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*Research article*

# Generalized existence and uniqueness results for nonlinear Caputo fractional boundary value problems

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**Abstract:** In this paper, we study a class of nonlinear fractional boundary value problems involving the Caputo fractional derivative of order $\tau \in (1, 2]$. We establish sufficient conditions for the existence and uniqueness of solutions. Our analysis is based on Krasnoselskii's fixed-point theorem and the Banach contraction principle in appropriate Banach spaces. The obtained results not only guarantee the solvability of the proposed equation but also generalize and improve several related results available in the literature. An example is provided in several orders to validate the results.

**Keywords:** fractional differential equations; differential equations; fractional derivatives; existence; contraction principle
**Mathematics Subject Classification:** 26A33, 34A08, 34A12, 34A34, 34K37

## 1. Introduction

Fractional derivatives (FDs) have become an essential tool for modeling complex real-world problems. The involvement of nonlocal behavior in different research areas makes it more attractive for scientists and researchers. The study of nonlinear fractional differential equations (FDEs) is of great importance because it provides a rigorous mathematical framework for modeling real-world systems that exhibit memory, heredity, and nonlocal effects including phenomena that classical integer-order models fail to capture [1, 2]. Fractional derivatives enable a more realistic description of complex dynamical behaviors observed in physics, biology, engineering, and finance, such as anomalous diffusion, viscoelasticity, population dynamics with delayed responses, and long-range dependence in financial markets. Understanding the existence and uniqueness of solutions to such systems is crucial to ensure the well-posedness of models, validate analytical predictions, and guarantee that numerical simulations are meaningful and reliable [3, 4].

Ahmad and Nieto [5] were among the first to establish existence results for Caputo-type FDEs with antiperiodic boundary conditions, employing topological techniques such as the Leray–Schauder degree. Their work underscored the significance of nonlocal constraints in the study of fractional systems and demonstrated their relevance across various areas of physics and applied mathematics. Subsequently, Agarwal, Ahmad, and Nieto [6] advanced the field by formulating nonlocal parametric antiperiodic boundary conditions. This approach generalized the classical antiperiodic framework by introducing an intermediate point in addition to fixed endpoints, and the authors proved existence results using fixed-point theory.

Fixed-point theorems play a central role in the analysis of fractional differential equations, particularly in establishing the existence and uniqueness of solutions for nonlinear boundary value problems. Due to the nonlocal and integral nature of fractional operators, explicit solutions are rarely obtainable, making functional analytic techniques essential. By transforming fractional differential equations into equivalent integral equations, classical tools such as Banach's contraction principle, Schauder's fixed-point theorem, and Krasnoselskii's fixed-point theorem can be applied to prove solvability under suitable continuity, compactness, and Lipschitz conditions. These approaches provide a rigorous and flexible framework for handling nonlinearities and complex boundary conditions and have become standard methods in the modern theory of fractional differential equations [7–9].

This paper studies the existence of a solution of the following problem:

$$\begin{cases} {}^cD^\tau y(x) = Y(x, y(x)),\ \tau \in (1,2],\ 0 \le x \le z,\ w \in (0,z), \\ \alpha y(0) + \beta y(w) = -\gamma y(z),\ \alpha,\beta,\gamma \ge 0,\ \alpha+\beta+\gamma \ne 0, \\ \delta y'(0) + \epsilon y'(w) = -\zeta y'(z),\ \delta,\epsilon,\zeta \ge 0,\ \delta+\epsilon+\zeta \ne 0, \end{cases} \tag{1.1}$$

where ${}^cD^\tau$ is the Caputo derivative of order $\tau$, and $Y : [0, z] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. This study is based on the Krasnoselskii fixed-point theorem.

This study is of high importance, as it develops a generalized framework for establishing existence and uniqueness results for nonlinear fractional boundary value problems with mixed-boundary conditions. By employing a combination of Krasnoselskii's fixed-point theorem and the Banach contraction principle, the study provides analytically verifiable criteria that extend and unify previous results in the literature. This research work will strengthen the theoretical foundation of fractional calculus and will facilitate its application to a wide range of practical problems, including viscoelastic material modeling, anomalous transport phenomena, neuronal signal processing, and geophysical wave propagation. Thus, the present work enhances both the mathematical framework of fractional differential equations and the construction of practical, memory-inclusive models for real-world applications, paving the way for future research in the analysis, control, and simulation of complex memory-dependent systems across diverse scientific and engineering domains.

