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*Research article*

## Optimal control of a class of semilinear fractional elliptic equations

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**Abstract:** In this paper, a class of semilinear fractional elliptic equations associated with the spectral or integral fractional Dirichlet Laplace operator was considered. Unlike most contributions to fractional optimal control, which assume that the control enters the state equation linearly, we addressed space-fractional equations in which the control appears in a nonlinear form. We proved existence and regularity results for the state equation and, using a measurable selection argument, established the existence of optimal controls. We then derived a pointwise Pontryagin-type minimum principle and first-order necessary optimality conditions. Second-order conditions for optimality are also obtained for  $L^\infty$ , and  $L^2$ -local solutions under some structural assumptions.

**Keywords:** nonlinear fractional equation; optimal solutions; Pontryagin principle; first- and second-order optimality conditions

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### 1. Introduction

Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 2$ ) be an open and bounded domain with boundary  $\partial\Omega$ . In this paper, we are interested in the following control problem. Find

$$\inf_{u \in \mathcal{U}} J(u) := \int_{\Omega} L(x, y(x), u(x)) dx, \tag{1.1}$$

subject to the constraints that  $y$  solves the semilinear fractional elliptic diffusion equation

$$\begin{cases} (-\Delta_D)^s y = F(x, y, u) & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

and the set of admissible controls is given by

$$\mathcal{U} := \{v \in L^\infty(\Omega) : \alpha \leq v(x) \leq \beta \text{ for a.e. } x \in \Omega\} \quad (1.3)$$

with  $-\infty < \alpha < \beta < +\infty$ . In (1.1),  $L : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a given function satisfying suitable conditions. In (1.2),  $y$  denotes the state,  $u$  is the control function,  $(-\Delta_D)^s$  denotes the spectral fractional Dirichlet Laplace operator of order  $0 < s < 1$ , that is, the fractional  $s$  powers of the realization in  $L^2(\Omega)$  of the Laplace operator  $-\Delta$  with a zero Dirichlet boundary condition on  $\partial\Omega$ , and  $F : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a given function satisfying suitable conditions. We notice that the boundary condition in (1.2) can be dropped since it is already included in the definition of the operator  $(-\Delta_D)^s$ . The precise assumptions on the functions  $L$  and  $F$  will be given in Section 2.2. We may also replace (1.2) with the system

$$\begin{cases} (-\Delta)^s y = F(x, y, u) & \text{in } \Omega, \\ y = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.4)$$

and our results remain valid. In (1.4),  $(-\Delta)^s$  ( $0 < s < 1$ ) denotes the integral fractional Laplace operator that we will introduce in the next section.

The main purpose of this paper is to solve the optimal control problems (1.1)–(1.2) and (1.1)–(1.4). The optimal control of nonlinear systems with control appearing nonlinearly in the state equation has been considered before by many researchers (see, e.g., [13–15, 22, 33] and their references). In [34], Raymond and Zidani addressed the optimal control of problems governed by semilinear parabolic equations. The controls appeared in a nonlinear form in the state equation. After some regularity results, they obtained the optimality conditions using the Hamiltonian Pontryagin principle. In [16], Casas and Tröltzsch considered the optimal control of a semilinear elliptic equation with Neumann boundary condition in space dimension  $N = 2$  or  $N = 3$ , where the control appears nonlinearly in the state equation. Let us also mention that their control was not explicitly included in the objective functional. Using a measurable selection technique, they proved the existence of optimal controls. They also derived the first- and second-order conditions for optimality. The case of second-order elliptic operators with a Dirichlet boundary condition was previously studied by Casas and Yong [13]. All the above-mentioned papers were dealing with integer-order operators, that is, classical second-order elliptic operators with standard boundary conditions (Dirichlet and Neumann).

In the case of fractional-order operators, only a few papers considered it, to the best of our knowledge. The authors in [37] considered a general linear evolution equation involving a fractional diffusion operator. They derived existence results as well as first-order necessary and second-order sufficient optimality conditions for a minimization problem on the order of the fractional operator. In [30], Kien et al. considered a fractional differential equation of order  $0 < \alpha < 1$  with control constraints. The control appeared in a nonlinear form in the state equation and they established the first- and second-order optimality conditions for local optimal solutions. A theory of no-gap second-order conditions was also obtained in the case where  $1/2 < \alpha < 1$ . Kamocki [29] considered an optimal control problem containing a control system described by a nonlinear partial differential equation with

the fractional Dirichlet-Laplacian associated with an integral cost. He proved by means of Filippov's theorem the existence of optimal solutions. The papers [5, 32] considered similar problems where the control appears linearly in the state equation and the underlying operator can be the fractional power of second-order uniformly elliptic operators with a Dirichlet boundary condition or the integral fractional Laplace operator with a zero Dirichlet exterior condition. The linear case with a non zero Dirichlet exterior condition was first investigated in [3].

In this paper, we consider the optimal control problems (1.1)–(1.2) and (1.1)–(1.4). The systems are nonlinear with respect to both the state  $y$  and the control  $u$ . Such problems are not only motivated by the beauty of mathematics. A rather incomplete list of problems, where fractional derivatives and fractional diffusion operators appear, includes mechanics [6], where they have been used to model viscoelastic behavior [20], turbulence [18], and the hereditary properties of materials [24], diffusion processes [1], in particular, processes in disordered media where the disorder may change the laws of Brownian motion and thus leads to anomalous diffusion [8], nonlocal electrostatics [28], finance [31], image processing [23], biophysics [11], chaotic dynamical systems, and many others [10]. Optimal control problems arise naturally in these applications (see also [2, 4, 5, 9, 21] and their references for more details). Optimal control problems, where their main objective are to minimize a cost functional subject to some state and/or control constraints and to characterize the associated optimality systems and conditions, have been intensively studied in the past and are still a main field of research of several researchers due to the numerous applications of optimization, optimal control, controllability, and stabilization. In mainstream papers of optimal control problems, the control appears linearly in the state equation and often in a quadratic Tikhonov regularization of the objective functional. In several real-world applications, the state equation is semilinear, quasilinear, or fully nonlinear and the control often appears nonlinearly that we consider here.

We prove some existence and regularity results for the state equations (1.2) and (1.4), and obtain the existence of optimal solutions to the control problems (1.1)–(1.2) and (1.1)–(1.4). The latter result is obtained by using a measurable selection theorem. Next, we derive the first-order optimality conditions and obtain a pointwise Pontryagin principle. Finally, we derive the second-order conditions for optimality. The main results and novelties of the present paper can be summarized as follows.

- Under suitable assumptions on the nonlinearly  $F$ , we have obtained some regularity results for all  $0 < s < 1$  (Theorem 3.3) of weak solutions to problem (1.2) or (1.4) that are crucial in the study of the existence of optimal solutions and the characterization of the optimality conditions for the control problems.
- We notice that we are dealing with non-convex optimization problems and the control appears nonlinearly in our state equations. Such control problems are, in general, very hard to solve. To the best of our knowledge in the fractional case, the only papers dealing with such problems are the recent papers [29, 30]. [30] considered fractional-in-time ordinary differential equation. In [29], the author studied the problem (1.1)–(1.2). With different assumptions on the linearities  $L$  and  $F$  and with different proofs, the author showed the existence of optimal solutions but the first- or second-order optimality conditions were not investigated. In fact [29], assumed that there exist  $A \geq 0$  and  $a \in L^2(\Omega)$ ,  $a \geq 0$ , such that  $|F(x, y, u)| \leq A|y| + a(x)$  for a.e  $x \in \Omega$  and all  $(y, u) \in \mathbb{R} \times \mathbb{R}$ .
- In the present paper, compared to [30], we have been able to deal with space fractional PDEs. More precisely, taking advantage of suitable kinds of convexity conditions on  $L$  and  $F$  (see (2.23)

and (2.24)) and a classical measurable selection theorem, we have been able to prove the existence of optimal solutions to our control problems for all  $0 < s < 1$  (Theorem 4.1). But we notice that, depending on  $0 < s \leq 1/2$  or  $1/2 < s < 1$ , the spectral fractional Laplacian requires different regularity assumptions (that are so far optimal) on the domain  $\Omega$ . Regarding the integral fractional Laplace operator, for all  $0 < s < 1$ , we have the same regularity assumption on  $\Omega$ . We observe that (2.23) or (2.24) does not mean that  $L$  or  $F$  is convex with respect to the third variable.

