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Research article

Some reverse dynamic inequalities of Hilbert-type using mean inequality

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Abstract: This paper contains some new reverse dynamic inequalities of Hilbert-type on delta time scale calculus using mean inequality by applying reverse Hölder's inequality, chain rule on time scales, integration by parts, and the mean inequality. As special cases of our results, we get the discrete, continuous, and quantum analogs of inequalities, i.e., when $\mathbb{T} = \mathbb{N}$, $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = q^{\mathbb{N}}$ for $q > 1$.

Keywords: Hilbert-type inequalities; time scales delta calculus; reverse Hölder's inequality; mean inequality
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1. Introduction

In 1998, Pachpatte [1] showed an inequality of Hilbert-type and showed that if $F(\varrho) : \{0, 1, 2, ..., \gamma\} \subset \mathbb{N} \to \mathbb{R}$ and $b(\vartheta) : \{0, 1, 2, ..., \ell\} \subset \mathbb{N} \to \mathbb{R}$ with $F(0) = b(0) = 0$, then

$$\sum_{\varrho=1}^{\gamma}\sum_{\vartheta=1}^{\ell}\frac{\left|F_{\varrho}\right|\left|b_{\vartheta}\right|}{\varrho+\vartheta} \le C(\gamma,\ell)\left(\sum_{\varrho=1}^{\gamma}(\gamma-\varrho+1)\left|\nabla F_{\varrho}\right|^{2}\right)^{\frac{1}{2}}\left(\sum_{\vartheta=1}^{\ell}(\ell-\vartheta+1)\left|\nabla b_{\vartheta}\right|^{2}\right)^{\frac{1}{2}}, \tag{1.1}$$

where $\nabla F_{\varrho} = F_{\varrho} - F_{\varrho-1}$, $\nabla b_{\vartheta} = b_{\vartheta} - b_{\vartheta-1}$ and

$$C(\gamma,\ell) = \frac{1}{2}\sqrt{\gamma\ell}.$$

In 2000, Pachpatte [2] extended (1.1) and showed that if $\lambda, \mu > 1$ with $1/\lambda + 1/\mu = 1$, $F(\varrho) : \{0, 1, 2, ..., \gamma\} \subset \mathbb{N} \to \mathbb{R}$ and $b(\vartheta) : \{0, 1, 2, ..., \ell\} \subset \mathbb{N} \to \mathbb{R}$ with $F(0) = b(0) = 0$, then

$$\sum_{j=1}^{\gamma}\sum_{\vartheta=1}^{\ell}\frac{\left|F_j\right||b_\vartheta|}{\mu j^{\lambda-1}+\lambda\vartheta^{\mu-1}} \leq D(\lambda,\mu,\gamma,\ell)\left(\sum_{j=1}^{\gamma}(\gamma-j+1)\left|\nabla F_j\right|^{\lambda}\right)^{\frac{1}{\lambda}}\left(\sum_{\vartheta=1}^{\ell}(\ell-\vartheta+1)\left|\nabla b_\vartheta\right|^{\mu}\right)^{\frac{1}{\mu}}, \quad (1.2)$$

where

$$D(\lambda,\mu,\gamma,\ell) = \frac{1}{\lambda\mu}\gamma^{\frac{\lambda-1}{\lambda}}\ell^{\frac{\mu-1}{\mu}}.$$

In 2002, Kim et al. [3] expanded (1.2) and showed that if $\lambda, \mu > 1$, $F(\varrho) : \{0, 1, 2, ..., \gamma\} \subset \mathbb{N} \to \mathbb{R}$, and $b(\vartheta) : \{0, 1, 2, ..., \ell\} \subset \mathbb{N} \to \mathbb{R}$ with $F(0) = b(0) = 0$, then

$$\sum_{j=1}^{\gamma}\sum_{\vartheta=1}^{\ell}\frac{\left|F_j\right||b_\vartheta|}{\mu j^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}$$

$$\leq D^{*}(\lambda,\mu,\gamma,\ell)\left(\sum_{j=1}^{\gamma}(\gamma-j+1)\left|\nabla F_j\right|^{\lambda}\right)^{\frac{1}{\lambda}}\left(\sum_{\vartheta=1}^{\ell}(\ell-\vartheta+1)\left|\nabla b_\vartheta\right|^{\mu}\right)^{\frac{1}{\mu}}, \quad (1.3)$$

where

$$D^{*}(\lambda,\mu,\gamma,\ell) = \frac{1}{\lambda+\mu}\gamma^{\frac{\lambda-1}{\lambda}}\ell^{\frac{\mu-1}{\mu}}.$$

Also, the authors [3] showed that if $\lambda, \mu > 1$ and $f : (0, x) \to \mathbb{R}$, $g : (0, y) \to \mathbb{R}$ for $x, y \in (0, \infty)$ such that $f(0) = g(0) = 0$, then

$$\int_0^x\int_0^y\frac{|f(\varrho)||g(\tau)|}{\mu\varrho^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}d\varrho d\tau$$

$$\leq M(\lambda,\mu,x,y)\left(\int_0^x(x-\varrho)\left|f'(\varrho)\right|^{\lambda}d\varrho\right)^{\frac{1}{\lambda}}\left(\int_0^y(y-\tau)\left|g'(\tau)\right|^{\mu}d\tau\right)^{\frac{1}{\mu}}, \quad (1.4)$$

where

$$M(\lambda,\mu,x,y) = \frac{1}{\lambda+\mu}x^{\frac{\lambda-1}{\lambda}}y^{\frac{\mu-1}{\mu}}.$$

In 2011, Zhao et al. [4] broadened (1.1) and showed that if $\gamma_s > 1$ such that $1/\gamma_s + 1/\ell_s = 1$ and $F_s(\varrho_s)$ are real sequences defined for $\varrho_s = 0, 1, 2, ..., m_s$, where m_s are natural numbers, $F_s(0) = 0$, and $s = 1, 2, ..., h$. Define the operator ∇ by $\nabla F_s(\varrho_s) = F_s(\varrho_s) - F_s(\varrho_s - 1)$ for any function $F_s(\varrho_s)$, $s = 1, 2, ..., h$. Then,

$$\sum_{\varrho_1=1}^{m_1}\sum_{\varrho_2=1}^{m_2}\cdots\sum_{\varrho_h=1}^{m_h}\frac{\prod_{s=1}^{h}|F_s(\varrho_s)|}{\left(\sum_{s=1}^{h}\varrho_s/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}} \leq M\prod_{s=1}^{h}\left(\sum_{\varrho_s=1}^{m_s}(m_s-\varrho_s+1)\left|\nabla F_s(\varrho_s)\right|^{\gamma_s}\right)^{\frac{1}{\gamma_s}}, \quad (1.5)$$

where

$$M = M(m_1, m_2, ..., m_h) = \left(h-\sum_{s=1}^{h}\frac{1}{\gamma_s}\right)^{\sum_{s=1}^{h}1/\gamma_s-h}.\prod_{s=1}^{h}m_s^{1/\ell_s}.$$

Also, the authors [4] showed that if $h_s \geq 1$, $\gamma_s > 1$ are constants and $1/\gamma_s + 1/\ell_s = 1$ and $f_s(\varrho_s)$ are real-valued differentiable functions defined on $[0, x_s)$, where $x_s \in (0, \infty)$. Suppose $f_s(0) = 0$ for $s = 1, 2, ..., h$. Then,

$$\int_0^{x_1} ... \int_0^{x_h} \frac{\prod_{s=1}^{h} \left| f_s^{h_s}(\varrho_s) \right|}{\left(\sum_{s=1}^{h} \varrho_s / \ell_s \right)^{\sum_{s=1}^{h} 1/\ell_s}} d\varrho_h ... d\varrho_1$$

$$\leq K \prod_{s=1}^{h} \left(\int_0^{x_s} (x_s - \varrho_s) \left| f_s^{h_s - 1}(\varrho_s) . f_s'(\varrho_s) \right|^{\gamma_s} d\varrho_s \right)^{\frac{1}{\gamma_s}}, \tag{1.6}$$

where

$$K = K(x_1, ..., x_h) = \left(h - \sum_{s=1}^{h} \frac{1}{\gamma_s} \right)^{\sum_{s=1}^{h} 1/\gamma_s - h} . \prod_{s=1}^{h} h_s x_s^{1/\ell_s}.$$

Also, they showed that if $\gamma_s, \ell_s > 1$ such that $1/\gamma_s + 1/\ell_s = 1$ and $F_s(\varrho_s, \tau_s)$ are real sequences defined for (ϱ_s, τ_s), where $\varrho_s = 0, 1, 2, ..., m_s$, $\tau_s = 0, 1, 2, ..., h_s$, and m_s, h_s $(s = 1, 2, ..., h)$ are natural numbers. Assume that $F_s(0, \tau_s) = F_s(\varrho_s, 0) = 0$, $\forall s = 1, 2, ..., h$. Define the operator ∇_1 and ∇_2 by

$$\nabla_1 F_s(\varrho_s, \tau_s) = F_s(\varrho_s, \tau_s) - F_s(\varrho_s - 1, \tau_s),$$
$$\nabla_2 F_s(\varrho_s, \tau_s) = F_s(\varrho_s, \tau_s) - F_s(\varrho_s, \tau_s - 1).$$

Then,

$$\sum_{\varrho_1=1}^{m_1} \sum_{\tau_1=1}^{h_1} ... \sum_{\varrho_h=1}^{m_h} \sum_{\tau_h=1}^{h_h} \frac{\prod_{s=1}^{h} |F_s(\varrho_s, \tau_s)|}{\left(\sum_{s=1}^{h} \varrho_s \tau_s / \ell_s \right)^{\sum_{s=1}^{h} 1/\ell_s}}$$

$$\leq L \prod_{s=1}^{h} \left(\sum_{\varrho_s=1}^{m_s} \sum_{\tau_s=1}^{h_s} (m_s - \varrho_s + 1)(h_s - \tau_s + 1) |\nabla_2 \nabla_1 F_s(\varrho_s, \tau_s)|^{\gamma_s} \right)^{\frac{1}{\gamma_s}}, \tag{1.7}$$

where

$$L = \left(h - \sum_{s=1}^{h} \frac{1}{\gamma_s} \right)^{\sum_{s=1}^{h} 1/\gamma_s - h} . \prod_{s=1}^{h} (m_s h_s)^{1/\ell_s}.$$

Over recent decades, considerable focus has been placed on developing discrete counterparts to continuous findings across various branches of analysis. The growing interest in the discrete domain stems from the notable divergence in behavior between discrete operators and their continuous analogs. This paper aims to derive discrete inequalities within a broader framework termed a "time scale". Here, a time scale, denoted by $\mathbb{T}$, represents any nonempty closed subset of the real number $\mathbb{R}$. For more details about the dynamic inequalities on time scales, see the papers [5–9].