The rest of the paper is organized as follows: In Section 2, we present the basic theory, concepts and results in the form of Theorems and Lemmas for the analysis of our work. The main results regarding the existence and uniqueness of the solution has been proved through analytical skills in Section 3. In Section 4, we presented examples in the support of main theoretical results. Finally, the concluding remarks along with future directions are presented in Section 5.

## 2. Preliminary

**Definition 2.1.** [10] The Caputo fractional derivative (CFD) of order $\tau > 0$, denoted ${}^cD^\tau$, for a given function $y \in C^k([0,z])$, is defined by

$$ {}^cD^\tau y(x) = \frac{1}{\Gamma(k-\tau)} \int_0^x (x-\nu)^{k-\tau-1} y^{(k)}(\nu) d\nu. $$

**Definition 2.2.** [10] The Riemann-Liouville fractional integral $I^\tau$ of a function $y(x)$, is defined by

$$ I^\tau y(x) = \frac{1}{\Gamma(\tau)} \int_0^x (x-\nu)^{\tau-1} y(\nu) d\nu. $$

**Theorem 2.1.** *[11] Let $\mathcal{A}$ be a closed, nonempty, convex subset of a Banach space $\Omega$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be operators on $\mathcal{A}$ and satisfying (i) for all $y_1, y_2 \in \mathcal{A}$, $\mathcal{F}_1 y_1 + \mathcal{F}_2 y_2 \in \mathcal{A}$; (ii) the operator $\mathcal{F}_1$ is compact and continuous; and (iii) $\mathcal{F}_2$ is a contraction. Then, there exists a $y \in \mathcal{A}$ such that $y = \mathcal{F}_1 y + \mathcal{F}_2 y$.*

**Lemma 2.2.** *[10] $y(x) = \sum_{i=0}^n e_i x^i$ is the general solution of ${}^cD^\tau y(x) = 0$, $\tau > 0$, $n = [\tau] - 1$, and $e_i \in \mathbb{R}$, where $[\tau]$ is the integer part of the real number $\tau$.*

**Lemma 2.3.** *For any $f \in C[0,z]$, the unique solution of*

$$ \begin{cases} {}^cD^\tau y(x) = f(x), \ \tau \in (1,2], \ 0 \le x \le z, \ w \in (0,z), \\ \alpha y(0) + \beta y(w) = -\gamma y(z), \quad \alpha, \beta, \gamma \ge 0, \alpha + \beta + \gamma \ne 0, \\ \delta y'(0) + \epsilon y'(w) = -\zeta y'(z), \quad \delta, \epsilon, \zeta \ge 0, \quad \delta + \epsilon + \zeta \ne 0, \end{cases} \tag{2.1} $$

*is given by*

$$ \begin{aligned} y(x) = & \int_0^x \frac{(x-\nu)^{\tau-1}}{\Gamma(\tau)} f(\nu)\, d\nu - \frac{1}{\alpha+\beta+\gamma} \Big[ \frac{\beta}{\Gamma(\tau)} \int_0^w (w-\nu)^{\tau-1} f(\nu)\, d\nu + \frac{\gamma}{\Gamma(\tau)} \int_0^z (z-\nu)^{\tau-1} f(\nu)\, d\nu \Big] \\ & + \frac{(\beta w + \gamma z) - (\alpha+\beta+\gamma)x}{(\alpha+\beta+\gamma)(\delta+\epsilon+\zeta)} \Big[ \frac{\epsilon}{\Gamma(\tau-1)} \int_0^w (w-\nu)^{\tau-2} f(\nu)\, d\nu + \frac{\zeta}{\Gamma(\tau-1)} \int_0^z (z-r)^{\tau-2} f(\nu)\, d\nu \Big]. \end{aligned} \tag{2.2} $$

*Proof.* Using Lemma (2.2), the solution of (2.1) is given by

$$ y(x) = \frac{1}{\Gamma(\tau)} \int_0^x (x-\nu)^{\tau-1} f(\nu) d\nu - e_0 - e_1 x. $$

