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Research article

On the p-arithmetic-mean norm of operator pairs and its applications

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Abstract: The main purpose of this paper is to introduce and study the so-called p-arithmetic-mean norm for pairs of bounded linear operators on a complex Hilbert space H. Specifically, for a pair (A, B) of bounded linear operators on H, with $\nu \in [0, 1]$ and $p > 0$, we define the following:

$$\|(A, B)\|_{p,\nu} := \sup_{\|x\|=1} \Big((1-\nu)\|Ax\|^p + \nu\|Bx\|^p\Big)^{1/p}.$$

We establish, among other results, that for $\nu \in (0, 1]$ and $p \in (0, 1]$,

$$\|(A, B)\|_{2p,\nu}^{2p} \leq R\|A - B\|^{2p} + \min\Big\{\|A\|^{2(1-\nu)p}\|B\|^{2\nu p},\ \|(1-\nu)|A|^2 + \nu|B|^2\|^p\Big\}.$$

Applications are given to off-diagonal operator matrices, and to the particular cases $(A, B) = (T, T^*)$ and $(A, B) = \Big(\mathrm{Re}(T), \mathrm{Im}(T)\Big)$, where T is a bounded linear operator on H, T^* denotes its adjoint, $\mathrm{Re}\, T = \frac{1}{2}(T + T^*)$ is its real part, and $\mathrm{Im}\, T = \frac{1}{2i}(T - T^*)$ is its imaginary part.

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1. Introduction

Mathematical inequalities play an important role in the operator theory, functional analysis, and their applications. They provide fundamental tools to estimate operator norms, numerical radii, and spectral properties of linear operators. Inequalities help us understand the behavior of operators in complex Hilbert spaces and have applications in matrix analysis, quantum mechanics, and numerical computations. For example, classical results on positive definite matrices and inequalities for the numerical radii have been influential in the operator theory [1–3].

Recent work has focused on refining these inequalities and extending them to operator matrices, off-diagonal operators, and pairs of operators. Notable contributions include studies on numerical radius inequalities for operator matrices [4, 5]. In the broader context of Hilbert space operators, Bhunia and Paul [6] and Omidvar and Moradi [7] have furthered the development of numerical radius inequalities. Techniques that involved the generalized Aluthge transform were explored in [8], while applications to the estimation of polynomial zeros were detailed in [9]. Moreover, extensions of Euclidean operator radius inequalities have also been investigated [10]. More recently, specific attention has been given to operator pairs, with improved bounds for the Euclidean numerical radius and power vector inequalities presented in [11, 12]. These refinements provide sharper bounds and reveal relationships between different operator norms and numerical radii, thus enhancing the tools available for both theoretical and applied research. The topic of inequalities for pairs of operators, including the Euclidean operator radius, has also seen significant developments [13–15].

Interested readers can find comprehensive treatments of numerical radius inequalities and their applications in standard references and surveys [16,17]. For example, Kittaneh [18] established several classical numerical radius inequalities, while Bag et al. [19] provided bounds using the t-Aluthge transform. Further developments and equality characterizations were studied in [20–22]. Earlier works on bounds for the numerical radii of matrices and operator norms are given in [23, 24]. One such generalization of the numerical radius and unitary invariants in the multivariable operator theory was widely studied [25].

In this paper, we introduce and study the so-called p-arithmetic-mean (AM) norm for pairs of bounded linear operators in a complex Hilbert space, with $p > 0$. This new concept allows us to derive novel bounds for operator pairs and provides several applications, including off-diagonal operator matrices.

2. Preliminaries

To derive our results, we need to recall some well-known results. The famous Young's inequality for scalars states that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu)\, a + \nu b, \tag{2.1}$$

with equality if and only if $a = b$. The inequality (2.1) is also called the ν-weighted arithmetic-geometric mean inequality.

We use the following double inequality, obtained by Kittaneh and Manasrah [26, 27], that provides a refinement and an additive reverse for Young's inequality:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \leq (1-\nu)\, a + \nu b - a^{1-\nu} b^{\nu} \leq R\left(\sqrt{a} - \sqrt{b}\right)^2, \tag{2.2}$$

where $a, b \geq 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$, and $R = \max\{1-\nu, \nu\}$.

We recall the following vector inequality for positive operators $A \geq 0$, obtained by McCarthy [28]:

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle, \ p \geq 1,$$

for $x \in H$, $\|x\| = 1$, and Buzano's inequality [29],

$$\left|\langle x, e\rangle\langle e, y\rangle\right| \leq \frac{1}{2}\left[\|x\|\|y\| + \left|\langle x, y\rangle\right|\right],$$

that holds for any $x, y, e \in H$, with $\|e\| = 1$.

Let us introduce the following definition that includes so-called p-arithmetic-mean norm for pairs of bounded linear operators, with $p > 0$.

Definition 2.1. *For $\nu \in [0,1]$, $p > 0$, and $A, B \in \mathcal{B}(H)$, we define the p-arithmetic-mean norm for the pair of operators (A, B) by*

$$\|(A,B)\|_{p,\nu} := \sup_{\|x\|=1}\left((1-\nu)\|Ax\|^p + \nu\|Bx\|^p\right)^{1/p}$$

and

$$\|(A,B)\|_p := \sup_{\|x\|=1}\left(\|Ax\|^p + \|Bx\|^p\right)^{1/p}.$$

We observe that

$$\|(A,B)\|_{p,1/2}^p = \frac{1}{2}\|(A,B)\|_p^p.$$

Additionally, we consider

$$\begin{aligned}\|(A,B)\|_{e,\nu} := \|(A,B)\|_{2,\nu} &= \sup_{\|x\|=1}\left((1-\nu)\|Ax\|^2 + \nu\|Bx\|^2\right)^{1/2} \\ &= \sup_{\|x\|=1}\left\langle\left((1-\nu)|A|^2 + \nu|B|^2\right)x, x\right\rangle^{1/2} \\ &= \left\|(1-\nu)|A|^2 + \nu|B|^2\right\|^{1/2}\end{aligned}$$

and

$$\|(A,B)\|_\nu := \|(A,B)\|_{1,\nu} := \sup_{\|x\|=1}\left((1-\nu)\|Ax\| + \nu\|Bx\|\right).$$

Moreover, we put

$$\|(A,B)\|_e := \|(A,B)\|_2 = \sup_{\|x\|=1}\left(\|Ax\|^2 + \|Bx\|^2\right)^{1/2} = \left\||A|^2 + |B|^2\right\|^{1/2}$$

and

$$\|(A,B)\| := \|(A,B)\|_1 = \sup_{\|x\|=1}\left(\|Ax\| + \|Bx\|\right).$$

Additionally, we observe that

$$\max\left\{(1-\nu)^{1/p}\|A\|, \nu^{1/p}\|B\|\right\} \leq \|(A,B)\|_{p,\nu} \leq \left((1-\nu)\|A\|^p + \nu\|B\|^p\right)^{1/p},$$

for $\nu \in [0,1]$, $p \geq 1$.

