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Research article

Self-normalized Cramér moderate deviations for a supercritical Galton-Waston process with immigration in random environments

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Abstract: In this paper, we mainly investigated the self-normalized Cramér-type moderate deviations for the Galton-Watson process with immigration in random environments. Our central approach was to establish a self-normalized moderate deviation principle for martingales related to the Lotka-Nagaev estimator under a set of relatively broad conditions.

Keywords: Galton-Watson process; immigration; self-normalized; Cramér-type; moderate deviations; random environment
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1. Introduction

Branching processes constitute a fundamental class of stochastic models with significant theoretical and applied importance in probability theory. Originating from the classical Galton-Watson model in 1873, this framework has been extended to include various advanced formulations, such as branching processes with immigration, continuous-time versions, and more general random environment adaptations.

For the research of deviations, Cramér [1] introduced nonlinear transform techniques to approximate abnormal distributions by the normal distribution, which became foundational for the development of deviation research. Building upon Cramér's foundational contributions, researchers have established moderate deviations across various probabilistic frameworks. For instance, Fan and Shao [2] investigated tail probability asymptotics for martingales with conditional variance normalization. Doukhan [3] rigorously derived asymptotic expansions of moderate deviations for both standardized and normalized Lotka-Nagaev estimators.

Without immigration, the model degenerates into a pure branching process in a random environment (i.e., BPRE). Significant progress has been made in BPRE theory, including central limit theorems [4], large deviations [5], Berry-Esseen bounds [6], extinction probabilities [7], and moderate deviations and

moments [8], encompassing their integrated treatment and subsequent refinements in recent studies.

Recently, research on branching processes with immigration in random environments (i.e., BPIREs) has also made significant theoretical advances. For instance, Huang and Wang [9] characterized the asymptotic behavior of transition probabilities. Wang and Liu [10] revealed the effect of immigration on large deviation rate functions. The research by Wang and Liu [11] demonstrated the regulatory role of immigration in convergence rates. Wang and Huang [12] established a systematic moment theory framework. These achievements collectively advanced the theoretical system of BPIREs.

Compared with classical branching process research, the innovation of this study lies in introducing immigration, which substantially enhances both the model complexity and applicability. By constructing martingale difference sequences satisfying moderate deviation conditions, we extend martingale theory to Galton-Watson processes with immigration terms, thereby establishing a self-normalized Cramér-type moderate deviation principle.

Let

$$\xi = (\xi_i)_{i\geq 0}$$

denote a sequence of independent and identically distributed (i.i.d.) random environments. Let $(Z_i)_{i\geq 0}$ denote the branching process with immigration $(Y_i)_{i\geq 0}$ in a random environment ξ, with the iterative presentation as follows:

$$Z_0 = 1, \quad Z_{i+1} = Y_i + \sum_{k=1}^{Z_i} X_{i,k}, \quad i = 0, 1, 2, \cdots.$$

Here, $X_{i,k}$ denotes the number of offspring produced by the k-th particle in the i-th generation, and Y_i represents the number of immigrants in the i-th generation. Given the environment ξ, the sequence $(X_{i,k}, Y_i, i = 0, 1, \cdots, k = 1, 2, \cdots)$ forms a set of independent random variables. For each fixed i, all $X_{i,k}$ $(k = 1, 2, \cdots)$ share the same distribution, which depends on the environment ξ_i at time i and is denoted as

$$\mathbf{p}(\xi_i) = (p_n(\xi_i) : n \in \mathbf{N}).$$

Similarly, the distribution of Y_i also depends on ξ_i and is denoted by

$$\widehat{\mathbf{p}}(\xi_i) = (\widehat{p_n}(\xi_i) : n \in \mathbf{N}).$$

For simplicity, we conventionally denote $X_0 = X_{0,1}$ (whose conditional distribution, given the environment ξ, is $\mathbf{p}(\xi_0)$).

Let (Γ, P_ξ) denote the probability space in which the process is defined under a given environment ξ, where P_ξ is commonly referred to as the *quenched* law. The state space of the random environment ξ_n is denoted by Θ, a compete space.

One can construct a complete probability space as $\Gamma \times \Theta^{\mathbf{N}}$ equipped with measure P, defined by:

$$P(dx, d\xi) = P(dx \mid \xi)\, \zeta(d\xi),$$

and then P is referred to as the total law of the process $(Z_i)_{i\geq 0}$, called the annealed law.

The key property is that integration satisfies:

$$\mathbf{E}_P[g] = \int \zeta(d\xi) \int g(x, \xi)\, P_\xi(dx),$$

for measurable $g \geq 0$.

Throughout the paper, we denote the probability distribution of ξ by ζ. The measure P_ξ can be regarded as the conditional probability distribution of P under the given environment ξ. We denote the corresponding expectations by E_ξ and E, with respect to the quenched law P_ξ and the annealed law P.