- In addition to the first-order necessary optimality conditions and a minimum principle of Pontryagin type obtained in Theorem 5.8, we have been able to obtain second-order necessary and sufficient conditions for local optimality in Theorems 6.1 and 6.8, respectively.

The rest of the paper is organized as follows. In Section 2, we introduce the function spaces and state the assumptions needed in the sequel. Section 3 deals with the existence and regularity results for the state equations (1.2) and (1.4), where the main results are obtained in Theorems 3.2 and 3.3. In Section 4, we prove the existence of optimal solutions (Theorem 4.1). Section 5 is devoted to the first-order necessary optimality conditions and the minimum principle of Pontryagin type that are given in Theorem 5.8. In Section 6, we derive the second-order necessary and sufficient conditions for local optimality.

## 2. Preliminary results and assumptions

For the sake of completeness, we give some well-known results that are used throughout the paper and we state the assumptions on the nonlinearities  $L$  and  $F$ . Throughout the paper,  $\Omega \subset \mathbb{R}^N$  is an arbitrary bounded open set. We shall specify if any regularity on  $\Omega$  is needed.

### 2.1. Preliminaries

Let  $\mathcal{D}(\Omega)$  be the space of test functions on  $\Omega$ , that is, the space of all infinitely continuously differentiable functions with compact support in  $\Omega$ . We let

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}$$

where

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \int_{\Omega} |\nabla u|^2 dx < \infty \right\}$$

is the first-order Sobolev space endowed with the norm

$$\|u\|_{H^1(\Omega)} = \left( \int_{\Omega} |u|^2 dx + \int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

Let  $-\Delta_D$  be the realization on  $L^2(\Omega)$  of the Laplace operator  $-\Delta$  with the zero Dirichlet boundary condition. That is,  $-\Delta_D$  is the positive and self-adjoint operator on  $L^2(\Omega)$  associated with the closed, bilinear, coercive, and symmetric form

$$\mathcal{A}_D(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx, \quad u, v \in H_0^1(\Omega),$$

in the sense that

$$\begin{cases} D(\Delta_D) = \{u \in H_0^1(\Omega) : \exists w \in L^2(\Omega), \mathcal{A}_D(u, v) = (w, v)_{L^2(\Omega)}, \forall v \in H_0^1(\Omega)\}, \\ -\Delta_D u = w. \end{cases} \quad (2.1)$$

For instance, if  $\Omega$  has a smooth boundary, say of class  $C^2$ , then  $D(\Delta_D) = H^2(\Omega) \cap H_0^1(\Omega)$ , where

$$H^2(\Omega) = \{u \in H^1(\Omega) : \partial_{x_j} u \in H^1(\Omega), j = 1, 2, \dots, N\}.$$

It is well-known that  $-\Delta_D$  has a compact resolvent and its eigenvalues form a non-decreasing sequence  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$  of real numbers satisfying  $\lim_{n \rightarrow \infty} \lambda_n = \infty$ . We denote by  $(\varphi_n)$  the orthonormal basis of eigenfunctions associated with the eigenvalues  $(\lambda_n)$ .

For  $0 < s < 1$ , we define the fractional-order Sobolev space

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\}$$

and we endow it with the norm given by

$$\|u\|_{H^s(\Omega)} = \left( \int_{\Omega} |u|^2 dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}.$$

We set

$$H_0^s(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^s(\Omega)},$$

and

$$H_{00}^{\frac{1}{2}}(\Omega) := \left\{ u \in H^{\frac{1}{2}}(\Omega) : \int_{\Omega} \frac{u^2(x)}{\text{dist}(x, \partial\Omega)} dx < \infty \right\}$$

where  $\text{dist}(x, \partial\Omega)$ ,  $x \in \Omega$ , denotes the distance from  $x$  to  $\partial\Omega$ .

We have that  $H_0^s(\Omega)$  is a Hilbert space and if  $1/2 < s < 1$ , then the norm on  $H_0^s(\Omega)$  is equivalent to the norm

$$\|w\|_{H_0^s(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{(w(x) - w(y))^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}. \quad (2.2)$$

Since  $\Omega$  is assumed to be bounded, we have the following continuous embedding:

$$H_0^s(\Omega) \hookrightarrow \begin{cases} L^{\frac{2N}{N-2s}}(\Omega) & \text{if } N > 2s, \\ L^p(\Omega), p \in [1, \infty) & \text{if } N = 2s, \\ C^{0, s - \frac{N}{2}}(\bar{\Omega}) & \text{if } N < 2s. \end{cases} \quad (2.3)$$

We also introduce the Hilbert space

$$\tilde{H}_0^s(\Omega) := \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$$

that we endow with the norm given by

$$\|u\|_{\tilde{H}_0^s(\Omega)}^2 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^s}{|x - y|^{N+2s}} dx dy.$$

Notice that the embedding (2.3) also holds with  $H_0^s(\Omega)$  replaced with  $\widetilde{H}_0^s(\Omega)$ . The precise relationship between the two spaces  $H_0^s(\Omega)$  and  $\widetilde{H}_0^s(\Omega)$  has been given in [19].

For any  $s \geq 0$ , we also introduce the fractional-order Sobolev space

$$\mathbb{H}^s(\Omega) := \left\{ u = \sum_{i=1}^{\infty} u_n \varphi_n \in L^2(\Omega) : \|u\|_{\mathbb{H}^s(\Omega)}^2 := \sum_{i=1}^{\infty} \lambda_n^s u_n^2 < \infty \right\},$$

where we recall that  $(\lambda_n)$  are the eigenvalues of  $-\Delta_D$  with associated normalized eigenfunctions  $(\varphi_n)$  and

$$u_n := (u, \varphi_n)_{L^2(\Omega)} = \int_{\Omega} u \varphi_n \, dx.$$

It is well-known that

$$\mathbb{H}^s(\Omega) = \begin{cases} H_0^s(\Omega) & \text{if } s \neq \frac{1}{2}, \\ H_{00}^{\frac{1}{2}}(\Omega) & \text{if } s = \frac{1}{2}. \end{cases} \quad (2.4)$$

It follows from (2.4) that the embedding (2.3) holds with  $H_0^s(\Omega)$  replaced by  $\mathbb{H}^s(\Omega)$ . For more details on fractional-order Sobolev spaces, we refer to [21, 25, 39] and their references.

**Definition 2.1.** The spectral fractional Dirichlet Laplacian of order  $s \geq 0$  is defined by

$$D((-\Delta_D)^s) = \mathbb{H}^s(\Omega), \quad (-\Delta_D)^s u = \sum_{n=1}^{\infty} \lambda_n^s u_n \varphi_n \quad \text{with } u_n = \int_{\Omega} u \varphi_n \, dx.$$

We notice that, in this case, we have that

$$\|u\|_{\mathbb{H}^s(\Omega)} = \|(-\Delta_D)^{\frac{s}{2}} u\|_{L^2(\Omega)}. \quad (2.5)$$

Note that  $\mathcal{D}(\Omega) \hookrightarrow \mathbb{H}^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow \mathbb{H}^{-s}(\Omega)$ , so, the operator  $(-\Delta_D)^s$  is unbounded, densely defined, and with bounded inverse  $(-\Delta_D)^{-s}$  in  $L^2(\Omega)$ . But it can also be viewed as a bounded operator from  $\mathbb{H}^s(\Omega)$  into its dual  $\mathbb{H}^{-s}(\Omega) := (\mathbb{H}^s(\Omega))^*$ .

Next, we consider the linear elliptic problem

$$(-\Delta_D)^s \varphi + b(x)\varphi = f, \quad (2.6)$$

where we assume that  $b \in L^\infty(\Omega)$  and is nonnegative.