Let $\nu \in [0,1]$, $p \geq 1$. Observe that for all $(A,B) \in \mathcal{B}(H) \times \mathcal{B}(H)$, we have $\|(A,B)\|_{p,\nu} \geq 0$, and if $\|(A,B)\|_{p,\nu} = 0$, then $(A,B) = (0,0)$. Moreover,

$$\begin{aligned}\|\alpha\,(A,B)\|_{p,\nu} &= \|(\alpha A, \alpha B)\|_{p,\nu} = \sup_{\|x\|=1} \left((1-\nu)\,\|\alpha A x\|^2 + \nu\,\|\alpha B x\|^2\right)^{1/2} \\ &= |\alpha| \sup_{\|x\|=1} \left((1-\nu)\,\|Ax\|^2 + \nu\,\|Bx\|^2\right)^{1/2} = |\alpha|\,\|(A,B)\|_{p,\nu}\,.\end{aligned}$$

By the weighted Minkowski's inequality, we have the following:

$$\begin{aligned}\|(A,B)+(C,D)\|_{p,\nu} &= \|(A+C, B+D)\|_{p,\nu} \\ &= \sup_{\|x\|=1} \left((1-\nu)\,\|Ax+Cx\|^p + \nu\,\|Bx+Dx\|^p\right)^{1/p} \\ &\leq \sup_{\|x\|=1} \left[\left((1-\nu)\,\|Ax\|^p + \nu\,\|Bx\|^p\right)^{1/p} + \left((1-\nu)\,\|Cx\|^p + \nu\,\|Dx\|^p\right)^{1/p}\right] \\ &\leq \sup_{\|x\|=1} \left[\left((1-\nu)\,\|Ax\|^p + \nu\,\|Bx\|^p\right)^{1/p}\right] + \sup_{\|x\|=1} \left((1-\nu)\,\|Cx\|^p + \nu\,\|Dx\|^p\right)^{1/p} \\ &= \|(A,B)\|_{p,\nu} + \|(C,D)\|_{p,\nu}\,,\end{aligned}$$

which shows that $\|\cdot,\cdot\|_{p,\nu}$ is a norm on $\mathcal{B}(H) \times \mathcal{B}(H)$. Additionally, the above quantity $\|\cdot,\cdot\|_{p,\nu}$ can be extended for $p \in (0,1)$; however, in this case, it is no longer a norm.

In recent years, the study of norms for operator pairs has garnered significant attention, yet most key research results have focused on symmetric weights (e.g., $\nu = 1/2$) or single p-values (e.g., $p = 2$). The design presented here, which features an adjustable weight $\nu \in (0,1]$ and a flexible $p > 0$, constitutes a breakthrough in this field. It allows for the precise characterization of the synergy within asymmetric operator pairs, particularly in scenarios where components such as the real and imaginary parts have unbalanced contributions. Moreover, a systematic comparison with classical L_p-type operator pair norms highlights the distinct characteristics of the proposed definition, especially for $p \in (0,1]$. While classical norms typically enforce convexity and lose structural integrity in this range, the proposed weighted quasi-norm preserves essential analytical properties. The incorporation of the weight ν enables a continuous transition between symmetric and asymmetric configurations, a feature absent in standard L_p models. This allows for a deeper investigation into the geometric structure of operator pairs in non-locally convex spaces, thus distinguishing the current work from traditional approaches.

In 1952, Kato [30] proved the following celebrated generalization of the Schwarz inequality for any bounded linear operator T on H:

$$\left|\langle Tx, y\rangle\right|^2 \leq \langle (T^*T)^{\alpha}\, x, x\rangle \langle (TT^*)^{1-\alpha}\, y, y\rangle, \tag{2.3}$$

for any $x, y \in H$, $\alpha \in [0,1]$. Utilizing the modulus notation introduced before, we can write (2.3) as follows:

$$\left|\langle Tx, y\rangle\right|^2 \leq \langle |T|^{2\alpha}\, x, x\rangle \langle |T^*|^{2(1-\alpha)}\, y, y\rangle,$$

for any $x, y \in H$, $\alpha \in [0,1]$.

In this paper, we prove, among other results, that if $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$, then

$$\|(A, B)\|_{2p,\nu}^{2p} \leq R \|A - B\|^{2p} + \min\left\{\|A\|^{2(1-\nu)p} \|B\|^{2\nu p}, \left\|(1-\nu)|A|^2 + \nu|B|^2\right\|^p\right\},$$

for $\nu \in (0, 1]$ and $p \in (0, 1]$. Applications for off-diagonal operator matrices and in the case when $(A, B) = (T, T^*)$ or $(A, B) = (\mathrm{Re}T, \mathrm{Im}T)$ are also given.

3. Main results

In this section, we establish our results. In order to prove the first one, we need the following elementary lemma.

Lemma 3.1. *For $r \in (0, 1]$ and $a, b \geq 0$, we have the following:*

$$(a + b)^r \leq a^r + b^r. \tag{3.1}$$

Proof. Let $r \in (0, 1)$. Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^r + 1 - (x + 1)^r$. We have

$$f'(x) = r\left[x^{r-1} - (x + 1)^{r-1}\right] = r\frac{(x + 1)^{1-r} - x^{1-r}}{(x + 1)^{1-r} x^{1-r}} > 0,$$

for $x \in (0, \infty)$, namely f is strictly increasing on $(0, \infty)$. Therefore, $f(x) \geq f(0) = 0$, which gives $x^r + 1 - (x + 1)^r \geq 0$ for $x \geq 0$. By taking $x = \frac{b}{a}$ $(a \neq 0)$, we obtain (3.1), which also holds for $a = 0$.

