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Research article

Filippov-type existence theorems for q-fractional differential inclusions with nonlinear q-integral conditions in Banach spaces

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Abstract: This study focused on boundary value problems involving fractional q-difference inclusions subject to nonlinear q-integral conditions:

$$\begin{gathered} {}^{c}D_{q,\gamma}\left(\nu\left(\iota\right)-h\left(\iota,\nu\left(\iota\right)\right)\right)\in F\left(\iota,\nu\left(\iota\right)\right),\ \iota\in\left[0,\ell\right],1<\gamma\leq 2,\\ \nu\left(0\right)-\nu'\left(0\right)=a\left(\iota\right)\int_0^{\ell}\mathcal{G}_1\left(\tau,\nu\left(\tau\right)\right)d_q\tau,\\ \nu\left(\ell\right)-\nu'\left(\ell\right)=b\left(\iota\right)\int_0^{\ell}\mathcal{G}_2\left(\tau,\nu\left(\tau\right)\right)d_q\tau. \end{gathered}$$

By applying appropriate fixed point theorems, we established the existence of solutions to the problem considered. In addition, a Filippov-type result was provided. This work continued the study recently presented in [N. Allouch, J. R. Graef, S. Hamani, Boundary value problem for fractional q-difference equations with integral conditions in Banach spaces, *Fractal Fract.*, **6** (2022), 237].

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1. Introduction

Fractional differential inclusions generalize conventional differential equations by incorporating both fractional derivatives and set-valued functions, making them appropriate for modeling intricate systems in domains such as physics, engineering, economics, biology, control theory, and anomalous diffusion (see [1–3] and their citations). The presence of solutions to these problems is usually established by employing tools like fixed point theorems, measures of non-compactness, and selection theorems (see [2–4]). These techniques have been applied across diverse definitions of fractional derivatives, encompassing the Caputo and Riemann-Liouville types.

The q-fractional derivative, which merges fractional calculus with q-calculus, further expands this framework and is especially useful in modeling systems with discrete or quantum behavior, for instance, in quantum physics and control theory (see [5, 6]). The analysis of existence results for q-fractional differential equations and inclusions in Banach spaces has received increasing attention. Higher-order and multipoint q-integral boundary value problems were investigated in [7, 8]. Nonlinear impulsive q-difference equations with closed boundary conditions were studied in [9, 10]. Filippov-type results for fractional differential inclusions were established in [11–13], while the existence and topological structure of solution sets in Banach spaces were discussed in [14–16].

In 2020, Alqahtani et al. [17] studied the following q-fractional differential inclusions:

$$\begin{gathered} {}^{c}D_{q,\gamma}\nu(\iota) \in F(\iota, \nu(\iota)), \ \iota \in [0, \ell], 0 < \gamma \leq 1, \\ \nu(0) = \nu_0 \in \mathcal{E}. \end{gathered}$$

Through the framework of set-valued analysis, the measure of non-compactness, and the fixed point theory (specifically Darbo's and Mönch's fixed point theorems), we establish existence results under appropriate compactness and continuity conditions.

Recently, in 2022, Allouch et al. [18] used techniques involving measures of non-compactness and Mönch's fixed point theorem in order to demonstrate the existence of solutions to boundary value problems involving fractional q-difference equations with nonlinear integral conditions

$$\begin{gathered} {}^{c}D_{q,\gamma}\nu(\iota) = f(\iota, \nu(\iota)), \ \iota \in [0, \ell], 1 < \gamma \leq 2, \\ \nu(0) - \nu'(0) = \int_0^{\ell} \mathcal{G}_1(\tau, \nu(\tau))\, d\tau, \\ \nu(\ell) - \nu'(\ell) = \int_0^{\ell} \mathcal{G}_2(\tau, \nu(\tau))\, d\tau, \end{gathered}$$

where $0 < q < 1$, the functions $\mathcal{G}_1, \mathcal{G}_2 : [0, \ell] \times \mathcal{E} \to \mathcal{E}$ and $f : [0, \ell] \times \mathcal{E} \to \mathcal{E}$ are continuous, where $\mathcal{E}$ is a Banach space. ${}^{c}D_{q,\gamma}$ represents Caputo fractional q-difference derivative of order γ.

Motivated by the aforementioned works, our objective is to examine the existence of solutions and provide a Filippov-type result to a boundary value problem (BVP) for fractional q-difference inclusions with nonlinear q-integral conditions

$$\begin{gathered} {}^{c}D_{q,\gamma}(\nu(\iota) - h(\iota, \nu(\iota))) \in F(\iota, \nu(\iota)), \ \iota \in [0, \ell], 1 < \gamma \leq 2, \\ \nu(0) - \nu'(0) = a(\iota) \int_0^{\ell} \mathcal{G}_1(\tau, \nu(\tau))\, d_q\tau, \\ \nu(\ell) - \nu'(\ell) = b(\iota) \int_0^{\ell} \mathcal{G}_2(\tau, \nu(\tau))\, d_q\tau, \end{gathered} \tag{1.1}$$

where $q \in (0, 1)$ and a, b:$[0, \ell] \to \mathbb{R}$ are two real functions, $\mathcal{G}_1, \mathcal{G}_2$ and $h : [0, \ell] \times \mathcal{E} \to \mathcal{E}$ are continuous functions, $F : [0, \ell] \times \mathcal{E} \to \mathcal{P}(\mathcal{E})$ is a multivalued map, and $\mathcal{P}(\mathcal{E})$ is the family of all subsets of $\mathcal{E}$. ${}^{c}D_{q,\gamma}$ represents the Caputo fractional q-difference derivative of order γ.

This study extends previous research on fractional q-difference equations by considering higher-order ($1 < \gamma \leq 2$) multivalued (Filippov-type) inclusions with nonlinear q-integral boundary conditions. Unlike earlier works [6, 7], which addressed either first-order multivalued or higher-order single-valued cases, our approach unifies and generalizes these frameworks. Using Caputo fractional q-derivatives, set-valued analysis, selection theorems, and fixed point theory, we establish existence results to the problem (1.1), which extend the applicability of the theory to thermostat models, population dynamics, and switching systems. Owing to the multivalued nature of the righthand side, uniqueness and continuous dependence of solutions cannot generally be expected under mild assumptions and can be guaranteed only under stronger conditions such as single-valuedness, Lipschitz continuity with respect to the Hausdorff metric, or suitable monotonicity properties; the investigation of these aspects is therefore left for future research.

The structure of this paper is as follows: In Section 2, we introduce several essential definitions, theorems, and lemmas that form the foundation of our analysis. The main existence results discussed in Section 3 are obtained using fixed point techniques, and we conclude our study by presenting a Filippov-type result to the problem (1.1). The results obtained here extend the results (single-valued map) that have been presented in [18].