This paper aims to get reversed analogs similar to inequalities of Hilbert-type (1.5)–(1.7) on delta time scales. These inequalities will be established by applying reverse Hölder's inequality, mean inequality, and integration by parts on time scales.

The organization of the paper is follows: In Section 2, we show some lemmas on time scales needed for Section 3 where we prove our results. These results show special cases when $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \mathbb{R}$ give the reverse analogs on delta calculus of inequalities (1.5) and (1.7), and (1.6), respectively. Also, we can get other inequalities on different time scales, like $\mathbb{T} = q^{\mathbb{N}}$ for $q > 1$.

2. Preliminaries and basic lemmas

Instead of repitition about the basic facts of time scales and time scale notation, we refer to the book [10] by Bohner and Peterson which summarizes and organizes much of the theory on time scales.

A time scale $\mathbb{T}$ is defined by a nonempty closed subset of the real numbers $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ as follows:

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

while the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is given by $\mu(t) = \sigma(t) - t$.

In particular, if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\rho(t) = t$, and $\mu(t) = \sigma(t) - t = 0$.

If $\mathbb{T} = \mathbb{N}$, then $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and $\mu(t) = \sigma(t) - t = 1$.

If $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, then $\sigma(t) = qt$, $\rho(t) = \frac{t}{q}$, and $\mu(t) = \sigma(t) - t = (q - 1)t$.

Definition 2.1 ([10]). *We set*

$$\inf \phi = \sup \mathbb{T}, \quad \sup \phi = \inf \mathbb{T}.$$

For $t \in \mathbb{T}$, we have the following cases (also see Figures 1 and 2):

(1) If $\sigma(t) > t$, then we say that t is right-scattered.
(2) If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then we say that t is right-dense.
(3) If $\rho(t) < t$, then we say that t is left-scattered.
(4) If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then we say that t is left-dense.
(5) If t is left-scattered and right-scattered at the same time, then we say that t is isolated (i.e., $\rho(t) < t < \sigma(t)$).
(6) If t is left-dense and right-dense at the same time, then we say that t is dense (i.e., $\rho(t) = t = \sigma(t)$).

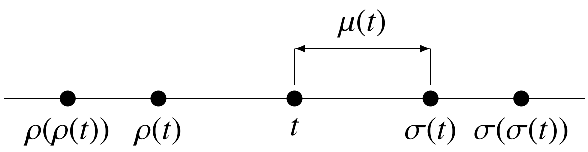

Figure 1. Forward and backward jump operators and the graininess function on a time scales.

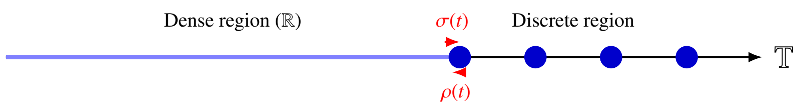

Figure 2. Right scattered points, left scattered points, isolated points, right dense points, left dense points, and dense points.

Theorem 2.1 (Chain rule [10, Theorem 1.90]). *Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ with*

$$(f \circ g)^{\Delta}(t) = f'(g(c))\, g^{\Delta}(t). \tag{2.1}$$

Lemma 2.1 (Integration by parts [11])**.** *If* F, $b \in \mathbb{T}$ *and* u, $v \in C_{rd}([F,b]_{\mathbb{T}}, \mathbb{R})$, *then*

$$\int_F^b u(\tau)v^{\Delta}(\tau)\Delta\tau = [u(\tau)v(\tau)]_F^b - \int_F^b u^{\Delta}(\tau)v^{\sigma}(\tau)\Delta\tau. \tag{2.2}$$

Lemma 2.2 (Hölder's inequality [11])**.** *If* F, $b \in \mathbb{T}$ *and* f, $g \in C_{rd}([F,b]_{\mathbb{T}}, \mathbb{R})$, *then*

$$\int_F^b |f(\tau)g(\tau)|\Delta\tau \leq \left[\int_F^b |f(\tau)|^{\alpha}\Delta\tau\right]^{\frac{1}{\alpha}} \left[\int_F^b |g(\tau)|^{\nu}\Delta\tau\right]^{\frac{1}{\nu}},$$

where $\alpha > 1$, *and* $1/\alpha + 1/\nu = 1$. *In addition, if* $\alpha < 0$ *and* $1/\alpha + 1/\nu = 1$, *then*

$$\int_F^b |f(\tau)g(\tau)|\Delta\tau \geq \left[\int_F^b |f(\tau)|^{\alpha}\Delta\tau\right]^{\frac{1}{\alpha}} \left[\int_F^b |g(\tau)|^{\nu}\Delta\tau\right]^{\frac{1}{\nu}}. \tag{2.3}$$

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be time scales. Let CC_{rd} denote the set of functions $f(\tau_1,\tau_2)$ on $\mathbb{T}_1 \times \mathbb{T}_2$, where f is $rd-$ continuous in τ_1 and τ_2. Let CC_{rd}^1 denote the set of all functions CC_{rd} for which both the Δ_1 partial derivative and Δ_2 partial derivative exist and are in CC_{rd}. The following theorem is adapted from [12], when $\alpha = 0$ and $h(\tau, x) = 1$.

Lemma 2.3 (Hölder's inequality in two dimensions [12])**.** *Assume that* F, $b \in \mathbb{T}$ *with* $F < b$, f, $g \in CC_{rd}([F,b]_{\mathbb{T}} \times [F,b]_{\mathbb{T}}, \mathbb{R})$ *and* α, $\nu > 1$ *such that* $1/\alpha + 1/\nu = 1$. *Then,*

$$\begin{aligned}&\int_F^b \int_F^b |f(\tau,x)g(\tau,x)|\Delta_1\tau\Delta_2 x\\ &\leq \left[\int_F^b \int_F^b |f(\tau,x)|^{\alpha}\Delta_1\tau\Delta_2 x\right]^{\frac{1}{\alpha}} \left[\int_F^b \int_F^b |g(\tau,x)|^{\nu}\Delta_1\tau\Delta_2 x\right]^{\frac{1}{\nu}}.\end{aligned}$$

This inequality is reversed when $\nu < 0$ *such that* $1/\alpha + 1/\nu = 1$, *and it becomes*

$$\begin{aligned}&\int_F^b \int_F^b |f(\tau,x)g(\tau,x)|\Delta_1\tau\Delta_2 x \qquad (2.4)\\ &\geq \left[\int_F^b \int_F^b |f(\tau,x)|^{\alpha}\Delta_1\tau\Delta_2 x\right]^{\frac{1}{\alpha}} \left[\int_F^b \int_F^b |g(\tau,x)|^{\nu}\Delta_1\tau\Delta_2 x\right]^{\frac{1}{\nu}}.\end{aligned}$$

Lemma 2.4 (Fubini's theorem [13])**.** *If* $F, b, c, \beta \in \mathbb{T}$ *and* $\Phi \in CC_{rd}([F,b]_{\mathbb{T}} \times [c,\beta]_{\mathbb{T}}, \mathbb{R})$ *is* $\Delta-$*integrable, then*

$$\int_F^b \left(\int_c^{\beta} \Phi(\tau,x)\Delta_2 x\right)\Delta_1\tau = \int_c^{\beta}\left(\int_F^b \Phi(\tau,x)\Delta_1\tau\right)\Delta_2 x. \tag{2.5}$$

Before starting our main results, we prove the following auxiliary lemma which will be needed in the proof of these results.

Lemma 2.5. *Assume* $F_s, b_s \in \mathbb{T}$ *and either* $\lambda_s \in C_{rd}([F_s,b_s]_{\mathbb{T}}, (-\infty, 0])$ *is a nonincreasing function, or* $\lambda_s \in C_{rd}([F_s,b_s]_{\mathbb{T}}, [0,\infty))$ *is a nondecreasing function with* $\lambda_s(F_s) = 0$, $s = 1,2,...,h$. *Then,*

$$\int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\tau_s)\right|\Delta\tau_s = |\lambda_s(x_s)|, \quad x_s \in [F_s,b_s]_{\mathbb{T}}. \tag{2.6}$$

Proof. First, if $\lambda_s \in C_{rd}([F_s, b_s]_{\mathbb{T}}, (-\infty, 0])$ is a nonincreasing function where $\lambda_s(F_s) = 0$, then we see that $\lambda_s^{\Delta}(\tau_s) \le 0$, and then

$$\int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\tau_s)\right| \Delta\tau_s = -\int_{F_s}^{x_s} \lambda_s^{\Delta}(\tau_s)\Delta\tau_s = -\left[\lambda_s(x_s) - \lambda_s(F_s)\right] = -\lambda_s(x_s) = \left|\lambda_s(x_s)\right|. \tag{2.7}$$