By applying the boundary conditions, we get

$$ \begin{aligned} e_0 = & \frac{1}{\alpha+\beta+\gamma} \Big[ \frac{\beta}{\Gamma(\tau)} \int_0^w (w-r)^{\tau-1} f(\nu) d\nu + \frac{\gamma}{\Gamma(\tau)} \int_0^z (z-\nu)^{\tau-1} f(\nu) d\nu \Big] \\ & - \frac{\beta w + \gamma z}{(\alpha+\beta+\gamma)(\delta+\epsilon+\zeta)} \Big[ \frac{\epsilon}{\Gamma(\tau-1)} \int_0^w (w-\nu)^{\tau-2} f(\nu) d\nu + \frac{\zeta}{\Gamma(\tau-1)} \int_0^z (z-\nu)^{\tau-2} f(\nu) d\nu \Big], \\ e_1 = & \frac{1}{\delta+\epsilon+\zeta} \Big[ \frac{\epsilon}{\Gamma(\tau-1)} \int_0^w (w-\nu)^{\tau-2} f(\nu) d\nu + \frac{\zeta}{\Gamma(\tau-1)} \int_0^z (z-\nu)^{\tau-2} f(\nu) d\nu \Big]. \end{aligned} $$

Substituting $e_0$ and $e_1$ into the solution, we arrive at (2.2). □

## 3. Main results

Let $C = C([0, z], \mathbb{R})$ denote the Banach space of all continuous function, and $Y : [0, z] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose the following:
$(S_1)$ $|Y(x, y) - Y(x, \hat{y})| \leq L|y - \hat{y}|$, $\forall x \in [0, z]$, $y, \hat{y} \in \mathbb{R}$, and $L > 0$.
$(S_2)$ $|Y(x, y)| \leq \sigma(x)$ for all $(x, y) \in [0, z] \times \mathbb{R}$, and $\sigma \in L^1([0, z], \mathbb{R}^+)$.
Define the operator $(\mathcal{F}y)(x) : C \to C$ as

$$\begin{aligned}(\mathcal{F}y)(x) = &\int_0^x \frac{(x-\nu)^{\tau-1}}{\Gamma(\tau)} Y(\nu, y(\nu))\, d\nu \\ &- \frac{1}{\alpha+\beta+\gamma}\Big[\frac{\beta}{\Gamma(\tau)}\int_0^w (w-\nu)^{\tau-1} Y(\nu, y(\nu))\, d\nu + \frac{\gamma}{\Gamma(\tau)}\int_0^z (z-\nu)^{\tau-1} Y(\nu, y(\nu))\, d\nu\Big] \\ &+ \frac{(\beta w + \gamma z) - (\alpha+\beta+\gamma)x}{(\alpha+\beta+\gamma)(\delta+\epsilon+\zeta)}\Big[\frac{\epsilon}{\Gamma(\tau-1)}\int_0^w (w-\nu)^{\tau-2} Y(\nu, y(\nu))\, d\nu \\ &+ \frac{\zeta}{\Gamma(\tau-1)}\int_0^z (z-\nu)^{\tau-2} Y(\nu, y(\nu))\, d\nu\Big]. \end{aligned} \tag{3.1}$$

**Theorem 3.1.** *Assume that $Y : [0, z] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $(S_1)$ holds. Then, (1.1) has a unique solution on $[0, z]$ if $LN_1 < 1$, where*

$$N_1 = \frac{(\alpha+\beta+2\gamma)(\delta+\epsilon+\zeta)z^\tau + \beta(\delta+\epsilon+\zeta)w^\tau + M(\tau\epsilon w^{\tau-1} + \tau\zeta z^{\tau-1})}{\Gamma(\tau+1)(\alpha+\beta+\gamma)(\delta+\epsilon+\zeta)}$$

*and $M = \max_{x\in[0,z]}\{\beta w + \gamma z, \alpha z + \beta(z - w)\}$.*

*Proof.* Suppose $\sup_{x\in[0,z]} |Y(x, 0)| = K$ and define $\mathcal{F}$ as in (3.1). Also, let $B_r = \{Y \in C : \|Y\| \leq r\}$, where $r \geq KN_1/(1 - LN_1)$ is a fixed radius. Therefore, we obtain