2. Concepts and materials

Let $C([0,\ell], \mathcal{E})$ represent the Banach space of all continuous functions $\nu : [0,\ell] \to \mathcal{E}$, with the supremum norm

$$\|\nu\|_\infty = \sup \{\|\nu(\iota)\|, \text{for all } \iota \in [0,\ell]\},$$

where $\mathcal{E}$ is a separable Banach space.

Let $L^1([0,\ell], \mathcal{E})$ denote the space of Bochner integrable functions $\varphi : [0,\ell] \to \mathcal{E}$, equipped with the norm

$$\|\varphi\|_{L^1} = \int_{[0,\ell]} \|\varphi(\iota)\| \, d\iota.$$

Let $\left(\mathcal{X}, \widetilde{d}\right)$ be a metric space induced by the norm $\|.\|$ on $\mathcal{X}$, and define the following collections of nonempty subsets of $\mathcal{X}$:

$$\begin{aligned}
\mathcal{P}_0(\mathcal{E}) &= \{A \in \mathcal{P}(\mathcal{E}) : A \neq \emptyset\}, \\
cl(\mathcal{E}) &= \{A \in \mathcal{P}_0(\mathcal{E}) : A \text{ is closed}\}, \\
bnd(\mathcal{E}) &= \{A \in \mathcal{P}_0(\mathcal{E}) : A \text{ is bounded}\}, \\
cpt(\mathcal{E}) &= \{A \in \mathcal{P}_0(\mathcal{E}) : A \text{ is compact}\}, \\
cv(\mathcal{E}) &= \{A \in \mathcal{P}_0(\mathcal{E}) : A \text{ is convex}\}.
\end{aligned}$$

Define the Hausdorff-type functional $\mathcal{H}_{\widetilde{d}} : \mathcal{P}(\mathcal{E}) \times \mathcal{P}(\mathcal{E}) \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{H}_{\widetilde{d}}(X, Y) = \max\left\{\sup_{x\in X} \widetilde{d}(x, Y), \sup_{y\in Y} \widetilde{d}(y, X)\right\},$$

where

$$\widetilde{d}(x, Y) = \inf_{y\in Y} \widetilde{d}(x, y) \text{ and } \widetilde{d}(y, X) = \inf_{x\in X} \widetilde{d}(x, y).$$

Then, $\left(\mathcal{P}_{b,cl}(\mathcal{E}), \mathcal{H}_{\widetilde{d}}\right)$ is a metric space and $\left(\mathcal{P}_{cl}(\mathcal{E}), \mathcal{H}_{\widetilde{d}}\right)$ is a generalized metric space (see [19]).

We now state the definitions and lemmas that will be referenced in the remainder of this paper.

Assume $\mathcal{E}$ is a separable Banach space and $\mathcal{Y} \subset \mathcal{E}$ is a nonempty and closed set. $\Xi : \mathcal{Y} \to \mathcal{P}_{cl}(\mathcal{E})$ is a multivalued operator.

Definition 2.1. *If there is $x \in \mathcal{Y}$ such that $x \in \Xi(x)$, then Ξ has a fixed point.*

Definition 2.2. *If the function $\Lambda : [0, \ell] \to \mathbb{R}$ defined by*

$$\Lambda(\iota) = \widetilde{d}(x, \Xi(t)) = \inf\{\|x - y\| : y \in \Xi(t)(\iota)\} \text{ for every } x \in \mathcal{E},$$

is measurable, then Ξ is said to be measurable.

Lemma 2.1. *[20] Let $\mathcal{X}$ be a separable metric space and $G : \mathcal{X} \to cl(\mathcal{X})$ be a multivalued map. Then, G has a measurable selection.*

Definition 2.3. *A multivalued map $\mathcal{T} : \mathcal{E} \to cl(\mathcal{E})$ is κ-Lipschitz if and only if there exists $\kappa > 0$ such that*

$$\mathcal{H}_{\widetilde{d}}(\mathcal{T}x, \mathcal{T}x') \leq \kappa \widetilde{d}(x, x'), \text{ for each } x, x' \in \mathcal{E}.$$

If $\kappa < 1$, then $\mathcal{T}$ is called a contraction multivalued map.

Lemma 2.2. *[9] Let $\left(\mathcal{X}, \widetilde{d}\right)$ be a complete metric space. If $\Omega : \mathcal{X} \to \mathcal{X}$ is a contraction, then the set of fixed points* Fix Ω *is nonempty.*

Extensive background on multivalued maps and differential inclusions can be found in the monograph of Aubin and Frankowska [21], where the foundations of set-valued analysis are developed. The theory of multivalued differential equations is systematically treated in Deimling [22], while comprehensive results on topological and variational methods are presented in Hu and Papageorgiou [23]. Let $q \in (0, 1)$ and $\alpha, \beta, \gamma \in \mathbb{R}$,

$$[\alpha]_q = \frac{q^\alpha - 1}{q - 1} = 1 + q + q^2 ... + q^{\alpha - 1}.$$

The q-analogue of the power function $(\alpha - \beta)^{(n)}$ with $n \in \mathbb{N}$ is

$$(\alpha - \beta)^0 = 1, \ (\alpha - \beta)^{(n)} = \prod_{i=0}^{n-1} \left(\alpha - \beta q^i\right), \ \alpha, \beta \in \mathbb{R}, \ n \in \mathbb{N}.$$

In general terms,

$$(\alpha - \beta)^{(\gamma)} = \alpha^\gamma \prod_{i=0}^{\infty} \left(\frac{\alpha - \beta q^i}{\alpha - \beta q^{\gamma + i}}\right).$$

Note that, if $\beta = 0$, then

$$\alpha^{(\gamma)} = \alpha^\gamma.$$

The q-gamma function is defined by

$$\Gamma_q(\iota) = \frac{(1 - q)^{(\iota - 1)}}{(1 - q)^{\iota - 1}}, \ x \in \mathbb{R} \backslash \{0, -1, -2, ...\}, \ 0 < q < 1,$$

and satisfies

$$\Gamma_q(\iota+1)=[\iota]_q\Gamma_q(\iota).$$

The q-derivative of a function $\mathfrak{f}:[0,\ell]\to\mathcal{E}$ is defined by

$$D_q\mathfrak{f}(\iota)=\frac{d_q\mathfrak{f}(\iota)}{d_q\iota}=\frac{\mathfrak{f}(q\iota)-\mathfrak{f}(\iota)}{(q-1)\iota},\ \iota\neq 0,\ D_q\mathfrak{f}(0)=\lim_{\iota\to 0}D_q\mathfrak{f}(\iota),$$

and q-derivatives of higher order are defined by

$$D_{q,m}\mathfrak{f}(\iota)=\begin{cases}\mathfrak{f}(\iota), & \text{if } m=0,\\ D_qD_{q,m-1}\mathfrak{f}(\iota), & \text{if } m\in\mathbb{N}.\end{cases}$$