Let μ denotes the mean number of offspring per individual, commonly referred to as the *offspring mean*. Clearly, we have

$$\mu = \mathbf{E}Z_1 = \mathbf{E}X_{i,k} = \sum_{n=1}^{\infty} n\mathbf{E}p_n(\xi_{i,k}), \quad k \in \mathbf{N},$$
$$a = \mathbf{E}(Y_i),$$

where p_k is the probability that an individual produces k offspring. Thus, μ represents both the mean of Z_1 and the common mean of all $X_{i,k}$. Set

$$\mu_i := \mu_i(\xi_i) = \mathbf{E}_\xi X_{i,k} = \sum_{k=1}^{\infty} kp_k(\xi_i), \quad i \geq 0,$$
$$m = \mathbf{E}\ln\mu_0, \quad \nu^2 = \mathbf{E}(\ln\mu_0 - m)^2, \quad \text{and} \quad v^2 = \mathbf{E}(Z_1 - \mu_0)^2.$$

In this article, suppose

$$p_0(\xi_0) = 0, \text{ a.s.}$$

Then $(Z_i)_{i\geq 0}$ is supercritical, since each individual in the system has at least one offspring. Symbols σ, τ, and $\bar{v}$ represent the standard deviations of Z_1, μ_0, and Y_1, respectively:

$$\sigma^2 = \mathbf{E}(Z_1 - \mu)^2, \quad \tau^2 = \mathbf{E}(\mu_0 - \mu)^2, \quad \bar{v}^2 = \mathbf{E}(Y_1 - a)^2. \tag{1.1}$$

We require $\sigma, \tau, \bar{v} \in (0, \infty)$ to avoid degenerate cases, which guarantees that m, ν^2, and v are all finite. Consider a sequence of nonnegative i.i.d. random variables $(\mu_i)_{i\geq 1}$. The distribution of each μ_i depends on its corresponding ξ_i, but is independent of $(Z_j)_{1\leq j\leq i}$ for all $i \geq 1$. Clearly, all these random variables have the same expectation, denoted by

$$\mu = \mathbf{E}\mu_i.$$

The structure of this paper is organized as follows. In Section 1, we first review the research progress on deviation problems in branching processes, summarizing key achievements and theoretical developments. Subsequently, the mathematical models used in this paper are formally defined, and a unified notation system is established. In Section 2, we state the main theorems of this paper, provide complete mathematical proofs, and construct a comprehensive theoretical framework based on these results. In Section 3, we focus on analyzing the impact mechanism of immigration on conclusions, revealing the regulatory role of variance terms under supercritical conditions.

2. Main results

For the sake of discussion, we first provide some prelinminaries. Consider a sequence of martingale differences $\left(\eta_j, \mathcal{G}_j\right)_{j\geq 0}$ on $(\Omega, \mathcal{G}, \mathbf{P})$, where $\eta_0 = 0$ and the σ-fields form a filtration:

$$\{\emptyset, \Omega\} = \mathcal{G}_0 \subseteq \cdots \subseteq \mathcal{G}_j \subseteq \mathcal{G}.$$

Denote

$$T_0 = 0, \quad T_i = \sum_{j=1}^{i} \eta_j, \ i \geq 1,$$

and then

$$T = (\eta_i, \mathcal{G}_i)_{i\geq 0}$$

is a martingale. Let

$$[T]_0 = 0, \ [T]_i = \sum_{j=1}^{i} \eta_j^2, \quad \langle T\rangle_0 = 0,$$

and

$$\langle T\rangle_i = \sum_{j=1}^{i} \mathbf{E}\left[\eta_j^2 \mid \mathcal{G}_{j-1}\right].$$

Consider three sequences of positive numbers, denoted by $(A_i)_{i\geq 1}$, $(\lambda_i)_{i\geq 1}$, and $(\gamma_i)_{i\geq 1}$. Throughout this paper, we suppose that $\max\left\{A_i^{-1}, \ \lambda_i, \ \gamma_i\right\} \to 0$ as $i \to \infty$. On the basis of the above notations, we make the following assumptions.

(B1) The following holds for all $x > 0$, for some A_i, and $\lambda_i \in (0, \frac{1}{4}]$:

$\mathbf{P}\Big(\left|\langle T\rangle_i - A_i^2\right| \geq xA_i^2\Big) \leq c\exp\Big\{-x\lambda_i^{-2}\Big\}$;

(B2) There exist $q \in (0, 1]$ and $\gamma_i \ \in \left(0, \dfrac{1}{4}\right]$ such that $\forall 1 \leq j \leq i$,

$\mathbf{E}\left[\ |\eta_j|^{2+q} \ \middle| \ \mathcal{G}_{j-1}\right] \leq (\gamma_i A_i)^q \, \mathbf{E}\left[\eta_j^2 \ \middle| \ \mathcal{G}_{j-1}\right]$;

(B3) If there exists $\epsilon_i \in \left(0, \dfrac{1}{4}\right], \epsilon_i \to 0,$ such that $\forall 1 \leq j \leq i$,

$$\mathbf{E}\left[|\eta_j|^{\rho}\middle|\mathcal{G}_{j-1}\right] \leq \frac{1}{2}\rho!(A_i\epsilon_i)^{\rho-2}\mathbf{E}\left[\eta_j^2\middle|\mathcal{G}_{j-1}\right], \quad \rho \geq 2.$$