Second, if $\lambda_s \in C_{rd}([F_s, b_s]_{\mathbb{T}}, [0, \infty))$ is a nondecreasing function with $\lambda_s(F_s) = 0$, then we get $\lambda_s^{\Delta}(\tau_s) \ge 0$, and then

$$\int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\tau_s)\right| \Delta\tau_s = \int_{F_s}^{x_s} \lambda_s^{\Delta}(\tau_s)\Delta\tau_s = \lambda_s(x_s) - \lambda_s(F_s) = \lambda_s(x_s) = \left|\lambda_s(x_s)\right|. \tag{2.8}$$

From (2.7) and (2.8), we have, for the two cases of λ_s, that

$$\int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\tau_s)\right| \Delta\tau_s = \left|\lambda_s(x_s)\right|,$$

which is (2.6). □

Lemma 2.6 (Mean inequality [14])**.** *If $\alpha_s, \beta_s > 0$ for $s = 1, 2, ..., h$, then*

$$\prod_{s=1}^{h} \alpha_s^{\beta_s} \le \frac{\left(\sum_{s=1}^{h} \alpha_s\beta_s\right)^{\sum_{s=1}^{h}\beta_s}}{\left(\sum_{s=1}^{h} \beta_s\right)^{\sum_{s=1}^{h}\beta_s}}. \tag{2.9}$$

Lemma 2.7. *Let $\ell_s < 0$ with $1/\gamma_s + 1/\ell_s = 1$ and $\varrho_s > 0$, $s = 1, 2, ..., h$. Then,*

$$\prod_{s=1}^{h} \varrho_s^{1/\ell_s} \ge \frac{\left(\sum_{s=1}^{h} \varrho_s/\ell_s\right)^{\sum_{s=1}^{h} 1/\ell_s}}{\left(h - \sum_{s=1}^{h} 1/\gamma_s\right)^{\left(h - \sum_{s=1}^{h} 1/\gamma_s\right)}}. \tag{2.10}$$

Proof. Applying Lemma 2.6 with $\alpha_s = \varrho_s$ and $\beta_s = -1/\ell_s$, we observe that

$$\prod_{s=1}^{h} \varrho_s^{1/\ell_s} \ge \left(\frac{\sum_{s=1}^{h} \varrho_s/\ell_s}{\sum_{s=1}^{h} 1/\ell_s}\right)^{\sum_{s=1}^{h} 1/\ell_s}. \tag{2.11}$$

Since $1/\ell_s = 1 - 1/\gamma_s$, we see that

$$\sum_{s=1}^{h} 1/\ell_s = \sum_{s=1}^{h} (1 - 1/\gamma_s) = h - \sum_{s=1}^{h} 1/\gamma_s,$$

thus (2.11) can be written as the following:

$$\prod_{s=1}^{h} \varrho_s^{1/\ell_s} \ge \frac{\left(\sum_{s=1}^{h} \varrho_s/\ell_s\right)^{\sum_{s=1}^{h} 1/\ell_s}}{\left(h - \sum_{s=1}^{h} 1/\gamma_s\right)^{h - \sum_{s=1}^{h} 1/\gamma_s}},$$

which is (2.10). □

3. Main results

Throughout the paper we assume that the integrals considered are assumed to exist. Now, we can state and prove our results.

Theorem 3.1. *Let* $F_s, \varepsilon_s \in \mathbb{T}$, $0 < \gamma_s < 1$, *and* $\ell_s < 0$ *such that* $1/\gamma_s + 1/\ell_s = 1$, $\lambda_s \in \mathrm{C}_{rd}([F_s, \varepsilon_s]_{\mathbb{T}}, \mathbb{R})$ *with* $\lambda_s(F_s) = 0$ *and either* $\lambda_s \in \mathrm{C}_{rd}([F_s, b_s]_{\mathbb{T}}, (-\infty, 0])$ *be a nonincreasing function, or* $\lambda_s \in \mathrm{C}_{rd}([F_s, b_s]_{\mathbb{T}}, [0, \infty))$ *be a nondecreasing function for* $s = 1, 2, ..., h$. *Then,*

$$\int_{F_1}^{\varepsilon_1} \cdots \int_{F_h}^{\varepsilon_h} \frac{\left(h - \sum_{s=1}^{h} 1/\gamma_s\right)^{h - \sum_{s=1}^{h} 1/\gamma_s} \prod_{s=1}^{h} |\lambda_s(x_s)|}{\left(\sum_{s=1}^{h} (x_s - F_s)/\ell_s\right)^{\sum_{s=1}^{h} 1/\ell_s}} \Delta x_h ... \Delta x_1$$
$$\geq \prod_{s=1}^{h} (\varepsilon_s - F_s)^{\frac{1}{\ell_s}} \prod_{s=1}^{h} \left(\int_{F_s}^{\varepsilon_s} [\varepsilon_s - \sigma(x_s)] \left|\lambda_s^{\Delta}(x_s)\right|^{\gamma_s} \Delta x_s \right)^{\frac{1}{\gamma_s}}. \tag{3.1}$$

Proof. Applying (2.6), we have for $x_s \in [F_s, b_s]_{\mathbb{T}}$ that $\int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\zeta_s)\right| \Delta\zeta_s = |\lambda_s(x_s)|$, and so

$$\prod_{s=1}^{h} |\lambda_s(x_s)| = \prod_{s=1}^{h} \int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\zeta_s)\right| \Delta\zeta_s. \tag{3.2}$$

Applying (2.3) on $\int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\zeta_s)\right| \Delta\zeta_s$, with $\ell_s < 0$, $f(\zeta_s) = \left|\lambda_s^{\Delta}(\zeta_s)\right|$, and $g(\zeta_s) = 1$, we see that

$$\int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\zeta_s)\right| \Delta\zeta_s \geq \left(\int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s} \Delta\zeta_s \right)^{\frac{1}{\gamma_s}} \left(\int_{F_s}^{x_s} \Delta\zeta_s \right)^{\frac{1}{\ell_s}} = (x_s - F_s)^{\frac{1}{\ell_s}} \left(\int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s} \Delta\zeta_s \right)^{\frac{1}{\gamma_s}},$$

and then

$$\prod_{s=1}^{h} \int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\zeta_s)\right| \Delta\zeta_s \geq \prod_{s=1}^{h} (x_s - F_s)^{\frac{1}{\ell_s}} \prod_{s=1}^{h} \left(\int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s} \Delta\zeta_s \right)^{\frac{1}{\gamma_s}}. \tag{3.3}$$

Substituting (3.3) into (3.2), we see that

$$\prod_{s=1}^{h} |\lambda_s(x_s)| \geq \prod_{s=1}^{h} (x_s - F_s)^{\frac{1}{\ell_s}} \prod_{s=1}^{h} \left(\int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s} \Delta\zeta_s \right)^{\frac{1}{\gamma_s}}. \tag{3.4}$$

Applying (2.10) with $\varrho_s = x_s - F_s$, we obtain

$$\prod_{s=1}^{h} (x_s - F_s)^{1/\ell_s} \geq \frac{\left(\sum_{s=1}^{h} (x_s - F_s)/\ell_s\right)^{\sum_{s=1}^{h} 1/\ell_s}}{\left(h - \sum_{s=1}^{h} 1/\gamma_s\right)^{h - \sum_{s=1}^{h} 1/\gamma_s}}. \tag{3.5}$$

Substituting (3.5) into (3.4), we have

$$\prod_{s=1}^{h} |\lambda_s(x_s)| \geq \frac{\left(\sum_{s=1}^{h} (x_s - F_s)/\ell_s\right)^{\sum_{s=1}^{h} 1/\ell_s}}{\left(h - \sum_{s=1}^{h} 1/\gamma_s\right)^{h - \sum_{s=1}^{h} 1/\gamma_s}} \prod_{s=1}^{h} \left(\int_{F_s}^{x_s} \left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s} \Delta\zeta_s \right)^{\frac{1}{\gamma_s}}. \tag{3.6}$$

Multiplying (3.6) by

$$\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}/\left(\sum_{s=1}^{h}\left(x_s-F_s\right)/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s},$$

we observe that

$$\begin{aligned}&\int_{F_1}^{\varepsilon_1}...\int_{F_h}^{\varepsilon_h}\frac{\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}}{\left(\sum_{s=1}^{h}\left(x_s-F_s\right)/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}}\prod_{s=1}^{h}\left|\lambda_s(x_s)\right|\Delta x_h...\Delta x_1\\&\geq\int_{F_1}^{\varepsilon_1}...\int_{F_h}^{\varepsilon_h}\prod_{s=1}^{h}\left(\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s\right)^{\frac{1}{\gamma_s}}\Delta x_h...\Delta x_1=\prod_{s=1}^{h}\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s\right)^{\frac{1}{\gamma_s}}\Delta x_s.\end{aligned}\tag{3.7}$$

Again by applying (2.3) on

$$\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s\right)^{\frac{1}{\gamma_s}}\Delta x_s,$$

with $\ell_s<0$, $f\left(x_s\right)=\left(\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s\right)^{\frac{1}{\gamma_s}}$ and $g\left(x_s\right)=1$, we have that

$$\begin{aligned}\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s\right)^{\frac{1}{\gamma_s}}\Delta x_s&\geq\left(\int_{F_s}^{\varepsilon_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s\Delta x_s\right)^{\frac{1}{\gamma_s}}\left(\int_{F_s}^{\varepsilon_s}\Delta x_s\right)^{\frac{1}{\ell_s}}\\&=\left(\varepsilon_s-F_s\right)^{\frac{1}{\ell_s}}\left(\int_{F_s}^{\varepsilon_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s\Delta x_s\right)^{\frac{1}{\gamma_s}},\end{aligned}$$

and then

$$\prod_{s=1}^{h}\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s\right)^{\frac{1}{\gamma_s}}\Delta x_s\geq\prod_{s=1}^{h}\left(\varepsilon_s-F_s\right)^{\frac{1}{\ell_s}}\prod_{s=1}^{h}\left(\int_{F_s}^{\varepsilon_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s\Delta x_s\right)^{\frac{1}{\gamma_s}}.\tag{3.8}$$