**Definition 2.2.** A function  $\varphi \in \mathbb{H}^s(\Omega)$  is called a weak solution of (2.6) if the equality

$$\int_{\Omega} (-\Delta)^{s/2} \varphi (-\Delta)^{s/2} \phi \, dx + \int_{\Omega} b(x) \varphi \phi \, dx = \langle f, \phi \rangle_{\mathbb{H}^{-s}(\Omega), \mathbb{H}^s(\Omega)} \quad (2.7)$$

holds for every  $\phi \in \mathbb{H}^s(\Omega)$ .

We have the following result of existence and regularity of weak solutions. We refer to [4, 5, 36] and their references for the complete proof.

**Theorem 2.3.** For every  $f \in \mathbb{H}^{-s}(\Omega)$ , Eq (2.6) has a unique weak solution  $\varphi \in \mathbb{H}^s(\Omega)$  and there is a constant  $C = C(N, s, \Omega) > 0$  such that

$$\|\varphi\|_{L^2(\Omega)} \leq \|\varphi\|_{\mathbb{H}^s(\Omega)} \leq C\|f\|_{\mathbb{H}^{-s}(\Omega)}. \quad (2.8)$$

In addition, the following assertions hold.

(a) If  $N < 2s$ , then  $\varphi \in C^{0,s-\frac{N}{2}}(\bar{\Omega})$  and there is a constant  $C > 0$  such that

$$\|\varphi\|_{C^{0,s-\frac{N}{2}}(\bar{\Omega})} \leq C\|f\|_{\mathbb{H}^{-s}(\Omega)}. \quad (2.9)$$

(b) If  $N = 2s$ , then  $\varphi \in L^p(\Omega)$  for every  $p \in [1, \infty)$  and there is a constant  $C > 0$  such that

$$\|\varphi\|_{L^p(\Omega)} \leq C\|f\|_{\mathbb{H}^{-s}(\Omega)}. \quad (2.10)$$

(c) If  $f \in L^p(\Omega)$  for some  $p > N/2s$ , then  $\varphi \in L^\infty(\Omega)$  and there is a constant  $C > 0$  such that

$$\|\varphi\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \quad (2.11)$$

(d) If  $f \in L^r(\Omega)$ ,  $N > 2s$ , and  $1 \leq r < \frac{N}{N-2s}$ , then  $\varphi \in L^r(\Omega)$  and there is a constant  $C > 0$  such that

$$\|\varphi\|_{L^r(\Omega)} \leq C\|f\|_{L^1(\Omega)}. \quad (2.12)$$

*Proof.* The existence and uniqueness of solutions together with the estimate (2.8) is a simple application of the classical Lax-Milgram lemma. parts (a) and (b) follow from (2.8) and the continuous embedding (2.3). The proof of part (c) can be found in [4, 5] and the references therein. For Part (d) we refer to [36, Proposition 1.4].  $\square$

Next, we consider the integral fractional Laplacian formally given by

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where  $C_{N,s}$  is a normalization constant and P.V. stands for the Cauchy principal value.

Let  $\mathcal{E} : \tilde{H}_0^s(\Omega) \times \tilde{H}_0^s(\Omega) \rightarrow \mathbb{R}$  be the bilinear, closed, and coercive form given by

$$\mathcal{E}(u, v) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy, \quad u, v \in \tilde{H}_0^s(\Omega).$$

Then, the operator  $(-\Delta)_\Omega^s$  associated with  $\mathcal{E}$  in the sense that

$$\begin{cases} D((-\Delta)_\Omega^s) = \{u \in \tilde{H}_0^s(\Omega) : \exists f \in L^2(\Omega) : \mathcal{E}(u, v) = (f, v)_{L^2(\Omega)} \forall v \in \tilde{H}_0^s(\Omega)\}, \\ (-\Delta)_\Omega^s u = f \end{cases}$$

is the realization in  $L^2(\Omega)$  of  $(-\Delta)^s$  with the zero Dirichlet exterior condition  $u = 0$  in  $\mathbb{R}^N \setminus \Omega$ . We refer to [5, 19, 35] and their references for more details about this operator. With this definition, system (1.4) can be rewritten as

$$(-\Delta)_\Omega^s y = F(x, y, u) \quad \text{in } \Omega.$$

Next, we consider the following linear elliptic problem:

$$(-\Delta)_\Omega^s \varphi + b(x)\varphi = f, \quad (2.13)$$

where we assume that  $b \in L^\infty(\Omega)$  and is nonnegative.

**Definition 2.4.** A function  $\varphi \in \widetilde{H}_0^s(\Omega)$  is called a weak solution of (2.13) if the equality

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\varphi(x) - \varphi(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} dx dy + \int_{\Omega} b(x)\varphi\phi dx = \langle f, \phi \rangle_{\widetilde{H}^{-s}(\Omega), \widetilde{H}_0^s(\Omega)} \quad (2.14)$$

holds, for every  $\phi \in \widetilde{H}_0^s(\Omega)$ .

**Remark 2.5.** All the results obtained in Theorem 2.3 with  $\mathbb{H}^s(\Omega)$  and  $\mathbb{H}^{-s}(\Omega)$  replaced with  $\widetilde{H}_0^s(\Omega)$  and  $(\widetilde{H}_0^s(\Omega))^*$ , respectively, also hold true for problem (2.13). The proof is contained in the references given in the proof of Theorem 2.3.

We also have the following result of existence and regularity of weak solutions. We refer to [35] for the proof.

**Theorem 2.6.** Assume that  $\Omega$  is of class  $C^{1,1}$  and  $f \in L^\infty(\Omega)$ . Then the Dirichlet problem (2.13) has a unique weak solution  $\varphi \in \widetilde{H}_0^s(\Omega) \cap C^{0,s}(\mathbb{R}^N)$  and there is a constant  $C > 0$  such that

$$\|\varphi\|_{C^{0,s}(\overline{\Omega})} \leq C\|f\|_{L^\infty(\Omega)}. \quad (2.15)$$

## 2.2. General assumptions on the nonlinearities

We make the following assumptions on the function  $F$  involved in the state equations (1.2) and (1.4). However, they will not be used at the same time.

**Assumption 2.7.** The measurable function  $F : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function of class  $C^2$  with respect to the last two components and satisfies the following conditions.

- $F(\cdot, 0, 0) \in L^{\tilde{p}}(\Omega)$  for some  $\tilde{p} > N/2s$  and

$$\frac{\partial F}{\partial t}(x, t, \xi) \leq 0 \quad \text{for a.e. } x \in \Omega \quad \text{and for all } (t, \xi) \in \mathbb{R}^2. \quad (2.16)$$

- $\forall M > 0, \exists C_{F,M} > 0$  such that

$$\sum_{1 \leq i+j \leq 2} \left| \frac{\partial^{i+j} F}{\partial t^i \partial \xi^j}(x, t, u) \right| \leq C_{F,M} \quad (2.17)$$

for a.e.  $x \in \Omega$  and for all  $(t, \xi) \in \mathbb{R}^2$ , with  $|t| \leq M$  and  $\xi \in [\alpha, \beta]$ .

- $\forall \varepsilon > 0$  and  $M > 0, \exists \rho > 0$  such that, if  $|t_1|, |t_2| \leq M, |t_1 - t_2| < \rho, \xi_1, \xi_2 \in [\alpha, \beta]$  with  $|\xi_1 - \xi_2| < \rho$ , then

$$\sum_{i+j=2} \left| \frac{\partial^{i+j} F}{\partial t^i \partial \xi^j}(x, t_2, \xi_2) - \frac{\partial^{i+j} F}{\partial t^i \partial \xi^j}(x, t_1, \xi_1) \right| \leq \varepsilon \quad \text{for a.a. } x \in \Omega. \quad (2.18)$$

We conclude this section by giving the assumptions on the nonlinearity  $L$  involved in the functional  $J$  given in (1.1).

**Assumption 2.8.** The measurable function  $L : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is assumed to be a Carathéodory function of class  $C^2$  with respect to the last two components and satisfies the following conditions.

