Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(2): 4571–4585.
DOI: 10.3934/math.2026184
Received: 18 December 2025
Revised: 02 February 2026
Accepted: 04 February 2026
Published: 24 February 2026

Research article

On Jordan σ-centralizers and related linear maps in algebras

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Abstract: Let $\mathcal{B}$ be an algebra over a commutative ring with identity S, and let $\sigma\colon \mathcal{B} \to \mathcal{B}$ be an algebra homomorphism. In this paper, we study linear operators $\Delta\colon \mathcal{B} \to \mathcal{B}$ that are constrained by zero-product conditions involving the Jordan product $u \circ v = uv + vu$. In particular, we consider mappings that satisfy

$$uv = 0 \Rightarrow \Delta(u \circ v) = \Delta(u) \circ \sigma(v), \quad uv = 0 \Rightarrow \Delta(u \circ v) = \sigma(u) \circ \Delta(v),$$

and

$$uv = 0 \Rightarrow \Delta(u \circ v) = \Delta(u) \circ \sigma(v) = \sigma(u) \circ \Delta(v).$$

Assuming the endomorphism σ is bijective, we prove that the scenario essentially simplifies to the identity case $\sigma = \mathrm{id}_{\mathcal{B}}$. Such a simplification enables a comprehensive the forms of these linear operators. As a result, we obtain precise expressions for these operators across diverse algebraic structures, including generalized matrix algebras, upper-triangular algebras, von Neumann algebras, standard operator algebras, and nest algebras. Moreover, this approach produces analogous results for Jordan σ-centralizers, thus extending and integrating various prior findings in the field.

Keywords: Jordan σ-centralizers; zero-product; algebras isomorphism; generalized matrix algebras; upper-triangular algebras; von Neumann algebras; standard operator algebras; nest algebras
Mathematics Subject Classification: 7B40, 15A78, 46L10, 47L10, 47L35

1. Introduction

This present work is carried out in the framework of associative algebras; however it is worth noting that many techniques related to Jordan-type identities, zero-product conditions, and structural descriptions of linear maps extend naturally beyond associativity. In particular, *alternative algebras* form a prominent and well-studied class of *non-associative algebras* where analogous problems concerning additivity, multiplicativity, and Jordan-type mappings have been extensively investigated.

Several results on zero-product-determined structures and additive mappings on alternative rings indicate that phenomena similar to those established in this paper may also hold in a broader non-associative setting. We refer the interested reader to the monograph of Brešar [1].

Suppose $\mathcal{B}$ represents an algebra over a commutative ring with identity S. A linear operator Δ: $\mathcal{B} \to \mathcal{B}$ satisfies as a *Jordan centralizer*

$$\Delta(u \circ v) = \Delta(u) \circ v, \quad u, v \in \mathcal{B},$$

with the Jordan operation defined as $u \circ v = uv + vu$. This relation can alternatively be written as

$$\Delta(u \circ v) = u \circ \Delta(v), \quad u, v \in \mathcal{B}.$$

Such Jordan centralizers are crucial for exploring Jordan-related properties in associative algebras and have been extensively studied in contexts like operator algebras and generalized matrix algebras. A typical instance emerges as follows. Let $k \in \mathcal{Z}(\mathcal{B})$ and η: $\mathcal{B} \to \mathcal{Z}(\mathcal{B})$ be a linear operator satisfying

$$\eta(u \circ v) = 0, \quad u, v \in \mathcal{B}.$$

The operator

$$\Delta(u) = ku + \eta(u)$$

defines a Jordan centralizer, often termed a standard one. An important question, which has been addressed in several algebraic settings, asks whether every Jordan (Lie) centralizer must necessarily be of this standard form. Positive answers to this question are known for many classes of algebras; see, for instance, [2–5]. Motivated by zero-product theory, researchers have also considered mappings that satisfy the Jordan centralizer identity only under the condition that the product of two elements is zero. More precisely, for a linear map Δ: $\mathcal{B} \to \mathcal{B}$, the following relations have been studied:

$$\begin{aligned} uv = 0 &\Rightarrow \Delta(u \circ v) = \Delta(u) \circ v, && (J_1), \\ uv = 0 &\Rightarrow \Delta(u \circ v) = u \circ \Delta(v), && (J_2), \\ uv = 0 &\Rightarrow \Delta(u \circ v) = \Delta(u) \circ v = u \circ \Delta(v), && (J). \end{aligned}$$

If $k \in \mathcal{Z}(\mathcal{B})$ and η: $\mathcal{B} \to \mathcal{Z}(\mathcal{B})$ obeys

$$\eta(u \circ v) = 0$$

for $uv = 0$, then

$$\Delta(u) = ku + \eta(u)$$

serves as a *canonical solution* for each of these conditions. Identifying when operators satisfying (J_1), (J_2), or (J) must be canonical has been a focal issue, addressed in [6–9]. To broaden this concept, let σ: $\mathcal{B} \to \mathcal{B}$ be a algebra endomorphism. A linear operator Δ: $\mathcal{B} \to \mathcal{B}$ is termed a *Jordan σ-centralizer* if

$$\Delta(u \circ v) = \Delta(u) \circ \sigma(v), \quad u, v \in \mathcal{B},$$

or equivalently,

$$\Delta(u \circ v) = \sigma(u) \circ \Delta(v).$$

For $\sigma = \mathrm{id}_{\mathcal{B}}$, this aligns with the standard Jordan centralizer. A Jordan σ-centralizer expressed as

$$\Delta(u) = k\,\sigma(u) + \eta(u),$$

where $k \in \mathcal{Z}(\mathcal{B})$ and $\eta\colon \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ satisfies $\eta(u \circ v) = 0$, is deemed *proper*. Lately, these operators have been probed in [2, 3]. Mirroring the zero-product strategy above, we introduce these modifications:

$$\begin{aligned} uv = 0 &\Rightarrow \Delta(u \circ v) = \Delta(u) \circ \sigma(v), && (J_1^\sigma),\\ uv = 0 &\Rightarrow \Delta(u \circ v) = \sigma(u) \circ \Delta(v), && (J_2^\sigma),\\ uv = 0 &\Rightarrow \Delta(u \circ v) = \Delta(u) \circ \sigma(v) = \sigma(u) \circ \Delta(v), && (J^\sigma). \end{aligned}$$