Substituting (3.8) into (3.7), we see that

$$\begin{aligned}&\int_{F_1}^{\varepsilon_1}...\int_{F_h}^{\varepsilon_h}\frac{\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}}{\left(\sum_{s=1}^{h}\left(x_s-F_s\right)/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}}\prod_{s=1}^{h}\left|\lambda_s(x_s)\right|\Delta x_h...\Delta x_1\\&\geq\prod_{s=1}^{h}\left(\varepsilon_s-F_s\right)^{\frac{1}{\ell_s}}\prod_{s=1}^{h}\left(\int_{F_s}^{\varepsilon_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s\Delta x_s\right)^{\frac{1}{\gamma_s}}.\end{aligned}\tag{3.9}$$

Using (2.2) on

$$\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s\right)\Delta x_s,$$

with $f(x_s)=\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s$ and $g^{\Delta}(x_s)=1$, we get

$$\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s\right)\Delta x_s=\left(\int_{F_s}^{x_s}\left|\lambda_s^{\Delta}(\zeta_s)\right|^{\gamma_s}\Delta\zeta_s\right)g(x_s)\Big|_{F_s}^{\varepsilon_s}-\int_{F_s}^{\varepsilon_s}\left|\lambda_s^{\Delta}(x_s)\right|^{\gamma_s}g^{\sigma}(x_s)\Delta x_s,\tag{3.10}$$

where $g(x_s) = x_s - \varepsilon_s$. Since $g(\varepsilon_s) = 0$, from (3.10) we have

$$\int_{F_s}^{\varepsilon_s} \left(\int_{F_s}^{x_s} \left| \lambda_s^{\Delta}(\zeta_s) \right|^{\gamma_s} \Delta\zeta_s \right) \Delta x_s = \int_{F_s}^{\varepsilon_s} \left| \lambda_s^{\Delta}(x_s) \right|^{\gamma_s} [\varepsilon_s - \sigma(x_s)] \, \Delta x_s. \tag{3.11}$$

Substituting (3.11) into (3.9), we obtain

$$\begin{aligned} &\int_{F_1}^{\varepsilon_1} \cdots \int_{F_h}^{\varepsilon_h} \frac{\left(h - \sum_{s=1}^{h} 1/\gamma_s\right)^{h-\sum_{s=1}^{h} 1/\gamma_s} \prod_{s=1}^{h} |\lambda_s(x_s)|}{\left(\sum_{s=1}^{h} (x_s - F_s)/\ell_s\right)^{\sum_{s=1}^{h} 1/\ell_s}} \Delta x_h \ldots \Delta x_1 \\ &\geq \prod_{s=1}^{h} (\varepsilon_s - F_s)^{\frac{1}{\ell_s}} \prod_{s=1}^{h} \left(\int_{F_s}^{\varepsilon_s} [\varepsilon_s - \sigma(x_s)] \left| \lambda_s^{\Delta}(x_s) \right|^{\gamma_s} \Delta x_s \right)^{\frac{1}{\gamma_s}}, \end{aligned}$$

which is the desired inequality (3.1). □

Remark 3.1. *If* $\mathbb{T} = \mathbb{N}_0$, $\sigma(x_s) = x_s + 1$, *and* $F_s = 0$ *for* $s = 1, 2, ..., h$, $0 < \gamma_s < 1$, *and* $\ell_s < 0$ *such that* $1/\gamma_s + 1/\ell_s = 1$. *Then (3.1) reduces to the following:*

$$\begin{aligned} &\sum_{x_1=0}^{\varepsilon_1-1} \cdots \sum_{x_h=0}^{\varepsilon_h-1} \frac{\left(h - \sum_{s=1}^{h} 1/\gamma_s\right)^{h-\sum_{s=1}^{h} 1/\gamma_s} \prod_{s=1}^{h} |\lambda_s(x_s)|}{\left(\sum_{s=1}^{h} x_s/\ell_s\right)^{\sum_{s=1}^{h} 1/\ell_s}} \\ &\geq \prod_{s=1}^{h} \varepsilon_s^{\frac{1}{\ell_s}} \prod_{s=1}^{h} \left(\sum_{x_s=0}^{\varepsilon_s-1} [\varepsilon_s - x_s - 1] \, |\Delta\lambda_s(x_s)|^{\gamma_s} \right)^{\frac{1}{\gamma_s}}, \end{aligned} \tag{3.12}$$

where

$$\Delta\lambda(x) = \frac{\lambda(\sigma(x)) - \lambda(x)}{\sigma(x) - x} = \lambda(x+1) - \lambda(x).$$

Example 1. *Determining* $\mathbb{T} = \mathbb{N}_0$, $h = 2$, $s = 1, 2$, $\gamma_1 = \frac{1}{2}$, $\ell_1 = -1$, $\gamma_2 = \frac{1}{3}$, $\ell_2 = \frac{-1}{2}$, $\lambda_1(x) = \lambda_2(x) = x$, $\varepsilon_1 = 3$, *and* $\varepsilon_2 = 2$. *Then, the left side of inequality* (3.12) *becomes*

$$\begin{aligned} &\sum_{x_1=0}^{\varepsilon_1-1} \cdots \sum_{x_h=0}^{\varepsilon_h-1} \frac{\left(h - \sum_{s=1}^{h} 1/\gamma_s\right)^{h-\sum_{s=1}^{h} 1/\gamma_s} \prod_{s=1}^{h} |\lambda_s(x_s)|}{\left(\sum_{s=1}^{h} x_s/\ell_s\right)^{\sum_{s=1}^{h} 1/\ell_s}} \\ &= \sum_{x_1=0}^{2} \sum_{x_2=0}^{1} \frac{\left(2 - \sum_{s=1}^{2} 1/\gamma_s\right)^{2-\sum_{s=1}^{2} 1/\gamma_s} \prod_{s=1}^{2} \lambda_s(x_s)}{\left(\sum_{s=1}^{2} x_s/\ell_s\right)^{\sum_{s=1}^{2} 1/\ell_s}} = \frac{155}{27}. \end{aligned}$$

In this case, the right side of (3.12) *gives*

$$\prod_{s=1}^{h} \varepsilon_s^{\frac{1}{\ell_s}} \prod_{s=1}^{h} \left(\sum_{x_s=0}^{\varepsilon_s-1} [\varepsilon_s - x_s - 1] \, |\Delta\lambda_s(x_s)|^{\gamma_s} \right)^{\frac{1}{\gamma_s}} = \prod_{s=1}^{2} \varepsilon_s^{\frac{1}{\ell_s}} \prod_{s=1}^{2} \left(\sum_{x_s=0}^{\varepsilon_s-1} [\varepsilon_s - x_s - 1] \right)^{\frac{1}{\gamma_s}} = \frac{3}{4},$$

and this indicates the validity of (3.12).

Remark 3.2. *If $\mathbb{T}=\mathbb{R}, \sigma(x_s)=x_s, F_s=0, 0<\gamma_s<1$, and $\ell_s<0$ such that $1/\gamma_s+1/\ell_s=1$. Then, (3.1) reduces to the following:*

$$\int_0^{\varepsilon_1}\dots\int_0^{\varepsilon_h}\frac{\left(h-\sum_{s=1}^h 1/\gamma_s\right)^{h-\sum_{s=1}^h 1/\gamma_s}\prod_{s=1}^h|\lambda_s(x_s)|}{\left(\sum_{s=1}^h x_s/\ell_s\right)^{\sum_{s=1}^h 1/\ell_s}}dx_h...dx_1$$
$$\geq\prod_{s=1}^h\varepsilon_s^{\frac{1}{\ell_s}}\prod_{s=1}^h\left(\int_0^{\varepsilon_s}[\varepsilon_s-x_s]\left|\lambda_s'(x_s)\right|^{\gamma_s}dx_s\right)^{\frac{1}{\gamma_s}}.$$

Remark 3.3. *If $\mathbb{T}=\beta^{\mathbb{N}_0}$ for $\beta>1$, $\sigma(x_s)=\beta x_s$, $\mu(x_s)=\sigma(x_s)-x_s=(\beta-1)x_s$, $0<\gamma_s<1$, and $\ell_s<0$ such that $1/\gamma_s+1/\ell_s=1$. Then, (3.1) reduces to the following:*

$$\sum_{x_1=F_1}^{\varepsilon_1/\beta}\dots\sum_{x_h=F_h}^{\varepsilon_h/\beta}\frac{\left(h-\sum_{s=1}^h 1/\gamma_s\right)^{h-\sum_{s=1}^h 1/\gamma_s}\prod_{s=1}^h(\beta-1)x_s|\lambda_s(x_s)|}{\left(\sum_{s=1}^h(x_s-F_s)/\ell_s\right)^{\sum_{s=1}^h 1/\ell_s}}$$
$$\geq\prod_{s=1}^h(\varepsilon_s-F_s)^{\frac{1}{\ell_s}}\prod_{s=1}^h\left(\sum_{x_s=F_s}^{\varepsilon_s/\beta}(\beta-1)x_s[\varepsilon_s-\beta x_s]\left|\lambda_s^{\Delta}(x_s)\right|^{\gamma_s}\right)^{\frac{1}{\gamma_s}},$$

where
$$\lambda_s^{\Delta}(x_s)=\Delta_\beta\lambda_s(x_s)=\frac{\lambda_s(\beta x_s)-\lambda_s(x_s)}{(\beta-1)x_s}.$$