Lemma 1. *Conditions (B1) and (B2) imply that*

$$\left|\ln \frac{\mathbf{P}\left(T_i/\sqrt{[T]_i} \geq x\right)}{1-\Phi(x)}\right| \leq c_q\left(x^{2+q}\gamma_i^q + \lambda_i(x^2 + |\ln\lambda_i|) + (1+x)\widehat{\gamma}_i(x,q)\right) \tag{2.1}$$

uniformly for

$$0 \leq x = o(\min\left\{\gamma_i^{-1}, \lambda_i^{-1}\right\}),$$

where

$$\widehat{\gamma}_i(x,q) = \frac{\gamma_i^{q(2-q)/4}}{1 + x^{q(2+q)/4}}.$$

Lemma 2. *Under conditions (B1) and (B3), for all*

$$0 \leq x = o(\ \min\{\epsilon_i^{-1}, \lambda_i^{-1}\}),$$

we get

$$\left|\ln \frac{\mathbf{P}\left(T_i > xA_i\right)}{1-\Phi(x)}\right| \leq c_q\left(x^3\epsilon_i + x^2\lambda_i + (1+x)\left(\lambda_i\left|\ln\lambda_i\right| + \epsilon_i\left|\ln\epsilon_i\right|\right)\right).$$

Lemma 3. *For all* $0 < x \leq m\sqrt{i}/\nu$, *it holds that*

$$\mathbf{P}\left(\left|\frac{\ln Z_i - im}{\sqrt{i}\nu}\right| \geq x\right) \leq c_q^{-1} \exp\left\{-c_q x^2\right\}.$$

Assume that the total population of all generations $\left(Z_j\right)_{j\geq 0}$ is observable. For any $i \geq 1$, consider

$$M_i = \frac{\sqrt{i}\left(\frac{1}{i}\sum\limits_{j=1}^{i}\frac{Z_{j+1}-a}{Z_j} - \mu\right)}{\sqrt{\frac{1}{i-1}\sum\limits_{j=1}^{i}\left(\frac{Z_{j+1}-a}{Z_j} - \frac{1}{i}\sum\limits_{j=1}^{i}\frac{Z_{j+1}-a}{Z_j}\right)^2}}.$$

It is easily seen that the Lotka-Nagaev estimator is employed to construct a standardized statistic M_i, based on observed population generation data $(Z_j)_{j\geq 0}$, for testing the stability of the growth rate parameter.

Theorem 1. *Suppose that*

$$\mathbf{E}|Z_1 - \mu_0|^{2+q} + \mathbf{E}|\mu_0 - \mu|^{2+q} + \mathbf{E}|Y_1 - a|^{2+q} < \infty$$

for some $q \in (0, 1]$, *and then*

$$\left|\ln \frac{\mathbf{P}\left(M_i \geq x\right)}{1 - \Phi(x)}\right| = O\left(\frac{x^{2+q}}{i^{q/2}} + (1 + x)\frac{i^{-q(2-q)/8}}{1 + x^{q(2+q)/4}}\right) \tag{2.2}$$

uniformly for

$$0 \leq x = o((i/\ln i)^{\frac{1}{2}}),$$

where $\Phi(x)$ *is the distribution function of the standard normal law.*

Theorem 1 establishes the optimal convergence range. To enhance practical applications, we further provide a slightly weaker but more verifiable convergence condition in the following remark.

Remark 1. *Specifically, we have that*

$$\frac{\mathbf{P}\left(M_i \geq x\right)}{1 - \Phi(x)} = 1 + o(1)$$

holds uniformly for

$$0 \leq x = o\left(i^{q/(4+2q)}\right)$$

as $i \to \infty$. *The same holds when substituting* $\frac{\mathbf{P}(M_i \geq x)}{1-\Phi(x)}$ *for* $\frac{\mathbf{P}(M_i \leq -x)}{\Phi(-x)}$.

The proof of Theorem 1 will be completed based on the estimates provided in Lemmas 1 and 3 (see Fan and Shao [13]).

Proof. Denote

$$\tilde{\alpha}_{j+1} = \frac{Z_{j+1} - a}{Z_j} - \mu. \tag{2.3}$$

Define

$$\mathfrak{Q}_1 = \{\emptyset, \Omega\}$$

and

$$\mathfrak{Q}_{j+1} = \sigma\{\alpha_{k-1}, Z_k : 1 \le k \le j+1\}$$

for all $j \ge 1$. It follows directly that

$$\begin{aligned}
\mathbf{E}\left[\tilde{\alpha}_{j+1} \mid \mathfrak{Q}_j\right] &= Z_j^{-1}\mathbf{E}\left[Z_{j+1} - \mu Z_j - a \mid \mathfrak{Q}_j\right] \\
&= Z_j^{-1}\sum_{k=1}^{Z_j}\mathbf{E}\left[(X_{j,k} - \mu) + (Y_j - a) \mid \mathfrak{Q}_j\right] \\
&= 0.
\end{aligned}$$