- $\frac{\partial L}{\partial t}(\cdot, 0, 0), \frac{\partial L}{\partial \xi}(\cdot, 0, 0) \in L^\infty(\Omega)$ .

- There exist a function  $\phi_0 \in L^1(\Omega)$  and nonnegative constants  $c_0, c_1, \beta_0, \beta_1$  such that

$$\phi_0(x) - c_0|t|^{\beta_0} - c_1|\xi|^{\beta_1} \leq L(x, t, \xi) \text{ for a.e. } x \in \Omega \text{ and all } (t, \xi) \in \mathbb{R}^2. \quad (2.19)$$

- $\forall M > 0, \exists C_{L,M} > 0$  such that

$$|L(x, t, \xi)| \leq C_{L,M} \quad (2.20)$$

and

$$\sum_{1 \leq i+j \leq 2} \left| \frac{\partial^{i+j} L}{\partial t^i \partial \xi^j}(x, t, \xi) \right| \leq C_{L,M} \quad (2.21)$$

for a.e.  $x \in \Omega$  and for all  $(t, \xi) \in \mathbb{R}^2$ , with  $|t| \leq M$  and  $\xi \in [\alpha, \beta]$ .

- $\forall \varepsilon > 0$  and  $M > 0, \exists \rho > 0$  such that, if  $|t_1|, |t_2| \leq M, |t_1 - t_2| < \rho, \xi_1, \xi_2 \in [\alpha, \beta]$  with  $|\xi_1 - \xi_2| < \rho$ , then

$$\sum_{i+j=2} \left| \frac{\partial^{i+j} L}{\partial t^i \partial \xi^j}(x, t_2, \xi_2) - \frac{\partial^{i+j} L}{\partial t^i \partial \xi^j}(x, t_1, \xi_1) \right| \leq \varepsilon \text{ for a.a. } x \in \Omega. \quad (2.22)$$

Finally, we give an assumption that both  $F$  and  $L$  must satisfy.

**Assumption 2.9.** For all  $(x, t) \in \Omega \times \mathbb{R}$ , the mappings  $F(x, t, \cdot)$  and  $L(x, t, \cdot)$  satisfy the following convexity condition. For any  $\xi, v \in [\alpha, \beta]$  and  $\lambda \in [0, 1]$ , there exists  $w \in [\alpha, \beta]$  such that

$$F(x, t, w) = \lambda F(x, t, \xi) + (1 - \lambda)F(x, t, v) \quad (2.23)$$

and

$$L(x, t, w) \leq \lambda L(x, t, \xi) + (1 - \lambda)L(x, t, v). \quad (2.24)$$

**Remark 2.10.** We observe the following.

- The first two conditions in Assumptions 2.7 ((2.16) and (2.17)) are essentially used to obtain the existence and enough regularity of solutions to the state equation (1.2).
- The other conditions in Assumptions 2.7 and all the conditions in Assumptions 2.8 are used to prove the existence of optimal solutions of the control problem and to obtain the first- and second-order optimality conditions.

**Remark 2.11.** We notice that the kind of convexity conditions (2.23) and (2.24) are important for the existence of optimal solutions of our control problem for the case where  $F$  is nonlinear with respect to the control  $u$ . These conditions were used to establish optimality conditions for mathematical programming problems (see, e.g., [27, Theorem 3]). Another variant was also used in [17, Theorem 9.3.i] to prove the existence of optimal solutions to optimal control problems governed by ordinary differential equations.

**Example 2.12.** We have the following.

- (1) All the conditions in Assumption 2.8 are obviously satisfied by the classical tracking cost functional

$$L(x, y, u) = \frac{1}{2}(y(x) - y_d(x))^2 + \frac{1}{2}u^2(x) \text{ provided that } y_d \in L^2(\Omega).$$

(2) All the conditions in Assumption 2.7 are obviously satisfied by the function

$$F(x, y, u) = e^{-y(x)}u^2(x) + e^{2u(x)} \quad \text{or} \quad F(x, y, u) = e^{-y(x)}u^2(x) + u(x).$$

(3) Let  $b : \Omega \rightarrow (0, \infty)$  be a function in  $L^\infty(\Omega)$ ,  $p$  an odd integer, and  $0 \leq \alpha < \beta$ . Then, all the conditions in Assumption 2.7 are satisfied by the function

$$F(x, y, u) = -b(x)y^p(x) + u^3(x).$$

(4) Next, we give some examples of functions  $L$  and  $F$  satisfying (2.23) and (2.24).

- If  $L(x, t, u)$  is convex with respect to  $u$ , and  $F(x, t, u) = A(x, t) + B(x, t)u$ , then for any  $u, v \in \mathcal{U}$  and  $\lambda \in [0, 1]$ , we may choose  $w = \lambda u + (1 - \lambda)v$ .
- If  $L(x, t, u) = \varphi(x, t) + u^2$ ,  $F(x, t, u) = A(x, t) + u^2$ , and  $0 \leq \alpha < \beta$ , then for any  $u, v \in \mathcal{U}$  and  $\lambda \in [0, 1]$ , we may choose  $w = \sqrt{\lambda u^2 + (1 - \lambda)v^2}$ .
- If  $L(x, t, u) = \varphi(x, t) + u^3$ ,  $F(x, t, u) = A(x, t) + u^3$ , and  $0 \leq \alpha < \beta$ , then for any  $u, v \in \mathcal{U}$  and  $\lambda \in [0, 1]$ , we may choose  $w = (\lambda u^3 + (1 - \lambda)v^3)^{1/3}$ .

In addition, for smooth  $A$ ,  $B$ , and  $\varphi$ , all the conditions in Assumptions 2.7 and 2.8 are satisfied.

### 3. Existence and regularity of solutions to the state equation

From now on, we simplify the notations by setting

$$\mathbb{V} := \mathbb{H}^s(\Omega) \quad \text{and} \quad \mathbb{V}^* := \mathbb{H}^{-s}(\Omega), \quad (3.1)$$

and we let  $\langle \cdot, \cdot \rangle_{\mathbb{V}^*, \mathbb{V}}$  denote the duality mapping between  $\mathbb{V}^*$  and  $\mathbb{V}$ .

**Definition 3.1.** Let  $u \in L^\infty(\Omega)$ .

(a) We say that  $y \in \mathbb{V}$  is a weak solution of (1.2), if the equality

$$\mathcal{F}(y, \phi) := \int_{\Omega} (-\Delta_D)^{\frac{s}{2}} y (-\Delta_D)^{\frac{s}{2}} \phi \, dx = \int_{\Omega} F(x, y, u) \phi \, dx \quad (3.2)$$

holds, for every  $\phi \in \mathbb{V}$ .

(b) We say that  $y \in \widetilde{H}_0^s(\Omega)$  is a weak solution of (1.4), if the equality

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(y(x) - y(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} \, dx dy = \int_{\Omega} F(x, y, u) \phi \, dx \quad (3.3)$$

holds, for every  $\phi \in \widetilde{H}_0^s(\Omega)$ .

We have the following existence and regularity result.

**Theorem 3.2.** Let  $s \in (0, 1)$ ,  $\tilde{p} > N/2s$ , and suppose that (2.16) and (2.17) hold. Then the following assertions hold.

(a) For every  $u \in L^\infty(\Omega)$ , Eq (1.2) has a unique weak solution  $y \in \mathbb{V} \cap L^\infty(\Omega)$  and there is a constant  $C := C(N, s, \Omega, \tilde{p}) > 0$  such that

$$\|y\|_{\mathbb{V}} + \|y\|_{L^\infty(\Omega)} \leq C (\|F(\cdot, 0, 0)\|_{L^{\tilde{p}}(\Omega)} + \|u\|_{L^\infty(\Omega)}). \quad (3.4)$$

Moreover, there is a constant  $C := C(N, s, \Omega, \tilde{p}, \alpha, \beta) > 0$  such that

$$\|y\|_{\mathbb{V}} + \|y\|_{L^\infty(\Omega)} \leq C (\|F(\cdot, 0, 0)\|_{L^{\tilde{p}}(\Omega)} + 1), \quad \forall u \in \mathcal{U}. \quad (3.5)$$

(b) For every  $u \in L^\infty(\Omega)$ , Eq (1.4) has a unique weak solution  $y \in \tilde{H}_0^s(\Omega) \cap L^\infty(\Omega)$ . In addition, the estimates (3.4) and (3.5) hold with  $\mathbb{V}$  replaced with  $\tilde{H}_0^s(\Omega)$ .

