [4, 11–14]. Our proofs rely on a sophisticated synthesis of potential theory, a detailed analysis of Hessian measure convergence in the Cegrell classes, and a judicious use of the properties of m-maximal functions.

The article is structured as follows. Section 2 gathers the necessary preliminaries, recalling the definitions of m-hyperconvex domains, m-sh functions, the Cegrell classes $\mathcal{E}^0_m, \mathcal{F}_m, \mathcal{E}_m$, and the notion of subextension. Section 3 is the core of the article and presents our main results on the properties of the maximal subextension. Section 4 investigates the convergence properties of maximal subextensions for sequences of m-sh functions. Finally, Section 5 concludes the study by summarizing the key findings and discussing future works.

2. Preliminaries

2.1. m-hyperconvex domains and m-sh functions

We employ conventional differential operators: $d^c = -i(\partial - \bar{\partial})$, $d = \bar{\partial} + \partial$ and set $\omega = dd^c|z|^2$.

We first review the notion of $(1, 1)$-forms satisfying the m-positivity condition, following [2]:

Definition 1. *A real $(1, 1)$-form α on a domain Ω is termed m-positive if at each point of Ω, the inequality*

$$\alpha^j \wedge \omega^{n-j} \geq 0$$

holds for every $j = 1, \ldots, m$.

Building on this, the work [2] developed the theory of m-sh functions:

Definition 2. *A function v defined on Ω with value in $\mathbb{R} \cup \{-\infty\}$ is called be $m-sh$ if it fulfills two conditions: first, it must be subharmonic, and second,*

$$dd^c v \wedge \omega^{n-m} \wedge \eta_1 \wedge \cdots \wedge \eta_{m-1} \geq 0 \quad \text{for all } m\text{-positive forms } \eta_1, \ldots, \eta_{m-1}.$$

The set formed by these functions on Ω will be denoted by $\mathcal{SH}_m(\Omega)$.

Example 1. *Take*

$$v(z_1, z_2, z_3) = \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2 - \frac{1}{4}|z_3|^2.$$

One can verify that $v \in \mathcal{SH}_2(\mathbb{C}^3)$, but $v \notin \mathcal{SH}_3(\mathbb{C}^3)$ since its restriction to $(0, 0, z_3)$ fails to be subharmonic.

Definition 3. *A domain $\Omega \subset \mathbb{C}^n$ is m-hyperconvex if it is bounded and there exists a function φ: $\Omega \to \mathbb{R}^-$, called an m-sh defining function, with the following properties:*

(1) φ is continuous
(2) φ is m-subharmonic
(3) for all $t < 0$ one has $\{\varphi < t\} \Subset \Omega$.

Definition 4. *A subset $P \subset \Omega$ is m-polar if there is a function $u \in \mathcal{SH}_m(\Omega)$ with $P \subset \{u = -\infty\}$.*

Additional properties of m-sh functions appear in [2, 3].

Consider m functions $v_1, \ldots, v_m \in \mathcal{SH}_m(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$. The foundational work of Bedford and Taylor provides a method to define the wedge product of their complex Hessians, resulting in a well-defined positive current of bidegree (n, n):

$$dd^c v_1 \wedge \cdots \wedge dd^c v_m \wedge \omega^{n-m}.$$

This current is called the complex Hessian measure associated with the tuple $(v_1, \ldots, v_m)$. A particularly important case is when all functions are identical,

$$v_1 = \cdots = v_m = v.$$

Then

$$\mathrm{H}_m(v) := (dd^c v)^m \wedge \omega^{n-m}.$$

In what follows, we will work with two m-hyperconvex domains, which we denote by Ω and $\widehat{\Omega}$ satisfying $\Omega \subset \widehat{\Omega} \subset \mathbb{C}^n$.

To investigate the Hessian equation, Chinh [3] defined these categories of m-sh functions, extending the Cegrell classes [15] for $m = n$.

Definition 5. *Define the classes:*

$$\mathcal{E}^0_m(\Omega) := \left\{ v \in L^\infty(\Omega) \cap \mathcal{SH}^-_m(\Omega) : \lim_{z \to \partial\Omega} v(z) = 0, \ \int_\Omega \mathcal{H}_m(v) < +\infty \right\},$$

$$\mathcal{F}_m(\Omega) := \left\{ v \in SH^-_m(\Omega) : \exists\, (v_k) \subset \mathcal{E}^0_m, \ v_k \searrow v, \ \sup_k \int_\Omega H_m(v_k) < +\infty \right\},$$

and

$$\mathcal{E}_m(\Omega) := \{ v \in \mathcal{SH}^-_m(\Omega) : \forall U \Subset \Omega, \exists\, v_U \in \mathcal{F}_m(\Omega); \ v_U = v \text{ on } U \}.$$

One can check that these inclusions hold:

$$\mathcal{E}_m^0(\Omega) \subset \mathcal{F}_m(\Omega) \subset \mathcal{E}_m(\Omega).$$

Definition 6.

(1) For any Borel subset $L \subseteq \Omega$, the m-capacity of a compact set L relative to Ω, denoted by $Cap_m(L, \Omega)$, measures its size by taking the supremum of the Hessian mass $\int_L H_m(v)$ over all m-sh functions v on Ω normalized between -1 and 0.

(2) We say that (v_k) converges in m-capacity to v, if for all compact $L \Subset \Omega$ and $r > 0$,

$$\lim_{k\to\infty} Cap_m\left(\{|v_k - v| > r\} \cap L,\ \Omega\right) = 0.$$

Definition 7. *Following [3], the relative m-extremal function of a subset $L \subset \Omega \subset \mathbb{C}^n$ is*

$$h_{m,L,\Omega} := \sup\{v \in \mathcal{SH}_m(\Omega) : v \leq 0,\ v|_L \leq -1\}.$$

Note that its regularization $h^*_{m,L,\Omega}$ is a negative m-sh function on Ω. As stated in [3, Proposition 2.17], if $L \Subset \Omega$ is open, then $h_{m,L,\Omega} \in \mathcal{E}_m^0(\Omega)$.

Definition 8. *A function $v \in \mathcal{SH}_m(\Omega)$ is said to be m-maximal if for every $w \in \mathcal{SH}_m(\Omega)$ such that if $w \leq v$ outside a compact subset of Ω, then $w \leq v$ in Ω.*

Blocki [2] established that $\mathrm{H}_m(v) = 0$ for any m-maximal function v in the class $\mathcal{E}_m(\Omega)$. We write $\mathcal{MSH}_m(\Omega)$ for the set of all m-maximal functions within $\mathcal{SH}_m(\Omega)$.

Definition 9. *An exhaustion sequence of bounded m-hyperconvex domains (L_j) of Ω is said to be a fundamental sequence associated to Ω if $L_j \Subset L_{j+1} \Subset \Omega$ for all j and $\bigcup_{j=1}^{\infty} L_j = \Omega$.*

Definition 10. *Let $f \in \mathcal{E}_m(\Omega)$. A function $v \in \mathcal{SH}_m^-(\Omega)$ lies in $\mathcal{F}_m(\Omega, f)$ if there exists $\xi \in \mathcal{F}_m(\Omega)$ with*

$$f \geq v \geq \xi + f.$$

Observe that if $\xi = 0$ everywhere, then $\mathcal{F}_m(\Omega, \xi) = \mathcal{F}_m(\Omega)$.

For $f \in \mathcal{E}_m(\Omega)$, we set

$$\mathcal{F}_m^a(\Omega) := \{v \in \mathcal{F}_m(\Omega) : \mathrm{H}_m(v)(P) = 0 \text{ for all } m\text{-polar sets } P\}$$

and

$$\mathcal{F}_m^a(\Omega, f) := \{v \in \mathcal{E}_m(\Omega) : \exists \xi \in \mathcal{F}_m^a(\Omega), f \geq v \geq \xi + f\}.$$

Finally, we review the Lelong number for m-sh functions from [16]. Let

$$\Theta_m(\xi) = -\|\xi - a\|^{2-\frac{2n}{m}}$$

for $a \in \Omega$.