The q-integral of a function $\mathfrak{f}$:$[0,\beta]\to\mathcal{E}$ is given by

$$\int_0^\iota\mathfrak{f}(\tau)d_q\tau=\iota(1-q)\sum_{m=0}^{\infty}\mathfrak{f}(\iota q^n)q^n,\ 0\leq|q|<1,\ t\in[0,\beta].$$

If $\alpha\in[0,\beta]$ and $\mathfrak{f}$ are defined in the interval $[0,\beta]$, its integral from α to β is defined by

$$\int_\alpha^\beta\mathfrak{f}(\tau)d_q\tau=\int_0^\beta\mathfrak{f}(\tau)d_q\tau-\int_0^\alpha\mathfrak{f}(\tau)d_q\tau.$$

Analogously to derivatives, the operator $\mathcal{I}_{q,n}$ can be stated as follows:

$$(\mathcal{I}_{q,0}\mathfrak{f})(\iota)=\mathfrak{f}(\iota)\ \text{and}\ (\mathcal{I}_{q,m}\mathfrak{f})(\iota)=\mathcal{I}_q(\mathcal{I}_{q,m-1}\mathfrak{f})(\iota),\ m\in\mathbb{N}.$$

According to the fundamental theorem of calculus, the operators $\mathcal{I}_q$ and D_q satisfy the relation:

$$D_q\left(\mathcal{I}_q\mathfrak{f}\right)(\iota)=\mathfrak{f}(\iota),$$

and if $\mathfrak{f}$ is continuous at $\iota=0$, then:

$$\mathcal{I}_q\left(D_q\mathfrak{g}\right)(\iota)=\mathfrak{g}(\iota)-\mathfrak{g}(0).$$

For classical results on fractional q-integrals and derivatives, see Agarwal [24]. The development of q-analogues of Caputo derivatives and Mittag-Leffler functions was studied in Rajković et al. [25]. For recent existence results in nonlinear fractional q-difference problems, see Agarwal et al. [26].

Definition 2.4. *Let $\gamma\geq 0$ and $\mathfrak{f}$ be a function defined on $[0,\ell]$. The Riemann-Liouville fractional q-integral is given as follows:*

$$\mathcal{I}_{q,\gamma}\mathfrak{f}(\iota)=\begin{cases}\mathfrak{f}(\iota), & \text{if } \gamma=0,\\ \frac{1}{\Gamma_q(\gamma)}\int_0^\iota(\iota-q\tau)^{(\gamma-1)}\mathfrak{f}(\tau)d_q\tau, & \text{if } \gamma>0,\end{cases}\quad \iota\in[0,\ell].$$

Definition 2.5. *The Riemann-Liouville fractional q-derivative of order $\gamma\geq 0$ is defined by*

$$\mathrm{D}_{q,\gamma}\mathfrak{f}(\iota)=\begin{cases}\mathfrak{f}(\iota), & \text{if } \gamma=0,\\ \left(\mathrm{D}_{q,[\gamma]}\mathcal{I}_{q,[\gamma]-\gamma}\mathfrak{f}\right)(\iota), & \text{if } \gamma>0,\end{cases}\quad \iota\in[0,\ell],$$

where $[\gamma]$ denotes the smallest integer greater than or equal to γ.

Definition 2.6. *The Caputo fractional q-derivative of order $\gamma \geq 0$ is defined by*

$$ {}^{c}\mathrm{D}_{q,\gamma}\mathfrak{f}(\iota) = \begin{cases} \mathfrak{f}(\iota), & \text{if } \gamma = 0, \\ \left(\mathcal{I}_{q,[\gamma]-\gamma}\mathrm{D}_{q,[\gamma]}\mathfrak{f}\right)(\tau), & \text{if } \gamma > 0, \end{cases} \quad \iota \in [0, \ell]. $$

Lemma 2.3. *[25] Let $\gamma \geq 0$. Consequently, we have the following equality:*

$$ \left(\mathcal{I}^{c}_{q,\gamma}\mathrm{D}_{q,\gamma}\mathfrak{f}\right)(\iota) = \mathfrak{f}(\iota) - \sum_{j=0}^{[\gamma]-1} \frac{\iota^{j}}{\Gamma_q(j+1)}\left(\mathrm{D}_{q,\gamma}\mathfrak{f}\right)(0). $$

Lemma 2.4. *Let $\gamma \in (1, 2]$ and $q \in (0, 1)$. For $\nu, \mathcal{M}, h_1, h_2 \in C([0, \ell], \mathcal{E})$ and $a, b{:}[0, \ell] \to \mathbb{R}$ are two real functions. The following system with nonlinear integral conditions:*

$$ \begin{aligned} {}^{c}\mathrm{D}_{q,\gamma}\nu(\iota) &= \mathcal{M}(\iota),\ \iota \in [0, \ell], 1 < \gamma \leq 2, \\ \nu(0) - \nu'(0) &= a(\iota)\int_0^{\ell} h_1(\tau)\, d_q\tau, \\ \nu(\ell) - \nu'(\ell) &= b(\iota)\int_0^{\ell} h_2(\tau)\, d_q\tau, \end{aligned} \tag{2.1} $$

has a solution that is shown below:

$$ \nu(\iota) = \mathcal{K}(\iota) + \int_0^{\ell} G(\iota, \tau)\, \mathcal{M}(\tau)\, d_q\tau. \tag{2.2} $$

The functions $\mathcal{K}(\iota)$ and $G(\iota, \tau)$ are given by

$$ \mathcal{K}(\iota) = \frac{\ell - \iota - 1}{\ell} a(\iota)\int_0^{\ell} h_1(\tau)\, d_q\tau + \frac{\iota + 1}{\ell} b(\iota)\int_0^{\ell} h_2(\tau)\, d_q\tau, \tag{2.3} $$

and

$$ G(\iota, \tau) = \frac{1}{\Gamma_q(\gamma)} \begin{cases} (\iota - q\tau)^{(\gamma-1)} + \frac{\iota+1}{\ell}\left((\gamma - 1)(\ell - q\tau)^{(\gamma-2)} - (\ell - q\tau)^{(\gamma-1)}\right), & 0 < \tau < \iota < \ell, \\ \frac{\iota+1}{\ell}\left((\gamma - 1)(\ell - q\tau)^{(\gamma-2)} - (\ell - q\tau)^{(\gamma-1)}\right), & 0 < \iota < \tau < \ell. \end{cases} \tag{2.4} $$

Proof. On the equation ${}^{c}D_{q,\gamma}\nu(\iota) = \mathcal{M}(\iota)$, we can apply the operator $\mathcal{I}_{q,\gamma}$, and we find

$$ \nu(\iota) = \frac{1}{\Gamma_q(\gamma)}\int_0^{\iota} (\iota - q\tau)^{(\gamma-1)}\, \mathcal{M}(\tau)\, d_q\tau + \iota\rho_1 + \rho_2, \tag{2.5} $$

where $\rho_1, \rho_2 \in \mathbb{R}$ are arbitrary constant.