*Proof.* Let  $u \in L^\infty(\Omega)$ . We claim that  $F(\cdot, 0, u) \in L^{\tilde{p}}(\Omega)$ . Using the mean value theorem, we write

$$F(x, 0, u(x)) = F(x, 0, 0) + \frac{\partial F}{\partial u}(x, 0, \epsilon(x)u(x))u(x),$$

where  $\epsilon : \Omega \rightarrow [0, 1]$  is a measurable function. Therefore, using (2.17) with  $M = 1$ , we deduce that

$$\begin{aligned} |F(x, 0, u(x))| &\leq |F(x, 0, 0)| + \left| \frac{\partial F}{\partial u}(x, 0, \epsilon(x)u(x)) \right| |u(x)| \\ &\leq |F(x, 0, 0)| + C_{F,1} \|u\|_{L^\infty(\Omega)}. \end{aligned}$$

Hence,

$$\|F(\cdot, 0, u)\|_{L^{\tilde{p}}(\Omega)} \leq C(N, s, \Omega, \tilde{p}) (\|F(\cdot, 0, 0)\|_{L^{\tilde{p}}(\Omega)} + \|u\|_{L^\infty(\Omega)}), \quad (3.6)$$

and so the claim is proved.

(a) Using similar arguments as in [5, Section 3] or [32, Theorem 3.1], we can deduce that (1.2) has a unique weak solution  $y \in \mathbb{V} \cap L^\infty(\Omega)$ . In addition, the following estimate holds:

$$\|y\|_{\mathbb{V}} + \|y\|_{L^\infty(\Omega)} \leq C \|F(\cdot, 0, u)\|_{L^{\tilde{p}}(\Omega)}. \quad (3.7)$$

Combining (3.6)–(3.7) we get (3.4). If  $u \in \mathcal{U}$ , we can also deduce (3.5).

(b) The proof of this part follows similarly as in part (a) by making the necessary modifications and using the results obtained in [5]. We omit the details for the sake of brevity.  $\square$

Next, we have the following regularity result for weak solutions to (1.2) and (1.4).

**Theorem 3.3.** *Let  $s \in (0, 1)$  and  $u \in L^\infty(\Omega)$ . Let Assumption 2.7 hold. Then, the following assertions hold:*

(a) *Let  $\Omega$  be of class  $C^1$ . If  $\frac{1}{2} < s < 1$  and  $\frac{N}{2s} < \tilde{p} < \frac{N}{2s-1}$ , or if  $0 < s \leq \frac{1}{2}$  and  $\tilde{p} < \infty$ , then the weak solution  $y$  of (1.2) belongs to  $C^{0,\sigma}(\bar{\Omega})$ , where  $\sigma = 2s - \frac{N}{\tilde{p}}$ .*

(b) *Let  $s > \frac{1}{2}$  and  $\tilde{p} > \frac{N}{2s-1}$ . If  $\Omega$  is of class  $C^{1,\sigma}$  for  $0 < \sigma := 2s - \frac{N}{\tilde{p}} - 1 < 1$ , then the weak solution  $y$  of (1.2) belongs to  $C^{0,\sigma}(\bar{\Omega})$ .*

(c) *In both cases (a) and (b), there is a constant  $C := C(N, s, \Omega) > 0$  such that*

$$\|y\|_{C^{0,\sigma}(\bar{\Omega})} \leq C (\|y\|_{\mathbb{V}} + \|F(\cdot, y(\cdot), u(\cdot))\|_{L^{\tilde{p}}(\Omega)}). \quad (3.8)$$

(d) Let  $\Omega$  be of class  $C^{1,1}$ . If  $\tilde{p} = \infty$ , then the weak solution  $y$  of (1.4) belongs to  $C^{0,s}(\bar{\Omega})$  and the estimate (3.8) holds with  $\sigma = s$  and  $\mathbb{V}$  replaced with  $\tilde{H}_0^s(\Omega)$ .

*Proof.* We can write system (1.2) as

$$(-\Delta_D)^s y = F(x, y, u) \quad \text{in } \Omega. \quad (3.9)$$

Let  $y \in \mathbb{V} \cap L^\infty(\Omega)$  be the weak solution of (1.2), equivalently, of (3.9). According to [4, 26], it suffices to show that  $F(\cdot, y(\cdot), u(\cdot)) \in L^{\tilde{p}}(\Omega)$ . Indeed, using (2.17) and (3.4), we have, from the mean value theorem that there is a measurable function  $\epsilon : \Omega \rightarrow [0, 1]$  such that

$$\begin{aligned} |F(x, y(x), u(x))| &\leq |F(x, 0, u(x))| + \left| \frac{\partial F}{\partial y}(x, \epsilon(x)y(x), u(x)) \right| |y(x)| \\ &\leq |F(x, 0, u(x))| + \left| \frac{\partial F}{\partial y}(x, \epsilon(x)y(x), u(x)) \right| \|y\|_{L^\infty(\Omega)} \\ &\leq |F(x, 0, u(x))| + C_{F,M} M, \end{aligned} \quad (3.10)$$

where  $M = C(N, s, \Omega, p) (\|F(\cdot, 0, 0)\|_{L^{\tilde{p}}(\Omega)} + \|u\|_{L^\infty(\Omega)}) > 0$ . Hence, using (3.6) and (3.10), we can deduce that

$$\|F(\cdot, y(\cdot), u(\cdot))\|_{L^{\tilde{p}}(\Omega)} \leq 2C_{F,M} M. \quad (3.11)$$

Therefore,  $F(\cdot, y(\cdot), u(\cdot)) \in L^{\tilde{p}}(\Omega)$ . Applying the results of [4, 12, 26], we can deduce that parts (a) and (b) hold. It follows from [12] that

$$\|y\|_{C^{0,\sigma}(\bar{\Omega})} \leq C \left( \|y\|_{L^2(\Omega)} + \|y\|_{\mathbb{V}} + \|F(\cdot, y(\cdot), u(\cdot))\|_{L^{\tilde{p}}(\Omega)} \right). \quad (3.12)$$

Thanks to (2.3), we have the continuous embedding  $\mathbb{V} \hookrightarrow L^2(\Omega)$ . Hence,

$$\|y\|_{L^2(\Omega)} \leq C(N, s, \Omega) \|y\|_{\mathbb{V}}. \quad (3.13)$$

Combining (3.12)–(3.13) leads us to (3.8).

The proof of part (d) follows similarly by using Theorem 2.6.  $\square$

**Remark 3.4.** We observe the following facts that will be useful.

- (a) Let  $s > \frac{1}{2}$ ,  $\tilde{p} > \frac{N}{2s-1}$ , and  $\Omega$  be of class  $C^{1,\sigma}$  for  $0 < \sigma := 2s - \frac{N}{\tilde{p}} - 1 < 1$ . In the case where  $0 < s \leq 1/2$ , assume that  $\tilde{p} < \infty$ ,  $\sigma = 2s - N/\tilde{p}$ , and  $\Omega$  is of class  $C^1$ . Then from (3.4), (3.5), and (3.8), we can deduce the following estimates for the solution  $y$  of equation (1.2):

$$\|y\|_{\mathbb{V}} + \|y\|_{C^{0,\sigma}(\bar{\Omega})} \leq C (\|F(\cdot, 0, 0)\|_{L^{\tilde{p}}(\Omega)} + \|u\|_{L^\infty(\Omega)}) \quad \forall u \in L^\infty(\Omega) \quad (3.14)$$

where  $C := C(N, s, \Omega, \tilde{p}) > 0$ , and

$$\|y\|_{\mathbb{V}} + \|y\|_{C^{0,\sigma}(\bar{\Omega})} \leq C (\|F(\cdot, 0, 0)\|_{L^{\tilde{p}}(\Omega)} + 1) \quad \forall u \in \mathcal{U} \quad (3.15)$$

where  $C := C(N, s, \Omega, \tilde{p}, \alpha, \beta) > 0$ .