Definition 11. *Let $y \in \Omega$ and $u \in \mathcal{SH}_m^-(\Omega)$. The Lelong number of u at y is*

$$\nu_u(y) = \int_{\{y\}} dd^c u \wedge (dd^c \Theta_m)^{m-1} \wedge \omega^{n-m}.$$

2.2. Subextension and maximal subextension of m-sh functions

We now recall the concept of subextension for m-sh functions.

Definition 12. *Consider u to be an m-sh function on Ω. An m-sh function $\widetilde{u}$ on the larger domain $\widehat{\Omega}$ is termed a subextension of u if it satisfies $\widetilde{u}(z) \leq u(z)$ for every $z \in \Omega$.*

In [8], the authors established the existence of the subextension in the Cegrell class $\mathcal{F}_m(\widehat{\Omega})$ and gave its characterizations. More precisely, they proved the following:

Theorem 2. *If $u \in \mathcal{F}_m(\Omega)$, then there exists a subextension $\widetilde{u}$ in $\mathcal{F}_m(\widehat{\Omega})$ of u such that*

$$\int_{\widehat{\Omega}} H_m(\widetilde{u}) \leq \int_{\Omega} H_m(u).$$

Definition 13. *(Maximal subextension) For a given function u in the class $\mathcal{E}_m(\Omega)$, its maximal subextension to $\widehat{\Omega}$ is defined by taking the supremum over all negative m-sh functions on the larger domain that are dominated by u on the original one:*

$$\widehat{u}(x) = \sup\left\{h(x) : h \in \mathcal{SH}_m^-(\widehat{\Omega}) \text{ and } h(x) \leq u(x), \forall x \in \Omega\right\}.$$

The following fundamental property of the maximal subextension was established in [12]:

Remark 1. *Given $u \in \mathcal{F}_m(\Omega)$, one can always construct its maximal subextension $\widehat{u}$, which satisfies $H_m(\widehat{u}) \leq \mathbf{1}_\Omega H_m(u)$, where $\mathbf{1}_\Omega$ denotes the indicator function of the set Ω.*

3. Properties of maximal subextensions

The following proposition collects key properties of maximal subextensions for functions in $\mathcal{E}_m(\Omega)$.

Proposition 1. *The maximal subextension operator on $\mathcal{SH}_m(\Omega)$ is monotonic and satisfies the following inequalities for any m-sh functions u, v defined in Ω and $t > 0$:*

(i) (Monotonicity) $v \geq u \Rightarrow \widehat{v} \geq \widehat{u}$;
(ii) (Scaling) $\widehat{tu} = t\widehat{u}$;
(iii) (Maximum) $\max(\widehat{v}, \widehat{u}) \leq \widehat{\max(v, u)}$.

Proof. (i) By definition, $\widehat{u}|_\Omega \leq u \leq v$. Since $\widehat{v}$ is the supremum of all m-sh functions on $\widehat{\Omega}$ that are bounded above by v on Ω, and $\widehat{u}$ is one such function, it follows that $\widehat{u} \leq \widehat{v}$.

(ii) Note that $\widehat{tu} \in \mathcal{SH}_m(\widehat{\Omega})$ and $\widehat{tu}|_\Omega \leq tu$. By the maximality of $\widehat{tu}$, we conclude $\widehat{tu} \leq t\widehat{u}$. For the converse sense, we have $\frac{1}{t}\widehat{tu} \leq u$ on Ω so by the maximality of u we get $\widehat{tu} \leq t\widehat{u}$. The result follows.

(iii) An application of part (i) on the functions u, v and $\max(u, v)$ implies that $\widehat{\max(u, v)} \geq \widehat{u}$ and $\widehat{\max(u, v)} \geq \widehat{v}$. Hence, $\widehat{\max(u, v)} \geq \max(\widehat{u}, \widehat{v})$. □

The above properties were proven in the case $m = n$ for plurisubharmonic in [14]. A fundamental tool in our analysis is the following comparison principle for $\mathcal{F}_m^a(\Omega, f)$.

Lemma 1. *Assume that $\Omega \Subset \mathbb{C}^n$ and take $f \in \mathcal{E}_m(\Omega)$. If $u, v \in \mathcal{F}_m^a(\Omega, f)$ such that:*

(i) $H_m(u)$ *vanishes on the set* $\{v > u > -\infty\}$*;*
(ii) the total mass $\int_\Omega H_m(u)$ *is finite;*

then it follows that $v \leq u$ *throughout* Ω*.*

Proof. Define $w := \max(v, u)$. Then $w \in \mathcal{F}_m^a(\Omega, f)$ and $w \geq u$ in Ω. Consequently,

$$\int_\Omega H_m(w) \leq \int_\Omega H_m(u) < +\infty.$$

On one hand, [17, Proposition 3.5] implies:

$$\mathbf{1}_{\{f=-\infty\}} H_m(f) = \mathbf{1}_{\{u=-\infty\}} H_m(u) = \mathbf{1}_{\{w=-\infty\}} H_m(w) \text{ on } \Omega.$$

On the other hand, let E be a compact set included in $\{u > -\infty\} \cap \{u = w\}$. Since $E \subset \{u + \frac{1}{j} > w\}$, Theorem 7 for $T = 1$ in [18] yields:

$$\int_E H_m(u) = \lim_{j\to\infty} \int_E H_m\left(\max\left(w, u + \frac{1}{j}\right)\right) \leq \int_E H_m(\max(w, u)) = \int_E H_m(w).$$

Thus,

$$H_m(w) \geq H_m(u) \text{ on } \{u = w\} \cap \{u > -\infty\}.$$

Moreover,

$$H_m(u) = 0 \text{ on } \{-\infty < u < w\} = \{-\infty < u < v\}.$$

We conclude that

$$H_m(w) \geq H_m(u) \text{ on } \Omega.$$

According to [17, Proposition 3.6], we get $w \leq u$ in Ω. Hence $u \geq v$ in Ω, completing the proof. □

The following proposition was proved in the particular case $m = n$ in [19]. We improve it here and extend it here to the general case.

Proposition 2. *For every* $u \in \mathcal{F}_m^a(\Omega)$*, one has*

(1) $\widehat{u} \in \mathcal{F}_m^a(\widehat{\Omega})$*;*
(2) $\displaystyle\int_{(\widehat{\Omega}\setminus\Omega)\cup\left((\widehat{u}<u)\cap\Omega\right)} H_m(\widehat{u}) = 0.$

Proof. 1) From [12, Lemma 3.2], we know $\widehat{u} \in \mathcal{F}_m(\widehat{\Omega})$ and $H_m(\widehat{u}) \leq \mathbf{1}_\Omega H_m(u)$. It suffices to show that $H_m(\widehat{u})$ does not charge m-polar sets. Let $P \subset \widehat{\Omega}$ be m-polar. Then

$$\mathbf{1}_P H_m(\widehat{u}) \leq \mathbf{1}_{P\cap\Omega} H_m(u) = 0,$$

where the last equality holds because $u \in \mathcal{F}_m^a(\Omega)$. Thus,

$$\int_P H_m(\widehat{u}) = 0$$

proving $\widehat{u} \in \mathcal{F}_m^a(\widehat{\Omega})$.

2) Using [12, Lemma 3.2], we infer that

$$\int_{\Omega\cap\{\widehat{u}<u\}} H_m(\widehat{u}) = 0.$$

Combining this with the inequality $\mathbf{1}_\Omega H_m(u) \geq H_m(\widehat{u})$, we obtain that

$$\int_{(\widehat{\Omega}\setminus\Omega)\cup\left(\widehat{(u}<u)\cap\Omega\right)} H_m(\widehat{u}) = \int_{\widehat{\Omega}\setminus\Omega} H_m(\widehat{u}) + \int_{\widehat{(u}<u)\cap\Omega} H_m(\widehat{u}) = 0.$$

The result follows. □

The converse of the previous proposition also holds:

Theorem 3. *Let $u \in \mathcal{F}_m^a(\Omega)$, and suppose that there exists $v \in \mathcal{F}_m^a(\widehat{\Omega})$ satisfying $u \geq v$ on Ω and*

$$\int_{(\{v<u\}\cap\Omega)\cup(\widehat{\Omega}\setminus\Omega)} H_m(v) = 0.$$

Then $\widehat{u}(z) = v(z)$ $\forall z \in \widehat{\Omega}$.