$$\begin{aligned}\|(\mathcal{F}y)(x)\| \leq \max_{x\in[0,z]}\Big[\Big[&\int_0^x \frac{(x-\nu)^{\tau-1}}{\Gamma(\tau)} \left(|Y(\nu, y) - Y(\nu, 0)| + |Y(\nu, 0)|\right) d\nu\Big] \\ &+ \frac{1}{\alpha+\beta+\gamma}\Big[\frac{\beta}{\Gamma(\tau)}\int_0^w (w-\nu)^{\tau-1} \left(|Y(\nu, y) - Y(\nu, 0)| + |Y(\nu, 0)|\right) d\nu \\ &\quad + \frac{\gamma}{\Gamma(\tau)}\int_0^z (z-\nu)^{\tau-1} \left(|Y(\nu, y) - Y(\nu, 0)| + |Y(\nu, 0)|\right) d\nu\Big] \\ &+ \frac{|(\beta w + \gamma z) - (\alpha+\beta+\gamma)x|}{(\alpha+\beta+\gamma)(\delta+\epsilon+\zeta)}\Big[\frac{\epsilon}{\Gamma(\tau-1)}\int_0^w (w-\nu)^{\tau-2} \left(|Y(\nu, y) - Y(\nu, 0)| + |Y(\nu, 0)|\right) d\nu \\ &\quad + \frac{\zeta}{\Gamma(\tau-1)}\int_0^z (z-\nu)^{\tau-2} \left(|Y(\nu, y) - Y(\nu, 0)| + |Y(\nu, 0)|\right) d\nu\Big]\Big].\end{aligned}$$

Thus, we find

$$\|(\mathcal{F}y)(x)\| \leq \frac{(Lr+K)\Big[(\alpha+\beta+2\gamma)(\delta+\epsilon+\zeta)z^\tau + \beta(\delta+\epsilon+\zeta)w^\tau + M\Big(\tau\epsilon w^{\tau-1} + \tau\zeta z^{\tau-1}\Big)\Big]}{\Gamma(\tau+1)(\alpha+\beta+\gamma)(\delta+\epsilon+\zeta)},$$

which yields

$$\|(\mathcal{F}y)(x)\| \leq (Lr+K)N_1 \leq r.$$

Now, for each $x \in [0, z]$ and $y_1, y_2 \in C$, we have

$$\begin{aligned}
\|(\mathcal{F}y_1)(x) - (\mathcal{F}y_2)(x)\| \leq \max_{x\in[0,z]}\Bigg[\Big[ & \int_0^x \frac{(x-\nu)^{\tau-1}}{\Gamma(\tau)} (|Y(\nu, y_1(\nu)) - Y(\nu, y_2(\nu))|)\, d\nu\Big] \\
& + \frac{1}{\alpha+\beta+\gamma}\Big[\frac{\beta}{\Gamma(\tau)} \int_0^w (w-\nu)^{\tau-1} (|Y(\nu, y_1(\nu)) - Y(\nu, y_2(\nu))|)\, d\nu \\
& \quad + \frac{\gamma}{\Gamma(\tau)} \int_0^z (z-\nu)^{\tau-1} (|Y(\nu, y_1(\nu)) - Y(\nu, y_2(\nu))|)\, d\nu\Big] \\
& + \frac{|(\beta w + \gamma z) - (\alpha+\beta+\gamma)x|}{(\alpha+\beta+\gamma)(\delta+\epsilon+\zeta)}\Big[\frac{\epsilon}{\Gamma(\tau-1)} \int_0^w (w-\nu)^{\tau-2} (|Y(\nu, y_1(\nu)) - Y(\nu, y_2(\nu))|)\, d\nu \\
& \quad + \frac{\zeta}{\Gamma(\tau-1)} \int_0^z (z-\nu)^{\tau-2} (|Y(\nu, y_1(\nu)) - Y(\nu, y_2(\nu))|)\, d\nu\Big]\Bigg]
\end{aligned}$$

$$\leq \frac{L\|y_1 - y_2\|\Big[(\alpha+\beta+2\gamma)(\delta+\epsilon+\zeta)z^\tau + \beta(\delta+\epsilon+\zeta)w^\tau + M\Big(\tau\epsilon w^{\tau-1} + \tau\zeta z^{\tau-1}\Big)\Big]}{\Gamma(\tau+1)(\alpha+\beta+\gamma)(\delta+\epsilon+\zeta)} = LN_1\|y_1 - y_2\|.$$

Now, as $LN_1 < 1$, then $\mathcal{F}$ is a contraction. Therefore, we have proved the Theorem 3.1 based on the contraction mapping principle. □