With $k \in \mathcal{Z}(\mathcal{B})$ and suitable $\eta\colon \mathcal{B} \to \mathcal{Z}(\mathcal{B})$, the operator

$$\Delta(u) = k\,\sigma(u) + \eta(u)$$

yields canonical solutions for (J_1^σ), (J_2^σ), and (J^σ). Note that these conditions are typically not interchangeable. The primary objective here is to characterize all linear operators satisfying (J_1^σ), (J_2^σ), and (J^σ), and to pinpoint when they must be canonical. We show that a ring $\mathcal{B}$ possesses the canonical property for the σ-conditions exactly when it does for the standard relations (J_1), (J_2), and (J). Thus, for various key algebras like generalized matrix algebras, upper-triangular algebras, von Neumann algebras, standard operator algebras, and nest algebras, each Jordan σ-centralizer proves to be proper. Lastly, the methods here naturally apply to related areas, encompassing σ-centralizers, Lie σ-centralizers, and Jordan triple σ-centralizers, offering a cohesive structure that extends several established outcomes. In this direction, several important contributions are available in the literature; see, for instance, Burgos and Ortega [6], Dales [7], Davidson [8], Fadaee and Ghahramani [9], and Kadison and Ringrose [10] for more detailed treatments.

Remark 1.1. We emphasize that throughout this paper, the identities

$$uv = 0 \Longrightarrow \Delta(u \circ v) = \Delta(u) \circ \sigma(v)$$

and

$$uv = 0 \Longrightarrow \Delta(u \circ v) = \sigma(u) \circ \Delta(v),$$

together with their variants, are understood strictly as *one-sided implications*. That is, the zero-product condition $uv = 0$ is assumed *a priori*, and the corresponding Jordan-type identities are required only under this hypothesis. No converse implication is imposed or claimed.

In particular, we do not assume that an identity of the form

$$\Delta(u \circ v) = \Delta(u) \circ \sigma(v)$$

for arbitrary $u, v \in B$, forces $uv = 0$, nor do we attempt to characterize endomorphisms σ or linear maps Δ for which such reverse implications might hold. As illustrated in Example 2.1, even when Δ satisfies (J^σ), (J_1^σ), or (J_2^σ), the bijectivity of σ is essential for reducing the problem to the classical Jordan centralizer setting.

Moreover, in the present work, the mapping Δ is assumed to be linear only. Investigating Jordan σ-centralizers under additional algebraic constraints on Δ, such as multiplicativity, homomorphic behavior, or nonlinear variants, constitutes an interesting direction for further research. Such problems fall outside the scope of the current paper and will be addressed elsewhere.

2. Alternative characterizations

Here, we develop a linkage between the adjusted conditions (J_1^σ), (J_2^σ), and (J^σ) and their standard counterparts (J_1), (J_2), and (J), under the assumption that σ is an algebra's isomorphism. In this analysis, all algebras are unitary over a commutative domain S with unity, where 2 is invertible. The Jordan operation remains

$$u \circ v = uv + vu.$$

Theorem 2.1. *For an algebra $\mathcal{B}$ and an isomorphism σ of $\mathcal{B}$, these assertions hold equivalently:*

(i) Any linear operator Δ: $\mathcal{B} \to \mathcal{B}$ obeying (J_1) takes the shape

$$\Delta(u) = ku + \eta(u), \quad u \in \mathcal{B},$$

with $k \in \mathcal{Z}(\mathcal{B})$ and linear η: $\mathcal{B} \to \mathcal{Z}(\mathcal{B})$ such that $\eta(u \circ v) = 0$ when $uv = 0$.

(ii) Any linear operator Δ: $\mathcal{B} \to \mathcal{B}$ obeying (J_1^σ) has the form

$$\Delta(u) = k\,\sigma(u) + \eta(u), \quad u \in \mathcal{B},$$

for certain $k \in \mathcal{Z}(\mathcal{B})$ and linear η: $\mathcal{B} \to \mathcal{Z}(\mathcal{B})$ satisfying $\eta(u \circ v) = 0$ when $uv = 0$.

Proof. (i) $\Rightarrow$ (ii): Let Δ obey (J_1^σ). Introduce

$$\Psi = \sigma^{-1} \circ \Delta.$$

When $uv = 0$,

$$\Psi(u \circ v) = \sigma^{-1}(\Delta(u \circ v)) = \sigma^{-1}(\Delta(u) \circ \sigma(v)) = \sigma^{-1}(\Delta(u)) \circ v = \Psi(u) \circ v,$$

indicating Ψ satisfies (J_1). Per (i), exist $k_0 \in \mathcal{Z}(\mathcal{B})$ and linear η_0: $\mathcal{B} \to \mathcal{Z}(\mathcal{B})$ with $\eta_0(u \circ v) = 0$ for $uv = 0$ so that

$$\Psi(u) = k_0 u + \eta_0(u).$$

Composing with σ produces

$$\Delta(u) = \sigma(\Psi(u)) = \sigma(k_0)\sigma(u) + \sigma(\eta_0(u)) =: k\,\sigma(u) + \eta(u),$$

where

$$k = \sigma(k_0) \in \mathcal{Z}(\mathcal{B})$$

and

$$\eta = \sigma \circ \eta_0 : \mathcal{B} \to \mathcal{Z}(\mathcal{B}).$$

Evidently, $\eta(u \circ v) = 0$ for u, v with $uv = 0$.