Proof. We start with the proof that $v \geq \widehat{u}$. To show this we will use Lemma 1. By Proposition 2, we get that $\widehat{u} \in \mathcal{F}_m^a(\widehat{\Omega})$. Using [3], we get $\int_{\widehat{\Omega}} H_m(v) < +\infty$. Hence. it suffices to prove that $\int_{\{-\infty<v<\widehat{u}\}} H_m(v) = 0$.

$$\begin{aligned}
\int_{\{-\infty<v<\widehat{u}\}} H_m(v) &= \int_{\{-\infty<v<\widehat{u}\}\cap\Omega} H_m(v) + \int_{\{-\infty<v<\widehat{u}\}\cap(\widehat{\Omega}\setminus\Omega)} H_m(v) \\
&= \int_{\{-\infty<v<\widehat{u}\}\cap\Omega} H_m(v) \\
&\leq \int_{\{-\infty<v<u\}\cap\Omega} H_m(v) \\
&\leq \int_{\{v<u\}\cap\Omega} H_m(v) = 0.
\end{aligned}$$

It follows that $v \geq \widehat{u}$ on $\widehat{\Omega}$. Conversely, the maximality of $\widehat{u}$ ensures that $v \leq \widehat{u}$. The result follows. □

Proposition 3. *Suppose that $\Omega \Subset \widehat{\Omega}$. If the function $u \in \mathcal{SH}_m^-(\Omega)$ admits a subextension $\widetilde{u} \in \mathcal{SH}_m^-(\widehat{\Omega})$ with $\widetilde{u} \not\equiv -\infty$, then:*

(1) If $m \neq n$, then there exists a constant $0 < c < \frac{n}{n-m}$ such that $\int_\Omega |u|^c d\lambda < +\infty$;
(2) $\sup_{x\in\Omega} \nu_u(x) < +\infty$.

Proof. 1) Since $\Omega \Subset \widehat{\Omega}$ and $\widetilde{u}$ is not identically $-\infty$, then using [2, Proposition 1.3], there exists $0 < c < \frac{n}{n-m}$ such that $\int_\Omega |\widetilde{u}|^c d\lambda < +\infty$. Because $\widetilde{u}(z) \leq u(z)$ for all $z \in \Omega$, we get that

$$\int_\Omega |u|^c d\lambda \leq \int_\Omega |\widetilde{u}|^c d\lambda \leq \int_{\overline{\Omega}} |\widetilde{u}|^c d\lambda < +\infty.$$

2) Assume, for contradiction, that $\sup_{x\in\Omega} \nu_u(x) = +\infty$. Then there exists a sequence $(y_j) \subset \Omega$ such that $\nu_u(y_j) \geq j$. Since $\Omega \Subset \widehat{\Omega}$, by selecting an appropriate subsequence if needed, we can assume that $y_j \to y \in \overline{\Omega} \subset \widehat{\Omega}$. As $\widetilde{u}(z) \leq u(z)$ for all $z \in \Omega$, then by [20, Proposition 5.2], we get $\nu_{\widetilde{u}}(y_j) \geq \nu_u(y_j) \geq j$. By the upper semicontinuity of the Lelong number,

$$+\infty = \limsup_{j\to\infty} \nu_u(y_j) \leq \limsup_{j\to\infty} \nu_{\widetilde{u}}(y_j) \leq \nu_{\widetilde{u}}(y),$$

which is a contradiction because $\nu_{\widetilde{u}}(y)$ is finite. □

Remark 2. *Let $L \Subset \Omega$ be compact such that $Cap_m(L,\Omega) \neq 0$. The m-extremal function of L relative to Ω is defined as*

$$h_{m,L,\Omega} = \sup\{w \in \mathcal{SH}_m^-(\Omega) : w|_L \leq -1\}.$$

We claim that

$$\widehat{h^*_{m,L,\Omega}} = h^*_{m,L,\widehat{\Omega}}.$$

Indeed, by definition,

$$\widehat{h^*_{m,L,\Omega}} = \sup\{w \in \mathcal{SH}_m^-(\widehat{\Omega}) : w \leq h^*_{m,L,\Omega} \text{ on } \Omega\}.$$

Since $Cap_m(L,\Omega) \neq 0$, then using [21, Theorem 1.4.12], we get that any w in this set satisfies

$$w_{|L} \leq h^*_{m,L,\Omega}|_L \leq -1,$$

hence $w \leq h_{m,L,\widehat{\Omega}}$. Conversely,

$$h_{m,L,\widehat{\Omega}}|_\Omega \leq h_{m,L,\Omega} \leq h^*_{m,L,\Omega},$$

so

$$h_{m,L,\widehat{\Omega}} \leq \widehat{h^*_{m,L,\Omega}}.$$

Taking the upper semicontinuous regularization (which preserves the inequality for m-sh functions) gives the result.

Now we present results concerning the subextension of functions in $\mathcal{E}_m(\Omega)$.

Proposition 4. *Let $u \in \mathcal{E}_m(\Omega)$. The following statements hold:*

(i) There exists $v \in \mathcal{E}_m(\Omega)$ with $H_m(v) = 0$ such that $u \geq \widehat{u} \geq u + v$ on Ω;
(ii) If $\widehat{u} \in \mathcal{E}_m(\widehat{\Omega})$, then $\mathbf{1}_{\{\widehat{u}=-\infty\}\cap\Omega} H_m(\widehat{u}) = \mathbf{1}_{\{u=-\infty\}} H_m(u)$ on Ω.

Proof. (i) Using the local property of $\mathcal{E}_m(\Omega)$ (see [3]), we obtain that the function $\widehat{u}$ belongs to $\mathcal{E}_m(\Omega)$. Define

$$v = \sup\left\{f \in \mathcal{SH}_m^-(\Omega) : \limsup_{z\to\delta} f(z) \leq \widehat{u}(\delta) \text{ for all } \delta \in \partial\Omega\right\}.$$

To prove that $v \in \mathcal{E}_m(\Omega)$, consider a sequence $(u_j) \subset \mathcal{E}_{0,m}(\Omega) \cap C(\overline{\Omega})$ decreasing to u. According to [10, Lemma 3.2], the corresponding maximal subextensions $(\widehat{u_j}) \subset \mathcal{E}_{0,m}(\widehat{\Omega})$ also form a decreasing sequence converging to $\widehat{u}$ by Proposition 1. This implies the boundary estimate

$$\limsup_{z\to\delta} v(z) \leq \widehat{u_j}(\delta),$$

which, upon taking the limit $j \to \infty$, yields

$$\limsup_{z\to\delta} v(z) \le \widehat{u}(\delta).$$

Consequently, the upper semicontinuous regularization satisfies

$$\limsup_{z\to\delta} v^\star(z) \le \widehat{u}(\delta)$$

for every $\delta \in \partial\Omega$, from which it follows that $v^\star = v$, and therefore $v \in \mathcal{SH}_m^-(\Omega)$. The inequality $v \ge \widehat{u}_{|\Omega}$ finally allows us to conclude that $v \in \mathcal{E}_m(\Omega)$. Furthermore, one can show that v is maximal ($H_m(v) = 0$) by testing against functions that are bounded above by v outside a compact set.

We now define the function $\widehat{v}$ on the extended domain $\widehat{\Omega}$ as follows:

$$\widehat{v}(x) = \begin{cases} \max(u(x) + v(x), \widehat{u}(x)), & \text{for } x \in \Omega, \\ \widehat{u}(x), & \text{for } x \in \widehat{\Omega} \setminus \Omega. \end{cases}$$

For $\delta \in \partial\Omega \cap \tilde{\Omega}$, we have

$$\limsup_{x\to\delta}(u + v)(x) \le \limsup_{x\to\delta} v(x) \le \widehat{u}(x).$$

Then by the glued principle of m-sh functions, we get that the function $\widehat{v}$ is a negative m-sh function on $\widehat{\Omega}$, with $\widehat{v} \ge \widehat{u}$ on $\widehat{\Omega}$ and $u \ge \widehat{v}$ on Ω; the latter follows from the conditions $v \le 0$ and $\widehat{u} \le u$. The maximality of $\widehat{u}$ then implies $\widehat{v} = \widehat{u}$. Consequently, the identity $\widehat{u} = \max(\widehat{u}, u + v)$ must hold on Ω, which yields $u + v \le \widehat{u}$ there. The converse inequality $\widehat{u} \le u$ is true by definition.