**Theorem 3.2.** *Assume that* $Y : [0, z] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function, and* $(S_1)$ *and* $(S_2)$ *hold. Then, the problem (1.1) has at least one solution on* $[0, z]$ *if* $LN_2 < 1$*, where*

$$N_2 = \frac{\beta(\delta+\epsilon+\zeta)w^\tau + \gamma(\delta+\epsilon+\zeta)z^\tau + \epsilon\tau M w^{\tau-1} + \zeta\tau M z^{\tau-1}}{\Gamma(\tau+1)(\alpha+\beta+\gamma)(\delta+\epsilon+\zeta)}$$

*and* $M = \max_{x\in[0,z]}\{\beta w + \gamma z, \alpha z + \beta(z - w)\}$.

*Proof.* Let $B_r = \{y \in C : \|y\| \leq r\}$, where $r \geq \|\sigma\| N_1$ is a fixed radius. Define two operators

$$\begin{aligned}
(\mathcal{F}_1 y)(x) &= \int_0^x \frac{(x-\nu)^{\tau-1}}{\Gamma(\tau)} Y(\nu, y(\nu))\, d\nu, \\
(\mathcal{F}_2 y)(x) &= -\frac{1}{\alpha+\beta+\gamma}\Big[\frac{\beta}{\Gamma(\tau)} \int_0^w (w-\nu)^{\tau-1} Y(\nu, y(\nu))\, d\nu + \frac{\gamma}{\Gamma(\tau)} \int_0^z (z-\nu)^{\tau-1} Y(\nu, y(\nu))\, d\nu\Big] \\
&\quad + \frac{(\beta w + \gamma z) - (\alpha+\beta+\gamma)x}{(\alpha+\beta+\gamma)(\delta+\epsilon+\zeta)}\Big[\frac{\epsilon}{\Gamma(\tau-1)} \int_0^w (w-\nu)^{\tau-2} Y(\nu, y(\nu))\, d\nu \\
&\qquad + \frac{\zeta}{\Gamma(\tau-1)} \int_0^z (z-\nu)^{\tau-2} Y(\nu, y(\nu))\, d\nu\Big].
\end{aligned}$$

For any $y_1$ and $y_2 \in B_r$, we have

$$\|\mathcal{F}_1 y_1 + \mathcal{F}_2 y_2\| \leq \frac{\|\sigma\|\Big[(\alpha+\beta+2\gamma)(\delta+\epsilon+\zeta)z^\tau + \beta(\delta+\epsilon+\zeta)w^\tau + M\Big(\tau\epsilon w^{\tau-1} + \tau\zeta z^{\tau-1}\Big)\Big]}{\Gamma(\tau+1)(\alpha+\beta+\gamma)(\delta+\epsilon+\zeta)} \leq r.$$

Therefore, $\mathcal{F}_1 y_1 + \mathcal{F}_2 y_2 \in B_r$.

Also, $\mathcal{F}_2$ is a contraction by the assumption $LN_2 < 1$. Moreover, the operator $\mathcal{F}_1$ is continuous due to the continuity of $Y$ and uniformally bounded because

$$\mathcal{F}_1 y \leq \frac{z^\tau \|\sigma\|}{\Gamma(\tau+1)}.$$

Finally, we prove the compactness of the operator $\mathcal{F}_1$. We suppose $\sup_{(x,y)\in[0,z]\times B_r} \|Y(x,y)\| = Y_{max} < \infty$. So, taking $0 \leq x_1 < x_2 \leq z$, we have

$$\|(\mathcal{F}_1 y)(x_1) - \mathcal{F}_1 y)(x_2)\| = \|\frac{1}{\Gamma(\tau)} \int_0^{x_1} [(x_1-\nu)^{\tau-1} - (x_2-\nu)^{\tau-1}] Y(\nu, y(\nu)) d\nu + \frac{1}{\Gamma(\tau)} \int_{x_1}^{x_2} (x_2-\nu)^{\tau-1} Y(\nu, y(\nu)) d\nu\|,$$

which implies that

$$\|(\mathcal{F}_1 y)(x_1) - \mathcal{F}_1 y)(x_2)\| \leq \frac{Y_{max}}{\Gamma(\tau+1)} \left[ 2(x_2 - x_1)^\tau + x_1^\tau - x_2^\tau \right].$$

The norm tends to zero as $x_2 \to x_1$. Hence, $\mathcal{F}_1(B_r)$ is equicontinuous uniformly with respect to $y \in B_r$. Moreover, $\mathcal{F}_1(B_r)$ is uniformly bounded. Therefore, by the Arzelà–Ascoli theorem, $\mathcal{F}_1(B_r)$ is relatively compact in $C([0,z], \mathbb{R})$. Consequently, $\mathcal{F}_1$ is a compact operator on $B_r$.