Applying the boundary conditions specified in (2.1), we obtain

$$ \rho_2 - \rho_1 = a(\iota)\int_0^{\ell} h_1(\tau)\, d_q\tau, $$

and

$$ \frac{1}{\Gamma_q(\gamma)}\int_0^{\ell} (\ell - q\tau)^{(\gamma-1)}\, \mathcal{M}(\tau)\, d_q\tau + \ell\rho_1 + \rho_2 $$

$$= \frac{\gamma-1}{\Gamma_q(\gamma)}\int_0^{\ell}(\ell-q\tau)^{(\gamma-2)}\mathcal{M}(\tau)\,d_q\tau+\rho_1+b(\iota)\int_0^{\ell}h_2(\tau)\,d_q\tau.$$

With a simple calculation, ρ_1 and ρ_2 are given by

$$\begin{aligned}\rho_1 &= \frac{1}{\ell\Gamma_q(\gamma)}\left(\int_0^{\ell}\left((\gamma-1)(\ell-q\tau)^{(\gamma-2)}-(\ell-q\tau)^{(\gamma-1)}\right)\mathcal{M}(\tau)\,d_q\tau\right)\\ &\quad+\frac{1}{\ell}\left(-a(\iota)\int_0^{\ell}h_1(\tau)\,d_q\tau+b(\iota)\int_0^{\ell}h_2(\tau)\,d_q\tau\right),\\ \rho_2 &= \frac{1}{\ell\Gamma_q(\gamma)}\left(\int_0^{\ell}\left((\gamma-1)(\ell-q\tau)^{(\gamma-2)}-(\ell-q\tau)^{(\gamma-1)}\right)\mathcal{M}(\tau)\,d_q\tau\right)\\ &\quad+\frac{1}{\ell}\left((\ell-1)a(\iota)\int_0^{\ell}h_1(\tau)\,d_q\tau+b(\iota)\int_0^{\ell}h_2(\tau)\,d_q\tau\right).\end{aligned}$$

By changing the values of ρ_1 and ρ_2 in Eq (2.5),

$$\begin{aligned}\nu(\iota) &= \frac{1}{\Gamma_q(\gamma)}\int_0^{\iota}(\iota-q\tau)^{(\gamma-1)}\mathcal{M}(\tau)\,d_q\tau\\ &\quad+\frac{\iota+1}{\ell\Gamma_q(\gamma)}\int_0^{\ell}\left((\gamma-1)(\ell-q\tau)^{(\gamma-2)}-(\ell-q\tau)^{(\gamma-1)}\right)\mathcal{M}(\tau)\,d_q\tau\\ &\quad+\frac{\ell-\iota-1}{\ell}a(\iota)\int_0^{\ell}h_1(\tau)\,d_q\tau+\frac{\iota+1}{\ell}b(\iota)\int_0^{\ell}h_2(\tau)\,d_q\tau\\ &= \frac{1}{\Gamma_q(\gamma)}\int_0^{\iota}(\iota-q\tau)^{(\gamma-1)}\mathcal{M}(\tau)\,d_q\tau\\ &\quad+\frac{\iota+1}{\ell\Gamma_q(\gamma)}\int_0^{\iota}\left((\gamma-1)(\ell-q\tau)^{(\gamma-2)}-(\ell-q\tau)^{(\gamma-1)}\right)\mathcal{M}(\tau)\,d_q\tau\\ &\quad+\frac{\iota+1}{\ell\Gamma_q(\gamma)}\int_{\iota}^{\ell}\left((\gamma-1)(\ell-q\tau)^{(\gamma-2)}-(\ell-q\tau)^{(\gamma-1)}\right)\mathcal{M}(\tau)\,d_q\tau\\ &\quad+\frac{\ell-\iota-1}{\ell}a(\iota)\int_0^{\ell}h_1(\tau)\,d_q\tau+\frac{\iota+1}{\ell}b(\iota)\int_0^{\ell}h_2(\tau)\,d_q\tau\\ &= \frac{1}{\Gamma_q(\gamma)}\left(\int_0^{\iota}\left((\iota-q\tau)^{(\gamma-1)}+\frac{\iota+1}{\ell}\left((\gamma-1)(\ell-q\tau)^{(\gamma-2)}-(\ell-q\tau)^{(\gamma-1)}\right)\right)\mathcal{M}(\tau)\,d_q\tau\right)\\ &\quad+\frac{1}{\Gamma_q(\gamma)}\int_{\iota}^{\ell}\frac{\iota+1}{\ell}\left((\gamma-1)(\ell-q\tau)^{(\gamma-2)}-(\ell-q\tau)^{(\gamma-1)}\right)\mathcal{M}(\tau)\,d_q\tau\\ &\quad+\frac{\ell-\iota-1}{\ell}a(\iota)\int_0^{\ell}h_1(\tau)\,d_q\tau+\frac{\iota+1}{\ell}b(\iota)\int_0^{\ell}h_2(\tau)\,d_q\tau.\end{aligned}$$

So,

$$\nu(\iota)=\mathcal{K}(\iota)+\int_0^{\ell}G(\iota,\tau)\mathcal{M}(\tau)\,d_q\tau,$$

where $\mathcal{K}(\iota)$ and $G(\iota,\tau)$ are given by (2.3) and (2.4), respectively. □

Lemma 2.5. *Let $\ell \geq 1$, then the Green function $G(\iota,\tau)$ has the following proporties:*

$$|G(\iota,\tau)| \leq \mathcal{A}_1, \tag{2.6}$$

where

$$\mathcal{A}_1 = \frac{1}{\Gamma_q(\gamma)}\left(3\ell^{(\gamma-1)} + 2(\gamma-1)\ell^{(\gamma-2)}\right).$$

Proof. For $\iota,\tau \in [0,\ell]$, we assume that

$$A = (\ell - q\tau)^{(\gamma-1)} \text{ and } B = (\ell - q\tau)^{(\gamma-2)},$$

then, we have:

If $\tau < \iota$,

$$\begin{aligned} |G(\iota,\tau)| &= \frac{1}{\Gamma_q(\gamma)}\left|(\iota - q\tau)^{(\gamma-1)} + \frac{\iota+1}{\ell}\left((\gamma-1)B - A\right)\right|, \\ &\leq \frac{1}{\Gamma_q(\gamma)}\left(\ell^{(\gamma-1)} + \frac{\ell+1}{\ell}\left((\gamma-1)\ell^{(\gamma-2)} + \ell^{(\gamma-1)}\right)\right) \\ &\leq \frac{1}{\Gamma_q(\gamma)}\left(3\ell^{(\gamma-1)} + 2(\gamma-1)\ell^{(\gamma-2)}\right) = \mathcal{A}_1. \end{aligned}$$

If $\tau > \iota$,

$$\begin{aligned} |G(\iota,\tau)| &= \frac{1}{\Gamma_q(\gamma)}\left|\frac{\iota+1}{\ell}\left((\gamma-1)B - A\right)\right| \\ &\leq \frac{2}{\Gamma_q(\gamma)}\left((\gamma-1)\ell^{(\gamma-2)} + \ell^{(\gamma-1)}\right), \\ &\leq \mathcal{A}_1. \end{aligned}$$

Then, $|G(\iota,\tau)| \leq \mathcal{A}_1$. □

Finally, we present the solution's definition related to the system (1.1).