Theorem 3.2. *Assume that $F_s, \varepsilon_s, \upsilon_s\in\mathbb{T}$, $0<\gamma_s<1$, and $\ell_s<0$ such that $1/\gamma_s+1/\ell_s=1$ and $\lambda_s\in CC'_{rd}([F_s,\varepsilon_s]_{\mathbb{T}}\times[F_s,\upsilon_s]_{\mathbb{T}},\mathbb{R}^+\cup\{0\})$ with $\lambda_s^{\Delta_2\Delta_1}(\tau_s,x_s)\geq 0$, $\lambda_s(F_s,x_s)=\lambda_s(\tau_s,F_s)=0$ for $x_s\in[F_s,\varepsilon_s]_{\mathbb{T}}$ and $\tau_s\in[F_s,\upsilon_s]_{\mathbb{T}}$, $s=1,2,...,h$. Then,*

$$\int_{F_1}^{\upsilon_1}\int_{F_h}^{\upsilon_h}\dots\int_{F_1}^{\varepsilon_1}\int_{F_h}^{\varepsilon_h}\frac{\left(h-\sum_{s=1}^h 1/\gamma_s\right)^{h-\sum_{s=1}^h 1/\gamma_s}\prod_{s=1}^h|\lambda_s(\tau_s,x_s)|}{\left(\sum_{s=1}^h(\tau_s-F_s)(x_s-F_s)/\ell_s\right)^{\sum_{s=1}^h 1/\ell_s}}\Delta_2x_h\Delta_2x_1...\Delta_1\tau_h\Delta_1\tau_1$$
$$\geq\prod_{s=1}^h(\upsilon_s-F_s)^{\frac{1}{\ell_s}}(\varepsilon_s-F_s)^{\frac{1}{\ell_s}}\prod_{s=1}^h\left(\int_{F_s}^{\varepsilon_s}\int_{F_s}^{\upsilon_s}(\upsilon_s-\sigma(\tau_s))(\varepsilon_s-\sigma(x_s))\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,x_s)\right|^{\gamma_s}\Delta_1\tau_s\Delta_2x_s\right)^{\frac{1}{\gamma_s}}. \tag{3.13}$$

Here the Δ_1-derivative of the function $\lambda(\tau,x)$ is the $\Delta-$derivative with respect to the first variable τ, and the Δ_2-derivative of the function $\lambda(\tau,x)$ is the $\Delta-$derivative with respect to the second variable x.

Proof. Applying (2.5) and the hypothesis $\lambda_s^{\Delta_2\Delta_1}(\tau_s,x_s)\geq 0$ and $\lambda_s(F_s,x_s)=\lambda_s(\tau_s,F_s)=0$ for $x_s\in[F_s,\varepsilon_s]_{\mathbb{T}}$ and $\tau_s\in[F_s,\upsilon_s]_{\mathbb{T}}$, we see that

$$\begin{aligned}\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(\zeta_s,\vartheta_s)\right|\Delta_2\vartheta_s\Delta_1\zeta_s &= \int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\lambda_s^{\Delta_2\Delta_1}(\zeta_s,\vartheta_s)\Delta_2\vartheta_s\Delta_1\zeta_s\\ &= \int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left[\lambda_s^{\Delta_2}(y_s,\vartheta_s)\right]^{\Delta_1}\Delta_2\vartheta_s\Delta_1y_s\end{aligned}$$

$$\begin{aligned} &= \int_{F_s}^{x_s}\left(\int_{F_s}^{\tau_s}\left[\lambda_s^{\Delta_2}(y_s,\vartheta_s)\right]^{\Delta_1}\Delta_1 y_s\right)\Delta_2\vartheta_s \\ &= \int_{F_s}^{x_s}\left(\lambda_s^{\Delta_2}(\tau_s,\vartheta_s)-\lambda_s^{\Delta_2}(F_s,\vartheta_s)\right)\Delta_2\vartheta_s \\ &= \lambda_s(\tau_s,x_s)-\lambda_s(\tau_s,F_s)-\lambda_s(F_s,x_s)+\lambda_s(F_s,F_s) \\ &= \lambda_s(\tau_s,x_s)=|\lambda_s(\tau_s,x_s)|, \end{aligned}$$

and then

$$\prod_{s=1}^{h}|\lambda_s(\tau_s,x_s)|=\prod_{s=1}^{h}\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|\Delta_2\vartheta_s\Delta_1 y_s. \tag{3.14}$$

Applying (2.4) on $\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|\Delta_2\vartheta_s\Delta_1 y_s$, with $\ell_s < 0$, $f(y_s,\vartheta_s) = 1$ and $g(y_s,\vartheta_s) = \left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|$, we observe that

$$\begin{aligned} &\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|\Delta_2\vartheta_s\Delta_1 y_s \\ \geq\ & \left(\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1 y_s\right)^{\frac{1}{\gamma_s}}\left(\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\Delta_2\vartheta_s\Delta_1 y_s\right)^{\frac{1}{\ell_s}} \\ =\ & (\tau_s-F_s)^{\frac{1}{\ell_s}}(x_s-F_s)^{\frac{1}{\ell_s}}\left(\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1 y_s\right)^{\frac{1}{\gamma_s}}, \end{aligned}$$

and then

$$\begin{aligned} &\prod_{s=1}^{h}\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|\Delta_2\vartheta_s\Delta_1 y_s \\ &\geq \prod_{s=1}^{h}(\tau_s-F_s)^{\frac{1}{\ell_s}}(x_s-F_s)^{\frac{1}{\ell_s}}\prod_{s=1}^{h}\left(\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1 y_s\right)^{\frac{1}{\gamma_s}}. \end{aligned} \tag{3.15}$$

Substituting (3.15) into (3.14), we see that

$$\prod_{s=1}^{h}|\lambda_s(\tau_s,x_s)|\geq\prod_{s=1}^{h}(\tau_s-F_s)^{\frac{1}{\ell_s}}(x_s-F_s)^{\frac{1}{\ell_s}}\prod_{s=1}^{h}\left(\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1 y_s\right)^{\frac{1}{\gamma_s}}. \tag{3.16}$$

Applying (2.10) with $\varrho_s = (\tau_s - F_s)(x_s - F_s)$, we have that

$$\prod_{s=1}^{h}(\tau_s-F_s)^{\frac{1}{\ell_s}}(x_s-F_s)^{\frac{1}{\ell_s}}\geq\frac{\left(\sum_{s=1}^{h}(\tau_s-F_s)(x_s-F_s)/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}}{\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}}. \tag{3.17}$$

Substituting (3.17) into (3.16), we obtain

$$\prod_{s=1}^{h}|\lambda_s(\tau_s,x_s)| \geq \frac{\left(\sum_{s=1}^{h}(\tau_s-F_s)(x_s-F_s)/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}}{\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}} \times\prod_{s=1}^{h}\left(\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1 y_s\right)^{\frac{1}{\gamma_s}}. \tag{3.18}$$

Multiplying (3.18) by

$$\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}\Big/\left(\sum_{s=1}^{h}(\tau_s-F_s)(x_s-F_s)/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s},$$

we observe that

$$\begin{aligned}
&\int_{F_1}^{\upsilon_1}\int_{F_h}^{\upsilon_h}\cdots\int_{F_1}^{\varepsilon_1}\int_{F_h}^{\varepsilon_h}\frac{\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}\prod_{s=1}^{h}|\lambda_s(\tau_s,x_s)|}{\left(\sum_{s=1}^{h}(\tau_s-F_s)(x_s-F_s)/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}}\Delta_2x_h\Delta_2x_1\ldots\Delta_1\tau_h\Delta_1\tau_1\\
&\geq\int_{F_1}^{\upsilon_1}\int_{F_h}^{\upsilon_h}\cdots\int_{F_1}^{\varepsilon_1}\int_{F_h}^{\varepsilon_h}\prod_{s=1}^{h}\left(\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1y_s\right)^{\frac{1}{\gamma_s}}\Delta_2x_h\Delta_2x_1\ldots\Delta_1\tau_h\Delta_1\tau_1 \qquad (3.19)\\
&=\prod_{s=1}^{h}\int_{F_s}^{\upsilon_s}\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1y_s\right)^{\frac{1}{\gamma_s}}\Delta_2x_s\Delta_1\tau_s.
\end{aligned}$$

Applying (2.4) on

$$\int_{F_s}^{\upsilon_s}\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1y_s\right)^{\frac{1}{\gamma_s}}\Delta_2x_s\Delta_1\tau_s,$$

with $\ell_s<0$, $f(x_s,\tau_s)=1$ and

$$g(x_s,\tau_s)=\left(\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1y_s\right)^{\frac{1}{\gamma_s}},$$

we have that

$$\begin{aligned}
&\int_{F_s}^{\upsilon_s}\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1y_s\right)^{\frac{1}{\gamma_s}}\Delta_2x_s\Delta_1\tau_s\\
&\geq\left(\int_{F_s}^{\upsilon_s}\int_{F_s}^{\varepsilon_s}\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1y_s\Delta_2x_s\Delta_1\tau_s\right)^{\frac{1}{\gamma_s}}\left(\int_{F_s}^{\upsilon_s}\int_{F_s}^{\varepsilon_s}\Delta_2x_s\Delta_1\tau_s\right)^{\frac{1}{\ell_s}}\\
&=(\upsilon_s-F_s)^{\frac{1}{\ell_s}}(\varepsilon_s-F_s)^{\frac{1}{\ell_s}}\left(\int_{F_s}^{\upsilon_s}\int_{F_s}^{\varepsilon_s}\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1y_s\Delta_2x_s\Delta_1\tau_s\right)^{\frac{1}{\gamma_s}},
\end{aligned}$$

and then

$$\prod_{s=1}^{h}\int_{F_s}^{\upsilon_s}\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1 y_s\right)^{\frac{1}{\gamma_s}}\Delta_2 x_s\Delta_1\tau_s$$
$$\geq\prod_{s=1}^{h}(\upsilon_s-F_s)^{\frac{1}{\ell_s}}(\varepsilon_s-F_s)^{\frac{1}{\ell_s}} \tag{3.20}$$
$$\times\prod_{s=1}^{h}\left(\int_{F_s}^{\upsilon_s}\int_{F_s}^{\varepsilon_s}\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1 y_s\Delta_2 x_s\Delta_1\tau_s\right)^{\frac{1}{\gamma_s}}.$$