Since $X_{j,k}$ is independent of Z_j and hence $\left(\tilde{\alpha}_j, \mathfrak{Q}_j\right)_{j=1,\cdots,i}$ forms a finite martingale difference sequence, this enables us to derive the following:

$$\begin{aligned}
\sum_{j=1}^{i}\mathbf{E}\left[\tilde{\alpha}_{j+1}^2 \mid \mathfrak{Q}_j\right] &= \sum_{j=1}^{i} Z_j^{-2}\mathbf{E}\left[\left(Z_{j+1} - \mu Z_j - a\right)^2 \mid \mathfrak{Q}_j\right] \\
&= \sum_{j=1}^{i} Z_j^{-2}\mathbf{E}\left[\left(\sum_{k=1}^{Z_j}\left(X_{j,k} - \mu\right)\right)^2 + 2\sum_{k=1}^{Z_j}\left(X_{j,k} - \mu\right)(Y_j - a) + (Y_j - a)^2 \;\middle|\; \mathfrak{Q}_j\right] \\
&= \sum_{j=1}^{i} Z_j^{-2}\mathbf{E}\left[\left(\sum_{k=1}^{Z_j}\left(X_{j,k} - \mu_j + \mu_j - \mu\right)\right)^2 + 2\sum_{k=1}^{Z_j}\left(X_{j,k} - \mu\right)(Y_j - a)\right. \\
&\quad \left. + (Y_j - a)^2 \;\middle|\; \mathfrak{Q}_j\right] \\
&= \sum_{j=1}^{i} Z_j^{-2}\left(\mathbf{E}\left[\left(\sum_{k=1}^{Z_j}\left(X_{j,k} - \mu_j\right)\right)^2 \;\middle|\; \mathfrak{Q}_j\right] + \mathbf{E}\left[\left(\sum_{k=1}^{Z_j}\left(\mu_j - \mu\right)\right)^2 \;\middle|\; \mathfrak{Q}_j\right]\right. \\
&\quad + 2\mathbf{E}\left[\left(\sum_{k=1}^{Z_j}\left(X_{j,k} - \mu_j\right)\right)\left(\sum_{k=1}^{Z_j}\left(\mu_j - \mu\right)\right) + \sum_{k=1}^{Z_j}\left(X_{j,k} - \mu\right)(Y_j - a) \;\middle|\; \mathfrak{Q}_j\right] \\
&\quad \left. + \mathbf{E}\left[(Y_j - a)^2 \;\middle|\; \mathfrak{Q}_j\right]\right).
\end{aligned}$$

Observe that the random variables $(X_{j,k})_{k\ge1}$ are i.i.d. conditionally on ξ_j together with the standard

deviations of Z_1, μ_0, and Y_1 as defined in (1.1), and we can deduce that

$$\begin{aligned}\mathbf{E}\left[\left(\sum_{k=1}^{Z_j}\left(X_{j,k}-\mu_j\right)\right)^2\middle|\mathfrak{Q}_j\right] &= \mathbf{E}\left[\mathbf{E}\left[\left(\sum_{k=1}^{Z_j}\left(X_{j,k}-\mu_j\right)\right)^2\middle|\xi_j,\mathfrak{Q}_j\right]\middle|\mathfrak{Q}_j\right]\\ &= \mathbf{E}\left[\sum_{k=1}^{Z_j}\mathbf{E}\left[\left(X_{j,k}-\mu_j\right)^2\middle|\xi_j,\mathfrak{Q}_j\right]\middle|\mathfrak{Q}_j\right]\\ &= \sum_{k=1}^{Z_j}\mathbf{E}\left[\left(X_{j,k}-\mu_j\right)^2\mid\mathfrak{Q}_j\right]\\ &= \sum_{k=1}^{Z_j}\mathbf{E}\left(X_{j,k}-\mu_j\right)^2\\ &= Z_j\sigma^2.\end{aligned}$$

Similarly, we get

$$\mathbf{E}\left[\left(\sum_{k=1}^{Z_j}\left(\mu_j-\mu\right)\right)^2\middle|\mathfrak{Q}_j\right] = {Z_j}^2\mathbf{E}\left[\left(\mu_j-\mu\right)^2\mid\mathfrak{Q}_j\right] = {Z_j}^2\mathbf{E}\left(\mu_0-\mu\right)^2 = {Z_j}^2\tau^2$$

and

$$\mathbf{E}\left[\left(\sum_{k=1}^{Z_j}\left(X_{j,k}-\mu_j\right)\right)\left(\sum_{k=1}^{Z_j}\left(\mu_j-\mu\right)\right)\middle|\mathfrak{Q}_j\right] = 0,$$

and also

$$\mathbf{E}\left[\left(\sum_{k=1}^{Z_j}\left(X_{j,k}-\mu\right)\right)\left(Y_j-a\right)\middle|\mathfrak{Q}_j\right] = 0.$$

Hence, we have

$$\sum_{j=1}^{i}\mathbf{E}\left[\tilde{\alpha}_{j+1}^2\mid\mathfrak{Q}_j\right] = \sum_{j=1}^{i}(\tau^2+Z_j^{-1}\sigma^2+Z_j^{-2}\bar{v}^2) \tag{2.4}$$

and

$$\mathbf{E}\left[\tilde{\alpha}_{j+1}^2\mid\mathfrak{Q}_j\right] \geq \mathbf{E}\left(\mu_0-\mu\right)^2 = \tau^2. \tag{2.5}$$