Lemma 3.2. *For $r \in (0, 1]$ and $x, y \geq 0$, we have the following:*

$$|x^r - y^r| \leq |x - y|^r. \tag{3.2}$$

Proof. For $x \geq y$, by (3.1), we have

$$|x - y|^r + y^r = (x - y)^r + y^r \geq x^r,$$

which gives

$$|x - y|^r \geq x^r - y^r = |x^r - y^r|.$$

For $y \geq x$, the argument is similar and the inequality (3.2) also holds.

Lemma 3.3. *For $p \in (0, 1]$, $\nu \in [0, 1]$, and $u, z \geq 0$, we have the following:*

$$(1 - \nu) u^{2p} + \nu z^{2p} \leq u^{2(1-\nu)p} z^{2\nu p} + R|u - z|^{2p}, \quad R = \max\{\nu, 1 - \nu\}. \tag{3.3}$$

Proof. From the second Kittaneh-Manasrah inequality (2.2) for $a = u^{2p}$ and $b = z^{2p}$, we have the following:

$$(1 - \nu) u^{2p} + \nu z^{2p} \leq u^{2(1-\nu)p} z^{2\nu p} + R|u^p - z^p|^2. \tag{3.4}$$

By (3.2), we have $(u^p - z^p)^2 \leq |u - z|^{2p}$, which, together with (3.4), we derive (3.3).

Our first main result is as follows.

Theorem 3.1. *Let $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$. For all $\nu \in [0, 1]$ and $p \in (0, 1]$, we have*

$$\|(A, B)\|_{2p,\nu}^{2p} \leq R\,\|A - B\|^{2p} + \|A\|^{2(1-\nu)p}\,\|B\|^{2\nu p} \tag{3.5}$$

and

$$\|(A, B)\|_{2p,\nu}^{2p} \leq R\,\|A - B\|^{2p} + \left\|(1-\nu)\,|A|^2 + \nu\,|B|^2\right\|^p. \tag{3.6}$$

Additionally, we have

$$\|(A, B)\|_{2p,\nu}^{2p} \leq R\,\|A - B\|^{2p} + \left\|\frac{|A|^{2/\nu} + |B|^{2/(1-\nu)}}{2}\right\|^{2(1-\nu)\nu p}, \tag{3.7}$$

for $\nu \in (0, 1)$.

Proof. If we take $u = \|Ax\|$ and $z = \|Bx\|$, $x \in H$ in (3.3), then for $p \in (0, 1]$, $\nu \in [0, 1]$, we get

$$(1-\nu)\,\|Ax\|^{2p} + \nu\,\|Bx\|^{2p} \leq \|Ax\|^{2(1-\nu)p}\,\|Bx\|^{2\nu p} + R\,\big|\|Ax\| - \|Bx\|\big|^{2p}, \tag{3.8}$$

for $x \in H$.

By the continuity property of the norm, we have $\big|\|Ax\| - \|Bx\|\big| \leq \|Ax - Bx\|$, and by (3.8), we obtain

$$(1-\nu)\,\|Ax\|^{2p} + \nu\,\|Bx\|^{2p} \leq \|Ax\|^{2(1-\nu)p}\,\|Bx\|^{2\nu p} + R\,\|Ax - Bx\|^{2p}, \tag{3.9}$$

for $x \in H$.

If $x \in H$ with $\|x\| = 1$, then

$$\|Ax\|^{2(1-\nu)p}\,\|Bx\|^{2\nu p} + R\,\|Ax - Bx\|^{2p} \leq \|A\|^{2(1-\nu)p}\,\|B\|^{2\nu p} + R\,\|A - B\|^{2p},$$

and by (3.9), we get

$$(1-\nu)\,\|Ax\|^{2p} + \nu\,\|Bx\|^{2p} \leq \|A\|^{2(1-\nu)p}\,\|B\|^{2\nu p} + R\,\|A - B\|^{2p},$$

for $x \in H$ with $\|x\| = 1$.

By the monotonicity property of the supremum, if $f(x) \leq C$ for all $x \in S$, where $S = \{x \in H : \|x\| = 1\}$ and C is a constant, then $\sup_{x\in S} f(x) \leq C$. Let $f(x) = (1-\nu)\|Ax\|^{2p} + \nu\|Bx\|^{2p}$ and $C = \|A\|^{2(1-\nu)p}\|B\|^{2\nu p} + R\|A - B\|^{2p}$. Hence, $\sup_{x\in S} f(x) = \|(A, B)\|_{2p,\nu}^{2p} \leq C$, that is, Eq (3.5) holds.

Now, if we take the supremum over $x \in H$ with $\|x\| = 1$, we deduce the desired result (3.5).

Observe, by the weighted A-G inequality, we have for $x \in H$ with $\|x\| = 1$ that

$$\begin{aligned}
\|Ax\|^{2(1-\nu)p}\,\|Bx\|^{2\nu p} &= \left(\|Ax\|^{2(1-\nu)}\,\|Bx\|^{2\nu}\right)^p \\
&= \left(\langle\,|A|^2\,x, x\rangle^{1-\nu}\langle\,|B|^2\,x, x\rangle^{\nu}\right)^p \\
&\leq \left((1-\nu)\,\langle\,|A|^2\,x, x\rangle + \nu\langle\,|B|^2\,x, x\rangle\right)^p \\
&= \langle\left[(1-\nu)\,|A|^2 + \nu\,|B|^2\right]x, x\rangle^p \\
&\leq \left\|(1-\nu)\,|A|^2 + \nu\,|B|^2\right\|^p.
\end{aligned}$$

By (3.9), we get

$$(1-\nu)\left\|Ax\right\|^{2p}+\nu\left\|Bx\right\|^{2p}\leq\left\|(1-\nu)\left|A\right|^{2}+\nu\left|B\right|^{2}\right\|^{p}+R\left\|A-B\right\|^{2p},$$

for $x\in H$ with $\|x\|=1$. Now, if we take the supremum over $x\in H$ with $\|x\|=1$, we deduce the desired result (3.6).