Substituting (3.20) into (3.19) and applying (2.5), we see that

$$\int_{F_1}^{\upsilon_1}\int_{F_h}^{\upsilon_h}\ldots\int_{F_1}^{\varepsilon_1}\int_{F_h}^{\varepsilon_h}\frac{\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}\prod_{s=1}^{h}|\lambda_s(\tau_s,x_s)|}{\left(\sum_{s=1}^{h}(\tau_s-F_s)(x_s-F_s)/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}}\Delta_2 x_h\Delta_2 x_1\ldots\Delta_1\tau_h\Delta_1\tau_1$$
$$\geq\prod_{s=1}^{h}(\upsilon_s-F_s)^{\frac{1}{\ell_s}}(\varepsilon_s-F_s)^{\frac{1}{\ell_s}}\prod_{s=1}^{h}\left(\int_{F_s}^{\upsilon_s}\int_{F_s}^{\varepsilon_s}\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1 y_s\Delta_2 x_s\Delta_1\tau_s\right)^{\frac{1}{\gamma_s}}$$
$$=\prod_{s=1}^{h}(\upsilon_s-F_s)^{\frac{1}{\ell_s}}(\varepsilon_s-F_s)^{\frac{1}{\ell_s}}$$
$$\times\prod_{s=1}^{h}\left(\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{\upsilon_s}\left[\int_{F_s}^{\tau_s}\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\Delta_1 y_s\right]\Delta_1\tau_s\right)\Delta_2 x_s\right)^{\frac{1}{\gamma_s}}. \tag{3.21}$$

Applying (2.2) on the term $\int_{F_s}^{\upsilon_s}\left(\int_{F_s}^{\tau_s}\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_1 y_s\right)\Delta_1\tau_s$ with

$$f(\tau_s)=\int_{F_s}^{\tau_s}\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_1 y_s \text{ and } g^{\Delta}(\tau_s)=1,$$

we get

$$\int_{F_s}^{\upsilon_s}\left(\int_{F_s}^{\tau_s}\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_1 y_s\right)\Delta_1\tau_s$$
$$= g(\tau_s)\left(\int_{F_s}^{\tau_s}\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_1 y_s\right)\Bigg|_{F_s}^{\upsilon_s}$$
$$-\int_{F_s}^{\upsilon_s}g^{\sigma}(\tau_s)\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_1\tau_s, \tag{3.22}$$

where $g(\tau_s)=\tau_s-\upsilon_s$. Since $g(\upsilon_s)=0$, from (3.22) we have

$$\int_{F_s}^{\upsilon_s}\left(\int_{F_s}^{\tau_s}\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_1 y_s\right)\Delta_1\tau_s$$
$$=\int_{F_s}^{\upsilon_s}[\upsilon_s-\sigma(\tau_s)]\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_1\tau_s. \tag{3.23}$$

Integrating (3.23) over x_s from F_s to ε_s and then by applying (2.5), we obtain

$$\begin{aligned}
& \int_{F_s}^{\varepsilon_s}\int_{F_s}^{\upsilon_s}\left(\int_{F_s}^{\tau_s}\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_1 y_s\right)\Delta_1\tau_s\Delta_2 x_s \\
= & \int_{F_s}^{\varepsilon_s}\int_{F_s}^{\upsilon_s}[\upsilon_s-\sigma(\tau_s)]\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_1\tau_s\Delta_2 x_s \\
= & \int_{F_s}^{\upsilon_s}\int_{F_s}^{\varepsilon_s}[\upsilon_s-\sigma(\tau_s)]\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_2 x_s\Delta_1\tau_s \\
= & \int_{F_s}^{\upsilon_s}[\upsilon_s-\sigma(\tau_s)]\left(\int_{F_s}^{\varepsilon_s}\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_2 x_s\right)\Delta_1\tau_s. \qquad (3.24)
\end{aligned}$$

Again by applying (2.2) on

$$\int_{F_s}^{\varepsilon_s}\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_2 x_s,$$

with

$$f(x_s)=\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s \quad \text{and} \quad g^{\Delta}(x_s)=1,$$

we see that

$$\begin{aligned}
& \int_{F_s}^{\varepsilon_s}\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_2 x_s \\
& = g(x_s)\left(\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right)\Bigg|_{F_s}^{\varepsilon_s} \\
& -\int_{F_s}^{\varepsilon_s} g^{\sigma}(x_s)\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,x_s)\right|^{\gamma_s}\Delta_2 x_s, \qquad (3.25)
\end{aligned}$$

where $g(x_s)=x_s-\varepsilon_s$. Since $g(\varepsilon_s)=0$, from (3.25) we have

$$\begin{aligned}
& \int_{F_s}^{\varepsilon_s}\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_2 x_s \\
= & \int_{F_s}^{\varepsilon_s}(\varepsilon_s-\sigma(x_s))\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,x_s)\right|^{\gamma_s}\Delta_2 x_s. \qquad (3.26)
\end{aligned}$$

Substituting (3.26) into (3.24) and applying (2.5), we obtain

$$\begin{aligned}
& \int_{F_s}^{\varepsilon_s}\int_{F_s}^{\upsilon_s}\left(\int_{F_s}^{\tau_s}\left[\int_{F_s}^{x_s}\left|\lambda_s^{\Delta_2\Delta_1}(y_s,\vartheta_s)\right|^{\gamma_s}\Delta_2\vartheta_s\right]\Delta_1 y_s\right)\Delta_1\tau_s\Delta_2 x_s \\
= & \int_{F_s}^{\upsilon_s}(\upsilon_s-\sigma(\tau_s))\left(\int_{F_s}^{\varepsilon_s}(\varepsilon_s-\sigma(x_s))\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,x_s)\right|^{\gamma_s}\Delta_2 x_s\right)\Delta_1\tau_s \\
= & \int_{F_s}^{\upsilon_s}\int_{F_s}^{\varepsilon_s}(\upsilon_s-\sigma(\tau_s))(\varepsilon_s-\sigma(x_s))\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,x_s)\right|^{\gamma_s}\Delta_2 x_s\Delta_1\tau_s \\
= & \int_{F_s}^{\varepsilon_s}\int_{F_s}^{\upsilon_s}(\upsilon_s-\sigma(\tau_s))(\varepsilon_s-\sigma(x_s))\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,x_s)\right|^{\gamma_s}\Delta_1\tau_s\Delta_2 x_s. \qquad (3.27)
\end{aligned}$$

Substituting (3.27) into (3.21), we see that

$$\int_{F_1}^{\upsilon_1}\int_{F_h}^{\upsilon_h}\cdots\int_{F_1}^{\varepsilon_1}\int_{F_h}^{\varepsilon_h}\frac{\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}\prod_{s=1}^{h}|\lambda_s(\tau_s,x_s)|}{\left(\sum_{s=1}^{h}(\tau_s-F_s)(x_s-F_s)/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}}\Delta_2x_h\Delta_2x_1...\Delta_1\tau_h\Delta_1\tau_1$$
$$\geq\prod_{s=1}^{h}(\upsilon_s-F_s)^{\frac{1}{\ell_s}}(\varepsilon_s-F_s)^{\frac{1}{\ell_s}}\,1\prod_{s=1}^{h}\left(\int_{F_s}^{\varepsilon_s}\int_{F_s}^{\upsilon_s}(\upsilon_s-\sigma(\tau_s))(\varepsilon_s-\sigma(x_s))\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,x_s)\right|^{\gamma_s}\Delta_1\tau_s\Delta_2x_s\right)^{\frac{1}{\gamma_s}},$$

which is (3.13). □

Remark 3.4. *If* $\mathbb{T}=\mathbb{N}_0$, $\sigma(x_s)=x_s+1$, *and* $F_s=0$, *we get the reverse analog of inequality (1.7) as follows:*

$$\sum_{\tau_1=0}^{\upsilon_1-1}\sum_{\tau_h=0}^{\upsilon_h-1}\cdots\sum_{x_1=0}^{\varepsilon_1-1}\sum_{x_h=0}^{\varepsilon_h-1}\frac{\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}\prod_{s=1}^{h}|\lambda_s(\tau_s,x_s)|}{\left(\sum_{s=1}^{h}\tau_sx_s/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}}$$
$$\geq\prod_{s=1}^{h}(\varepsilon_s\upsilon_s)^{\frac{1}{\ell_s}}\prod_{s=1}^{h}\left(\sum_{x_s=0}^{\varepsilon_s-1}\sum_{\tau_s=0}^{\upsilon_s-1}(\upsilon_s-\tau_s-1)(\varepsilon_s-x_s-1)|\Delta_2\Delta_1\lambda_s(\tau_s,x_s)|^{\gamma_s}\right)^{\frac{1}{\gamma_s}},$$

where

$$\Delta_1\lambda_s(\tau_s,x_s)=\lambda_s(\tau_s+1,x_s)-\lambda_s(\tau_s,x_s),$$
$$\Delta_2\lambda_s(\tau_s,x_s)=\lambda_s(\tau_s,x_s+1)-\lambda_s(\tau_s,x_s).$$