Definition 2.7. *Let $\gamma \in (1,2]$ and $q \in (0,1)$. A function $\nu \in C_{q,\gamma}([0,\ell],\mathcal{E})$ is a solution of the system* (1.1) *if ν satisfies the differential inclusion*

$${}^{c}\mathrm{D}_{q,\gamma}\left(\nu(\iota) - h(\iota,\nu(\iota))\right) \in F(\iota,\nu(\iota)), \ \textit{for all } \iota \in [0,\ell],$$

and the integral conditions

$$\begin{aligned} \nu(0) - \nu'(0) &= a(\iota)\int_0^\ell \mathcal{G}_1(\tau,\nu(\tau))\,d_q\tau, \\ \nu(\ell) - \nu'(\ell) &= b(\iota)\int_0^\ell \mathcal{G}_2(\tau,\nu(\tau))\,d_q\tau. \end{aligned}$$

3. Main results

We proceed to prove the existence of solutions to the problem (1.1), where the righthand side is non-convex-valued, by involving the fixed point theorem for multivalued mappings established by Covitz and Nadler [18].

In addition, for our result, we assume the following hypothesis:

Hypothesis (H):

(1) The function

$$h : [0, \ell] \times \mathcal{E} \to \mathcal{E} \tag{3.1}$$

is $\mathcal{A}_2$-Lipschitz.

(2) $F : [0, \ell] \times \mathcal{E} \to cpt(\mathcal{E})$ is a multifunction such that:

(a)

$$F(\cdot, u) : [0, \ell] \times \mathcal{E} \to cpt(\mathcal{E}) \tag{3.2}$$

is measurable for each $\iota \in [0, \ell]$.

(b) The map

$$\Gamma : \iota \to \widetilde{d}(\mathcal{M}(\iota), F(\iota v(\iota))) \tag{3.3}$$

is integrable.

(3) There exists $\sigma(t) \in C([0, \ell], \mathbb{R}^+)$ such that

$$\mathcal{H}_{\widetilde{d}}(F(t, v_1(\iota)), F(\iota, v_2(\iota))) \leq \sigma(t) \|v_1(\iota) - v_2(\iota)\| \text{ for almost all } v_1, v_2 \in \mathcal{E}. \tag{3.4}$$

To simplify the expressions that will be obtained, we set

$$\Gamma = \int_0^\ell \mathbf{\Gamma}(\tau)\, d_q\tau, \ \sigma = \int_0^\ell \sigma(\tau)\, d_q\tau.$$

3.1. Existence of solutions

Theorem 3.1. *Assume that Hypothesis* (**H**) *holds. Then, problem* (1.1) *admits at least one solution on* $[0, \ell]$ *if*

$$\mathcal{A}_1\sigma + \mathcal{A}_2 < 1. \tag{3.5}$$

Proof. Given $u \in C([0, \ell] \times \mathcal{E})$, we define the associated set of selections of F by

$$S_{F,u} := \{w \in L^1([0, \ell], \mathcal{E}) : w(t) \in F(t, u(t)) \text{ for a.e. } \iota \in [0, \ell]\}, \tag{3.6}$$

and for each $f \in S_{F,u}$, define the multivalued operator $\ominus : C([0, \ell] \times \mathcal{E}) \to cl(C([0, \ell] \times \mathcal{E}))$ by

$$\ominus(u) = \left\{ y \in C([0, \ell] \times \mathcal{E}) : y(t) = \mathcal{K}(\iota) + h(\iota, u(\iota)) + \int_0^\ell G(\iota, \tau) f(u)\, d_q\tau \right\}.$$

By the assumption (3.2), the set $S_{F,u} \neq \emptyset$ for each $u \in C([0, \ell] \times \mathcal{E})$. Hence, by Theorem III.6 in [20], F has a measurable selection.

Next, we verify that the operator $\ominus$ fulfills the requirements prescribed in Lemma 2.2. To show that $\ominus(u) \in cl(C([0,\ell]\times\mathcal{E}))$, for each $u \in C([0,\ell]\times\mathcal{E})$, let $\{v_n\}_{n\geq 0} \in \ominus(u)$ be such that $v_n \to v$ $(n\to\infty)$ in $C([0,\ell]\times\mathcal{E})$. Then, $v \in C([0,\ell]\times\mathcal{E})$, and there exists $w_n \in S_{F,u}$ such that, for each $\iota \in [0,\ell]$,

$$v_n(\iota) = \mathcal{K}(\iota) + h(\iota, u(\iota)) + \int_0^\ell G(\iota,\tau) w_n(\tau)\, d_q\tau. \tag{3.7}$$

As F has compact values, we may extract a subsequence such that w_n converges to w in $L^1([0,\ell]\times\mathcal{E})$. Consequently, $w \in S_{F,u}$, and for each $\iota \in [0,\ell]$, we have

$$v_n(\iota) \to v(\iota) = \mathcal{K}(\iota) + h(\iota, u(\iota)) + \int_0^\ell G(\iota,\tau) w(\tau)\, d_q\tau.$$

Hence, $v \in \ominus(u)$.

The next step is to verify that we can find a constant $\xi < 1$ such that

$$\mathcal{H}_{\widetilde{d}}(\ominus v_1, \ominus v_2) \leq \xi \|v_1 - v_2\| \text{ for each } v_1, v_2 \in C([0,\ell]\times\mathcal{E}).$$

Let $v_1, v_2 \in C([0,\ell]\times\mathcal{E})$, and $y_1 \in \ominus(v_1)$. Then, there exists $w_1(\iota) \in S_{F,v_1}$ such that, for each $\iota \in [0,\ell]$,

$$y_1(\iota) = \mathcal{K}(\iota) + h(\iota, v_1(\iota)) + \int_0^\ell G(\iota,\tau) w_1(\tau)\, d_q\tau.$$

By the assumption (3.4), we have

$$\mathcal{H}_{\widetilde{d}}(F(\iota, v_1), F(\iota, v_2)) \leq \sigma(\iota)\|v_1(\iota) - v_2(\iota)\|.$$

So, there exists $w \in S_{F,v_2}$ such that

$$\|w_1 - w\| \leq \sigma(\iota)\|v_1(\iota) - v_2(\iota)\|, \; \iota \in [0,\ell].$$