(ii) $\Rightarrow$ (i): Suppose Δ satisfies (J_1), and set $\Psi = \sigma \circ \Delta$. For $uv = 0$,

$$\Psi(u \circ v) = \sigma(\Delta(u \circ v)) = \sigma(\Delta(u) \circ v) = \sigma(\Delta(u)) \circ \sigma(v) = \Psi(u) \circ \sigma(v),$$

confirming Ψ obeys (J_1^σ). From (ii),

$$\Psi(u) = k_0\sigma(u) + \eta_0(u)$$

for some $k_0 \in \mathcal{Z}(\mathcal{B})$ and linear η_0: $\mathcal{B} \to \mathcal{Z}(\mathcal{B})$ with $\eta_0(u \circ v) = 0$ for $uv = 0$. Composing with σ^{-1} gives

$$\Delta(u) = \sigma^{-1}(k_0)u + \sigma^{-1}(\eta_0(u)) =: ku + \eta(u),$$

where $k \in \mathcal{Z}(\mathcal{B})$ and η: $\mathcal{B} \to \mathcal{Z}(\mathcal{B})$ preserves the zero-product trait. □

The next pair of results arises analogously by adapting the reasoning from Theorem 2.1 to the respective pairs $((J_2), (J_2^\sigma))$ and $((J), (J^\sigma))$.

Theorem 2.2. *For an algebra $\mathcal{B}$ and an isomorphism σ of $\mathcal{B}$, these hold equivalently:*

(i) Any linear operator $\Delta\colon \mathcal{B} \to \mathcal{B}$ satisfying (J_2) assumes

$$\Delta(u) = ku + \eta(u), \quad u \in \mathcal{B},$$

with $k \in \mathcal{Z}(\mathcal{B})$ and linear $\eta\colon \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ obeying $\eta(u \circ v) = 0$ when $uv = 0$.

(ii) Any linear operator $\Delta\colon \mathcal{B} \to \mathcal{B}$ satisfying (J_2^σ) assumes

$$\Delta(u) = k\,\sigma(u) + \eta(u), \quad u \in \mathcal{B},$$

with $k \in \mathcal{Z}(\mathcal{B})$ and linear $\eta\colon \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ satisfying $\eta(u \circ v) = 0$ when $uv = 0$.

Theorem 2.3. *For an algebra $\mathcal{B}$ and isomorphism σ of $\mathcal{B}$, the following are equivalent:*

(i) Any linear operator $\Delta\colon \mathcal{B} \to \mathcal{B}$ adhering to (J) is

$$\Delta(u) = ku + \eta(u), \quad u \in \mathcal{B},$$

for some $k \in \mathcal{Z}(\mathcal{B})$ and linear $\eta\colon \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ with $\eta(u \circ v) = 0$ when $uv = 0$.

(ii) Any linear operator $\Delta\colon \mathcal{B} \to \mathcal{B}$ adhering to (J^σ) is

$$\Delta(u) = k\,\sigma(u) + \eta(u), \quad u \in \mathcal{B},$$

where $k \in \mathcal{Z}(\mathcal{B})$ and $\eta\colon \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ is linear with $\eta(u \circ v) = 0$ when $uv = 0$.

Theorem 2.4. *For an algebra $\mathcal{B}$ and isomorphism σ, the following are equivalent:*

(i) Any Jordan centralizer $\Delta\colon \mathcal{B} \to \mathcal{B}$ has

$$\Delta(u) = ku + \eta(u), \quad u \in \mathcal{B},$$

for some $k \in \mathcal{Z}(\mathcal{B})$ and linear $\eta\colon \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ such that $\eta(u \circ v) = 0$ for all $u, v \in \mathcal{B}$.

(ii) Any Jordan σ-centralizer $\Delta\colon \mathcal{B} \to \mathcal{B}$ has

$$\Delta(u) = k\,\sigma(u) + \eta(u), \quad u \in \mathcal{B},$$

where $k \in \mathcal{Z}(\mathcal{B})$ and $\eta\colon \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ is linear satisfying $\eta(u \circ v) = 0$ for all $u, v \in \mathcal{B}$.

Directly from Theorems 2.1–2.4, it follows that for any algebras isomorphism σ, the set of canonical solutions for the adjusted identities $(J_1^\sigma), (J_2^\sigma), (J^\sigma)$ matches exactly that for their standard versions $(J_1), (J_2), (J)$. Similarly, the category of Jordan σ-centralizers parallels that of standard Jordan centralizers in structure.

Next, we offer two lemmas plus an example to illustrate that the prior equivalences do not persist when replacing the isomorphism σ with a general endomorphism.

Lemma 2.1. *Let $\mathcal{D}$ be a commutative unitary ring over S, and $\mathcal{E}$ a unitary algebra where every linear operator K: $\mathcal{E} \to \mathcal{E}$ satisfying (J_1) (or (J_2), (J)) decomposes as*

$$K(w) = fw + \iota(w), \quad f \in \mathcal{Z}(\mathcal{E}), \quad \iota : \mathcal{E} \to \mathcal{Z}(\mathcal{E}), \quad \iota(u \circ v) = 0$$

when $uv = 0$. For the algebras $\mathcal{B} = \mathcal{D} \times \mathcal{E}$, and any isomorphism σ of $\mathcal{B}$, any linear operator Δ: $\mathcal{B} \to \mathcal{B}$ satisfying (J_1^σ) (or (J_2^σ), (J^σ)) decomposes as

$$\Delta(U) = F\,\sigma(U) + \eta(U), \quad F \in \mathcal{Z}(\mathcal{B}), \quad \eta : \mathcal{B} \to \mathcal{Z}(\mathcal{B}), \quad \eta(U \circ V) = 0$$

when $UV = 0$.