Remark 3.5. *If* $\mathbb{T}=\mathbb{R}$, $\sigma(x_s)=x_s$, *and* $F_s=0$, γ_s, $\ell_s>1$ *such that* $1/\gamma_s+1/\ell_s=1$ *and* $\lambda_s\in C([0,\varepsilon_s]\times[0,\upsilon_s],\mathbb{R}^+\cup\{0\})$ *with* $\frac{\partial^2}{\partial x_s\partial\tau_s}\lambda_s(\tau_s,x_s)\geq 0$, $\lambda_s(0,x_s)=\lambda_s(\tau_s,0)=0$ *for* $x_s\in[0,\varepsilon_s]_{\mathbb{T}}$ *and* $\tau_s\in[0,\upsilon_s]_{\mathbb{T}}$, *where* $s=1,2,...,h$, *then*

$$\int_0^{\upsilon_1}\int_0^{\upsilon_h}\cdots\int_0^{\varepsilon_1}\int_0^{\varepsilon_h}\frac{\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}\prod_{s=1}^{h}|\lambda_s(\tau_s,x_s)|}{\left(\sum_{s=1}^{h}\tau_sx_s/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}}dx_hdx_1...d\tau_hd\tau_1$$
$$\geq\prod_{s=1}^{h}(\upsilon_s\varepsilon_s)^{\frac{1}{\ell_s}}\prod_{s=1}^{h}\left(\int_0^{\varepsilon_s}\int_0^{\upsilon_s}(\upsilon_s-\tau_s)(\varepsilon_s-x_s)\left|\frac{\partial^2}{\partial x_s\partial\tau_s}\lambda_s(\tau_s,x_s)\right|^{\gamma_s}d\tau_sdx_s\right)^{\frac{1}{\gamma_s}}.$$

Remark 3.6. *If* $\mathbb{T}=\beta^{\mathbb{N}_0}$ *for* $\beta>1$, $\sigma(y)=\beta y$ *for* $y\in\mathbb{T}$, γ_s, $\ell_s>1$ *such that* $1/\gamma_s+1/\ell_s=1$ *and* $\lambda_s:[F_s,\varepsilon_s]_{\mathbb{T}}\times[F_s,\upsilon_s]_{\mathbb{T}}\to\mathbb{R}^+\cup\{0\}$ *with* $\lambda_s^{\Delta_2\Delta_1}(\tau_s,x_s)\geq 0$, $\lambda_s(F_s,x_s)=\lambda_s(\tau_s,F_s)=0$ *for* $x_s\in[F_s,\varepsilon_s]_{\mathbb{T}}$ *and* $\tau_s\in[F_s,\upsilon_s]_{\mathbb{T}}$, *where* $s=1,2,...,h$, *then*

$$\sum_{\tau_1=F_1}^{\upsilon_1/\beta}\sum_{\tau_h=F_h}^{\upsilon_h/\beta}\cdots\sum_{x_1=F_1}^{\varepsilon_1/\beta}\sum_{x_h=F_h}^{\varepsilon_h/\beta}\frac{\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}\prod_{s=1}^{h}(\beta-1)^2\tau_sx_s|\lambda_s(\tau_s,x_s)|}{\left(\sum_{s=1}^{h}(\tau_s-F_s)(x_s-F_s)/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}}$$
$$\geq\prod_{s=1}^{h}(\upsilon_s-F_s)^{\frac{1}{\ell_s}}(\varepsilon_s-F_s)^{\frac{1}{\ell_s}}$$

$$\times \prod_{s=1}^{h}\left(\sum_{x_s=F_s}^{\varepsilon_s/\beta}\sum_{\tau_s=F_s}^{\upsilon_s/\beta}(\beta-1)^2\tau_s x_s(\upsilon_s-\beta\tau_s)(\varepsilon_s-\beta x_s)\left|\lambda_s^{\Delta_2\Delta_1}(\tau_s,x_s)\right|^{\gamma_s}\right)^{\frac{1}{\gamma_s}}.$$

Here the Δ_1-derivative of the function $\lambda(\tau,x)$ is the $\Delta-$derivative with respect to the first variable τ, and the Δ_2-derivative of the function $\lambda(\tau,x)$ is the $\Delta-$derivative with respect to the second variable x.

Theorem 3.3. *Let $F_s,\varepsilon_s\in\mathbb{T}$, $h_s\ge 1$, $0<\gamma_s<1$, and $\ell_s<0$ such that $1/\gamma_s+1/\ell_s=1$, and let $\lambda_s\in C_{rd}([F_s,\varepsilon_s]_{\mathbb{T}},\mathbb{R}^+)$ be a nonnegative and increasing function with $\lambda_s(F_s)=0$, for $s=1,2,...,h$. Then,*

$$\int_{F_1}^{\varepsilon_1}...\int_{F_h}^{\varepsilon_h}\frac{\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}\prod_{s=1}^{h}\lambda_s^{h_s}(x_s)}{\left(\sum_{s=1}^{h}(x_s-F_s)/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}}\Delta x_h...\Delta x_1$$
$$\ge \prod_{s=1}^{h}h_s(\varepsilon_s-F_s)^{\frac{1}{\ell_s}}\prod_{s=1}^{h}\left(\int_{F_s}^{\varepsilon_s}(\varepsilon_s-\sigma(x_s))\left([\lambda_s(x_s)]^{h_s-1}\lambda_s^{\Delta}(x_s)\right)^{\gamma_s}\Delta x_s\right)^{\frac{1}{\gamma_s}}. \tag{3.28}$$

Proof. Applying (2.1) on the term $\lambda_s^{h_s}(y_s)$, $h_s\ge 1$, we get

$$\left[\lambda_s^{h_s}(y_s)\right]^{\Delta}=h_s\lambda_s^{h_s-1}(c_s)\lambda_s^{\Delta}(y_s), \tag{3.29}$$

where $c_s\in[y_s,\sigma(y_s)]$. Since λ_s is an increasing function, $h_s\ge 1$, and $c_s\ge y_s$, from (3.29) we have

$$\left[\lambda_s^{h_s}(y_s)\right]^{\Delta}\ge h_s[\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s),$$

and then (where $\lambda_s(F_s)=0$), we observe that

$$h_s\int_{F_s}^{x_s}[\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\Delta y_s$$
$$\le \int_{F_s}^{x_s}\left[\lambda_s^{h_s}(y_s)\right]^{\Delta}\Delta y_s=\lambda_s^{h_s}(x_s)-\lambda_s^{h_s}(F_s)=\lambda_s^{h_s}(x_s).$$

Thus,

$$\prod_{s=1}^{h}h_s\int_{F_s}^{x_s}[\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\Delta y_s\le\prod_{s=1}^{h}\lambda_s^{h_s}(x_s). \tag{3.30}$$

Applying (2.3) on

$$\int_{F_s}^{x_s}[\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\Delta y_s,$$

with $\ell_s<0$, $f(y_s)=[\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)$ and $g(y_s)=1$, we have that

$$\int_{F_s}^{x_s}[\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\Delta y_s$$
$$\ge\left(\int_{F_s}^{x_s}\Delta y_s\right)^{\frac{1}{\ell_s}}\left(\int_{F_s}^{x_s}\left([\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\right)^{\gamma_s}\Delta y_s\right)^{\frac{1}{\gamma_s}}$$

$$= \quad (x_s - F_s)^{\frac{1}{\ell_s}} \left(\int_{F_s}^{x_s} \left([\lambda_s(y_s)]^{h_s-1} \lambda_s^{\Delta}(y_s) \right)^{\gamma_s} \Delta y_s \right)^{\frac{1}{\gamma_s}},$$

and then

$$\prod_{s=1}^{h} \int_{F_s}^{x_s} [\lambda_s(y_s)]^{h_s-1} \lambda_s^{\Delta}(y_s) \Delta y_s \geq \prod_{s=1}^{h} (x_s - F_s)^{\frac{1}{\ell_s}} \prod_{s=1}^{h} \left(\int_{F_s}^{x_s} \left([\lambda_s(y_s)]^{h_s-1} \lambda_s^{\Delta}(y_s) \right)^{\gamma_s} \Delta y_s \right)^{\frac{1}{\gamma_s}}. \tag{3.31}$$

Substituting (3.31) into (3.30), we get

$$\prod_{s=1}^{h} \lambda_s^{h_s}(x_s) \geq \prod_{s=1}^{h} h_s (x_s - F_s)^{\frac{1}{\ell_s}} \prod_{s=1}^{h} \left(\int_{F_s}^{x_s} \left([\lambda_s(y_s)]^{h_s-1} \lambda_s^{\Delta}(y_s) \right)^{\gamma_s} \Delta y_s \right)^{\frac{1}{\gamma_s}}. \tag{3.32}$$

Applying (2.10) with $\varrho_s = x_s - F_s$, we have that

$$\prod_{s=1}^{h} (x_s - F_s)^{1/\ell_s} \geq \frac{\left(\sum_{s=1}^{h} (x_s - F_s) / \ell_s \right)^{\sum_{s=1}^{h} 1/\ell_s}}{\left(h - \sum_{s=1}^{h} 1/\gamma_s \right)^{h - \sum_{s=1}^{h} 1/\gamma_s}}. \tag{3.33}$$

Substituting (3.33) into (3.32), we see that

$$\prod_{s=1}^{h} \lambda_s^{h_s}(x_s) \geq \frac{\left(\sum_{s=1}^{h} (x_s - F_s) / \ell_s \right)^{\sum_{s=1}^{h} 1/\ell_s}}{\left(h - \sum_{s=1}^{h} 1/\gamma_s \right)^{h - \sum_{s=1}^{h} 1/\gamma_s}} \prod_{s=1}^{h} h_s \left(\int_{F_s}^{x_s} \left([\lambda_s(y_s)]^{h_s-1} \lambda_s^{\Delta}(y_s) \right)^{\gamma_s} \Delta y_s \right)^{\frac{1}{\gamma_s}}. \tag{3.34}$$