Define $\mathcal{T} : [0,\ell] \to \mathcal{P}(\mathcal{E})$ by

$$\mathcal{T}(\iota) = \{w \in \mathcal{E} : \|w_1 - w\| \leq \sigma(\iota)\|v_1(\iota) - v_2(\iota)\|\}.$$

By the proposition III.4 in [20], the multivalued operator $\mathcal{V}(\iota) = \mathcal{T}(\iota) \cap F(\iota, v_2(\iota))$ is measurable. Therefore, there exists a measurable selection $w_2(\iota)$ of $\mathcal{V}$. Consequently, $w_2(\iota) \in S_{F,v_2}$, and for each $\iota \in [0,\ell]$, we have

$$\|w_1(\iota) - w_2(\iota)\| \leq \sigma(\iota)\|v_1(\iota) - v_2(\iota)\|. \tag{3.8}$$

For each $\iota \in [0,\ell]$, let us define

$$y_2(\iota) = \mathcal{K}(\iota) + h(\iota, v_2(\iota)) + \int_0^\ell G(\iota,\tau) w_2(\tau)\, d_q\tau.$$

Thus,

$$\|y_1(\iota) - y_2(\iota)\| \leq \|h(\iota, v_1(\iota)) - h(\iota, v_2(\iota))\| + \int_0^\ell |G(\iota,\tau)|\, \|w_2(\tau) - w_1(\tau)\|\, d_q\tau.$$

By the assumptions (3.1) and (3.8), we have

$$\|y_1(\iota) - y_2(\iota)\| \leq \mathcal{A}_2 \|v_2(\iota) - v_1(\iota)\| + \mathcal{A}_1 \int_0^{\ell} \sigma(t) \|v_2(\tau) - v_1(\tau)\| \, d_q\tau.$$

Hence,

$$\|y_1 - y_2\| \leq (\mathcal{A}_1\sigma + \mathcal{A}_2) \|v_1 - v_2\|.$$

Likewise, reversing the roles of v_1 and v_2, we arrive at

$$\mathcal{H}_{\widetilde{d}}(\ominus(v_1), \ominus(v_2)) \leq \xi \|v_1 - v_2\| \leq (\mathcal{A}_1\sigma + \mathcal{A}_2) \|v_1 - v_2\|.$$

By Lemma 2.2, the contractivity of $\ominus$ guarantees a fixed point v, which constitutes a solution to the problem (1.1). Hence, the proof is complete. □

The following example serves to demonstrate the conclusions of Theorem 3.1.

Example 3.1. Let the Banach space

$$\mathcal{E} = l^1 = \{x = (x_1, x_2, x_3, \ldots) | \sum_{k=1}^{\infty} |x_k| < \infty\}$$

equipped with the natural norm

$$\|x\|_1 = \sum_{k=1}^{\infty} |x_k|.$$

Let the continuous function $\varphi : [0,1] \to l^1$ defined by

$$\varphi(\iota) = \left(\frac{\sin(2\pi\iota)}{2}, \frac{\sin(2\pi\iota)}{2^2}, 0, 0, 0, ... \right),$$

and for all $(\iota, v) \in [0,1] \times l^1$,

$$\rho(\iota, v) = \frac{\|P_N(v)\|_1}{10},$$

with P_N the canonical projection onto the first N coordinates.

We consider the q-fractional differential inclusion with nonlinear integral conditions, given by

$$\begin{aligned} {}^cD_{0.5,1.4}(v(\iota) - h(\iota, v(\iota))) &\in F(\iota, v(\iota)), \ \iota \in [0,1], \\ v(0) - v'(0) &= a(\iota) \int_0^1 \mathcal{G}_1(\tau, v(\tau)) \, d_q\tau, \\ v(\ell) - v'(\ell) &= b(\iota) \int_0^1 \mathcal{G}_2(\tau, v(\tau)) \, d_q\tau, \end{aligned} \tag{3.9}$$

where

$$\begin{cases} q = 0.5, \quad \gamma = 1.4, \quad \ell = 1, \\ h(\iota, v(\iota)) = a_1 v(\iota) + a_2 \text{ with } a_1, a_2 \in \mathbb{R}. \end{cases}$$

The multivalued map $F : [0,1] \times l^1 \to \mathcal{P}_0\left(l^1\right)$ is defined by

$$F(\iota, v) = \overline{B}_{\mathbb{R}^N}(P_N\varphi(\iota), \rho(\iota, v)).$$

The functions $a, b, \mathcal{G}_1, \mathcal{G}_2$, are all chosen arbitrarily.

By simple calculation, we get

$$\mathcal{A}_1 = \frac{3.8}{\Gamma_{0.5}(1.4)} \simeq \frac{3.8}{0.9207} \simeq 4.1273.$$

On the other hand, the Hypothesis (**H**) is satisfied due to:

1) The function h is $|a_1|$-Lipschitz.

2) For each (ι, ν), $F(\iota, \nu)$ is a compact ball in a finite-dimensional space, hence compact in l^1, and the map $\iota \to F(\iota, \nu)$ is continuous, so, (2) of Hypothesis (**H**) is satisfied.

3) For almost all $\nu_1, \nu_2 \in l^1$ and $\iota \in [0, 1]$, we have

$$\mathcal{H}_d(F(t, \nu_1(\iota)), F(\iota, \nu_2(\iota))) = |\rho(\iota, \nu_1) - \rho(\iota, \nu_2)| \leq \frac{1}{10} \|\nu_1(\iota) - \nu_2(\iota)\|_1 .$$

Then, there exists $\sigma(t) \equiv \frac{1}{10} \in C([0, \ell], \mathbb{R}^+)$ such that $\mathcal{H}_d(F(t, \nu_1(\iota)), F(\iota, \nu_2(\iota))) \leq \frac{1}{10} \|\nu_1(\iota) - \nu_2(\iota)\|$, so

$$\sigma = \int_0^1 \sigma(\tau) d_q\tau = \frac{1}{10}.$$

The condition $\mathcal{A}_1\sigma + \mathcal{A}_2 < 1$ is verified if $\mathcal{A}_2 = |a_1| < 0.58727$. By Theorem 3.1, for all $\iota \in [0, 1]$, the problem (3.9) has at least one solution in $C_{0.5,1.4}\left([0, 1], l^1\right)$.

3.2. *Filippov's theorem*

In the following, we present a Filippov's result applicable to problem (1.1).