Consequently, by Theorem 2.1 the problem (1.1) has at least one solution on $[0,z]$. □

**Example 3.1.** For $\tau = 3/2$, $\tau = 5/3$, and $\tau = 7/4$, consider the following problem:

$$\begin{cases} {}^c D^\tau y(x) = \frac{1}{(x+3)^2} \frac{|y|}{1+|y|} + cos(x), x \in [0,2], \\ y(0) + y(1) = -y(2), \\ y'(0) + y'(1) = -y'(2). \end{cases} \tag{3.2}$$

First, observe that

$$|Y(x,y_1) - Y(x,y_2)| \leq \frac{1}{9} |y_1 - y_2|,$$

which implies that $L = 1/9$.

For $\tau = 3/2$, we obtain

$$LN_1 = \frac{2(19\sqrt{2}+5)}{81\sqrt{\pi}} \approx 0.444 < 1.$$

Hence, by Theorem 3.1, Eq (3.1) admits a unique solution on $[0,2]$. To illustrate the theoretical findings established in the previous section, we compute a numerical approximation of Eq (3.1) for the fractional order $\tau = 3/2$. The solution and its first derivative are presented in Figure 1 over the interval $[0,2]$. The numerical results demonstrate the smooth behavior of the solution and reflect the influence of the fractional-order operator on the system.

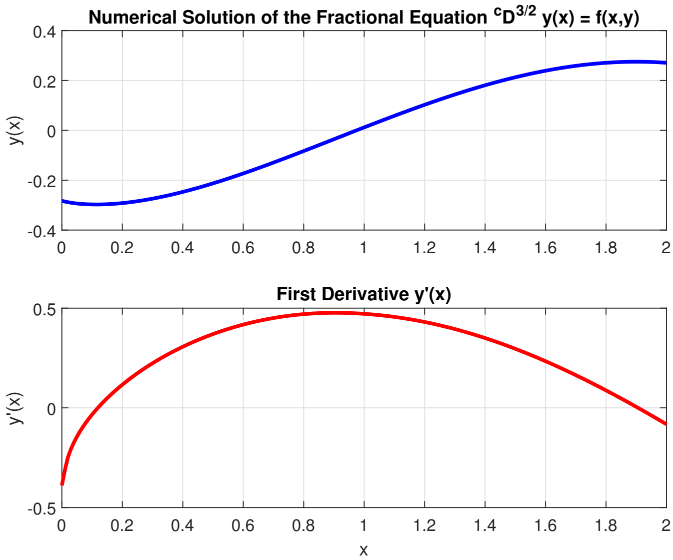


**Figure 1.** A numerical solution of (3.1 ) at $\tau = 3/2$ and its first derivative.

Similarly, for $\tau = 5/3$,

$$LN_1 = \frac{29\,(2^{2/3}) + 8}{90\,\Gamma\left(\frac{2}{3}\right)} \approx 0.44338 < 1,$$

and therefore, Theorem 3.1 again guarantees the existence and uniqueness of the solution on $[0, 2]$. To further examine the influence of the fractional order on the behavior of the solution, we compute a numerical approximation of Eq (3.1) for $\tau = 5/3$. The resulting solution profile and its first derivative are displayed in Figure 2. A comparison with the previous case highlights the sensitivity of the system dynamics to variations in the fractional parameter.