- (b) Let  $0 < s < 1$ ,  $\tilde{p} = \infty$ , and assume that  $\Omega$  is of class  $C^{1,1}$ . Then the solution  $y$  of (1.4) satisfies the estimates (3.14) and (3.15) with  $\sigma = s$  and  $\mathbb{V}$  replaced with  $\tilde{H}_0^s(\Omega)$ .

#### 4. Existence of optimal solutions

In this section, we are concerned with the existence of optimal solutions to the control problems (1.1)–(1.2) and (1.1)–(1.4). All the statements and proofs will be given for problem (1.1)–(1.2) but all our results remain true for the problem (1.1)–(1.4) by assuming that  $\Omega$  is of class  $C^{1,1}$ ,  $0 < s < 1$ ,  $\tilde{p} = \infty$ , and  $\sigma = s$ . The theorem on measurable selections of a measurable set-valued mapping (see, e.g., [7]) will play a crucial role.

**Theorem 4.1.** *Let  $s > \frac{1}{2}$ ,  $\tilde{p} > \frac{N}{2s-1}$ , and  $\Omega$  be of class  $C^{1,\sigma}$  for  $0 < \sigma = 2s - \frac{N}{\tilde{p}} - 1 < 1$ . In the case where  $0 < s \leq 1/2$ , assume that  $\tilde{p} < \infty$ ,  $\sigma = 2s - N/\tilde{p}$  and  $\Omega$  is of class  $C^1$ . Let Assumptions 2.7 and 2.8 hold. Then, there exists at least one solution  $(\bar{u}, \bar{y}) \in \mathcal{U} \times (\mathbb{V} \cap C(\bar{\Omega}))$  of the minimization problem (1.1)–(1.2).*

*Proof.* We first note that, from (2.19) and (2.20),  $j := \inf_{v \in \mathcal{U}} J(v)$  satisfies  $-\infty < j < +\infty$ . Let  $(v_k)_{k \geq 1} \subset \mathcal{U}$  be a minimizing sequence such that

$$\lim_{k \rightarrow \infty} J(v_k) = j.$$

Since  $\mathcal{U}$  is bounded in  $L^\infty(\Omega)$ , we have that the sequence  $(v_k)_{k \geq 1}$  is bounded. Since  $y_k := y(v_k)$  is the state associated with the control  $v_k$ , it follows from (3.2) that the equality

$$\int_{\Omega} (-\Delta_D)^{\frac{s}{2}} y_k (-\Delta_D)^{\frac{s}{2}} \phi \, dx = \int_{\Omega} F(x, y_k, v_k) \phi \, dx \quad (4.1)$$

holds, for every  $\phi \in \mathbb{V}$ . Moreover, it follows from (3.15) that there is a constant  $M_1 > 0$  independent of  $k$  such that

$$\|y_k\|_{\mathbb{V}} + \|y_k\|_{C^{0,\sigma}(\bar{\Omega})} \leq M_1. \quad (4.2)$$

From (4.2) and thanks to the compact embedding  $C^{0,\sigma}(\bar{\Omega}) \hookrightarrow C(\bar{\Omega})$ , there exists  $y \in \mathbb{V} \cap C^{0,\sigma}(\bar{\Omega})$  such that (up to a subsequence if necessary), as  $k \rightarrow \infty$ ,

$$y_k \rightharpoonup \bar{y} \text{ weakly in } \mathbb{V} \quad (4.3)$$

and

$$y_k \rightarrow \bar{y} \text{ strongly in } C(\bar{\Omega}). \quad (4.4)$$

We also have that

$$\|\bar{y}\|_{\mathbb{V}} + \|\bar{y}\|_{C(\bar{\Omega})} \leq M_1. \quad (4.5)$$

From (2.20) and using (4.2), we have that there exists  $C_{L,M_1} > 0$  such that

$$|L(x, y_k(x), v_k(x))| \leq C_{L,M_1} := M_2 \text{ for a.e. } x \in \Omega \text{ and for all } k \geq 1. \quad (4.6)$$

Using (3.11), we can deduce the existence of a constant  $M_3 > 0$  such that

$$\|F(\cdot, y_k(\cdot), v_k(\cdot))\|_{L^{\tilde{p}}(\Omega)} \leq M_3 \text{ for all } k \geq 1. \quad (4.7)$$

Hence, up to a subsequence, as  $k \rightarrow \infty$ ,  $L_k := L(\cdot, y_k(\cdot), v_k(\cdot))$  converges weak- $\star$  to  $\bar{L} \in L^\infty(\Omega)$  and  $F_k := F(\cdot, y_k(\cdot), v_k(\cdot))$  converges weakly to  $\bar{F} \in L^{\tilde{p}}(\Omega)$ . We note that  $(L^{\tilde{p}}(\Omega))^* = L^{\frac{\tilde{p}}{\tilde{p}-1}}(\Omega)$  and since  $\tilde{p} > \frac{N}{2s}$ , we can deduce that  $L^{\frac{2N}{N-2s}}(\Omega) \hookrightarrow L^{\frac{\tilde{p}}{\tilde{p}-1}}(\Omega)$ . Therefore, using (2.3), we obtain that

$$\lim_{k \rightarrow \infty} \int_{\Omega} F_k(x) \varphi(x) \, dx = \int_{\Omega} \bar{F}(x) \varphi(x) \, dx \text{ for all } \varphi \in \mathbb{V}. \quad (4.8)$$

Moreover, we also have that

$$\lim_{k \rightarrow \infty} \int_{\Omega} L_k(x) \psi(x) dx = \int_{\Omega} \bar{L}(x) \psi(x) dx \quad \text{for all } \psi \in L^2(\Omega). \quad (4.9)$$

Next, we define the set-valued map  $Q : \Omega \times [-M_1, M_1] \rightarrow \mathbb{R}^2$  by

$$Q(x, y) := \{(z, t) | M_2 \geq z \geq L(x, y, u), t = F(x, y, u) \text{ for some } u \in [\alpha, \beta]\}, \quad (4.10)$$

where  $M_2$  is given in (4.6). Then,  $Q(x, y)$  is nonempty, compact, and convex in  $\mathbb{R}^2$ . This follows from (4.6), the compactness of  $[\alpha, \beta]$ , the continuity of  $F$ , (2.23), and (2.24). In addition, the set-valued map  $Q$  is a Carathéodory multifunction (see [30, Lemma 3.1]). On the other hand, we also introduce the set-valued map  $\bar{Q} : \Omega \rightarrow \mathbb{R}^2$  defined by

$$\bar{Q}(x) := \{(z, t) | M_2 \geq z \geq L(x, \bar{y}(x), u), t = F(x, \bar{y}(x), u) \text{ for some } u \in [\alpha, \beta]\}, \quad (4.11)$$

where  $M_2$  is given in (4.6). Then, one can show as in [30, Lemma 3.2] that for a.e.  $x \in \Omega$ ,  $(\bar{L}(x), \bar{F}(x)) \in \bar{Q}(x)$ . Therefore, for a.e.  $x \in \Omega$ , there exists  $u \in [\alpha, \beta]$  such that

$$M_2 \geq \bar{L}(x) \geq L(x, \bar{y}(x), u), \quad \bar{F}(x) = F(x, \bar{y}(x), u). \quad (4.12)$$

Let  $\mathcal{P}(\mathbb{R})$  be the set of all subsets of  $\mathbb{R}$ . Define the multi-functions

$$\mathcal{H}_1 : \Omega \rightarrow \mathcal{P}(\mathbb{R}), \quad \mathcal{H}_1(x) = (-\infty, \bar{L}(x)] \times \{\bar{F}(x)\},$$

and

$$\mathcal{H}_2 : \Omega \rightarrow \mathcal{P}([\alpha, \beta]),$$

with

$$\mathcal{H}_2(x) = \{u \in [\alpha, \beta] : (L(x, \bar{y}(x), u), F(x, \bar{y}(x), u)) \in \mathcal{H}_1(x)\}.$$