Proof. We detail the (J_1^σ)-case; others proceed similarly. By Theorem 2.1, reducing to (J_1) suffices. Express elements of $\mathcal{B}$ as (d, w) with $d \in \mathcal{D}$, $w \in \mathcal{E}$, and

$$\Delta(d, w) = (\Delta_1(d) + K_1(w),\ \Delta_2(d) + K_2(w)).$$

Imposing (J_1) on $(0, w)$ and $(0, v)$ for $wv = 0$ implies

$$K_1(u \circ v) = 0$$

and K_2 satisfies (J_1) on $\mathcal{E}$. By premise,

$$K_2(w) = fw + \iota(w),$$

with $f \in \mathcal{Z}(\mathcal{E})$ and linear ι: $\mathcal{E} \to \mathcal{Z}(\mathcal{E})$ vanishing on $u \circ v$ for $uv = 0$. Applying (J_1) to $(d, 0)$ and $(0, w)$ yields $\Delta_2 = 0$, while on $(0, w_1)$ and $(d_2, 0)$ gives $K_1 = 0$. Set

$$\eta(d, w) = (\Delta_1(d),\ \iota(w)) \quad \text{and} \quad F = (0, f).$$

As

$$\mathcal{Z}(\mathcal{B}) = \mathcal{D} \times \mathcal{Z}(\mathcal{E}),$$

clearly $\eta(\mathcal{B}) \subseteq \mathcal{Z}(\mathcal{B})$, and $\eta(U \circ V) = 0$ for $UV = 0$. Hence,

$$\Delta(d, w) = F\,(d, w) + \eta(d, w),$$

finishing the proof. □

Lemma 2.2. *Using the setup and premises of Lemma 2.1, but for Jordan centralizers. For $\mathcal{B} = \mathcal{D} \times \mathcal{E}$ and for any isomorphism σ of $\mathcal{B}$, every Jordan σ-centralizer Δ: $\mathcal{B} \to \mathcal{B}$ satisfies*

$$\Delta(U) = F\,\sigma(U) + \eta(U), \quad F \in \mathcal{Z}(\mathcal{B}), \quad \eta : \mathcal{B} \to \mathcal{Z}(\mathcal{B}), \quad \eta(U \circ V) = 0.$$

Proof. The proof mirrors Lemma 2.1, substituting Theorem 2.4 for Theorem 2.1. □

Example 2.1. Take

$$\mathcal{B} = \mathbb{C} \times M_n(\mathbb{C})$$

for $n \geq 2$, so

$$\mathcal{Z}(\mathcal{B}) = \mathbb{C} \times \mathbb{C}I_n.$$

Define a (non-bijective) endomorphism $\sigma\colon \mathcal{B} \to \mathcal{B}$ via

$$\sigma((g, N)) = (g, 0),$$

and the linear operator $\Delta\colon \mathcal{B} \to \mathcal{B}$ by

$$\Delta((g, N)) = (g, \mathrm{tr}(N)E_{11}),$$

with E_{11} the initial matrix unit in $M_n(\mathbb{C})$. Direct verification confirms Δ obeys (J_1^{σ}) (and actually (J_2^{σ}), (J^{σ}) too). Assume, counterfactually, Δ fits the canonical

$$\Delta(u) = k \ \ \sigma(u) + \eta(u)$$

with $k \in \mathcal{Z}(\mathcal{B})$, $\eta\colon \mathcal{B} \to \mathcal{Z}(\mathcal{B})$. At $(0, E_{11})$,

$$(0, E_{11}) = (0, 0) + (\eta_1(0, E_{11}),\ \eta_2(0, E_{11})I_n),$$

impossible for $n \geq 2$. Thus, Δ lacks a canonical form, indicating the earlier equivalences fail for general endomorphism s.

Remark 2.1. The results of Section 2 reveal a structural principle underlying Jordan σ-centralizers and their zero-product variants. Namely, when σ is an algebra isomorphism, the classification of linear maps satisfying (J^{σ}), (J_1^{σ}), or (J_2^{σ}) is *equivalent* to the corresponding classical problems (J), (J_1), and (J_2) via composition with σ or σ^{-1}. This establishes a transfer principle showing that, in the bijective case, the σ-setting introduces no additional structural freedom beyond the identity case.

Example 2.1 demonstrates that this phenomenon is sharp: Once bijectivity of σ is lost, the transfer mechanism breaks down, and genuinely new, noncanonical solutions may arise. Hence, bijectivity of σ is not a technical assumption but an essential structural requirement.

From this viewpoint, the applications presented in Section 3 should be understood as systematic extensions of known classification results for Jordan centralizers and zero-product preserving maps to the σ-context. In particular, the results obtained for generalized matrix algebras, triangular algebras, von Neumann algebras, standard operator algebras, and nest algebras all unify and generalize earlier studies such as [2, 3, 6, 9] within a single coherent conceptual framework. Accordingly, the main contribution of this paper thus lies not in presenting isolated new examples, but in clarifying the precise scope, inherent limitations, and structural invariance of Jordan σ-centralizers across a broad and significant class of algebras.

3. Applications

In this part, we demonstrate applications of the characterization theorems from Section 2 to several prominent algebraic classes. In detail, we explore implications for generalized matrix algebras, upper-triangular algebras, von Neumann algebras, standard operator algebras, and nest algebras. Here, σ denotes an algebra isomorphism, and algebras are over a unitary commutative domain S with 2 invertible.

3.1. Generalized matrix algebras

We begin with *generalized matrix algebras*, central in equivalence theory and covealgebras wide algebras varieties. Let $(\mathcal{B}, \mathcal{D}, \mathcal{P}, \mathcal{Q}, \zeta_{\mathcal{P}\mathcal{Q}}, \psi_{\mathcal{Q}\mathcal{P}})$ form an equivalence context, $\mathcal{B}$ and $\mathcal{D}$ algebras, $\mathcal{P}$ a $(\mathcal{B}, \mathcal{D})$-bimodule, $\mathcal{Q}$ a $(\mathcal{D}, \mathcal{B})$-bimodule. The bimodule maps

$$\zeta_{\mathcal{P}\mathcal{Q}} : \mathcal{P} \otimes_{\mathcal{D}} \mathcal{Q} \to \mathcal{B}, \quad \psi_{\mathcal{Q}\mathcal{P}} : \mathcal{Q} \otimes_{\mathcal{B}} \mathcal{P} \to \mathcal{D}$$

fulfill usual compatibility:

$$\mathcal{P} \otimes_{\mathcal{D}} \mathcal{Q} \otimes_{\mathcal{B}} \mathcal{P} \underset{I_{\mathcal{P}} \otimes \psi_{\mathcal{Q}\mathcal{P}}}{\overset{\zeta_{\mathcal{P}\mathcal{Q}} \otimes I_{\mathcal{P}}}{\rightrightarrows}} \mathcal{B} \otimes_{\mathcal{B}} \mathcal{P} \cong \mathcal{P}, \quad \mathcal{Q} \otimes_{\mathcal{B}} \mathcal{P} \otimes_{\mathcal{D}} \mathcal{Q} \underset{I_{\mathcal{Q}} \otimes \zeta_{\mathcal{P}\mathcal{Q}}}{\overset{\psi_{\mathcal{Q}\mathcal{P}} \otimes I_{\mathcal{Q}}}{\rightrightarrows}} \mathcal{D} \otimes_{\mathcal{D}} \mathcal{Q} \cong \mathcal{Q}.$$