(ii) From (i), we have $v + u \le \widehat{u} \le u$ with $H_m(v) = 0$. Using the comparison of measures on m-polar sets (see [20, Proposition 5.2]), we get

$$\mathbf{1}_{\{u=-\infty\}} H_m(u) \le \mathbf{1}_{\{\widehat{u}=-\infty\}\cap\Omega} H_m(\widehat{u}) \le \mathbf{1}_{\{u+v=-\infty\}} H_m(v + u). \tag{1}$$

We analyze $H_m(u + v)$ on $\{u + v = -\infty\}$:

$$H_m(u + v) = \sum_{j=0}^{m} \binom{m}{j} (dd^c v)^j \wedge (dd^c u)^{m-j} \wedge \omega^{n-m}.$$

For $j \ge 1$, [20, Lemma 5.6] implies

$$\int_{\{v+u=-\infty\}} (dd^c v)^j \wedge (dd^c u)^{m-j} \wedge \omega^{n-m} \le \left(\int_{\{v+u=-\infty\}} H_m(v)\right)^{j/m} \left(\int_{\{v+u=-\infty\}} H_m(u)\right)^{(m-j)/m} = 0,$$

since $H_m(v) = 0$. Thus, only the $j = 0$ term survives on $\{u + v = -\infty\}$:

$$\mathbf{1}_{\{u+v=-\infty\}} H_m(v + u) = \mathbf{1}_{\{u+v=-\infty\}} H_m(u).$$

We have

$$\{u + v = -\infty\} = \{\{u = -\infty\} \cap \{u + v = -\infty\}\} \cup \{\{u > -\infty\} \cap \{u + v = -\infty\}\}.$$

Since $v + u \leq u$ on Ω and $\mathbf{1}_{\{u>-\infty\}}H_m(u)$ has no mass on m-polar sets, then

$$\mathbf{1}_{\{u+v=-\infty\}}H_m(v+u) = \mathbf{1}_{\{u=-\infty\}}H_m(u).$$

It follows, using the inequality (1), that

$$\mathbf{1}_{\{\widehat{u}=-\infty\}\cap\Omega}H_m(\widehat{u}) = \mathbf{1}_{\{u=-\infty\}}H_m(u).$$

This completes the proof. □

The Lelong number is preserved under maximal subextension.

Proposition 5. *For every $u \in \mathcal{E}_m(\Omega)$, one has*

$$\nu_{\widehat{u}}(y) = \nu_u(y) \quad \textit{for all } y \in \Omega.$$

Proof. Proposition 4 guarantees the existence of $v \in \mathcal{E}_m(\Omega)$ with $H_m(v) = 0$ and $u(x) \geq \widehat{u}(x) \geq v(x) + u(x)$ for all $x \in \Omega$. This implies $\nu_u(y) \leq \nu_{\widehat{u}}(y) \leq \nu_{u+v}(y)$ for $y \in \Omega$ [22, Lemma 1.9]. We have

$$\nu_{u+v}(y) = \int_{\{y\}} dd^c(u+v) \wedge (dd^c\theta_m)^{m-1} \wedge \omega^{n-m} = \nu_u(y) + \nu_v(y).$$

Using [23, Remark 1], we get

$$\nu_v(y) \leq (H_m(v)(\{y\}))^{\frac{1}{m}}.$$

Since $H_m(v) = 0$, we deduce that $\nu_v(y) = 0$, and hence $\nu_{u+v}(y) = \nu_u(y)$. Thus, $\nu_{\widehat{u}}(y) = \nu_u(y)$. □

Definition 14. *Consider φ to be an m-sh function defined on $\widehat{\Omega}$. The singular set of such a function φ is the set $\mathrm{Sg}_\varphi \subset \widehat{\Omega}$, where it is equal to $-\infty$.*

Remark 3. *The singular set of $\widehat{u}$ satisfies*

$$\mathrm{Sg}_u \subset \mathrm{Sg}_{\widehat{u}} \cap \Omega.$$

Indeed, if $u(z_0) = -\infty$, then $\widehat{u}(z_0) \leq u(z_0) = -\infty$, so $\{u = -\infty\} \subseteq \{\widehat{u} = -\infty\}$.

4. Convergence properties of maximal subextensions with fixed boundary data

Our focus in this section turns to the stability of maximal subextensions under limits. We will prove that the convergence in the L^1_{loc}-topology for sequences of m-sh functions with fixed boundary data is stable under taking the maximal subextension. Let us start by establishing the following technical result:

Proposition 6. *Let U be an open subset of $\Omega \Subset \mathbb{C}^n$. Suppose that $\Omega \subset \mathbb{C}^n$, $f, g, w \in \mathcal{E}_m(\Omega)$ and $\{u_j\}, \{v_j\} \subset \mathcal{E}_m(\Omega)$ such that:*

(i) $w \leq u_j$ for all $j \geq 1$;
(ii) The sequence u_j converges to u in Ω with respect to m-capacity;
(iii) $v_j \to v$ almost everywhere in Ω;
(iv) $H_m(u_j) = 0$ on $U \cap \{-\infty < u_j < v_j\} \cap \{f < g\}$.

Then, $H_m(u) = 0$ on $U \cap \{-\infty < u < v\} \cap \{f < g\}$.

Proof. Let $\alpha, \omega, \gamma > 0$ with $\alpha > \omega$. By [18, Proposition 7],

$$\begin{aligned}&\max(0, u_j + \alpha) \max(0, v_j + \omega) \max(0, g + \gamma) H_m(u_j)\\ &= \max(0, u_j + \alpha) \max(0, v_j + \omega) \max(0, g + \gamma) H_m(\max(u_j, -\alpha))\end{aligned}$$

in Ω. Since $\max(u_j, -\alpha)$ is locally bounded, then by [17, Lemma 3.8], we get the weak convergence of $\max(0, u_j + \alpha) \max(0, v_j + \omega) \max(0, g + \gamma) H_m(u_j)$ toward

$$\max(0, u + \alpha) \max(0, v + \omega) \max(0, g + \gamma) H_m(\max(u, -\alpha))$$

in Ω.

Define $h_j := (\sup_{k \geq j} u_k)^*$ for each $j \in \mathbb{N}^*$, then $h_j \geq u_j$ and $h_j \searrow u$ in Ω.

Take $j \geq k$. Since

$$\{u_j > -\alpha\} \cap \{h_k < -\omega\} \cap \{v_j > -\omega\} \subset \{-\alpha < u_j < v_j\}$$

and

$$\{f < -\gamma\} \cap \{g > -\gamma\} \subset \{f < g\},$$

we obtain

$$\max(0, u_j + \alpha) \max(0, v_j + \omega) \max(0, g + \gamma) H_m(u_j) = 0$$

in $U \cap \{h_k < -\omega\} \cap \{f < -\gamma\}$. It follows again from [18, Proposition 7] that

$$\max(0, u + \alpha) \max(0, v + \omega) \max(0, g + \gamma) H_m(u) = 0$$

in $U \cap \{h_k < -\omega\} \cap \{f < -\gamma\}$. Letting $k \to +\infty$, we conclude that

$$H_m(u) = 0 \text{ in } U \cap \{-\alpha < u < -\omega < v\} \cap \{f < -\gamma < g\}$$

for all $\alpha, \omega, \gamma > 0$ with $\alpha > \omega$. Since

$$\bigcup_{\alpha,\omega,\gamma>0} \{-\alpha < u < -\omega < v\} \cap \{f < -\gamma < g\} = \{-\infty < u < v\} \cap \{f < g\},$$

then

$$H_m(u) = 0 \text{ on } U \cap \{-\infty < u < v\} \cap \{f < g\},$$

as required. □

Definition 15. *Consider functions $f \in \mathcal{E}_m(\Omega)$, $g \in \mathcal{E}_m(\hat{\Omega})$, and let $u \in \mathcal{F}_m(\Omega, f)$ such that $g \leq f$ holds on Ω. We define the maximal subextension of u with respect to g as the upper envelope of all m-sh functions on $\hat{\Omega}$ that are dominated by g on the entire domain and by u specifically on Ω. Formally,*

$$\widehat{u^g} := \sup \left\{ \psi \in \mathcal{SH}_m(\hat{\Omega}) : \psi \leq g \text{ in } \hat{\Omega} \text{ and } \psi \leq u \text{ on } \Omega \right\}.$$

In the rest of this paper, unless otherwise stated, we assume that $\widehat{\Omega} \Subset \mathbb{C}^n$.