Moreover, we have

$$\left(\left\langle\left|A\right|^{2}x,x\right\rangle^{1-\nu}\left\langle\left|B\right|^{2}x,x\right\rangle^{\nu}\right)^{p}=\left(\left\langle\left|A\right|^{2}x,x\right\rangle^{1/\nu}\left\langle\left|B\right|^{2}x,x\right\rangle^{1/(1-\nu)}\right)^{(1-\nu)\nu p},$$

for $x\in H$. Since $1/\nu,1/(1-\nu)>1$, by McCarthy's inequality, we have the following:

$$\begin{aligned}
&\left(\left\langle\left|A\right|^{2}x,x\right\rangle^{1/\nu}\left\langle\left|B\right|^{2}x,x\right\rangle^{1/(1-\nu)}\right)^{(1-\nu)\nu p}\\
&\leq\left(\left\langle\left|A\right|^{2/\nu}x,x\right\rangle\left\langle\left|B\right|^{2/(1-\nu)}x,x\right\rangle\right)^{(1-\nu)\nu p}\\
&\leq\left(\left\langle\left|A\right|^{2/\nu}x,x\right\rangle^{1/2}\left\langle\left|B\right|^{2/(1-\nu)}x,x\right\rangle^{1/2}\right)^{2(1-\nu)\nu p}\\
&\leq\left(\frac{\left\langle\left|A\right|^{2/\nu}x,x\right\rangle+\left\langle\left|B\right|^{2/(1-\nu)}x,x\right\rangle}{2}\right)^{2(1-\nu)\nu p}\\
&=\left\langle\frac{\left|A\right|^{2/\nu}+\left|B\right|^{2/(1-\nu)}}{2}x,x\right\rangle^{2(1-\nu)\nu p}\leq\left\|\frac{\left|A\right|^{2/\nu}+\left|B\right|^{2/(1-\nu)}}{2}\right\|^{2(1-\nu)\nu p}.
\end{aligned}$$

By (3.9), we get

$$(1-\nu)\left\|Ax\right\|^{2p}+\nu\left\|Bx\right\|^{2p}\leq\left\|\frac{\left|A\right|^{2/\nu}+\left|B\right|^{2/(1-\nu)}}{2}\right\|^{2(1-\nu)\nu p}+R\left\|A-B\right\|^{2p},$$

for $x\in H$ with $\|x\|=1$.

Now, if we take the supremum over $x\in H$ with $\|x\|=1$, we deduce the desired result (3.7).

The case when $p=1$ is as follows.

Corollary 3.1. *Let $(A,B)\in\mathcal{B}(H)\times\mathcal{B}(H)$. For all $\nu\in(0,1)$, we have the following:*

$$\left\|(1-\nu)\left|A\right|^{2}+\nu\left|B\right|^{2}\right\|\leq R\left\|A-B\right\|^{2}+\left\{\begin{array}{l}\left\|A\right\|^{2(1-\nu)}\left\|B\right\|^{2\nu},\\ \left\|\frac{\left|A\right|^{2/\nu}+\left|B\right|^{2/(1-\nu)}}{2}\right\|^{2(1-\nu)\nu}.\end{array}\right.$$

Remark 3.1. *In the particular case when $\nu=1/2$, we get the simpler inequalities*

$$\left\|\left|A\right|^{2}+\left|B\right|^{2}\right\|\leq\left\|A-B\right\|^{2}+\left\{\begin{array}{l}2\left\|A\right\|\left\|B\right\|,\\ \sqrt{2}\left\|\left|A\right|^{4}+\left|B\right|^{4}\right\|^{1/2}.\end{array}\right.$$

The case when $p=1/2$ is as follows.

Corollary 3.2. *Let $(A,B)\in\mathcal{B}(H)\times\mathcal{B}(H)$. For all $\nu\in(0,1)$, we have the following:*

$$\left\|(A,B)\right\|_{\nu}\leq R\left\|A-B\right\|+\left\{\begin{array}{l}\left\|A\right\|^{1-\nu}\left\|B\right\|^{\nu},\\ \left\|(1-\nu)\left|A\right|^{2}+\nu\left|B\right|^{2}\right\|^{1/2},\\ \left\|\frac{\left|A\right|^{2/\nu}+\left|B\right|^{2/(1-\nu)}}{2}\right\|^{(1-\nu)\nu}.\end{array}\right.$$

Remark 3.2. *For $\nu = 1/2$, we obtain the following inequalities:*

$$\|(A,B)\| \leq \|A-B\| + 2 \times \begin{cases} \|A\|^{1/2}\|B\|^{1/2}, \\ \left\|\frac{|A|^2+|B|^2}{2}\right\|^{1/2}, \\ \left\|\frac{|A|^4+|B|^4}{2}\right\|^{1/4}. \end{cases}$$

Additionally, we have the following theorem.

Theorem 3.2. *Let $(A,B) \in \mathcal{B}(H) \times \mathcal{B}(H)$. For all $\nu \in [0,1]$ and $p \in (0,1]$, we have the following:*

$$\|(A,B)\|_{2p,\nu}^{2p} \leq R\|A-B\|^{2p} + \frac{1}{2^{(1-\nu)\nu p}}\left(w\left(|B|^{2/(1-\nu)}|A|^{2/\nu}\right) + \left\|\frac{|A|^{4/\nu}+|B|^{4/(1-\nu)}}{2}\right\|\right)^{(1-\nu)\nu p}. \tag{3.10}$$

Proof. For $x \in H$ with $\|x\| = 1$, by Buzano's inequality (taking $e = x$), we have the following:

$$\begin{aligned}
&\left(\langle |A|^{2/\nu}x, x\rangle\langle |B|^{2/(1-\nu)}x, x\rangle\right)^{(1-\nu)\nu p} \\
&\leq \left(\frac{1}{2}\langle |A|^{2/\nu}x, |B|^{2/(1-\nu)}x\rangle + \frac{1}{2}\left\||A|^{2/\nu}x\right\|\left\||B|^{2/(1-\nu)}x\right\|\right)^{(1-\nu)\nu p} \\
&\leq \left(\frac{1}{2}\langle |B|^{2/(1-\nu)}|A|^{2/\nu}x, x\rangle + \frac{1}{4}\left(\left\||A|^{2/\nu}x\right\|^2 + \left\||B|^{2/(1-\nu)}x\right\|^2\right)\right)^{(1-\nu)\nu p} \quad \text{(by AM-GM inequality)} \\
&= \left(\frac{1}{2}\langle |B|^{2/(1-\nu)}|A|^{2/\nu}x, x\rangle + \frac{1}{4}\left(\langle |A|^{4/\nu}x, x\rangle + \langle |B|^{4/(1-\nu)}x, x\rangle\right)\right)^{(1-\nu)\nu p} \\
&= \left(\frac{1}{2}\langle |B|^{2/(1-\nu)}|A|^{2/\nu}x, x\rangle + \frac{1}{2}\left\langle\left(\frac{|A|^{4/\nu}+|B|^{4/(1-\nu)}}{2}\right)x, x\right\rangle\right)^{(1-\nu)\nu p}.
\end{aligned}$$