Since  $L$  and  $F$  are continuous with respect to the last component, we have that for a.e.  $x \in \Omega$ ,  $\mathcal{H}_2(x)$  has a closed image. On the other hand, since  $\bar{L}$  and  $\bar{F}$  are measurable functions, we can deduce that  $\mathcal{H}_2$  is a measurable set-valued map. Using the measurable selection theorem [7, Theorem 8.1.3], we can deduce that there exists a measurable function  $\bar{u}$ , such that  $\bar{u}(x) \in \mathcal{H}_2(x)$  for a.e.  $x \in \Omega$ . Hence,  $\bar{u}(x) \in [\alpha, \beta]$  for a.e.  $x \in \Omega$  and

$$\bar{L}(x) \geq L(x, \bar{y}(x), \bar{u}(x)), \quad \bar{F}(x) = F(x, \bar{y}(x), \bar{u}(x)). \quad (4.13)$$

Taking the limit as  $k \rightarrow \infty$  in (4.1), while using (4.3), (4.8), and (4.13), we obtain that

$$\int_{\Omega} (-\Delta_D)^{\frac{s}{2}} \bar{y} (-\Delta_D)^{\frac{s}{2}} \phi dx = \int_{\Omega} F(x, \bar{y}, \bar{u}) \phi dx, \quad \forall \phi \in \mathbb{V}. \quad (4.14)$$

On the other hand, using (4.9) and (4.13), we can deduce that

$$\begin{aligned} j &= \lim_{k \rightarrow \infty} J(v_k) = \liminf_{k \rightarrow \infty} \int_{\Omega} L(x, y_k(x), v_k(x)) dx \\ &= \int_{\Omega} \bar{L}(x) dx \geq \int_{\Omega} L(x, \bar{y}(x), \bar{u}(x)) dx = J(\bar{u}). \end{aligned}$$

Therefore,  $(\bar{u}, \bar{y})$  is an optimal solution to the control problem (1.1)–(1.2). This completes the proof.  $\square$

We conclude this section with the following observation.

**Remark 4.2.** In Theorem 4.1, we only proved the existence of optimal solutions. Since the functional  $J$  is non-convex, in general, we cannot expect a unique solution to the minimization problem (1.1)–(1.2).

## 5. First-order necessary optimality conditions

Recall that by assumption,  $0 < s < 1$  and  $\tilde{p} > N/2s$ . In the remainder of the paper, if  $s > \frac{1}{2}$ , we assume that  $\tilde{p} > \frac{N}{2s-1}$ , and  $\Omega$  is of class  $C^{1,\sigma}$  for  $0 < \sigma = 2s - \frac{N}{\tilde{p}} - 1 < 1$ . In the case where  $0 < s \leq 1/2$ , we assume that  $\tilde{p} < \infty$ ,  $\sigma = 2s - N/\tilde{p}$ , and  $\Omega$  is of class  $C^1$ . In the case of the integral fractional Laplace operator, we assume that  $\tilde{p} = \infty$ ,  $\sigma = s$ , and  $\Omega$  is of class  $C^{1,1}$ . Then all our results also hold for the integral fractional Laplace operator, that is, if one replaces (1.2) with (1.4). But as in the previous sections, most of the statements and all the proofs will be given for the spectral fractional Laplacian.

Let us set  $\mathbb{Y} := \mathbb{V} \cap C^{0,\sigma}(\bar{\Omega})$  and define the control-to-state mapping

$$G : L^\infty(\Omega) \rightarrow \mathbb{Y}, \quad u \mapsto G(u) = y \quad (5.1)$$

which associates to each  $u \in L^\infty(\Omega)$  the unique weak solution  $y$  of (1.2).

The aim of this section is to derive the first-order necessary optimality conditions for the control problems (1.1)–(1.2) and (1.1)–(1.4), and to characterize the optimal control. But before going further, we need some regularity results for the control-to-state operator  $G$  given in (5.1). Let us introduce the vector space

$$\mathbb{X} := \left\{ y \in \mathbb{Y} : (-\Delta_D)^s y \in L^{\tilde{p}}(\Omega) \right\}. \quad (5.2)$$

Then,  $\mathbb{X}$  endowed with the graph norm

$$\|y\|_{\mathbb{X}} := \|y\|_{\mathbb{Y}} + \|(-\Delta_D)^s y\|_{L^{\tilde{p}}(\Omega)}, \quad y \in \mathbb{X}, \quad (5.3)$$

is a Banach space. Next, let us define the mapping

$$\mathcal{G} : \mathbb{X} \times L^\infty(\Omega) \rightarrow L^{\tilde{p}}(\Omega), \quad (y, u) \mapsto \mathcal{G}(y, u) := (-\Delta_D)^s y - F(x, y, u). \quad (5.4)$$

Then,  $\mathcal{G}$  is well defined. Indeed, from (3.11), we have that  $F(\cdot, y(\cdot), u(\cdot)) \in L^{\tilde{p}}(\Omega)$  for all  $(y, u) \in \mathbb{X} \times L^\infty(\Omega)$ . In addition, the state equation (1.2) can be viewed as  $\mathcal{G}(y, u) = 0$ . We have the following result.

**Lemma 5.1.** *The mapping  $\mathcal{G}$  defined in (5.4) is of class  $C^2$ .*

*Proof.* The first term of  $\mathcal{G}$  is linear and, from the definition of the norm (5.3), it is continuous from  $\mathbb{X}$  to  $L^{\tilde{p}}(\Omega)$ . Therefore, it is of class  $C^2$ . Assumption 2.7 gives that the second term is of class  $C^2$  from  $\mathbb{X}$  to  $L^{\tilde{p}}(\Omega)$ . This completes the proof.  $\square$

**Lemma 5.2.** *The mapping  $G$  given in (5.1) is of class  $C^2$ . In addition, under Assumption 2.7, the first and second directional derivatives of  $G$  are given by  $G'(u)v = z$  and  $\rho = G''(u)(v, w)$ , where  $u, v, w \in L^\infty(\Omega)$  and  $z, \rho \in \mathbb{V}$  are the unique weak solutions of*

$$(-\Delta_D)^s z = \frac{\partial F}{\partial y}(x, y, u)z + \frac{\partial F}{\partial u}(x, y, u)v \quad \text{in } \Omega, \quad (5.5)$$

and

$$(-\Delta_D)^s \rho = \frac{\partial F}{\partial y}(x, y, u)\rho + \frac{\partial^2 F}{\partial y^2}(x, y, u)G'(u)vG'(u)w$$

$$+ \frac{\partial^2 F}{\partial y \partial u}(x, y, u)(wG'(u)v + vG'(u)w) + \frac{\partial^2 F}{\partial u^2}(x, y, u)vw \text{ in } \Omega, \quad (5.6)$$

respectively. Moreover, for every  $u \in L^\infty(\Omega)$ , the linear mapping  $v \mapsto G'(u)v$  can be extended to a linear continuous mapping from  $L^2(\Omega) \rightarrow \mathbb{V}$ . More precisely, letting  $M = \|y\|_{C(\bar{\Omega})}$ , there are two constants  $C_{F,M} > 0$  and  $C = C(N, s, \Omega) > 0$  such that

$$\|z\|_{\mathbb{V}} = \|G'(u)v\|_{\mathbb{V}} \leq C_{F,M}C(N, s, \Omega)\|v\|_{L^2(\Omega)}. \quad (5.7)$$

*Proof.* Let  $u \in L^\infty(\Omega)$ . It follows from Lemma 5.1 that  $\mathcal{G}$  defined in (5.4) is of class  $C^2$ . Moreover,

$$\partial_y \mathcal{G}(y, u)\varphi = (-\Delta_D)^s \varphi - \partial_y F(x, y, u)\varphi.$$