The algebras

$$\mathcal{H} = \begin{bmatrix} \mathcal{B} & \mathcal{P} \\ \mathcal{Q} & \mathcal{D} \end{bmatrix} = \left\{ \begin{bmatrix} \beta & p \\ q & \delta \end{bmatrix} \;\middle|\; \beta \in \mathcal{B},\ \delta \in \mathcal{D},\ p \in \mathcal{P},\ q \in \mathcal{Q} \right\},$$

with standard matrix operations, is the *generalized matrix algebras* for this context. For unitary $\mathcal{B}$, $\mathcal{D}$ and bimodules, the unit of $\mathcal{H}$ is

$$I_{\mathcal{H}} = \begin{bmatrix} 1_{\mathcal{B}} & 0 \\ 0 & 1_{\mathcal{D}} \end{bmatrix}.$$

From [9, Lemma 1], the center of a unitary GMA is

$$\mathcal{Z}(\mathcal{H}) = \left\{ \begin{bmatrix} \beta & 0 \\ 0 & \delta \end{bmatrix} :\ \beta \in \mathcal{Z}(\mathcal{B}),\ \delta \in \mathcal{Z}(\mathcal{D}),\ \beta p = p\delta,\ q\beta = \delta q \text{ for all } p \in \mathcal{P},\ q \in \mathcal{Q} \right\}.$$

Denote projections by

$$\pi_{\mathcal{B}}\left(\begin{bmatrix} \beta & p \\ q & \delta \end{bmatrix} \right) = \beta, \quad \pi_{\mathcal{D}}\left(\begin{bmatrix} \beta & p \\ q & \delta \end{bmatrix} \right) = \delta,$$

so $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{H})) \subseteq \mathcal{Z}(\mathcal{B})$ and $\pi_{\mathcal{D}}(\mathcal{Z}(\mathcal{H})) \subseteq \mathcal{Z}(\mathcal{D})$.

Theorem 3.1. *Let*

$$\mathcal{H} = \begin{bmatrix} \mathcal{B} & \mathcal{P} \\ \mathcal{Q} & \mathcal{D} \end{bmatrix}$$

be a 2*-torsion-free unitary generalized matrix algebras, with σ an isomorphism of $\mathcal{H}$. Assume at least one of $\mathcal{P}$ or $\mathcal{Q}$ is loyal and*

$$\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{H})) = \mathcal{Z}(\mathcal{B}), \quad \pi_{\mathcal{D}}(\mathcal{Z}(\mathcal{H})) = \mathcal{Z}(\mathcal{D}).$$

Then any linear operator Δ: $\mathcal{H} \to \mathcal{H}$ satisfying any of (J_1^σ), (J_2^σ), or (J^σ) has

$$\Delta(W) = K\,\sigma(W) + \eta(W) \quad (W \in \mathcal{H}),$$

with $K \in \mathcal{Z}(\mathcal{H})$ and linear η: $\mathcal{H} \to \mathcal{Z}(\mathcal{H})$ such that $\eta(W \circ V) = 0$ when $WV = 0$.

Proof. As $\mathcal{H}$ is 2-torsion-free and one module $\mathcal{P}$ or $\mathcal{Q}$ loyal, the premises validate (i) of Theorems 2.1–2.3. By [2, Corollary 3.3], any linear operator on $\mathcal{H}$ satisfying (J_1), (J_2), or (J) is

$$\Delta(W) = KW + \eta(W),$$

with $K \in \mathcal{Z}(\mathcal{H})$ and $\eta(W \circ V) = 0$ when $WV = 0$. Theorems 2.1–2.3 then extend to the adjusted identities (J_i^σ) $(i = 1, 2)$ and (J^σ). □

Theorem 3.2. *Let*

$$\mathcal{H} = \begin{bmatrix} \mathcal{B} & \mathcal{P} \\ Q & \mathcal{D} \end{bmatrix}$$

be a 2*-torsion-free unitary GMA, with* σ *an isomorphism of* $\mathcal{H}$. *Require that*

$$\beta \in \mathcal{B},\ \beta\mathcal{P} = 0,\ Q\beta = 0 \Rightarrow \beta = 0, \quad \delta \in \mathcal{D},\ \mathcal{P}\delta = 0,\ \delta Q = 0 \Rightarrow \delta = 0,$$

and one of:

(1) $\pi_{\mathcal{D}}(\mathcal{Z}(\mathcal{H})) = \mathcal{Z}(\mathcal{D})$ *or* $[\mathcal{B}, \mathcal{B}] = \mathcal{B}$*;*
(2) $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{H})) = \mathcal{Z}(\mathcal{B})$ *or* $[\mathcal{D}, \mathcal{D}] = \mathcal{D}$*.*

Then any Jordan σ*-centralizer* $\Delta\colon \mathcal{H} \to \mathcal{H}$ *is*

$$\Delta(W) = K\,\sigma(W) + \eta(W) \quad (W \in \mathcal{H}),$$

with $K \in \mathcal{Z}(\mathcal{H})$ *and linear* $\eta\colon \mathcal{H} \to \mathcal{Z}(\mathcal{H})$ *obeying* $\eta(W \circ V) = 0$ *for all* $W, V \in \mathcal{H}$.