Proposition 7. *Take two functions, f in $\mathcal{E}_m(\Omega)$ and g in $\mathcal{E}_m(\widehat{\Omega})$, with the property that g does not exceed f at any point of Ω. If $u \in \mathcal{F}_m^a(\Omega, f)$, then $\widehat{u^g} \in \mathcal{F}_m^a(\hat{\Omega}, g)$.*

Proof. Select $\psi \in \mathcal{F}_m^a(\Omega)$ such that $\forall x \in \Omega$ one has

$$f(x) + \psi(x) \leq u(x) \leq f(x).$$

By Proposition 2, one has $\widehat{\psi^g} \in \mathcal{F}_m^a(\hat{\Omega})$. The definition of $\widehat{u^g}$ implies that $\widehat{u^g} \leq g$ on $\widehat{\Omega}$. It is easy to check $g + \widehat{\psi} \leq g$. Moreover, since $f \geq g$ and $\widehat{\psi} \leq \psi$ on Ω, then we obtain that

$$g + \widehat{\psi} \leq f + \psi \leq u.$$

We deduce that

$$\widehat{\psi^g} + g \leq \widehat{u^g} \leq g \text{ on } \hat{\Omega}.$$

Thus, $\widehat{u^g} \in \mathcal{F}_m^a(\hat{\Omega}, g)$, completing the proof. □

We require estimates on the Hessian measures of maximal subextensions, which is an improved version of [12, Proposition 3.4].

Lemma 2. *Take two functions, f in $\mathcal{E}_m(\Omega)$ and g in $\mathcal{E}_m(\widehat{\Omega})$, with the property that g does not exceed f at any point of Ω, and $u \in \mathcal{F}_m^a(\Omega, f)$. Assume that $u \geq g$ outside a compact set $K \subset \Omega$. Then*

$$H_m(\widehat{u^g}) \leq H_m(g) + \mathbf{1}_{K \cap \{\widehat{u^g} = u\}} H_m(u) \text{ in } \hat{\Omega}.$$

Moreover, $H_m(\widehat{u^g}) = 0$ in

$$((\hat{\Omega} \setminus K) \cap \{-\infty < \widehat{u^g} < g\}) \cup (\Omega \cap \{\widehat{u^g} < \min(u, g)\}).$$

Proof. Define:

$$v := \left(\sup\{\varphi \in \mathcal{SH}_m(\Omega) : \varphi \leq u \text{ on } K\}\right)^*.$$

It follows from the fact $u \in \mathcal{E}_m(\Omega)$ and the local property of the class $\mathcal{E}_m(\Omega)$ in $\mathcal{F}_m(\Omega)$ that $v = \widehat{u} \in \mathcal{F}_m(\Omega)$. The definition of v implies that $u(z) \leq v(z)$ $\forall z \in \Omega$. Moreover, using [2, Theorem 1.2], one has $H_m(v) = 0$ on $\Omega \setminus K$. On one hand we have $v \geq u$ in Ω, which implies that $\widehat{u^g} \leq \widehat{v^g}$. On the other hand, we notice that for any φ in the set defining $\widehat{v^g}$, one has that $\varphi(z) \leq u(z)$ $\forall z \in K$. Since $u(z) \geq g(z)$ $\forall z \in \Omega \setminus K$, then $\varphi \leq u$ on $\Omega \setminus K$. It follows that $\widehat{u^g} \geq \widehat{v^g}$ and hence $\widehat{u^g} = \widehat{v^g}$ in $\hat{\Omega}$. [12, Proposition 3.4] gives:

$$H_m(\widehat{v^g}) = 0 \text{ on } \Omega \cap \{\widehat{v^g} < \min(v, g)\}.$$

Since $\{\widehat{u^g} < \min(u, g)\} \subset \{\widehat{v^g} < \min(v, g)\}$,

$$H_m(\widehat{u^g}) = 0 \text{ on } \Omega \cap \{\widehat{u^g} < \min(u, g)\}. \tag{2}$$

By [3, Theorem 3.1], there exist sequences $\{g_j\} \subset \mathcal{E}_m^0(\hat{\Omega}) \cap C(\hat{\Omega})$ with $g_j \searrow g$ on $\hat{\Omega}$, and $\{u_j\} \subset \mathcal{E}_m^0(\Omega) \cap C(\Omega)$ with u_j decreases to u in Ω. Define:

$$h_j = \begin{cases} \min(u_j, g_j), & \text{on } \Omega, \\ g_j, & \text{on } \hat{\Omega} \setminus \Omega, \end{cases}$$

and

$$\hat{u}_j := (\sup\{\psi \in \mathcal{SH}_m^-(\hat{\Omega}) : \psi \leq h_j \text{ in } \hat{\Omega}\})^*.$$

Since $h_j \in L^\infty(\hat{\Omega})$, then $\hat{u}_j \in L^\infty(\hat{\Omega}) \cap \mathcal{SH}_m^-(\hat{\Omega})$ and $\hat{u}_j \searrow \widehat{u^g}$ in $\hat{\Omega}$ when j tends to ∞. Since $h_j \in C(\hat{\Omega})$, the [12, Lemma 3.1] implies:

$$H_m(\hat{u}_j) = 0 \text{ on } \{\hat{u}_j < h_j\}.$$

Since $u \geq g$ on $\Omega \setminus K$, we have $h_j(z) \geq g(z)$ for all $z \in \Omega \setminus K$. Consequently, this inequality extends to $\hat{\Omega} \setminus K$, so $h_j(z) \geq g(z)$ holds for every $z \in \hat{\Omega} \setminus K$. Consequently,

$$H_m(\hat{u}_j) = 0 \text{ on } \Big((\hat{\Omega} \setminus K) \cap \{\hat{u}_j < g\}\Big) \subset \{\hat{u}_j < h_j\}.$$

Since $\hat{u}_j \searrow \widehat{u^g}$ then $\hat{u}_j \geq \widehat{u^g}$ and $\hat{u}_j$ converges in m-capacity to $\widehat{u^g}$. Using Proposition 6, we obtain

$$H_m(\widehat{u^g}) = 0 \text{ on } (\hat{\Omega} \setminus K) \cap \{-\infty < \widehat{u^g} < g\}. \tag{3}$$

Using Eqs (2) and (3) one can obtain that

$$H_m(\widehat{u^g}) = 0 \text{ on } ((\hat{\Omega} \setminus K) \cap \{-\infty < \widehat{u^g} < g\}) \cup (\Omega \cap \{\widehat{u^g} < \min(u, g)\}).$$

To prove the first assertion, we observe again by [12, Proposition 3.4] that

$$H_m(\widehat{u^g}) \leq \mathbf{1}_\Omega H_m(v) + H_m(g).$$

Since

$$K \cap \{\widehat{u^g} < v\} \cap \{-\infty < v\} \cap \{-\infty < g\} \cap \{\widehat{u^g} < g\} \subset \Omega \cap \{\widehat{u^g} < min(v, g)\},$$

then

$$H_m(\widehat{u^g}) \leq \mathbf{1}_{K\cap\{v=-\infty\}} H_m(v) + \mathbf{1}_{K\cap\{\widehat{u^g}=v\}\cap\{v>-\infty\}} H_m(v) + \mathbf{1}_{K\cap\{\widehat{u^g}=g\}\cap\{g>-\infty\}} H_m(v) + H_m(g).$$

Since $u \leq v$ in Ω, [20, Lemma 5.6] gives:

$$H_m(v) \leq H_m(u) \text{ on } K \cap \{v = -\infty\}. \tag{4}$$

We now claim that

$$H_m(\widehat{u^g}) \leq H_m(u) \text{ on } K \cap \{\widehat{u^g} = v\} \cap \{v > -\infty\}. \tag{5}$$