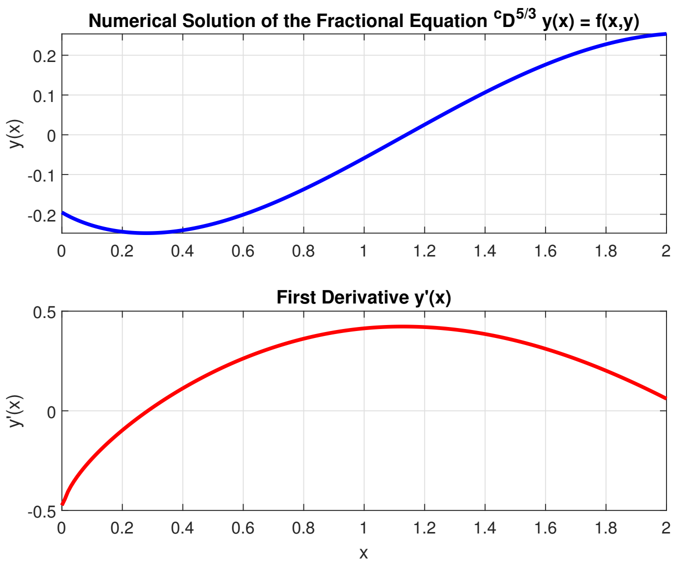


**Figure 2.** A numerical solution of (3.1) at $\tau = 5/3$ and its first derivative.

Finally, when $\tau = 7/4$, we have $LN_1 = \dfrac{39\,2^{3/4} + 11}{108\,\Gamma\left(\frac{11}{4}\right)} \approx 0.4408 < 1.$

Thus, Theorem 3.1 ensures that Eq (3.1) possesses a unique solution on the interval $[0, 2]$. Finally, we consider this case in order to further investigate the impact of the fractional order on the qualitative behavior of the solution. The corresponding numerical solution of Eq (3.1) and its first derivative are illustrated in Figure 3. The results confirm that variations in the fractional parameter significantly influence the curvature and growth rate of the solution, emphasizing the role of memory effects in the system dynamics.