In order to establish condition (B1), the following inequality must be estimated:

$$\mathbf{P}\left(\left|\frac{1}{i\tau^2}\sum_{j=1}^{i}\mathbf{E}\left[\tilde{\alpha}_{j+1}^2\mid\mathfrak{Q}_j\right]-1\right|\right) \leq \mathbf{P}\left(\frac{1}{i}\sum_{j=1}^{i}\frac{1}{Z_j}\geq x\right)+\mathbf{P}\left(\frac{1}{i}\sum_{j=1}^{i}\frac{1}{Z_j^2}\geq x\right).$$

Note that $Z_{j+1} \geq Z_j$, and whenever $x \geq \frac{8\ln i}{im}$, the conclusion follows:

$$\begin{aligned}
&\mathbf{P}\left(\frac{1}{i}\sum_{j=1}^{i}\frac{1}{Z_j} \geq x\right) \leq \mathbf{P}\left(\frac{1}{i}\sum_{j=0}^{i-1}\frac{1}{Z_j} \geq x\right) \\
&\leq \mathbf{P}\left(\frac{1}{i}\sum_{j=0}^{i-1}\frac{1}{Z_j} \geq x, Z_{\left[\frac{4}{5}ix\right]} \leq e^{m\left[\frac{1}{2}ix\right]}\right) + \mathbf{P}\left(\frac{1}{i}\sum_{j=0}^{i-1}\frac{1}{Z_j} \geq x, Z_{\left[\frac{4}{5}ix\right]} > e^{m\left[\frac{1}{2}ix\right]}\right) \\
&\leq \mathbf{P}\left(Z_{\left[\frac{4}{5}ix\right]} \leq e^{m\left[\frac{1}{2}ix\right]}\right) + \mathbf{P}\left(\frac{1}{i}\sum_{j=0}^{i-1}\frac{1}{Z_j} \geq x, Z_{\left[\frac{4}{5}ix\right]} > e^{m\left[\frac{1}{2}ix\right]}\right) \\
&\leq R_1 + R_2,
\end{aligned}$$

where

$$\begin{aligned}
R_1 &:= \mathbf{P}\left(\frac{\ln Z_{\left[\frac{4}{5}ix\right]} - m\left[\frac{4}{5}ix\right]}{\sqrt{\left[\frac{4}{5}ix\right]}\nu} \leq \frac{m\left[\frac{1}{2}ix\right] - m\left[\frac{4}{5}ix\right]}{\sqrt{\left[\frac{4}{5}ix\right]}\nu}\right), \\
R_2 &:= \mathbf{P}\left(\frac{\left[\frac{4}{5}ix\right]}{i} + \frac{i}{e^{\left[\frac{1}{2}ix\right]}} \geq x\right).
\end{aligned}$$

For R_1, Lemma 3 implies that for all $0 < x \leq 1$,

$$R_1 \leq c_q^{-1}\exp\left\{-c_q\left(\frac{m\left[\frac{1}{2}ix\right] - m\left[\frac{4}{5}ix\right]}{\sqrt{\left[\frac{4}{5}ix\right]}\nu}\right)^2\right\} \leq c_q^{-1}\exp\left\{-c_q' ix\right\}.$$

Regarding R_2, for all $x \geq \frac{8\ln i}{mi}$ and $i \geq 2 + \frac{5}{8}me^m$,

$$\frac{\left[\frac{4}{5}ix\right]}{i} + \frac{i}{e^{m\left[\frac{1}{2}ix\right]}} \leq \frac{4}{5}x + \frac{i}{e^{\frac{1}{2}8\ln i - m}} = \frac{4}{5}x + \frac{e^m}{i^3} < x.$$

This implies $R_2 = 0$. Therefore, we have for all $0 < x \leq 1$ and $i \geq 2 + \frac{5}{8}me^m$,

$$\begin{aligned}
\mathbf{P}\left(\frac{1}{i}\sum_{j=1}^{i}\frac{1}{Z_j} \geq x\right) &\leq \begin{cases} c_q^{-1}\exp\left\{-c_q' ix\right\}, & x \geq \frac{8\ln i}{im}, \\ 1, & 0 < x < \frac{8\ln i}{im}, \end{cases} \\
&\leq c_q^{-1}e^{8c_q'/m}\exp\left\{-c_q' i(\ln i)^{-1}x\right\}.
\end{aligned} \tag{2.6}$$

Similarly, we get

$$\mathbf{P}\left(\frac{1}{i}\sum_{j=1}^{i}\frac{1}{Z_j^2} \geq x\right) \leq c_q^{-1}e^{8c_q'/m}\exp\left\{-c_q'\, i(\ln i)^{-1}x\right\}. \tag{2.7}$$

Since both

$$\frac{1}{i}\sum_{j=1}^{i}\frac{1}{Z_j} \leq 1$$

and

$$\frac{1}{i}\sum_{j=1}^{i}\frac{1}{Z_j^2}\leq 1$$

hold, the convergence rate of the latter is significantly faster than that of the former. Consequently, from (2.6) and (2.7), for all $x>0$,

$$\mathbf{P}\left(\left|\frac{1}{i\tau^2}\sum_{j=1}^{i}\mathbf{E}\left[\tilde{\alpha}_{j+1}^2\mid\mathfrak{Q}_j\right]-1\right|\right)\leq c_q^{-1}e^{8c_q'/m}\exp\left\{-c_q'\,i(\ln i)^{-1}x\right\}. \tag{2.8}$$