Suppose E is a compact set contained in $K \cap \{\widehat{u^g} = v\} \cap \{v > -\infty\}$. Since $v \geq u \geq \widehat{u^g}$ holds on Ω, we have the inclusion $E \subset \{\widehat{u^g} + \frac{1}{j} > u\}$ for every positive integer j. An application of [18, Proposition 7] yields:

$$\int_E H_m(\widehat{u^g}) = \lim_{j\to\infty} \int_E H_m\left(\max\left(\widehat{u^g} + \frac{1}{j}, u\right)\right) \leq \int_E H_m(\max(\widehat{u^g}, u)) = \int_E H_m(u),$$

where the inequality above is justified using the fact that the sequence of measures $H_m(\max(\widehat{u^g} + \frac{1}{j}, u))$ converges toward $H_m(\max(\widehat{u^g}, u))$ (see [3]). Thus,

$$H_m(\widehat{u^g}) \leq H_m(u) \text{ on } K \cap \{\widehat{u^g} = v\} \cap \{v > -\infty\}$$

proving the claim. Using the same reasoning as above, but for any arbitrary compact $E \subset K \cap \{\widehat{u^g} = g\} \cap \{g > -\infty\}$, one can also deduce that

$$H_m(\widehat{u^g}) \leq H_m(g) \text{ on } K \cap \{\widehat{u^g} = g\} \cap \{g > -\infty\}. \tag{6}$$

Combining Eqs (4)–(6) and the fact that

$$\{K \cap \{v = -\infty\}\} \cup \{K \cap \{\widehat{u^g} = v\} \cap \{v > -\infty\}\} \subset \{K \cap \{\widehat{u^g} = u\}\},$$

we obtain:

$$H_m(\widehat{u^g}) \quad \leq \quad \mathbf{1}_{K\cap\{v=-\infty\}} H_m(u) \; + \; \mathbf{1}_{K\cap\{\widehat{u^g}=v\}\cap\{v>-\infty\}} H_m(u) \; + \; H_m(g) \quad \leq \quad \mathbf{1}_{K\cap\{\widehat{u^g}=u\}} H_m(u) \; + \; H_m(g)$$

completing the proof. □

Lemma 3. *Let $\{L_j\}$ be a fundamental sequence associated to Ω. For $u \in \mathcal{F}_m^a(\Omega)$, define:*

$$u_j := (\sup\{\varphi \in \mathcal{SH}_m^-(\Omega) : \varphi \leq u \text{ on } \Omega \setminus L_j\})^*.$$

Then $\widehat{u_j}$ increases almost everywhere to 0 in $\hat{\Omega}$ as $j \to \infty$.

Proof. We have $u_j \nearrow 0$ almost everywhere in Ω. Then by Proposition 1, we find that $\widehat{u_j}$ is increasing almost everywhere to $\varphi := (\sup_{j\geq 1} \widehat{u_j})^*$ in $\hat{\Omega}$. Since u belongs to $\mathcal{F}_m(\Omega)$ and $u_j \geq u$, so does u_j. Hence, [12, Lemma 3.2] implies

$$\mathbf{1}_{\Omega\cap\{\widehat{u}_j=u_j\}} H_m(u_j) \geq H_m(\widehat{u_j}) \text{ in } \hat{\Omega}. \tag{7}$$

As $u_j \to 0$ almost everywhere in Ω then $\max(u_j, -1)$ converges almost everywhere. to 0. Given that the complex Hessian operator $H_m(u)$ has the property of vanishing on all m-polar sets, then the work [17, Lemma 3.3] yields:

$$\lim_{j\to\infty} \int_\Omega - \max(u_j, -1) H_m(u) = 0. \tag{8}$$

By [20, Theorem 3.8], the sequence $\{\max(\varphi, -1)H_m(\widehat{u_j})\}$ converges weakly to $\max(\varphi, -1)H_m(\varphi)$. Using Eq (7) and [22, Lemma 2.7], we obtain:

$$\begin{aligned}\int_{\widehat{\Omega}} - \max(\varphi, -1) H_m(\varphi) &\leq \limsup_{j\to\infty} \int_{\widehat{\Omega}} - \max(\varphi, -1) H_m(\widehat{u_j}) \leq \limsup_{j\to\infty} \int_{\widehat{\Omega}} - \max(\widehat{u_j}, -1) H_m(\widehat{u_j}) \\ &\leq \limsup_{j\to\infty} \int_\Omega - \max(u_j, -1) H_m(u_j) \leq \limsup_{j\to\infty} \int_\Omega - \max(u_j, -1) H_m(u).\end{aligned}$$

Now by Eq (8), the last term of the above estimate vanishes. Hence, $-\max(\varphi, -1)H_m(\varphi) = 0$ in $\hat{\Omega}$. Finally, since $\varphi \in \mathcal{F}_m(\Omega)$, then by [20, Remark 5.4], one has $\varphi = 0$ in $\hat{\Omega}$, which completes the proof. □

Theorem 4. *Assume that $\hat{\Omega} \Subset \mathbb{C}^n$, and take $f \in \mathcal{E}_m(\Omega)$, $g \in \mathcal{E}_m(\hat{\Omega})$ and $w \in \mathcal{F}_m^a(\Omega, f)$ satisfying $g(z) \leq f(z)$ for all $z \in \Omega$ and*

$$\int_{\widehat{\Omega}} H_m(g) + \int_\Omega H_m(w) < +\infty.$$

Suppose $\{u_j\} \subset \mathcal{F}_m^a(\Omega, f)$ is a sequence bounded below by w for all j, and converging to u in m-capacity within Ω. Then, $\widehat{u_j^g} \to \widehat{u^g}$ in m-capacity in $\hat{\Omega}$.

Proof. The condition $u_j \geq w$ throughout Ω allows us to apply from [24, Lemma 2.5], which yields a subsequence $\{\widehat{u}^g_{j_k}\}$ with local L^1-limit ψ, where ψ is m-sh. Furthermore, by selecting an appropriate subsequence, we obtain $\widehat{u}^g_{j_k} \to \psi$ almost everywhere in $\hat{\Omega}$.

Set

$$\psi_k := \left(\sup_{l\geq k} \widehat{u}^g_{j_l}\right)^*$$

on $\hat{\Omega}$ and

$$v_k := \left(\sup_{l\geq k} u_{j_l}\right)^*$$

on Ω.

Take $\varphi \in C(\Omega) \cap \mathcal{E}^0_m(\Omega)$ and $\hat{\varphi} \in C(\hat{\Omega}) \cap \mathcal{E}^0_m(\hat{\Omega})$ with the property that $K \subset \Omega \cap \{\hat{\varphi} = \varphi\}$. Let $\lambda > 0$. Since $\widehat{u}^g_{j_k} \leq \psi_k \leq g$ in $\hat{\Omega}$,

$$\max\left(\frac{\psi_k}{\lambda}, \hat{\varphi}\right) = \max\left(\frac{\widehat{u}^g_{j_k}}{\lambda}, \hat{\varphi}\right) = \hat{\varphi} \text{ on } \hat{\Omega} \cap \{g = -\infty\}.$$

First, consider the case where $w \geq g$ outside some compact subset K of Ω. In this case, we get that $u_{j_k} \geq g$ on $\Omega \setminus K$, then by Lemma 2 we get

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{\hat{\Omega}} \left[\max\left(\frac{\psi_k}{\lambda}, \hat{\varphi}\right) - \max\left(\frac{\widehat{u}^g_{j_k}}{\lambda}, \hat{\varphi}\right)\right] H_m(\widehat{u}^g_{j_k}) \\
&\leq \limsup_{k\to\infty} \int_{K\cap\{\widehat{u}^g_{j_k} = u_{j_k}\}} \left[\max\left(\frac{\psi_k}{\lambda}, \hat{\varphi}\right) - \max\left(\frac{\widehat{u}^g_{j_k}}{\lambda}, \hat{\varphi}\right)\right] H_m(u_{j_k}) \\
&\quad + \limsup_{k\to\infty} \int_{\hat{\Omega}} \left[\max\left(\frac{\psi_k}{\lambda}, \hat{\varphi}\right) - \max\left(\frac{\widehat{u}^g_{j_k}}{\lambda}, \hat{\varphi}\right)\right] H_m(g) \\
&\leq \limsup_{k\to\infty} \int_{\Omega} \left[\max\left(\varphi, \frac{v_k}{\lambda}\right) - \max\left(\varphi, \frac{u_{j_k}}{\lambda}\right)\right] H_m(u_{j_k}) \\
&\quad + \limsup_{k\to\infty} \int_{\hat{\Omega}\cap\{g>-\infty\}} \left[\max\left(\frac{\psi_k}{\lambda}, \hat{\varphi}\right) - \max\left(\frac{\widehat{u}^g_{j_k}}{\lambda}, \hat{\varphi}\right)\right] H_m(g).
\end{aligned}$$