Proof. The premises imply, via [2, Corollary 4.3], that every Jordan centralizer on $\mathcal{H}$ is proper:

$$\Delta(W) = KW + \eta(W), \quad K \in \mathcal{Z}(\mathcal{H}), \quad \eta(W \circ V) = 0$$

when $WV = 0$. This validates Theorem 2.4 (i), so Theorem 2.4 yields the outcome. □

3.2. Upper-triangular algebras

Let $\mathcal{B}$ and $\mathcal{D}$ be unitary algebras over S, $\mathcal{P}$ a unitary $(\mathcal{B}, \mathcal{D})$-bimodule. The associated *upper-triangular algebras* are

$$\mathcal{T} = \mathrm{Tri}(\mathcal{B}, \mathcal{P}, \mathcal{D}) = \{(\beta, p, \delta) \mid \beta \in \mathcal{B},\ p \in \mathcal{P},\ \delta \in \mathcal{D}\},$$

with componentwise product

$$(\beta_1, p_1, \delta_1)(\beta_2, p_2, \delta_2) = (\beta_1\beta_2,\ \beta_1 p_2 + p_1\delta_2,\ \delta_1\delta_2).$$

Use $\pi_{\mathcal{B}}, \pi_{\mathcal{D}}$ for projections to $\mathcal{B}$, $\mathcal{D}$. The center is

$$\mathcal{Z}(\mathcal{T}) = \{(z_{\mathcal{B}}, 0, z_{\mathcal{D}}) \mid z_{\mathcal{B}} \in \mathcal{Z}(\mathcal{B}),\ z_{\mathcal{D}} \in \mathcal{Z}(\mathcal{D})\}.$$

Theorem 3.3. *Assume*

$$\mathcal{T} = \mathrm{Tri}(\mathcal{B}, \mathcal{P}, \mathcal{D})$$

is unitary with $\mathcal{P}$ *loyal as left* $\mathcal{B}$*-module and right* $\mathcal{D}$*-module. Let* σ *be an isomorphism of* $\mathcal{T}$*, and require:*

(a) Either $\pi_{\mathcal{D}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{D})$ *or* $[\mathcal{B}, \mathcal{B}] = \mathcal{B}$*;*
(b) Either $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{B})$ *or* $[\mathcal{D}, \mathcal{D}] = \mathcal{D}$*.*

Then any linear operator $\Delta\colon \mathcal{T} \to \mathcal{T}$ *satisfying one of* (J_1^σ), (J_2^σ), *or* (J^σ) *is*

$$\Delta(W) = K\,\sigma(W) + \eta(W) \quad (W \in \mathcal{T})$$

with $K \in \mathcal{Z}(\mathcal{T})$ *and linear* $\eta\colon \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ *such that* $\eta(W \circ V) = 0$ *when* $W, V \in \mathcal{T}$ *and* $WV = 0$.

Proof. The premises ensure upper-triangular algebras enforce standard Jordan-type forms (per known outcomes on such algebras). Thus, Δ adopts the form from Theorems 2.1–2.3, achieving the goal. □

Theorem 3.4. *Let*

$$\mathcal{T} = \mathrm{Tri}(\mathcal{B}, \mathcal{P}, \mathcal{D})$$

be a 2*-torsion-free unitary upper-triangular algebras with* $\mathcal{P}$ *loyal left and right. Assume (a) and (b) from Theorem 3.3. Then any Jordan* σ*-centralizer* $\Delta\colon \mathcal{T} \to \mathcal{T}$ *is*

$$\Delta(W) = K\,\sigma(W) + \eta(W) \quad (W \in \mathcal{T})$$

with $K \in \mathcal{Z}(\mathcal{T})$ *and linear* $\eta\colon \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ *such that* $\eta(W \circ V) = 0$ *for all* $W, V \in \mathcal{T}$.

Proof. Prior findings on Jordan centralizers in upper-triangular algebras confirm standard forms under these premises. Hence, Theorem 2.4 (i) applies, yielding the structure via Theorem 2.4. □

3.3. Von Neumann algebras

Let $\mathcal{K}$ be a complex Hilbert space, $\mathcal{N} \subseteq B(\mathcal{K})$ a von Neumann algebra. $\mathcal{N}$ is a unitary $*$-subalgebras of $B(\mathcal{K})$, weakly closed, with bicommutant $\mathcal{N}'' = \mathcal{N}$. A projection $Q \in \mathcal{N}$ ($Q = Q^2 = Q^*$) is *centrally commutative* if $Q \in \mathcal{Z}(\mathcal{N})$ and reduced algebras $Q\mathcal{N}Q$ are commutative. For $R \in \mathcal{N}$, the *central support* $\hat{R}$ is the minimal central projection E with $ER = R$. For self-adjoint $S \in \mathcal{N}$, the *kernel* $\check{S}$ is the supremum of central projections bounded by S. A projection Q is *kernel-free* if $\check{Q} = 0$. Every von Neumann algebras decomposes into a type I_1 summand and a remainder with a kernel-free projection.

Remark 3.1. Assume $\mathcal{N}$ lacks central type I_1 summands. If $Q \in \mathcal{N}$ is kernel-free with $\hat{Q} = I$, then

$$I \check{-} Q = 0, \;\; I \hat{-} Q = I.$$

Here, $Q\mathcal{N}$ and $(I - Q)\mathcal{N}$ act nontrivially on $\mathcal{K}$, so

$$QTQ = 0 \;\text{ and }\; T(I - Q) = 0 \implies T = 0.$$

Centers of corners are

$$\mathcal{Z}(Q\mathcal{N}Q) = Q\mathcal{Z}(\mathcal{N}), \quad \mathcal{Z}((I - Q)\mathcal{N}(I - Q)) = (I - Q)\mathcal{Z}(\mathcal{N}).$$

Theorem 3.5. *For any von Neumann algebras* $\mathcal{N}$ *and isomorphism* σ *of* $\mathcal{N}$*:*

(i) A linear operator $\Delta\colon \mathcal{N} \to \mathcal{N}$ *satisfying one of* (J_1^σ), (J_2^σ)*, or* (J^σ) *is*

$$\Delta(R) = K\,\sigma(R) + \eta(R), \quad R \in \mathcal{N},$$

with $K \in \mathcal{Z}(\mathcal{N})$*, linear* $\eta\colon \mathcal{N} \to \mathcal{Z}(\mathcal{N})$ *such that* $\eta(R \circ S) = 0$ *when* $RS = 0$.