By the definition of the numerical radius, $w(T) = \sup_{\|x\|=1}|\langle Tx, x\rangle|$ $(\forall T \in \mathcal{B}(H))$. Thus, for any unit vector x,

$$\langle |B|^{2/(1-\nu)}|A|^{2/\nu}x, x\rangle \leq |\langle |B|^{2/(1-\nu)}|A|^{2/\nu}x, x\rangle| \leq w(|B|^{2/(1-\nu)}|A|^{2/\nu}).$$

Furthermore, since $|A|, |B| \in \mathcal{B}(H)$, their powers $|A|^{2/\nu}$ and $|B|^{2/(1-\nu)}$ remain bounded linear operators; therefore, $|B|^{2/(1-\nu)}|A|^{2/\nu} \in \mathcal{B}(H)$ and the numerical radius definition is applicable.

$$\|A\|^{2(1-\nu)p}\|B\|^{2\nu p} \leq \left(\frac{1}{2}w\left(|B|^{2/(1-\nu)}|A|^{2/\nu}\right) + \frac{1}{2}\left\|\frac{|A|^{4/\nu}+|B|^{4/(1-\nu)}}{2}\right\|\right)^{(1-\nu)\nu p}.$$

Moreover, by (3.9), we get

$$(1-\nu)\|Ax\|^{2p} + \nu\|Bx\|^{2p} \leq R\|A-B\|^{2p} + \left(\frac{1}{2}w\left(|B|^{2/(1-\nu)}|A|^{2/\nu}\right) + \frac{1}{2}\left\|\frac{|A|^{4/\nu}+|B|^{4/(1-\nu)}}{2}\right\|\right)^{(1-\nu)\nu p},$$

for $x \in H$ with $\|x\| = 1$. Now, if we take the supremum over $x \in H$ with $\|x\| = 1$, we deduce the desired result (3.10).

For $p = 1$, we can state the following:

Corollary 3.3. *Let $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$. For all $\nu \in (0,1)$, we have the following:*

$$\left\|(1-\nu)|A|^2 + \nu|B|^2\right\| \leq R\|A-B\|^2 + \frac{1}{2^{(1-\nu)\nu}}\left(w\left(|B|^{2/(1-\nu)}|A|^{2/\nu}\right) + \left\|\frac{|A|^{4/\nu}+|B|^{4/(1-\nu)}}{2}\right\|\right)^{(1-\nu)\nu}.$$

In particular,

$$\left\||A|^2 + |B|^2\right\| \leq \|A-B\|^2 + 2^{3/4}\left(w\left(|B|^4|A|^4\right) + \left\|\frac{|A|^4+|B|^4}{2}\right\|\right)^{1/4}.$$

In the case when $p = 1/2$, we obtain the following:

Corollary 3.4. *Let $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$. For all $\nu \in (0,1)$, we have the following:*

$$\|(A,B)\|_\nu \leq R\|A-B\| + \frac{1}{2^{(1-\nu)\nu/2}}\left(w\left(|B|^{2/(1-\nu)}|A|^{2/\nu}\right) + \left\|\frac{|A|^{4/\nu}+|B|^{4/(1-\nu)}}{2}\right\|\right)^{(1-\nu)\nu/2}.$$

In particular,

$$\|(A,B)\| \leq \|A-B\| + 2^{7/8}\left(w\left(|B|^4|A|^4\right) + \left\|\frac{|A|^4+|B|^4}{2}\right\|\right)^{1/8}.$$

4. Applications

As applications of our results, we establish inequalities for the off-diagonal operator matrix.

Before that, we need to recall some notations. Consider the off-diagonal part $\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ of a 2×2 operator matrix $\begin{bmatrix} Z & X \\ Y & W \end{bmatrix}$ defined on $H \oplus H$.

It is well known that, for $X, Y \in \mathcal{B}(H)$,

$$w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & Y \\ X & 0 \end{bmatrix}\right),$$

$$w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & X \\ e^{i\theta}Y & 0 \end{bmatrix}\right) \text{ for } \theta \in \mathbb{R},$$

$$w\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) = w(Y) \text{ and } w\left(\begin{bmatrix} 0 & Y \\ Y^* & 0 \end{bmatrix}\right) = \|Y\|.$$

In 2011, Hirzallah, Kittaneh, and Shebrawi [31] proved, among other things, the following double inequality:

$$\frac{1}{2}\max\{w(X+Y), w(X-Y)\} \leq w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \frac{1}{2}[w(X+Y) + w(X-Y)].$$

Additionally, they showed that

$$w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \min\{w(X), w(Y)\} + \frac{1}{2}\min\{\|X+Y\|, \|X-Y\|\}.$$

Several other interesting inequalities of this type were also obtained.

In 2015, Kittaneh, Moslehian, and Yamazaki [32] showed that the following refinement for the triangle inequality holds:

$$\left\|\frac{X+Y}{2}\right\| \leq w\left(\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\right) \leq \frac{\|X\|+\|Y\|}{2},$$

for all $X, Y \in \mathcal{B}(H)$.

We use the following equality employed in the proof of Theorem 2.3 of [32]:

$$w\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) = \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|e^{i\theta}A + e^{-i\theta}B\right\|, \tag{4.1}$$

for all $A, B \in \mathcal{B}(H)$.