The first term of the last line vanishes because u_{j_K} converges to u in m-capacity, together with [17, Theorem A]. For the second term, it also vanishes using [17, Lemma 3.3]. We deduce that

$$\lim_{k\to\infty} \int_{\hat{\Omega}} \left[\max\left(\frac{\psi_k}{\lambda}, \hat{\varphi}\right) - \max\left(\frac{\widehat{u}^g_{j_k}}{\lambda}, \hat{\varphi}\right)\right] H_m(\widehat{u}^g_{j_k}) = 0.$$

Therefore, again by [17, Theorem A], it follows that $\widehat{u}^g_{j_k} \to H$ in capacity in $\hat{\Omega}$.

To finish this particular case, it suffices to prove that $\psi = \widehat{u}^g$. The first inequality: $\psi \leq \widehat{u}^g$ follows from the fact that $\psi_k \leq v_k$ on Ω. For the converse, we again apply Lemma 2 to get that

$$H_m(\widehat{u}^g_{j_k}) = 0 \text{ on } (\hat{\Omega} \setminus K) \cap \{\widehat{u}^g_{j_k} < g\}) \cup (\Omega \cap \{\widehat{u}^g_{j_k} < u_{j_k}\}),$$

for all $k \geq 1$. Since $\widehat{u}^g_{j_k} \geq \widehat{w}^g$ for all $k \geq 1$, according to Proposition 6, we have

$$H_m(\psi) = 0 \text{ on } (\hat{\Omega} \setminus K) \cap \{-\infty < \psi < g\}) \cup (\Omega \cap \{-\infty < \psi < u\}).$$

This, combined with the bounds $\widehat{u^g} \le u$ in Ω and $\widehat{u^g} \le g$ in $\hat{\Omega}$, leads to

$$H_m(\psi) = 0 \ \text{ on } \ \{-\infty < \psi < \widehat{u^g}\}.$$

Since $\widehat{u^g}, \psi \in \mathcal{F}_m^a(\hat{\Omega}, g)$, Lemma 1 implies that $\widehat{u^g} \le \psi$ in $\hat{\Omega}$, and hence $\widehat{u^g} = \psi$ in $\hat{\Omega}$. We deduce that $\widehat{u_{j_k}^g} \to \widehat{u^g}$ in capacity in $\hat{\Omega}$. This finishes the proof in the particular case when the function w dominates g outside some compact subset K of Ω.

To treat the general case, let $\{L_k\}$ be the fundamental sequence associated to Ω. The hypothesis $w \in \mathcal{F}_m^a(f, \Omega)$ implies the existence of a function $\xi \in \mathcal{F}_m^a(\Omega)$ satisfying the inequalities $\xi + f \le w \le f$ throughout Ω. Take

$$\xi_k := (\sup\{\varphi \in \mathcal{SH}_m^-(\Omega) : \varphi \le \psi \text{ on } \Omega\backslash L_k\})^*.$$

By Lemma 3 the sequence $\widehat{\xi_k}$ increases almost everywhere to 0 in $\hat{\Omega}$. Since

$$u_j \ge w \ge f + \xi_k \ge g + \widehat{\xi_k} \text{ on } \Omega\backslash L_k,$$

and using the reasoning above (made in the particular case), we deduce that $\widehat{u_j}^{g+\widehat{\xi_k}} \to \widehat{u^{g+\widehat{\xi_k}}}$ in m-capacity in $\hat{\Omega}$. Moreover one has

$$|\widehat{u_j}^g - \widehat{u^g}| = |\widehat{u_j}^g - \widehat{u_j}^{g+\widehat{\xi_k}} + \widehat{u_j}^{g+\widehat{\xi_k}} - \widehat{u^{g+\widehat{\xi_k}}} + \widehat{u^{g+\widehat{\xi_k}}} - \widehat{u^g}| \le |\widehat{u_j}^{g+\widehat{\xi_k}} - \widehat{u^{g+\widehat{\xi_k}}}| - 2\widehat{\xi_k} \text{ in } \hat{\Omega}$$

for all $k \ge 1$, hence $\widehat{u_j}^g \to \widehat{u^g}$ in capacity in $\hat{\Omega}$. This completes the proof. □

The following example demonstrates that the uniform lower boundedness hypothesis in Theorem 4 is essential even when $m = n = 1$, as its failure leads to the divergence of the maximal subextensions.

Example 2. *Let $\Omega = D(0,1)$, $\widehat{\Omega} = D(0,2)$ and $f = g \equiv 0$. Define*

$$u(z) = \log|z|, \qquad u_j(z) = \max(\log|z|, -j), \quad j \ge 1.$$

Then $\{u_j\} \subset \mathcal{F}_m(\Omega, f)$ converges to u in $L^1_{\text{loc}}(\Omega)$ and in m-capacity, but it is not uniformly bounded below.

Since u is radial, set $t = \log|z|$, and define $v(t) = t$ for $t < 0$. To find $\widehat{u^g}$, it remains to look for the largest convex function $\widehat{v}$ on $(-\infty, \log 2)$ satisfying $\widehat{v}(t) \le 0$ and $\widehat{v}(t) \le t$ for $t < 0$. The solution is the line through $(\log 2, 0)$ with slope 1:

$$\widehat{v}(t) = t - \log 2.$$

Returning to the z-variable gives the maximal subextension

$$\widehat{u^g}(z) = \log|z| - \log 2, \qquad z \in \widehat{\Omega}.$$

For $t = \log|z|$, let $v_j(t) = \max(t, -j)$. The largest convex function $\widehat{v_j}$ on $(-\infty, \log 2)$ with $\widehat{v_j} \le 0$ and $\widehat{v_j} \le v_j$ on $t < 0$ is piecewise linear: constant $-j$ for $t \le -j$, and the line joining $(-j, -j)$ to $(\log 2, 0)$ for $-j \le t \le \log 2$. In the z-variable, this yields

$$\widehat{u_j^g}(z) = \begin{cases} -j, & |z| \le e^{-j}, \\ \dfrac{j(\log|z| - \log 2)}{\log 2 + j}, & e^{-j} < |z| \le 2. \end{cases}$$

In particular, $\widehat{u_j^g}(0) = -j \to -\infty$ while $\widehat{u^g}(0) = -\log 2$; hence, $\widehat{u_j^g}$ does not converge to $\widehat{u^g}$ in capacity on $\widehat{\Omega}$.

Theorem 5. *Suppose that $\Omega \Subset \widehat{\Omega} \subset \mathbb{C}^n$. Take $f \in \mathcal{E}_m(\Omega)$ and $\{u_j\} \subset \mathcal{F}_m(\Omega, f)$ satisfying $u_j \geq w \in \mathcal{F}_m(\Omega, f)$ for all $j \geq 1$ and $g \in \mathcal{E}_m(\hat{\Omega})$ be such that $f \geq g$ on Ω and*

$$\int_\Omega H_m(w) + \int_{\widehat{\Omega}} H_m(g) < +\infty.$$

If $\{u_j\}$ converges to u in $L^1_{\mathrm{loc}}(\Omega)$, then $\widehat{u}_j^g$ converges to $\widehat{u}^g$ in $L^1_{\mathrm{loc}}(\widehat{\Omega})$.