(ii) Every Jordan σ*-centralizer* $\Delta\colon \mathcal{N} \to \mathcal{N}$ *has this form.*

Proof. Decompose $I = F_1 + F_2$, with $\mathcal{N}F_1$ commutative, $\mathcal{N}F_2$ holding kernel-free projection of support I. By Remark 3.1, $\mathcal{N}F_2$ exemplifies a generalized matrix algebra. Apply Theorems 3.1 and 3.2 to $\mathcal{N}F_2$. Then, Lemmas 2.1 and 2.2 extend via direct sum. □

3.4. Standard operator algebras

Let $\mathcal{K}$ be a complex Banach space. Let $B(\mathcal{K})$ be a bounded operator on $\mathcal{K}$, $F(\mathcal{K})$ be the finite-rank ideal in $B(\mathcal{K})$. A subalgebra $O \subseteq B(\mathcal{K})$ containing I and $F(\mathcal{K})$ is a *standard operator algebra*. Clearly, $B(\mathcal{K})$ is basic. Every standard operator algebra is prime, and has nontrivial idempotents. Its center is $\mathcal{Z}(O) = \mathbb{C}I$. The generalized center matches $\mathbb{C}I$. See [7] for details. We state:

Theorem 3.6. *For standard operator algebras O on Banach $\mathcal{K}$ and an isomorphism σ of O:*

(i) A linear operator Δ: $O \to O$ satisfying (J_1^σ), (J_2^σ), or (J^σ) is $\lambda \in \mathbb{C}$, linear η: $O \to \mathbb{C}I$ with

$$\Delta(R) = \lambda\,\sigma(R) + \eta(R), \quad R \in O,$$

$\eta(R \circ S) = 0$ when $RS = 0$.

(ii) Every Jordan σ-centralizer Δ: $O \to O$ has

$$\Delta(R) = \lambda\,\sigma(R) + \eta(R), \quad R \in O,$$

$\lambda \in \mathbb{C}$, linear η: $O \to \mathbb{C}I$ with $\eta(R \circ S) = 0$ for all $R, S \in O$.

Proof. (i) Assume Δ satisfies one of (J_1^σ), (J_2^σ), or (J^σ). Since O includes $F(\mathcal{K})$ and finite-ranks combine rank-one idempotents, [9] implies Δ centralizes O:

$$[\Delta(R), R] = 0 \quad (R \in O).$$

Primeness of O and [5, Theorem A] give

$$\Delta(R) = \lambda R + \eta(R), \quad R \in O,$$

λ in generalized center of O, additive η to it. An extended center $\mathbb{C}I$ implies $\lambda \in \mathbb{C}$, $\eta(O) \subseteq \mathbb{C}I$. Linearity of Δ makes η linear. For $R, S \in O$, $RS = 0$,

$$\eta(R \circ S) = \Delta(R \circ S) - \lambda(R \circ S) = [\Delta(R), S] + [R, \Delta(S)] - \lambda(R \circ S) = 0.$$

Theorems 2.1–2.3 apply, giving the form.

(ii) For Jordan σ-centralizer Δ, similarly Δ centralizes O. Part (i) with Theorem 2.4 concludes. □

Corollary 3.1. *For $B(\mathcal{K})$ on Banach $\mathcal{K}$ and isomorphism σ of $B(\mathcal{K})$:*

(i) Linear Δ: $B(\mathcal{K}) \to B(\mathcal{K})$ satisfying (J_1^σ), (J_2^σ), or (J^σ) is

$$\Delta(R) = \lambda\,\sigma(R) + \eta(R), \quad R \in B(\mathcal{K}),$$

$\lambda \in \mathbb{C}$, linear η: $B(\mathcal{K}) \to \mathbb{C}I$ with $\eta(R \circ S) = 0$ when $RS = 0$.

(ii) Every Jordan σ-centralizer Δ: $B(\mathcal{K}) \to B(\mathcal{K})$ has this form.

3.5. Nest algebras

Let $\mathcal{K}$ be a complex Banach. A family C of closed $\mathcal{K}$-subspaces forms a *chain* if:

- C totally ordered by inclusion;
- $\{0\}$, $\mathcal{K}$ in C;
- For $\{C_i\} \subseteq C$, $\bigcap_i C_i$, $\overline{\text{span}}\{C_i\}$ in C.

The *nest algebras* $\text{Alg}\, C$ comprise bounded $\mathcal{K}$-operators preserving each C-subspace. $\text{Alg}\, C$ weakly closed, nontrivial if $|C| > 2$. For $C \in C$,

$$C^- = \bigvee\{M \in C : M \subsetneq C\},$$

supremum of strict C-predecessors of C. See [8] for structure. We provide:

Theorem 3.7. *Let C chain on Banach $\mathcal{K}$ with:*

(a) Some nonzero C-element complemented in $\mathcal{K}$;
(b) Every $C \in C$ with $C^- = C$ complemented in $\mathcal{K}$.

For isomorphism σ of $\text{Alg}\, C$*:*

(i) Linear Δ: $\text{Alg}\, C \to \text{Alg}\, C$ *satisfying (J_1^σ), (J_2^σ), or (J^σ) is*

$$\Delta(R) = \lambda\, \sigma(R) + \eta(R) \quad (R \in \text{Alg}\, C),$$

$\lambda \in \mathbb{C}$, linear η: $\text{Alg}\, C \to \mathbb{C}I$ *with* $\eta(R \circ S) = 0$ *when* $RS = 0$.
(ii) Any Jordan σ-centralizer Δ on $\text{Alg}\, C$ *has this form.*

Proof. Per [9, Theorem 4.1], premises validate (i) of Theorems 2.1–2.3 for $\text{Alg}\, C$. Likewise, [9, Corollary 4.2] confirms Theorem 2.4 premises. The characterizations follow from these. □

For Hilbert $\mathcal{K}$, all closed subspaces are complemented, so Theorem 3.7 holds.