Notice that

$$\begin{aligned}\mathbf{E}\left[\left|\tilde{\alpha}_{j+1}\right|^{2+q}\mid\mathfrak{Q}_j\right]&=Z_j^{-2-q}\mathbf{E}\left[\left|Z_{j+1}-\mu Z_j\right|^{2+q}\mid\mathfrak{Q}_j\right]\\&=Z_j^{-2-q}\mathbf{E}\left[\left|\sum_{k=1}^{Z_j}\left(X_{j,k}-\mu\right)\right|^{2+q}+\left|Y_j-a\right|^{2+q}\Big|\mathfrak{Q}_j\right].\end{aligned} \tag{2.9}$$

From Minkowski's inequality, it follows that

$$\begin{aligned}&\mathbf{E}\left[\left|\sum_{k=1}^{Z_j}\left(X_{j,k}-\mu\right)\right|^{2+q}+\left|Y_j-a\right|^{2+q}\Big|\mathfrak{Q}_j\right]\\&\leq\left(\sum_{k=1}^{Z_j}\left(\mathbf{E}\left[\left|X_{j,k}-\mu\right|^{2+q}+\left|Y_j-a\right|^{2+q}\Big|\mathfrak{Q}_j\right]\right)^{\frac{1}{2+q}}\right)^{2+q}\\&=\left(\sum_{k=1}^{Z_j}\left(\mathbf{E}\left[\left|X_{j,k}-\mu_j+\mu_j-\mu\right|^{2+q}+\left|Y_j-a\right|^{2+q}\Big|\mathfrak{Q}_j\right]\right)^{\frac{1}{2+q}}\right)^{2+q}\\&\leq Z_j^{2+q}2^{1+q}\Big(\mathbf{E}\left[\left|X_{j,k}-\mu_j\right|^{2+q}\mid\mathfrak{Q}_j\right]+\mathbf{E}\left[\left|\mu_j-\mu\right|^{2+q}\mid\mathfrak{Q}_j\right]\\&\quad+\mathbf{E}\left[\left|Y_j-a\right|^{2+q}\mid\mathfrak{Q}_j\right]\Big)\\&=2^{1+q}Z_j^{2+q}\left(\mathbf{E}\left|Z_1-\mu_0\right|^{2+q}+\mathbf{E}\left|\mu_0-\mu\right|^{2+q}+\mathbf{E}\left|Y_1-a\right|^{2+q}\right).\end{aligned} \tag{2.10}$$

It can be seen that

$$\mathbf{E}\left[\tilde{\alpha}_{j+1}^2\mid\mathfrak{Q}_j\right]\geq\tau^2.$$

By Eqs (2.9) and (2.10), we can derive that

$$\begin{aligned}\mathbf{E}\left[\left|\tilde{\alpha}_{j+1}\right|^{2+q}\mid\mathfrak{Q}_j\right]&\leq 2^{1+q}\left(\mathbf{E}\left|Z_1-\mu_0\right|^{2+q}+\mathbf{E}\left|\mu_0-\mu\right|^{2+q}+\mathbf{E}\left|Y_1-a\right|^{2+q}\right)\\&\leq 2^{1+q}\left(\frac{\mathbf{E}\left|Z_1-\mu_0\right|^{2+q}+\mathbf{E}\left|\mu_0-\mu\right|^{2+q}+\mathbf{E}\left|Y_1-a\right|^{2+q}}{\mathbf{E}(\mu_0-\mu)^2}\right)\mathbf{E}\left[\tilde{\alpha}_{j+1}^2\mid\mathfrak{Q}_j\right].\end{aligned} \tag{2.11}$$

Let

$$\eta_j=\tilde{\alpha}_{j+1},\quad \mathcal{G}_j=\mathfrak{Q}_{j+1},$$

and thus assumptions (B1) and (B2) are derived from (2.8) and (2.11). Therefore, we construct a finite martingale difference sequence $(\eta_j, \mathcal{G}_j)_{j=1,\cdots,i}$. This enables

$$\lambda_i^2 \asymp i^{-1} \ln i \quad \text{and} \quad \gamma_i \asymp i^{-\frac{1}{2}}.$$

Similarly, $(-\eta_j, \mathcal{G}_j)_{j=1,\cdots,i}$ is also a finite sequence of martingale differences. Consequently, (2.2) holds when $\frac{\mathbf{P}(M_i \geq x)}{1-\Phi(x)}$ is replaced by $\frac{\mathbf{P}(M_i \leq -x)}{\Phi(-x)}$.