Proposition 4.1. *Let $A, B \in \mathcal{B}(H)$. For $1/2 \leq p \leq 1$ and $\nu \in [0,1]$, we have the following:*

$$w^{2p}\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B & 0 \end{bmatrix}\right) \leq \frac{1}{2^{2p}}R\left\|A - B^*\right\|^{2p} + \frac{1}{2^{2p}} \times \begin{cases} \|A\|^{2(1-\nu)p}\|B\|^{2\nu p}, \\ \left\|(1-\nu)|A|^2 + \nu|B^*|^2\right\|^p, \\ \left\|\frac{|A|^{2/\nu}+|B^*|^{2/(1-\nu)}}{2}\right\|^{2(1-\nu)\nu p}. \end{cases} \tag{4.2}$$

In particular,

$$w^{2p}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left\|A - B^*\right\|^{2p} + \begin{cases} \|A\|^p\|B\|^p, \\ \left\|\frac{|A|^2+|B^*|^2}{2}\right\|^p, \\ \left\|\frac{|A|^4+|B^*|^4}{2}\right\|^{p/2}. \end{cases}$$

Proof. For $r \geq 1$, by the convexity of the power function, we have that

$$\|(1-\nu)Cx + \nu Dx\|^r \leq ((1-\nu)\|Cx\| + \nu\|Dx\|)^r \leq (1-\nu)\|Cx\|^r + \nu\|Dx\|^r.$$

By taking the supremum over $x \in H$, $\|x\| = 1$, this gives that

$$\|(1-\nu)C + \nu D\|^r \leq \|(C,D)\|_{r,\nu}^r.$$

From the convexity of the norm, for $1/2 \leq p \leq 1$, we get that

$$\begin{aligned} \left\|\frac{(1-\nu)e^{i\theta}A + \nu e^{-i\theta}B^*}{2}\right\|^{2p} &= \frac{1}{2^{2p}}\left\|(1-\nu)e^{i\theta}A + \nu e^{-i\theta}B^*\right\|^{2p} \\ &\leq \frac{1}{2^{2p}}\left\|\left(e^{i\theta}A, e^{-i\theta}B^*\right)\right\|_{2p,\nu}^{2p} \\ &= \frac{1}{2^{2p}}\|(A,B^*)\|_{2p,\nu}^{2p}, \end{aligned}$$

for all $\theta \in \mathbb{R}$. By taking the supremum over $\theta \in \mathbb{R}$ and using (4.1), we get

$$w^{2p}\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B & 0 \end{bmatrix}\right) \leq \frac{1}{2^{2p}}\|(A,B^*)\|_{2p,\nu}^{2p}, \tag{4.3}$$

for $1/2 \leq p \leq 1$ and $\nu \in [0,1]$.

From Theorem 3.1, we have

$$\|(A, B^*)\|_{2p,\nu}^{2p} \leq R\,\|A - B^*\|^{2p} + \|A\|^{2(1-\nu)p}\,\|B\|^{2\nu p}$$

and

$$\|(A, B^*)\|_{2p,\nu}^{2p} \leq R\,\|A - B^*\|^{2p} + \left\|(1-\nu)\,|A|^2 + \nu\,|B^*|^2\right\|^p.$$

Additionally, we have

$$\|(A, B^*)\|_{2p,\nu}^{2p} \leq R\,\|A - B^*\|^{2p} + \left\|\frac{|A|^{2/\nu} + |B^*|^{2/(1-\nu)}}{2}\right\|^{2(1-\nu)\nu p}.$$

Then, by making use of (4.3), we deduce the desired result (4.2).

If we take $p = 1$ in Proposition 4.1, then we get the following:

$$w^2\left(\begin{bmatrix} 0 & (1-\nu)\,A \\ \nu B & 0 \end{bmatrix}\right) \leq \frac{1}{4}R\,\|A - B^*\|^2 + \frac{1}{4} \times \begin{cases} \|A\|^{2(1-\nu)}\,\|B\|^{2\nu}, \\ \left\|(1-\nu)\,|A|^2 + \nu\,|B^*|^2\right\|, \\ \left\|\frac{|A|^{2/\nu} + |B^*|^{2/(1-\nu)}}{2}\right\|^{2(1-\nu)\nu p}. \end{cases}$$

In particular,

$$w^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{1}{2}\,\|A - B^*\|^2 + \begin{cases} \|A\|\,\|B\|, \\ \left\|\frac{|A|^2 + |B^*|^2}{2}\right\|, \\ \left\|\frac{|A|^4 + |B^*|^4}{2}\right\|^{1/2}. \end{cases}$$

If we take $p = 1/2$ in Proposition 4.1 and take the square, then, for $\nu \in [0, 1]$, we get that

$$w\left(\begin{bmatrix} 0 & (1-\nu)\,A \\ \nu B & 0 \end{bmatrix}\right) \leq \frac{1}{2}R\,\|A - B^*\| + \frac{1}{2} \times \begin{cases} \|A\|^{1-\nu}\,\|B\|^{\nu}, \\ \left\|(1-\nu)\,|A|^2 + \nu\,|B^*|^2\right\|^{1/2}, \\ \left\|\frac{|A|^{2/\nu} + |B^*|^{2/(1-\nu)}}{2}\right\|^{(1-\nu)\nu}. \end{cases}$$

In particular,

$$w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{1}{2}\,\|A - B^*\| + \begin{cases} \|A\|^{1/2}\,\|B\|^{1/2}, \\ \left\|\frac{|A|^2 + |B^*|^2}{2}\right\|^{1/2}, \\ \left\|\frac{|A|^4 + |B^*|^4}{2}\right\|^{1/4}. \end{cases}$$

Additionally, we have the following proposition.

Proposition 4.2. *Let $A, B \in \mathcal{B}(H)$. For $1/2 \leq p \leq 1$ and $\nu \in [0, 1]$, we have the following:*

$$\begin{aligned} &w^{2p}\left(\begin{bmatrix} 0 & (1-\nu)\,A \\ \nu B & 0 \end{bmatrix}\right) \\ &\leq \frac{1}{2^{2p}}R\,\|A - B^*\|^{2p} + \frac{1}{2^{2p}} \times \frac{1}{2^{(1-\nu)\nu p}}\left(w\left(|B^*|^{2/(1-\nu)}\,|A|^{2/\nu}\right) + \left\|\frac{|A|^{4/\nu} + |B^*|^{4/(1-\nu)}}{2}\right\|\right)^{(1-\nu)\nu p}. \end{aligned} \tag{4.4}$$

In particular,

$$w^{2p}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{1}{2}\,\|A - B^*\|^{2p} + \frac{1}{2^{p/4}}\left(w\left(|B^*|^4\,|A|^4\right) + \left\|\frac{|A|^8 + |B^*|^8}{2}\right\|\right)^{p/4}.$$

Proof. From Theorem 3.2, we have the following:

$$\|(A,B^*)\|_{2p,\nu}^{2p} \le R\,\|A-B^*\|^{2p} + \frac{1}{2^{(1-\nu)\nu p}}\left(w\left(|B^*|^{2/(1-\nu)}\,|A|^{2/\nu}\right) + \left\|\frac{|A|^{4/\nu}+|B^*|^{4/(1-\nu)}}{2}\right\|\right)^{(1-\nu)\nu p}.$$

By making use of (4.3), we get (4.4).