Multiplying (3.34) on the term

$$\left(h - \sum_{s=1}^{h} 1/\gamma_s \right)^{h - \sum_{s=1}^{h} 1/\gamma_s} / \left(\sum_{s=1}^{h} (x_s - F_s) / \ell_s \right)^{\sum_{s=1}^{h} 1/\ell_s},$$

and integrating over x_s from F_s to ε_s, $s = 1, 2, ..., h$, we observe that

$$\begin{aligned} & \int_{F_1}^{\varepsilon_1} \cdots \int_{F_h}^{\varepsilon_h} \frac{\left(h - \sum_{s=1}^{h} 1/\gamma_s \right)^{h - \sum_{s=1}^{h} 1/\gamma_s} \prod_{s=1}^{h} \lambda_s^{h_s}(x_s)}{\left(\sum_{s=1}^{h} (x_s - F_s) / \ell_s \right)^{\sum_{s=1}^{h} 1/\ell_s}} \Delta x_h \ldots \Delta x_1 \\ \geq & \int_{F_1}^{\varepsilon_1} \cdots \int_{F_h}^{\varepsilon_h} \prod_{s=1}^{h} h_s \left(\int_{F_s}^{x_s} \left([\lambda_s(y_s)]^{h_s-1} \lambda_s^{\Delta}(y_s) \right)^{\gamma_s} \Delta y_s \right)^{\frac{1}{\gamma_s}} \Delta x_h \ldots \Delta x_1 \\ = & \prod_{s=1}^{h} h_s \int_{F_s}^{\varepsilon_s} \left(\int_{F_s}^{x_s} \left([\lambda_s(y_s)]^{h_s-1} \lambda_s^{\Delta}(y_s) \right)^{\gamma_s} \Delta y_s \right)^{\frac{1}{\gamma_s}} \Delta x_s. \end{aligned} \tag{3.35}$$

Applying (2.3) on the term

$$\int_{F_s}^{\varepsilon_s} \left(\int_{F_s}^{x_s} \left([\lambda_s(y_s)]^{h_s-1} \lambda_s^{\Delta}(y_s) \right)^{\gamma_s} \Delta y_s \right)^{\frac{1}{\gamma_s}} \Delta x_s,$$

with $\ell_s < 0$, $f(x_s) = 1$, and

$$g(x_s) = \left(\int_{F_s}^{x_s} \left([\lambda_s(y_s)]^{h_s-1} \lambda_s^{\Delta}(y_s) \right)^{\gamma_s} \Delta y_s \right)^{\frac{1}{\gamma_s}},$$

we have that

$$\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left([\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\right)^{\gamma_s}\Delta y_s\right)^{\frac{1}{\gamma_s}}\Delta x_s$$
$$\geq\quad(\varepsilon_s-F_s)^{\frac{1}{\ell_s}}\left(\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left([\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\right)^{\gamma_s}\Delta y_s\right)\Delta x_s\right)^{\frac{1}{\gamma_s}},$$

and then

$$\prod_{s=1}^{h}\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left([\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\right)^{\gamma_s}\Delta y_s\right)^{\frac{1}{\gamma_s}}\Delta x_s$$
$$\geq\prod_{s=1}^{h}(\varepsilon_s-F_s)^{\frac{1}{\ell_s}}\prod_{s=1}^{h}\left(\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left([\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\right)^{\gamma_s}\Delta y_s\right)\Delta x_s\right)^{\frac{1}{\gamma_s}}. \tag{3.36}$$

Substituting (3.36) into (3.35), we see that

$$\int_{F_1}^{\varepsilon_1}\cdots\int_{F_h}^{\varepsilon_h}\frac{\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}\prod_{s=1}^{h}\lambda_s^{h_s}(x_s)}{\left(\sum_{s=1}^{h}(x_s-F_s)/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}}\Delta x_h\ldots\Delta x_1$$
$$\geq\prod_{s=1}^{h}h_s(\varepsilon_s-F_s)^{\frac{1}{\ell_s}}\prod_{s=1}^{h}\left(\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left([\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\right)^{\gamma_s}\Delta y_s\right)\Delta x_s\right)^{\frac{1}{\gamma_s}}. \tag{3.37}$$

Applying (2.2) on

$$\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left([\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\right)^{\gamma_s}\Delta y_s\right)\Delta x_s,$$

with $f(x_s)=\int_{F_s}^{x_s}\left([\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\right)^{\gamma_s}\Delta y_s$ and $g^{\Delta}(x_s)=1$, we get

$$\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left([\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\right)^{\gamma_s}\Delta y_s\right)\Delta x_s$$
$$=\quad g(x_s)\left(\int_{F_s}^{x_s}\left([\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\right)^{\gamma_s}\Delta y_s\right)\Bigg|_{F_s}^{\varepsilon_s}-\int_{F_s}^{\varepsilon_s}g^{\sigma}(x_s)\left([\lambda_s(x_s)]^{h_s-1}\lambda_s^{\Delta}(x_s)\right)^{\gamma_s}\Delta x_s, \tag{3.38}$$

where $g(x_s)=x_s-\varepsilon_s$. Since $g(\varepsilon_s)=0$, from (3.38) we have

$$\int_{F_s}^{\varepsilon_s}\left(\int_{F_s}^{x_s}\left([\lambda_s(y_s)]^{h_s-1}\lambda_s^{\Delta}(y_s)\right)^{\gamma_s}\Delta y_s\right)\Delta x_s=\int_{F_s}^{\varepsilon_s}(\varepsilon_s-\sigma(x_s))\left([\lambda_s(x_s)]^{h_s-1}\lambda_s^{\Delta}(x_s)\right)^{\gamma_s}\Delta x_s. \tag{3.39}$$

Substituting (3.39) into (3.37), we observe that

$$\int_{F_1}^{\varepsilon_1}\cdots\int_{F_h}^{\varepsilon_h}\frac{\left(h-\sum_{s=1}^{h}1/\gamma_s\right)^{h-\sum_{s=1}^{h}1/\gamma_s}\prod_{s=1}^{h}\lambda_s^{h_s}(x_s)}{\left(\sum_{s=1}^{h}(x_s-F_s)/\ell_s\right)^{\sum_{s=1}^{h}1/\ell_s}}\Delta x_h\ldots\Delta x_1$$
$$\geq\prod_{s=1}^{h}h_s(\varepsilon_s-F_s)^{\frac{1}{\ell_s}}\prod_{s=1}^{h}\left(\int_{F_s}^{\varepsilon_s}(\varepsilon_s-\sigma(x_s))\left([\lambda_s(x_s)]^{h_s-1}\lambda_s^{\Delta}(x_s)\right)^{\gamma_s}\Delta x_s\right)^{\frac{1}{\gamma_s}},$$

which is the desired inequality (3.28), and this ends the proof. □

Remark 3.7. *If $\mathbb{T} = \mathbb{R}$, $\sigma(y) = y$ for any $y \in \mathbb{T}$, and $F_s = 0$ for $s = 1, 2, ..., h$, we have the reverse analog on delta calculus of (1.6) for the nonnegative increasing function λ with $\lambda_s(0) = 0$, $s = 1, 2, ..., h$.*

4. Conclusions

In this paper, we established some new reverse Hilbert-type inequalities on delta time scale calculus. To prove these inequalities, we needed to apply the dynamic reverse Hölder inequality. In addition, we presented some inequalities involving the first and second partial derivative on time scales and the special cases in continuous, discrete, and quantum calculus.

Author contributions

Ahmed I. Saied and Elkhateeb S. Aly formulated the main problem and derived the primary results. Said. Bourazza and Ahmed I. Saied presented the basic lemmas to prove the results. Sultanah Masmali contributed to the proofs and mathematical analysis. Elkhateeb S. Aly and Sultanah Masmali reviewed the literature and assisted in refining the manuscript. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declared that they have no conflicts of interest regarding the publication of this work.

References

1. B. G. Pachpatte, A note on Hilbert type inequality, *Tamkang J. Math.*, **29** (1998), 293–298.
2. B. G. Pachpatte, Inequalities similar to certain extensions of Hilbert's inequality, *J. Math. Anal. Appl.*, **243** (2000), 217–227. https://doi.org/10.1006/jmaa.1999.6646
3. Y. H. Kim, B. I. Kim, An analogue of Hilbert's inequality and its extensions, *Bull. Korean Math. Soc.*, **39** (2002), 377–388. https://doi.org/10.4134/BKMS.2002.39.3.377
4. C. J. Zhao, L. Y. Chen, W. S. Cheung, On some new Hilbert-type inequalities, *Math. Slovaca*, **61** (2011), 15–28. https://doi.org/10.2478/s12175-010-0056-0
5. M. Bohner, S. G. Georgiev, Multiple integration on time scales, In: *Multivariable dynamic calculus on time scales*, Springer, 2016, 449–515. https://doi.org/10.1007/978-3-319-47620-9_7

6. H. M. Rezk, G. AlNemer, A. I. Saied, O. Bazighifan, M. Zakarya, Some new generalizations of reverse Hilbert-type inequalities on time scales, *Symmetry*, **14** (2022), 750. https://doi.org/10.3390/sym14040750
7. M. Zakarya, G. AlNemer, A. I. Saied, R. Butush, O. Bazighifan, H. M. Rezk, Generalized inequalities of Hilbert-type on time scales nabla calculus, *Symmetry*, **14** (2022), 1512. https://doi.org/10.3390/sym14081512
8. M. Zakarya, A. I. Saied, G. AlNemer, H. M. Rezk, A study on some new reverse Hilbert-type inequalities and its generalizations on time scales, *J. Math.*, **2022** (2022), 6285367. https://doi.org/10.1155/2022/6285367
9. G. AlNemer, A. I. Saied, M. Zakarya, H. A. Abd El-Hamid, O. Bazighifan, H. M. Rezk, Some new reverse Hilbert's inequalities on time scales, *Symmetry*, **13** (2021), 2431. https://doi.org/10.3390/sym13122431
10. M. Bohner, A. Peterson, *Dynamic equations on time scales: an introduction with applications*, Birkhäuser, 2001.
11. R. P. Agarwal, D. O'Regan, S. H. Saker, *Dynamic inequalities on time scales*, Springer, 2014. https://doi.org/10.1007/978-3-319-11002-8
12. M. R. S. Ammi, D. F. Torres, Holder's and Hardy's two dimensional diamond-alpha inequalities on time scales, *arXiv*, 2009. https://doi.org/10.48550/arXiv.0910.4115
13. M. Bohner, G. S. Guseinov, Multiple integration on time scales, *Dyn. Syst. Appl.*, **14** (2005), 579–606.
14. G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge university press, 1952.