Let

$$\mu = \frac{1}{\sigma}\left(\frac{\Gamma}{1 - \frac{\mathcal{A}_1}{1-\mathcal{A}_2}\sigma}\right).$$

Let $\gamma \in (1, 2]$, $q \in (0, 1)$, the function $h : [0, \ell] \times \mathcal{E} \to \mathcal{E}$ be a contraction, $\mathcal{M} \in C([0, \ell], \mathcal{E})$, and let $\nu \in C([0, \ell] \times \mathcal{E})$ be a solution of the following problem with nonlinear integral conditions:

$$\begin{array}{c} {}^cD_{q,\gamma}(\nu(\iota) - h(\iota, \nu)) = \mathcal{M}(\iota),\ \iota \in [0, \ell], , 1 < \gamma \leq 2, \\ \nu(0) - \nu'(0) = a(\iota) \int_0^\ell h_1(\tau) d_q\tau, \\ \nu(\ell) - \nu'(\ell) = b(\iota) \int_0^\ell h_2(\tau) d_q\tau. \end{array} \tag{3.10}$$

The solution takes the form

$$\nu(\iota) = \mathcal{K}(\iota) + h(\iota, \nu(\iota)) + \int_0^\ell G(\iota, \tau) \mathcal{M}(\tau) d_q\tau. \tag{3.11}$$

Theorem 3.2. *Suppose that* (3.3) *and* (3.4) *hold, and*

$$\frac{\mathcal{A}_1}{1 - \mathcal{A}_2}\sigma < 1. \tag{3.12}$$

Consequently, the problem (1.1) *possesses at least one solution* y *satisfying, for all* $\iota \in [0, \ell]$, *the estimates*

$$\|\nu - y\| \leq \frac{\mathcal{A}_1}{1 - \mathcal{A}_2}(\mu\sigma + \Gamma). \tag{3.13}$$

Proof. Let $x_0 = {}^c D_{q,\gamma}\left(\nu(\iota) - h(\iota, \nu(\iota))\right)$ and $y_0 = \nu(\iota)$ for a.e. $\iota \in [0, \ell]$, i.e., then, by Lemma 2.4,

$$y_0(\iota) = \mathcal{K}(\iota) + h(\iota, y_0(\iota)) + \int_0^\ell G(\iota, \tau)\, x_0(\tau)\, d_q\tau.$$

Let $\mathcal{T}_1 : [0, \ell] \to \mathcal{P}(\mathcal{E})$ be given by

$$\mathcal{T}_1(\iota) = F(\iota, y_0(\iota)) \cap B(\mathcal{M}(\iota), \Gamma(\iota)).$$

By Proposition III.4 in [20], the measurability of $\mathcal{T}_1(t)$ ensures the existence of a measurable selection $\iota \to x_1(\iota)$ for $\mathcal{T}_1$. Let

$$y_1(\iota) = \mathcal{K}(\iota) + h(\iota, y_1(\iota)) + \int_0^\ell G(\iota, \tau)\, x_1(\tau)\, d_q\tau.$$

Then, by the assumptions (3.1) and (3.3) , we have

$$\begin{aligned} \|y_1(\iota) - y_0(\iota)\| &\le \|h(\iota, y_1(\iota)) - h(\iota, y_0(\iota))\| + \int_0^\ell |G(\iota, \tau)|\, \|x_1(\tau) - x_0(\tau)\|\, d_q\tau \\ &\le \mathcal{A}_2 \|y_1(\iota) - y_0(\iota)\| + \mathcal{A}_1 \int_0^\ell \Gamma(\tau)\, d_q\tau, \end{aligned}$$

which implies

$$\|y_1 - y_0\| \le \frac{\mathcal{A}_1}{1 - \mathcal{A}_2}\Gamma. \tag{3.14}$$

Similarly, the multivalued map

$$\mathcal{T}_2(\iota) = F(t, y_1(\iota)) \cap B(x_1(\iota), \sigma(\iota)\, \|y_1(\iota) - y_0(\iota)\|)$$

is measurable with nonempty closed values (see [20]). Hence, by Lemma 2.1, there exists a measurable selection x_2 such that $x_2(\iota) \in \mathcal{T}_2(\iota)$.

Let the function

$$y_2(\iota) = \mathcal{K}(\iota) + h(\iota, y_2(\iota)) + \int_0^\ell G(\iota, \tau)\, x_2(\tau)\, d_q\tau.$$

Then,

$$\begin{aligned} \|y_2(\iota) - y_1(\iota)\| &\le \|h(\iota, y_2(\iota)) - h(\iota, y_1(\iota))\| + \int_0^\ell |G(\iota, \tau)|\, \|x_2(\tau) - x_1(\tau)\|\, d_q\tau \\ &\le \mathcal{A}_2 \|y_2(\iota) - y_1(\iota)\| + \mathcal{A}_1 \int_0^\ell \sigma(\tau)\, d_q\tau\, \|y_1(\iota) - y_0(\iota)\| \\ &\le \frac{\mathcal{A}_1}{1 - \mathcal{A}_2}\sigma \|y_1(\iota) - y_0(\iota)\|, \end{aligned}$$

then,

$$\|y_2 - y_1\| \le \left(\frac{\mathcal{A}_1}{1 - \mathcal{A}_2}\right)^2 \sigma\Gamma. \tag{3.15}$$

As in the previous cases, the multivalued map

$$\mathcal{T}_3(\iota) = F(\iota, y_2) \cap B(x_2(\iota), \sigma(\iota)\, \|y_2(\iota) - y_1(\iota)\|)$$

is measurable. Consequently, applying the Kuratowski-Ryll-Nardzewski selection theorem, we obtain a measurable selection x_3 of $\mathcal{T}_3$.

Consider the function

$$y_3(\iota) = \mathcal{K}(\iota) + h(\iota, y_3(\iota)) + \int_0^{\ell} G(\iota, \tau)\, x_3(\tau)\, d_q\tau.$$

Then,

$$\begin{aligned} \|y_3(\iota) - y_2(\iota)\| &\leq \|h(\iota, y_3(\iota)) - h(\iota, y_2(\iota))\| + \int_0^{\ell} |G(\iota, \tau)|\, \|x_3(\tau) - x_2(\tau)\|\, d_q\tau \\ &\leq \mathcal{A}_1 \|y_3(\iota) - y_2(\iota)\| + \mathcal{A}_1 \int_0^{\ell} \sigma(\tau)\, d_q\tau\, \|y_2(\iota) - y_1(\iota)\|. \end{aligned}$$

So,

$$\|y_3 - y_2\| \leq \left(\frac{\mathcal{A}_1}{1 - \mathcal{A}_2}\right)^3 \sigma^2 \Gamma. \tag{3.16}$$

By iterating this construction for $n = 0, 1, 2, 3, \ldots$, we arrive at the inequality

$$\|y_n - y_{n-1}\| \leq \left(\frac{\mathcal{A}_1}{1 - \mathcal{A}_2}\right)^n \sigma^{n-1} \Gamma. \tag{3.17}$$

Suppose that (3.17) holds for some n, and it is now necessary to check that (3.17) is true for $n + 1$.