For $p = 1$ and $p = 1/2$, we can get some particular cases of interest; however, the details are not provided here.

Other applications for our results include inequalities for a single operator.

For $(A,B) = (T,T^*)$, for $r \ge 1$ and $\nu \in [0,1]$, we can introduce the following functional:

$$\sigma_{r,\nu}(T) := \|(T,T^*)\|_{r,\nu} := \sup_{\|x\|=1}\left((1-\nu)\,\|Tx\|^r + \nu\,\|T^*x\|^r\right)^{1/r}.$$

We observe that $\sigma_{r,\nu}(\cdot)$ is a norm on $\mathcal{B}(H)$. If T is normal, namely $TT^* = T^*T$, then $\|Tx\| = \|T^*x\|$ for all $x \in H$, which gives that $\sigma_{r,\nu}(T) = \|T\|$.

Additionally, we consider the following:

$$\sigma_r(T) := \|(T,T^*)\|_r := \sup_{\|x\|=1}\left(\|Tx\|^r + \|T^*x\|^r\right)^{1/r}.$$

For $r = 2$, we obtain

$$\sigma_{e,\nu}(T) := \sigma_{2,\nu}(T) = \left\|(1-\nu)\,|T|^2 + \nu\,|T^*|^2\right\|^{1/2}$$

for $\nu \in [0,1]$ and

$$\sigma_e(T) = \left\||T|^2 + |T^*|^2\right\|^{1/2}.$$

For $r = 1$, we denote

$$\sigma_\nu(T) := \|(T,T^*)\|_\nu := \sup_{\|x\|=1}\left((1-\nu)\,\|Tx\| + \nu\,\|T^*x\|\right)$$

and

$$\sigma(T) := \|(T,T^*)\| := \sup_{\|x\|=1}\left(\|Tx\| + \|T^*x\|\right).$$

From Theorem 3.1, for $(A,B) = (T,T^*)$ with $T \in \mathcal{B}(H)$, $\nu \in [0,1]$, and $p \in (0,1]$, we have that

$$\sigma_{2p,\nu}^{2p}(T) \le R\,\|T-T^*\|^{2p} + \left\{\begin{array}{l} \|T\|^{2p}, \\ \left\|(1-\nu)\,|T|^2 + \nu\,|T^*|^2\right\|^p, \\ \left\|\frac{|T|^{2/\nu}+|T^*|^{2/(1-\nu)}}{2}\right\|^{2(1-\nu)\nu p} \end{array}\right.$$

and

$$\sigma_{2p}^{2p}(T) \le \|T-T^*\|^{2p} + 2\times\left\{\begin{array}{l} \|T\|^{2p}, \\ \left\|\frac{|T|^2+|T^*|^2}{2}\right\|^p, \\ \left\|\frac{|T|^4+|T^*|^4}{2}\right\|^{p/2}. \end{array}\right.$$

The case when $p = 1$ in the two inequalities above gives

$$\sigma_{e,\nu}^2(T) \leq R\|T - T^*\|^2 + \left\{ \begin{array}{l} \|T\|^2, \\ \left\|(1-\nu)|T|^2 + \nu|T^*|^2\right\|, \\ \left\|\frac{|T|^{2/\nu} + |T^*|^{2/(1-\nu)}}{2}\right\|^{2(1-\nu)\nu} \end{array} \right.$$

and

$$\sigma_e^2(T) \leq \|T - T^*\|^2 + 2 \times \left\{ \begin{array}{l} \|T\|^2, \\ \left\|\frac{|T|^2 + |T^*|^2}{2}\right\|, \\ \left\|\frac{|T|^4 + |T^*|^4}{2}\right\|^{1/2} \end{array} \right..$$

Additionally, for $p = 1/2$, we get

$$\sigma_\nu(T) \leq R\|T - T^*\| + \left\{ \begin{array}{l} \|T\|, \\ \left\|(1-\nu)|T|^2 + \nu|T^*|^2\right\|^{1/2}, \\ \left\|\frac{|T|^{2/\nu} + |T^*|^{2/(1-\nu)}}{2}\right\|^{(1-\nu)\nu} \end{array} \right.$$

and

$$\sigma(T) \leq \|T - T^*\| + 2 \times \left\{ \begin{array}{l} \|T\|, \\ \left\|\frac{|T|^2 + |T^*|^2}{2}\right\|^{1/2}, \\ \left\|\frac{|T|^4 + |T^*|^4}{2}\right\|^{1/4}. \end{array} \right.$$

From Theorem 3.2, for all $\nu \in (0,1)$ and $p \in (0,1]$, we get that

$$\sigma_{2p,\nu}^{2p}(T) \leq R\|T - T^*\|^{2p} + \frac{1}{2^{(1-\nu)\nu p}}\left(w\left(|T^*|^{2/(1-\nu)}|T|^{2/\nu}\right) + \left\|\frac{|T|^{4/\nu} + |T^*|^{4/(1-\nu)}}{2}\right\|\right)^{(1-\nu)\nu p}$$

and

$$\sigma_{2p}^{2p}(T) \leq \|T - T^*\|^{2p} + 2^{1-p/2}\left(w\left(|T^*|^2|T|^2\right) + \left\|\frac{|T|^4 + |T^*|^4}{2}\right\|\right)^{p/2}.$$

For $p = 1$, we get that

$$\sigma_{e,\nu}^2(T) \leq R\|T - T^*\|^2 + \frac{1}{2^{(1-\nu)\nu}}\left(w\left(|T^*|^{2/(1-\nu)}|T|^{2/\nu}\right) + \left\|\frac{|T|^{4/\nu} + |T^*|^{4/(1-\nu)}}{2}\right\|\right)^{(1-\nu)\nu}$$

and

$$\sigma_e^2(T) \leq \|T - T^*\|^2 + 2^{3/4}\left(w\left(|T^*|^4|T|^4\right) + \left\|\frac{|T|^8 + |T^*|^8}{2}\right\|\right)^{1/4}.$$