This completes the proof. □

Assume that the total population of all generations $\left(Z_j\right)_{j\geq 0}$ is observable. Then, for any $i \geq 1$, we can utilize the Lotka-Nagaev estimator and construct the standardized statistic S_i as follows:

$$S_i = \frac{\sqrt{i}}{\tau}\left(\frac{1}{i}\sum_{j=1}^{i}\frac{Z_{j+1}-a}{Z_j} - \mu\right).$$

Theorem 2. *Provided that a constant $k \geq 0$, for any $\rho \geq 2$,*

$$\begin{aligned}
\mathbf{E}\left|Z_1-\mu_0\right|^{\rho} &\leq \frac{1}{2}\rho! k^{\rho-2}\mathbf{E}\left|Z_1-\mu_0\right|^2, \\
\mathbf{E}\left|\mu_0-\mu\right|^{\rho} &\leq \frac{1}{2}\rho! k^{\rho-2}\mathbf{E}\left|\mu_0-\mu\right|^2, \\
\mathbf{E}\left|Y_1-a\right|^{\rho} &\leq \frac{1}{2}\rho! k^{\rho-2}\mathbf{E}\left|Y_1-a\right|^2.
\end{aligned} \tag{2.12}$$

The estimate is valid uniformly when $0 \leq x = o((i/\ln i)^{\frac{1}{2}})$:

$$\left|\ln\frac{\mathbf{P}\left(S_i \geq x\right)}{1-\Phi(x)}\right| = O\left(x^3 i^{-\frac{1}{2}} + (1+x)i^{-\frac{1}{2}}\ln i\right). \tag{2.13}$$

Subsequently, a slightly weaker but more tractable version is presented.

Remark 2. *Specifically, we have that*

$$\frac{\mathbf{P}\left(S_i \geq x\right)}{1-\Phi(x)} = 1 + o(1),$$

which is valid uniformly for

$$0 \leq x = o(i^{\frac{1}{6}}).$$

Moreover, the result remains under the substitution of $\frac{\mathbf{P}(S_i \geq x)}{1-\Phi(x)}$ with $\frac{\mathbf{P}(S_i \leq -x)}{\Phi(-x)}$. These conditions (2.12) *correspond to Bernstein's conditions, which are mathematically equivalent to Cramér's condition, the existence of some $t_0 > 0$ such that*

$$\mathbf{E}e^{t_0|Z_1-\mu_0|} + \mathbf{E}e^{t_0|\mu_0-\mu|} + \mathbf{E}e^{t_0|Y_1-a|} < \infty.$$

We now formally establish Theorem 2 using Lemma 2. Lemma 2 (see Fan et al. [14]) provides a foundation for strengthening the proof of Theorem 2.

Proof. By definition of $\tilde{\alpha}_{j+1}$ and $\mathfrak{Q}_j$, it can be shown that for all integers $\rho \geq 2$,

$$\begin{aligned}
\mathbf{E}\left[\left|\tilde{\alpha}_{j+1}\right|^{\rho} \mid \mathfrak{Q}_j\right] &= Z_j^{-\rho}\mathbf{E}\left[\left|Z_{j+1} - \mu Z_j - a\right|^{\rho} \mid \mathfrak{Q}_j\right] \\
&= Z_j^{-\rho}\mathbf{E}\left[\left|\sum_{k=1}^{Z_j}\left(X_{j,k} - \mu\right) + \left(Y_j - a\right)\right|^{\rho}\middle|\mathfrak{Q}_j\right] \\
&= Z_j^{-\rho}\mathbf{E}\left[\left|\sum_{k=1}^{Z_j}\left(X_{j,k} - \mu_j\right) + \sum_{k=1}^{Z_j}\left(\mu_j - \mu\right) + \left(Y_j - a\right)\right|^{\rho}\middle|\mathfrak{Q}_{\mathrm{i}}\right] \\
&\leq 2^{\rho-1}Z_j^{-\rho}\Bigg(\mathbf{E}\left[\left|\sum_{k=1}^{Z_j}\left(X_{j,k} - \mu_j\right)\right|^{\rho}\middle|\mathfrak{Q}_j\right] + \mathbf{E}\left[\left|\sum_{k=1}^{Z_j}\left(\mu_j - \mu\right)\right|^{\rho}\middle|\mathfrak{Q}_j\right] \\
&\quad + \mathbf{E}\left[\left|Y_j - a\right|^{\rho} \mid \mathfrak{Q}_j\right]\Bigg).
\end{aligned}$$

Applying Minkowski's inequality to this case gives, for all $\rho \geq 2$,

$$\begin{aligned}
\mathbf{E}[\left|\tilde{\alpha}_{j+1}\right|^{\rho} \mid \mathfrak{Q}_j] &\leq 2^{\rho-1}Z_j^{-\rho}\Bigg(\Bigg(\sum_{k=1}^{Z_j}\Big(\mathbf{E}[|X_{j,k} - \mu_j|^{\rho} \mid \mathfrak{Q}_j]\Big)^{1/\rho}\Bigg)^{\rho} \\
&\quad + \Bigg(\sum_{k=1}^{Z_j}\Big(\mathbf{E}[|\mu_j - \mu|^{\rho} \mid \mathfrak{Q}_j]\Big)^{1/\rho}\Bigg)^{\rho} + \Big(\mathbf{E}[|Y_{j+1} - a|^{\rho} \mid \mathfrak{Q}_j]\Big)^{1/\rho}\Bigg)^{\rho}\Bigg) \\
&\leq 2^{\rho-1}\Big(\mathbf{E}\left|X_{j,k} - \mu_j\right|^{\rho} + \mathbf{E}\left|\mu_j - \mu\right|^{\rho} + Z_j^{-\rho}\mathbf{E}\left|Y_j - a\right|^{\rho}\Big) \\
&\leq 2^{\rho-1}\Big(\mathbf{E}\left|Z_1 - \mu_0\right|^{\rho} + \mathbf{E}\left|\mu_0 - \mu\right|^{\rho} + \mathbf{E}\left|Y_1 - a\right|^{\rho}\Big).
\end{aligned}$$