By Proposition III.4 in [20], the multivalued map

$$\mathcal{T}_{n+1}(\iota) = F(\iota, y_n) \cap B(x_n(\iota), \sigma(\iota)\, \|y_n(\iota) - y_{n-1}(\iota)\|)$$

is measurable, so it admits a measurable selection $\iota \to x_{n+1}(\iota) \in \mathcal{T}_{n+1}$.

Let us consider

$$y_{n+1}(\iota) = \mathcal{K}(\iota) + h(\iota, y_{n+1}(\iota)) + \int_0^{\ell} G(\iota, \tau)\, x_{n+1}(\tau)\, d_q\tau. \tag{3.18}$$

Then,

$$\begin{aligned} \|y_{n+1}(\iota) - y_n(\iota)\| &\leq \|h(\iota, y_{n+1}(\iota)) - h(\iota, y_n(\iota))\| + \int_0^{\ell} |G(\iota, \tau)|\, \|x_{n+1}(\tau) - x_n(\tau)\|\, d_q\tau \\ &\leq \frac{\mathcal{A}_1}{1 - \mathcal{A}_2} \sigma\, \|y_n(\iota) - y_{n-1}(\iota)\| \\ &\leq \left(\frac{\mathcal{A}_1}{1 - \mathcal{A}_2}\right)^2 \sigma^2\, \|y_{n-1}(\iota) - y_{n-2}(\iota)\| \\ &\;\;\vdots \\ &\leq \left(\frac{\mathcal{A}_1}{1 - \mathcal{A}_2}\right)^n \sigma^n\, \|y_1(\iota) - y_0(\iota)\|. \end{aligned}$$

So,

$$\|y_{n+1} - y_n\| \leq \left(\frac{\mathcal{A}_1}{1 - \mathcal{A}_2}\right)^{n+1} \sigma^n \Gamma. \tag{3.19}$$

The inequality $\frac{\mathcal{A}_1}{1-\mathcal{A}_2}\sigma < 1$ guarantees that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,\ell],\mathcal{E})$ that converges uniformly to a function $y \in C([0,\ell],\mathcal{E})$. From the definition of $\mathcal{T}_n$, we have

$$\|x_{n+1}(\iota) - x_n(\iota)\| \leq \sigma(\iota)\|y_n(\iota) - y_{n-1}(\iota)\|, \text{ for } n \in \mathbb{N}^*, a.e., \iota \in [0,\ell].$$

Hence, for almost every $\iota \in [0,\ell]$, the sequence $\{x_n(\iota), \iota \in [0,\ell]\}$ is a Cauchy sequence in $\mathcal{E}$. Therefore, there exists at least one function in $C([0,\ell],\mathcal{E})$ that solves problem (1.1) for all $\iota \in [0,\ell]$. So, the sequence $\{x_n(\iota), \iota \in [0,\ell]\}$ converges almost everywhere to a measurable function $x(.)$ in $\mathcal{E}$.

Additionally, because $x_0 =^c D_{q,\gamma}v(\iota) - h(\iota, v(\iota))$, by employing the least inequality, we deduce that

$$\begin{aligned} \|x_n(\iota) - x_0(\iota)\| &\leq \|x_n(\iota) - x_{n-1}(\iota)\| + \|x_{n-1}(\iota) - x_{n-2}(\iota)\| + ... + \|x_2(\iota) - x_1(\iota)\| + \|x_1(\iota) - x_0(\iota)\| \\ &\leq \sigma(\iota)\sum_{k=1}^{n-1}\|y_k(\iota) - y_{k-1}(\iota)\| + \|x_1(\iota) - x_0(\iota)\| \\ &\leq \sigma(\iota)\sum_{i=1}^{\infty}\left(\frac{\mathcal{A}_1}{1-\mathcal{A}_2}\right)^i\sigma^{i-1}\Gamma + \Gamma(\iota) \\ &= \frac{1}{\sigma}\left(\frac{\Gamma}{1-\frac{\mathcal{A}_1}{1-\mathcal{A}_2}\sigma}\right)\sigma(\iota) + \Gamma(\iota). \end{aligned}$$

Then, for all $n \in \mathbb{N}$,

$$\|x_n(\iota)\| \leq \mu\sigma(\iota) + \Gamma(\iota). \tag{3.20}$$

By (3.20), we deduce that x_n converges to x in $L^1([0,\ell],\mathcal{E})$. Consequently,

$$y(\iota) = \mathcal{K}(\iota) + h(\iota, y) + \int_0^\ell G(\iota,\tau)\,x(\tau)\,d_q\tau \tag{3.21}$$

solves problem (1.1) under the conditions

$$y(0) - y'(0) = a(\iota)\int_0^\ell \mathcal{G}_1(\tau, y(\tau))\,d_q\tau$$

and

$$y(\ell) - y'(\ell) = b(\iota)\int_0^\ell \mathcal{G}_2(\tau, y(\tau))\,d_q\tau.$$

Lastly, we establish that $y(\iota)$ fulfills the following estimate $\|v - y\| \leq \phi(\iota)$.

For all $\iota \in [0,\ell]$,

$$\begin{aligned} \|v(\iota) - y(\iota)\| &\leq \|h(\iota, v(\iota)) - h(\iota, y(\iota))\| + \int_0^\ell |G(\iota,\tau)|\,\|x_0(\tau) - x(\tau)\|\,d_q\tau \\ &\leq \mathcal{A}_2\|v(\iota) - y(\iota)\| + \mathcal{A}_1\int_0^\ell \|x_0(\tau) - x(\tau)\|\,d_q\tau \\ &\leq \mathcal{A}_2\|v(\iota) - y(\iota)\| + \mathcal{A}_1\int_0^\ell \|x_0(\tau) - x_n(\tau)\|\,d_q\tau + \mathcal{A}_1\int_0^\ell \|x(\tau) - x_n(\tau)\|\,d_q\tau. \end{aligned}$$

As $n \to \infty$, we conclude that

$$\begin{aligned} \|v(\iota) - y(\iota)\| &\leq \mathcal{A}_2 \|v(\iota) - y(\iota)\| + \mathcal{A}_1 \int_0^{\ell} \|x_0(\tau) - x_n(\tau)\| \, d_q\tau, \\ &\leq \mathcal{A}_2 \|v(\iota) - y(\iota)\| + \mathcal{A}_1 (\mu\sigma + \Gamma). \end{aligned}$$

Then,

$$\|v(\iota) - y(\iota)\| \leq \frac{\mathcal{A}_1}{1 - \mathcal{A}_2} (\mu\sigma + \Gamma).$$

The proof is therefore finished. □

Author contributions

Taher S. Hassan and Ali Rezaiguia: Supervision, Project administration, Writing–original draft, Writing–review and editing, Formal analysis, Investigation; Loredana Florentina Iambor: Funding, Formal analysis, Resources, Writing–review and editing; Ismoil Odinaev: Formal analysis, Resources, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

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