For $p = 1/2$, we obtain

$$\sigma_\nu(T) \leq R\|T - T^*\| + \frac{1}{2^{(1-\nu)\nu/2}}\left(w\left(|T^*|^{2/(1-\nu)}|T|^{2/\nu}\right) + \left\|\frac{|T|^{4/\nu} + |T^*|^{4/(1-\nu)}}{2}\right\|\right)^{(1-\nu)\nu/2}$$

and

$$\sigma(T) \leq \|T - T^*\| + 2^{7/8}\left(w\left(|T^*|^4\,|T|^4\right) + \left\|\frac{|T|^8 + |T^*|^8}{2}\right\|\right)^{1/8}.$$

For an operator $T \in \mathcal{B}(H)$, we consider the real and imaginary parts of T defined by the following:

$$\mathrm{Re}T := \frac{T + T^*}{2} \text{ and } \mathrm{Im}T = \frac{T - T^*}{2i}.$$

For $(A, B) = (\mathrm{Re}T, \mathrm{Im}T)$, $p \geq 1$ and $\nu \in [0, 1]$, we can introduce the following functional:

$$\rho_{p,\nu}(T) := \|(\mathrm{Re}T, \mathrm{Im}T)\|_{p,\nu} := \sup_{\|x\|=1}\left((1-\nu)\,\|\mathrm{Re}Tx\|^p + \nu\,\|\mathrm{Im}Tx\|^p\right)^{1/p}.$$

Since for a real number a, we have $\rho^p_{p,\nu}(aT) = |a|\,\rho^p_{p,\nu}(T)$, $T \in \mathcal{B}(H)$; then, we can conclude that $\rho^p_{p,\nu}(T)$ is a real norm on $\mathcal{B}(H)$.

Additionally,

$$\rho_p(T) := \|(\mathrm{Re}T, \mathrm{Im}T)\|_p := \sup_{\|x\|=1}\left(\|\mathrm{Re}Tx\|^p + \|\mathrm{Im}Tx\|^p\right)^{1/p}.$$

For $p = 2$, we have

$$\begin{aligned}\rho^2_{e,\nu}(T) := \|(\mathrm{Re}T, \mathrm{Im}T)\|_{2,\nu} &:= \sup_{\|x\|=1}\left((1-\nu)\,\|\mathrm{Re}Tx\|^2 + \nu\,\|\mathrm{Im}Tx\|^2\right)^{1/2} \\ &= \left\|(1-\nu)\,|\mathrm{Re}T|^2 + \nu\,|\mathrm{Im}T|^2\right\|^{1/2}\end{aligned}$$

and

$$\rho_e(T) := \|(\mathrm{Re}T, \mathrm{Im}T)\|_e := \sup_{\|x\|=1}\left(\|\mathrm{Re}Tx\|^2 + \|\mathrm{Im}Tx\|^2\right)^{1/2}.$$

For $p = 1$,

$$\rho_\nu(T) := \|(\mathrm{Re}T, \mathrm{Im}T)\|_\nu := \sup_{\|x\|=1}\left((1-\nu)\,\|\mathrm{Re}Tx\| + \nu\,\|\mathrm{Im}Tx\|\right)$$

and

$$\rho(T) := \|(\mathrm{Re}T, \mathrm{Im}T)\|_1 := \sup_{\|x\|=1}\left(\|\mathrm{Re}Tx\| + \|\mathrm{Im}Tx\|\right).$$

From Theorem 3.1, for $(A, B) = (\mathrm{Re}T, \mathrm{Im}T)$ and for all $\nu \in [0, 1]$ and $p \in (0, 1]$, we have that

$$\rho^{2p}_{2p,\nu}(T) \leq R\,\|\mathrm{Re}T - \mathrm{Im}T\|^{2p} + \begin{cases} \|\mathrm{Re}T\|^{2(1-\nu)p}\,\|\mathrm{Im}T\|^{2\nu p}, \\ \left\|(1-\nu)\,|\mathrm{Re}T|^2 + \nu\,|\mathrm{Im}T|^2\right\|^p. \end{cases}$$

Additionally, we have

$$\rho^{2p}_{2p,\nu}(T) \leq R\,\|\mathrm{Re}T - \mathrm{Im}T\|^{2p} + \left\|\frac{|\mathrm{Re}T|^{2/\nu} + |\mathrm{Im}T|^{2/(1-\nu)}}{2}\right\|^{2(1-\nu)\nu p},$$

for $\nu \in (0, 1)$.

For $p = 1$, we get

$$\rho^2_{e,\nu}(T) \leq R\,\|\mathrm{Re}T - \mathrm{Im}T\|^2 + \begin{cases} \|\mathrm{Re}T\|^{2(1-\nu)}\,\|\mathrm{Im}T\|^{2\nu}, \\ \left\|(1-\nu)\,|\mathrm{Re}T|^2 + \nu\,|\mathrm{Im}T|^2\right\| \end{cases}$$

and

$$\rho_{e,\nu}^{2}(T) \leq R\left\|\mathrm{Re}T-\mathrm{Im}T\right\|^{2}+\left\|\frac{\left|\mathrm{Re}T\right|^{2/\nu}+\left|\mathrm{Im}T\right|^{2/(1-\nu)}}{2}\right\|^{2(1-\nu)\nu},$$

for $\nu \in (0,1)$.

For $p = 1/2$, we obtain that

$$\rho_{\nu}(T) \leq R\left\|\mathrm{Re}T-\mathrm{Im}T\right\|+\left\{\begin{array}{l}\left\|\mathrm{Re}T\right\|^{1-\nu}\left\|\mathrm{Im}T\right\|^{\nu},\\ \left\|(1-\nu)\left|\mathrm{Re}T\right|^{2}+\nu\left|\mathrm{Im}T\right|^{2}\right\|^{1/2}.\end{array}\right.$$

Additionally, we have

$$\rho_{\nu}(T) \leq R\left\|\mathrm{Re}T-\mathrm{Im}T\right\|+\left\|\frac{\left|\mathrm{Re}T\right|^{2/\nu}+\left|\mathrm{Im}T\right|^{2/(1-\nu)}}{2}\right\|^{(1-\nu)\nu},$$

for $\nu \in (0,1)$.

Author contributions

Feryal Aladsani, Asmahan Alajyan, Silvestru Sever Dragomir and Kais Feki: Conceptualization, Visualization, Funding, Writing–review & editing, Formal analysis, Project administration, Validation, Investigation; Feryal Aladsani: Resources. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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