Observe that

$$\mathbf{E}\left[\tilde{\alpha}_{j+1}^2 \mid \mathfrak{Q}_j\right] \geq \tau^2.$$

Therefore, using condition (2.12), we have for all $\rho \geq 2$,

$$\begin{aligned}
\mathbf{E}\left[\left|\tilde{\alpha}_{j+1}\right|^{\rho} \mid \mathfrak{Q}_j\right] &\leq 2^{\rho-1}\frac{1}{2}\rho!k^{\rho-2}\left(\mathbf{E}\left(Z_1 - \mu_0\right)^2 + \mathbf{E}\left(\mu_0 - \mu\right)^2 + \mathbf{E}\left(Y_1 - a\right)^2\right) \\
&\leq 2^{\rho-1}\frac{1}{2}\rho!k^{\rho-2}\frac{1}{\tau^2}\left(\mathbf{E}(Z_1 - \mu_0)^2 + \mathbf{E}(\mu_0 - \mu)^2 + \mathbf{E}(Y_1 - a)^2\right)\mathbf{E}\left[\tilde{\alpha}_{j+1}^2 \mid \mathfrak{Q}_j\right] \\
&\leq \frac{1}{2}\rho!(2k)^{\rho-2}2K_{\tau}\mathbf{E}\left[\tilde{\alpha}_{j+1}^2 \mid \mathfrak{Q}_j\right] \\
&\leq \frac{1}{2}\rho!K^{\rho-2}\mathbf{E}\left[\tilde{\alpha}_{j+1}^2 \mid \mathfrak{Q}_j\right],
\end{aligned} \tag{2.14}$$

where

$$K_{\tau} = \frac{1}{\tau^2}\left(\mathbf{E}(Z_1 - \mu_0)^2 + \mathbf{E}(\mu_0 - \mu)^2 + \mathbf{E}(Y_1 - a)^2\right)$$

and

$$K = 1 + 2k + 2K_{\tau}.$$

Recalling inequality (2.8), it holds that

$$\mathbf{P}\left(\left|\frac{1}{i\tau^2}\sum_{j=1}^{i}\mathbf{E}\left[\tilde{\alpha}_{j+1}^2 \mid \mathfrak{Q}_j\right]-1\right|\right) \leq c_q^{-1}e^{8c_q'/m}\exp\left\{-c_q'\, i(\ln i)^{-1}x\right\}.$$

Let

$$\eta_j = \tilde{\alpha}_{j+1}$$

and

$$\mathcal{G}_j = \mathfrak{Q}_{j+1}.$$

Then, it can be seen that (2.8) and (2.14) satisfy conditions (B1) and (B3), respectively. Therefore, combining (2.8) with (2.14), the sequence $(\eta_j, \mathcal{G}_j)_{1\leq j\leq i}$ with

$$\eta_j = \frac{Z_{j+1}-a}{Z_j} - \mu$$

forms a martingale difference. We obtain

$$\lambda_i^2 \asymp i^{-1}\ln i \quad \text{and} \quad \gamma_i \asymp i^{-\frac{1}{2}}.$$

Therefore, it holds that

$$S_i = \frac{1}{\tau\sqrt{i}}\sum_{j=1}^{i}\eta_j,$$

and by applying Lemma 2 to the sequence $\left(\eta_j, \mathcal{G}_j\right)_{j=1,\cdots,i}$, we thus establish the following result, valid for all $0 \leq x = o((i/\ln i)^{1/2})$:

$$\left|\frac{\mathbf{P}(S_i \geq x)}{1-\Phi(x)}\right| = O\left(x^3 i^{-\frac{1}{2}} + (1+x)i^{-\frac{1}{2}}\ln i\right).$$

The argument can similarly be applied to $\left(-\eta_j, \mathcal{G}_j\right)_{j=1,\cdots,i}$, and it follows that Eq (2.13) remains valid upon substituting $\frac{\mathbf{P}(S_i\geq x)}{1-\Phi(x)}$ with $\frac{\mathbf{P}(S_i\leq -x)}{\Phi(x)}$. □

3. Conclusions

For BPIREs, this work establishes a self-normalized Cramér moderate deviation principle by constructing martingale conditions satisfying moderate deviations. Moreover, we prove that the standardized Lotka-Nagaev estimator is uniformly asymptotically normal for $0 \leq x = o((i/\ln i)^{1/2})$ under moment conditions. Compared to classical BPRE models, the key innovation lies in successfully handling the additional stochastic fluctuations induced by immigration terms Y_i. By establishing exponential inequalities for the martingale associated with immigration, we resolve technical challenges in conditional variance estimation.

For supercritical branching processes, the immigration variance $\bar{v}^2$ systematically regulates the convergence rate of self-normalized statistics by controlling higher-order moments, thereby establishing a quantitative framework for the influence of immigration effects in random environments. This directly enhances statistical inference for population dynamics.

Author contributions

J. Wang: conceptualization, writing–review and editing, funding acquisition; W. L. Xiao: conceptualization, writing–original draft preparation. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no potential conflicts of interest.

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