Corollary 3.2. *For Hilbert $\mathcal{K}$, nest* $\text{Alg}\, C$ *on $\mathcal{K}$. For an isomorphism σ of* $\text{Alg}\, C$*:*

(i) Linear Δ: $\text{Alg}\, C \to \text{Alg}\, C$ *satisfying (J_1^σ), (J_2^σ), or (J^σ) is*

$$\Delta(R) = \lambda\, \sigma(R) + \eta(R), \quad R \in \text{Alg}\, C,$$

$\lambda \in \mathbb{C}$, linear η: $\text{Alg}\, C \to \mathbb{C}I$ *with* $\eta(R \circ S) = 0$ *when* $RS = 0$.
(ii) Every Jordan σ-centralizer on $\text{Alg}\, C$ *has this.*

4. Conjectures

The results of this paper establish strong structural parallels between Jordan centralizers and Jordan σ-centralizers in associative algebras. Motivated by Theorems 2.1–2.4 and their applications, it is natural to ask whether analogous equivalences remain valid in broader algebraic settings.

In particular, alternative algebras form an important and well-studied class of non-associative algebrass in which Jordan-type identities and zero-product conditions have been investigated extensively. We conclude by proposing the following conjectures, which are intended to serve as a foundation and motivation for future research in this direction.

Conjecture 1. Let B be an alternative algebra, and let σ be an isomorphism of B. The following assertions are equivalent:

(i) Any linear operator $\Delta\colon B \to B$ satisfying (J_1) has the form

$$\Delta(u) = ku + \eta(u), \qquad u \in B,$$

where $k \in Z(B)$ and $\eta\colon B \to Z(B)$ is linear and satisfies $\eta(u \circ v) = 0$ whenever $uv = 0$.

(ii) Any linear operator $\Delta\colon B \to B$ satisfying (J_1^σ) has the form

$$\Delta(u) = k\sigma(u) + \eta(u), \qquad u \in B,$$

where $k \in Z(B)$ and $\eta\colon B \to Z(B)$ is linear and satisfies $\eta(u \circ v) = 0$ whenever $uv = 0$.

Conjecture 2. Let B be an alternative algebra, and let σ be an isomorphism of B. The following assertions are equivalent:

(i) Any linear operator $\Delta\colon B \to B$ satisfying (J_2) has the form

$$\Delta(u) = ku + \eta(u), \qquad u \in B,$$

where $k \in Z(B)$ and $\eta\colon B \to Z(B)$ is linear with $\eta(u \circ v) = 0$ whenever $uv = 0$.

(ii) Any linear operator $\Delta\colon B \to B$ satisfying (J_2^σ) has the form

$$\Delta(u) = k\sigma(u) + \eta(u), \qquad u \in B,$$

where $k \in Z(B)$ and $\eta\colon B \to Z(B)$ is linear with $\eta(u \circ v) = 0$ whenever $uv = 0$.

Conjecture 3. Let B be an alternative algebra, and let σ be an isomorphism of B. The following assertions are equivalent:

(i) Any linear operator $\Delta\colon B \to B$ satisfying (J) has the form

$$\Delta(u) = ku + \eta(u), \qquad u \in B,$$

where $k \in Z(B)$ and $\eta\colon B \to Z(B)$ is linear with $\eta(u \circ v) = 0$ whenever $uv = 0$.

(ii) Any linear operator $\Delta\colon B \to B$ satisfying (J^σ) has the form

$$\Delta(u) = k\sigma(u) + \eta(u), \qquad u \in B,$$

where $k \in Z(B)$ and $\eta\colon B \to Z(B)$ is linear with $\eta(u \circ v) = 0$ whenever $uv = 0$.

Conjecture 4. Let B be an alternative algebra, and let σ be an isomorphism of B. The following assertions are equivalent:

(i) Any Jordan centralizer $\Delta\colon B \to B$ has the form

$$\Delta(u) = ku + \eta(u), \qquad u \in B,$$

where $k \in Z(B)$ and $\eta\colon B \to Z(B)$ is linear, satisfying $\eta(u \circ v) = 0$ for all $u, v \in B$.

(ii) Any Jordan σ-centralizer $\Delta\colon B \to B$ has the form

$$\Delta(u) = k\sigma(u) + \eta(u), \qquad u \in B,$$

where $k \in Z(B)$ and $\eta\colon B \to Z(B)$ is linear, satisfying $\eta(u \circ v) = 0$ for all $u, v \in B$.

5. Conclusions and future directions

This study offers a cohesive viewpoint on Jordan σ-centralizers and their zero-product variants in associative algebras. By contrasting the revised identities (J_1^σ), (J_2^σ), (J^σ) with standard (J_1), (J_2), (J), we obtained equivalence theorems showing that, for σ an isomorphism, the structures of associated operators parallel those of usual Jordan centralizers. Key results indicate that any linear operator Δ satisfying a Jordan σ-centralizer identity adopts the canonical shape

$$\Delta(u) = k\,\sigma(u) + \eta(u),$$

k central in the algebras , η central-valued linear vanishing on Jordan null products. This integrates and advances multiple prior outcomes on Jordan centralizing operators and extensions. The methods apply across extensive algebra types: generalized matrix algebras, upper-triangular algebras, von Neumann algebras, standard operator algebras and nest algebras. Our structure broadens works like Ashraf-Ansari [2, 3] and Fadaee-Ghahramani [9], incorporating isomorphisms into Jordan-type centralizer studies.

Future investigations. This research suggests multiple avenues:

- Broadening to nonlinear, additive, or multiplicative operators under modified or near-Jordan σ-centralizer constraints.
- Probing connections of σ-centralizers to classes like σ-derivations, Lie σ-centralizers, higher Jordan triple σ-centralizers in associative/nonassociative environments.
- Employing the framework in Banach/C^*- algebras, adding topological/continuity/norm conditions for refined structures.
- Assessing Jordan σ-centralizer traits under algebraic operations: extensions, tensor products, quotients, and completions.
- Leveraging for incidence algebras, infinite-dimensional operator algebras, stressing analytic/spectral aspects.

Anticipated is the growth of a wider theory on σ-centralizing operators, super-algebras and operator-algebras.

Use of Generative-AI tools declaration

The author declares he has not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declare that there is no conflict of interest.

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