Proof. Define $v_k = (\sup\{u_j : j \geq k\})^*$ for $k \geq 1$. Then $v_k \in \mathcal{F}_m(\Omega, f)$ and $v_k \geq w$ on Ω for all $k \geq 1$. By the proof of [21, Lemma 1.8.3], the local L^1-convergence $u_j \to u$ implies $v_k \searrow u$ as $k \to \infty$, which in turn gives convergence in m-capacity. Theorem 4 ensures that $\widehat{v}_k^g \to \widehat{u}^g$ in m-capacity within $\widehat{\Omega}$, hence also in Lebesgue measure. By Proposition 1, we deduce that $\widehat{w}^g \leq \widehat{v}_k^g \leq 0$ and $|\widehat{v}_k^g| \leq |\widehat{w}^g|$. As $\widehat{w}^g \in L^1_{\mathrm{loc}}(\widehat{\Omega})$, then by the Lebesgue dominated convergence theorem, we deduce that for any compact subset $K \subset \widehat{\Omega}$, one has

$$\int_K |\widehat{v_k}^g - \widehat{u}^g| \, d\lambda = \int_K (\widehat{v_k}^g - \widehat{u}^g) \, d\lambda \to 0.$$

Since $\widehat{v_k}^g \geq \widehat{u_k}^g$ for all $k \geq 1$, it follows that for any compact $K \subset \widehat{\Omega}$,

$$0 \leq \int_K (\widehat{u_k}^g - \widehat{u}^g) \, d\lambda \leq \int_K (\widehat{v_k}^g - \widehat{u}^g) \, d\lambda \to 0$$

as $k \to \infty$, completing the proof. □

We deduce the following result in the classical Cegrell m-sh classes:

Corollary 1. *Let $\{u_j\}_{j\geq 1}$, $u \in \mathcal{F}_m(\Omega)$ satisfy $u_j \geq u$ for all $j \geq 1$. Suppose the sequence $\{u_j\}$ converges to u with respect to the L^1_{loc}-topology on Ω. Then the corresponding subextensions $\widehat{u}_j$ converge to $\widehat{u}$ in the L^1_{loc} sense on the larger domain $\widehat{\Omega}$.*

Theorem 6. *Let $\Omega \subset \hat{\Omega} \subset \mathbb{C}^n$, $f \in \mathcal{E}_m(\Omega)$, and $g \in \mathcal{E}_m(\widehat{\Omega}) \cap \mathcal{MSH}_m(\widehat{\Omega})$ with $f \geq g$ in Ω. Take $\{u_j\}_{j\geq 1}$, $u \in \mathcal{F}_m(\Omega, f)$ and $\widetilde{u}_j, \widetilde{u}$ such that*

$$H_m(\widetilde{u}_j) = \mathbf{1}_\Omega H_m(u_j)$$

and

$$H_m(\widetilde{u}) = \mathbf{1}_\Omega H_m(u).$$

Assume that $u_j \geq u$ for all $j \geq 1$, $\int_\Omega H_m(u) < \infty$, and $u_j \to u$ in m-capacity in Ω, then $H_m(\widetilde{u}_j)$ converges weakly to $H_m(\widetilde{u})$.

Proof. The existence of $\widetilde{u}_j, \widetilde{u}$ such that

$$H_m(\widetilde{u}_j) = \mathbf{1}_\Omega H_m(u_j)$$

and

$$H_m(\widetilde{u}) = \mathbf{1}_\Omega H_m(u)$$

follows directly from [11, Theorem 1.1].

Fix $\varphi \in C_0^\infty(\widehat{\Omega})$. Using [3, Lemma 3.10], there exists $\varphi_1, \varphi_2 \in \mathcal{E}_m^0(\widehat{\Omega})$ such that $\varphi = \varphi_1 - \varphi_2$. Hence,

$$\lim_{j\to\infty} \int_{\widehat{\Omega}} \varphi H_m(\widetilde{u}_j) = \lim_{j\to\infty} \int_\Omega \varphi_1 H_m(u_j) - \lim_{j\to\infty} \int_\Omega \varphi_2 H_m(u_j). \tag{9}$$

Now by [20, Theorem 3.8], we deduce that

$$\lim_{j\to\infty} \left(\int_\Omega \varphi_1 H_m(u_j) - \int_\Omega \varphi_2 H_m(u_j) \right) = \int_\Omega \varphi_1 H_m(u) - \int_\Omega \varphi_2 H_m(u).$$

The equality (9) becomes as follows

$$\lim_{j\to\infty} \int_{\widehat{\Omega}} \varphi H_m(\widetilde{u}_j) = \int_\Omega \varphi H_m(u) = \int_{\widehat{\Omega}} \varphi H_m(\widetilde{u}).$$

The result follows. □

The previous result fails if we omit the condition

$$H_m(\widetilde{u}_j) = \mathbf{1}_\Omega H_m(u_j),$$

as shown in the following example.

Example 3. *Let $\Omega = D(0,1) \subset \mathbb{C}$ and $\widehat{\Omega} = D(0,2)$. For each integer $j \geq 1$, consider the function $u_j(z) \equiv 0$ on Ω, so that $u_j \to u \equiv 0$ in C_1-capacity. Since*

$$|z - \frac{3}{2}e^{\frac{i}{j}}| \leq |z| + |\frac{3}{2}e^{\frac{i}{j}}| \leq 1 + \frac{3}{2} = \frac{5}{2}$$

for all $z \in \Omega$, then

$$\log|z - \frac{3}{2}e^{\frac{i}{j}}| - \log(\frac{5}{2}) \leq 0 = u_j(z).$$

Hence,

$$\widetilde{u}_j := \log|z - \frac{3}{2}e^{\frac{i}{j}}| - \log(\frac{5}{2})$$

is a subextension of u_j on $\widehat{\Omega}$.

We have

$$H_1(\widetilde{u}_j) = 2\pi\, \delta_{\frac{3}{2}e^{\frac{i}{j}}}.$$

Thus, the measures $H_1(\widetilde{u}_j)$ converge weakly to $2\pi\, \delta_{\frac{3}{2}}$.

The natural subextension of u in $\widehat{\Omega}$ is $\widetilde{u}(z) \equiv 0$, so $H_1(\widetilde{u}) = 0$. We deduce that $H_1(\widetilde{u}_j) \not\rightharpoonup H_1(\widetilde{u})$.

Corollary 2. *Let $\Omega \subset \hat{\Omega} \subset \mathbb{C}^n$. Take $\{u_j\}_{j\geq 1}$, $u_j \geq u$, and $u \in \mathcal{F}_m(\Omega)$ such that each of $H_m(u)$ and $H_m(\widehat{u}_j)$ is carried by an m-polar subset of Ω. Assume that $u_j \to u$ in m-capacity in Ω, then $H_m(\widehat{u}_j)$ converges weakly to $H_m(\widehat{u})$.*

Proof. Using in [12, Proposition 3.3], we deduce that

$$H_m(\widehat{u}) = \mathbf{1}_\Omega H_m(u)$$

and

$$H_m(\widehat{u}_j) = \mathbf{1}_\Omega H_m(u_j).$$

The result follows by applying Theorem 6. □

5. Conclusions

This paper advanced the theory of maximal subextensions for m-sh functions. We provided a complete characterization of the maximal subextension in $\mathcal{F}_m^a(\Omega)$, established its relationship with Lelong number invariance in $\mathcal{E}_m(\Omega)$, and proved strong convergence stability under capacity and L^1_{loc} limits. These results unify and extend previous work, offering new tools for the study of complex Hessian operators and m-pluripotential theory. The developed framework opens several natural research directions, such as studying boundary regularity of maximal subextensions, extending the results to Hermitian or almost complex manifolds, and investigating applications to complex Hessian equations with prescribed singularity.

Author contributions

Jawhar Hbil: conceptualization of the maximal subextension problem for m-subharmonic functions, development of the characterization theorem for $\mathcal{F}_m^a(\Omega)$ and the Lelong number preservation result in $\mathcal{E}_m(\Omega)$, and writing original draft; Mofareh Alhazmi: development of the comparison principle in $\mathcal{F}_m^a(\Omega, f)$, analysis of the singular sets and Hessian measure estimates; Mohamed Zaway: development of the convergence theorems for maximal subextensions with fixed boundary data, analysis of the stability under capacity and L^1_{loc} limits, and writing, review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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