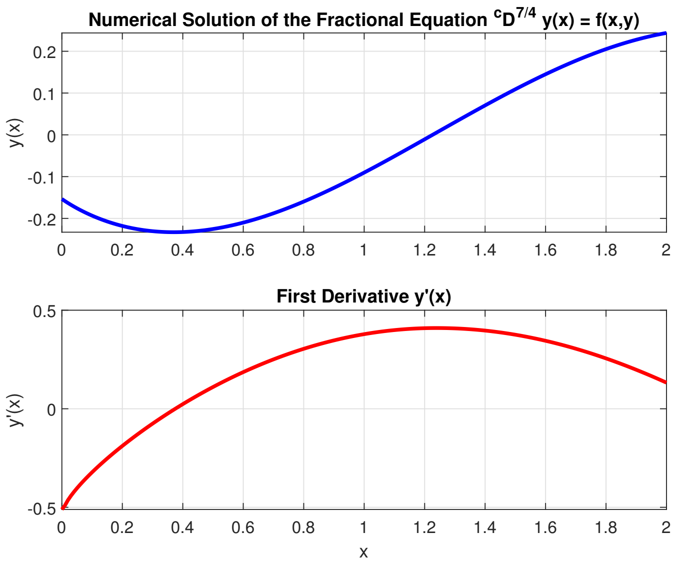


**Figure 3.** A numerical solution of (3.1 ) at $\tau = 7/4$ and its first derivative of $y(x)$.

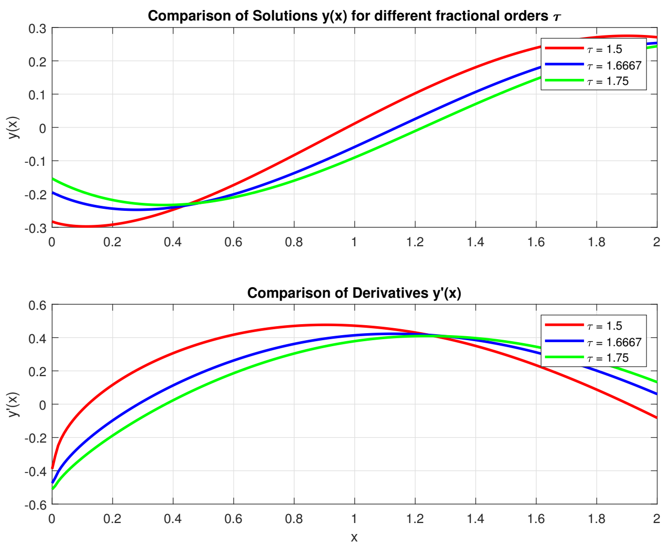


**Figure 4.** A numerical solution of (3.1) at different orders of $\tau$.

**Remark.** Over the interval $[0, 2]$, we see that Figure 4 illustrates the influence of the fractional order $\tau$ on both the solutions and its first derivative. The different values of $\tau$ lead to noticeably different dynamical behaviors. This difference highlights the crucial role of the fractional order in controlling the memory and smoothing effects inherent in the system. Although numerical simulations and graphical results provide valuable insights into the behavior of fractional-order systems, a theoretical analysis is indispensable to ensure that these results are mathematically meaningful. In particular, existence and uniqueness theorems guarantee that the plotted solution actually corresponds to a well-defined physical or mathematical state. Moreover, theoretical bounds establish how the solution depends on the fractional order $\tau$, which explains why small changes in $\tau$ lead to noticeable variations in smoothness and dynamics, as observed in the figures. Without such analytical foundations, numerical approximations could converge to spurious or unstable solutions. Therefore, the theoretical study provides reliability and a deeper understanding of how memory and nonlocal effects governed by fractional operators shape the system's behavior.

## 4. Conclusions

This paper offers significant advancements that both extend and generalize prior results in the study of FDEs. In particular, by adjusting the values of $\alpha, \beta, \gamma, \delta, \epsilon$, and $\zeta$, the integral presented in Eq (2.1) expands upon the findings of [5, 6], providing a broader framework for analysis. Furthermore, Theorems 3.1 and 3.2 extend and unify previous existence and uniqueness results for solutions of FDEs of order $\tau \in (1, 2]$, thereby consolidating and extending the contributions of earlier works [5, 6]. The problem (1.1) may be interpreted as a model for a viscoelastic beam whose response depends not only on its current state but also on its past behavior. The fractional derivative captures the material's memory, and the nonlocal boundary conditions describe how information measured at interior and boundary points is used to regulate the system. This type of model is especially useful for studying flexible structures with distributed sensing and feedback mechanisms, which may motivate future research and practical applications.

## Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.


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## Conflict of interest

The author declares no conflict of interest.

## References


1. S. Abbas, M. Benchohra, G. M. N'Guerekata, *Topics in fractional differential equations*, New York: Springer, 2012. https://doi.org/10.1007/978-1-4614-4036-9
2. K. Diethelm, *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*, Berlin, Heidelberg: Springer, 2010. https://doi.org/10.1007/978-3-642-14574-2
3. V. Gafiychuk, B. Datsko, V. Meleshko, Mathematical modeling of time fractional reaction–diffusion systems, *J. Comput. Appl. Math.*, **220** (2008), 215–225. https://doi.org/10.1016/j.cam.2007.08.011
4. I. Podlubny, *Fractional differential equations*, In: Mathematics in science and engineering, Academic Press, **198** (1999).
5. A. Bashir, J. Nieto, Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory, *Topol. Method. Nonl. An.*, **35** (2010), 259–304. Available from: `https://www.tmna.ncu.pl/static/published/2010/v35n2-06.pdf`.
6. R. P. Agarwal, A. Bashir, J. J. Nieto, Fractional differential equations with nonlocal (parametric type) anti-periodic boundary conditions, *Filomat*, **31** (2017), 1207–1214. Available from: `https://www.jstor.org/stable/24902217`.
7. C. K. Li, E. Pourhadi, Theoretical analysis on the nonlinear fractional differential equations and generalized heat equation, *Filomat*, **39** (2025), 7403–7425. Available from: `https://www.jstor.org/stable/27417965`.
8. S. S. Redhwan, S. L. Shaikh, M. S. Abdo, Implicit fractional differential equation with anti-periodic boundary condition involving Caputo–Katugampola type, *AIMS Math.*, **5** (2020), 3714–3730. https://doi.org/10.3934/math.2020240
9. S. Ramasamy, T. Senthilprabu, K. Karthikeyan, P. Geetha, S. Chasreechai, T. Sitthiwirattham, Existence, uniqueness and controllability results of nonlinear neutral implicit ABC fractional integro-differential equations with delay and impulses, *AIMS Math.*, **10** (2025), 4326–4354. https://doi.org/10.3934/math.2025200
10. S. Abbas, M. Benchohra, J. R. Graef, J. Henderson, *Implicit fractional differential and integral equations: Existence and stability*, Berlin, Boston: De Gruyter, **26** (2018). https://doi.org/10.1515/9783110553819
11. D. R. Smart, *Fixed point theorems*, Cambridge University Press, 1980.