For any  $v \in L^{\tilde{p}}(\Omega)$ , one can show, as in Theorem 3.2, that the problem

$$(-\Delta_D)^s \varphi - \partial_y F(x, y, u)\varphi = v \text{ in } \Omega$$

has a unique weak solution  $\varphi$  in  $\mathbb{X}$ , which depends continuously on  $v$ . Hence,  $\partial_y \mathcal{G}(y, u)$  defines an isomorphism from  $\mathbb{X}$  to  $L^{\tilde{p}}(\Omega)$ . Using the implicit function theorem, we can deduce that  $\mathcal{G}(y, u) = (0, 0)$  has a unique solution  $y = G(u)$ . Moreover, the operator  $G : u \mapsto y$  is itself of class  $C^2$ . Therefore, (5.5) and (5.6) follow easily. Note that if  $v \in L^2(\Omega)$ , then (5.5) still has a unique weak solution  $z \in \mathbb{V}$ . Let us show the estimate (5.7). If we multiply (5.5) with  $z$  and we integrate over  $\Omega$ , we obtain that

$$\|z\|_{\mathbb{H}^s(\Omega)}^2 = \int_{\Omega} \frac{\partial F}{\partial y}(x, y, u)z^2 \, dx + \int_{\Omega} \frac{\partial F}{\partial u}(x, y, u)vz \, dx.$$

Using (2.16), (2.17), and the continuous embedding  $\mathbb{V} \hookrightarrow L^2(\Omega)$ , we can deduce that

$$\|z\|_{\mathbb{V}}^2 \leq C_{F,M}C(N, s, \Omega)\|z\|_{\mathbb{V}}\|v\|_{L^2(\Omega)},$$

where  $M = \|y\|_{C(\bar{\Omega})}$ . This completes the proof.  $\square$

Next, let us introduce the adjoint state  $q \in \mathbb{V}$  as the unique weak solution of the adjoint equation

$$(-\Delta_D)^s q = \frac{\partial F}{\partial y}(x, y, u)q + \frac{\partial L}{\partial y}(x, y, u) \text{ in } \Omega. \quad (5.8)$$

**Proposition 5.3 (Existence of solutions to the adjoint equation).** *Let  $s > \frac{1}{2}$ ,  $\tilde{p} > \frac{N}{2s-1}$ , and  $\Omega$  be of class  $C^{1,\sigma}$  for  $0 < \sigma := 2s - \frac{N}{\tilde{p}} - 1 < 1$ . In the case where  $0 < s \leq 1/2$ , assume that  $\tilde{p} < \infty$ ,  $\sigma = 2s - N/\tilde{p}$ , and  $\Omega$  is of class  $C^1$ . Let Assumptions 2.7 and 2.8 hold and  $u \in L^\infty(\Omega)$ . Then there exists a unique weak solution  $q \in \mathbb{V} \cap C^{0,\sigma}(\bar{\Omega})$  to (5.8). Moreover, there exist two constants  $C = C(N, s, \Omega) > 0$  and  $C_{L,M} > 0$ , with  $M = C(\|F(\cdot, 0, 0)\|_{L^{\tilde{p}}(\Omega)} + 1)$ , such that*

$$\|q\|_{\mathbb{V}} + \|q\|_{C^{0,\sigma}(\bar{\Omega})} \leq C(N, s, \Omega)C_{L,M}. \quad (5.9)$$

*Proof.* From (2.21), we can deduce that there exists  $C_{L,M} > 0$ , where  $M := C(\|F(\cdot, 0, 0)\|_{L^{\tilde{p}}(\Omega)} + 1)$  such that

$$\left| \frac{\partial L}{\partial y}(x, y, u) \right| \leq C_{L,M}.$$

Therefore,  $\frac{\partial L}{\partial u}(\cdot, y, u) \in L^{\bar{p}}(\Omega)$ . Using the same arguments as in Theorems 3.2 and 3.3, we can deduce the existence of a unique weak solution  $q \in \mathbb{V} \cap C^{0,\sigma}(\bar{\Omega})$  to (5.8). Now, if we multiply (5.8) with  $q$  and we integrate over  $\Omega$ , we get, using (2.16) and the continuous embedding  $\mathbb{V} \hookrightarrow L^2(\Omega)$ , that

$$\|q\|_{\mathbb{V}} \leq C(N, s, \Omega)C_{L,M}.$$

Using (3.8), we get (5.9). This completes the proof.  $\square$

**Remark 5.4.** As in Lemma 5.2, one can prove using Assumptions 2.7 and 2.8 that the mapping  $u \mapsto q$  is of class  $C^1$ , where  $q$  is the weak solution to the adjoint state equation (5.8).

**Proposition 5.5 (Twice Fréchet differentiability of  $J$ ).** *Let  $u \in L^\infty(\Omega)$  and  $y$  be the weak solution of (1.2). Let Assumptions 2.7 and 2.8 hold. Under the hypothesis of Lemma 5.2, the functional  $J : L^\infty(\Omega) \rightarrow \mathbb{R}$  defined in (1.1) is twice continuously Fréchet differentiable and for every  $v, w \in L^\infty(\Omega)$ , we have that*

$$J'(u)v = \int_{\Omega} \left( \frac{\partial L}{\partial u}(x, y, u) + \frac{\partial F}{\partial u}(x, y, u)q \right) v \, dx, \quad (5.10)$$

and

$$\begin{aligned} J''(u)[v, w] &= \int_{\Omega} \frac{\partial^2 L}{\partial y \partial u}(x, y, u)(wG'(u)v + vG'(u)w) \, dx \\ &\quad + \int_{\Omega} \frac{\partial^2 L}{\partial y^2}(x, y, u)G'(u)vG'(u)w \, dx + \int_{\Omega} \frac{\partial^2 L}{\partial u^2}(x, y, u)vw \, dx \\ &\quad + \int_{\Omega} \frac{\partial^2 F}{\partial y \partial u}(x, y, u)(wG'(u)v + vG'(u)w)q \, dx \\ &\quad + \int_{\Omega} \left( \frac{\partial^2 F}{\partial y^2}(x, y, u)G'(u)vG'(u)w + \frac{\partial^2 F}{\partial u^2}(x, y, u)vw \right) q \, dx, \end{aligned} \quad (5.11)$$

where  $q$  is the unique weak solution of the adjoint equation (5.8).

*Proof.* First, we have that  $J$  is twice continuously Fréchet differentiable, since by Lemma 5.2,  $G$  has this property.

Second, let  $u, v, w \in L^\infty(\Omega)$ . After some straightforward calculations, we get

$$J'(u)v = \int_{\Omega} \frac{\partial L}{\partial u}(x, y, u)v \, dx + \int_{\Omega} \frac{\partial L}{\partial y}(x, y, u)G'(u)v \, dx. \quad (5.12)$$

Now, if we multiply (5.5) with  $q$ , the weak solution to the adjoint state (5.8), and we integrate by parts over  $\Omega$ , we can deduce that

$$\int_{\Omega} \frac{\partial L}{\partial y}(x, y, u)G'(u)v \, dx = \int_{\Omega} \frac{\partial F}{\partial u}(x, y, u)qv \, dx. \quad (5.13)$$

Combining (5.12)–(5.13) leads us to (5.10). On the other hand, after some calculations, we obtain that

$$\begin{aligned} J''(u)[v, w] &= \int_{\Omega} \frac{\partial^2 L}{\partial y^2}(x, y, u)G'(u)vG'(u)w \, dx + \int_{\Omega} \frac{\partial^2 L}{\partial u^2}(x, y, u)vw \, dx \\ &\quad + \int_{\Omega} \frac{\partial^2 L}{\partial y \partial u}(x, y, u)(wG'(u)v + vG'(u)w) \, dx \end{aligned}$$

$$+ \int_{\Omega} \frac{\partial L}{\partial y}(x, y, u) G''(u)[v, w] dx. \quad (5.14)$$

Multiplying (5.6) with  $q$ , the weak solution of (5.8), and after an integration by parts over  $\Omega$ , we get

$$\begin{aligned} \int_{\Omega} \frac{\partial L}{\partial y}(x, y, u) G''(u)[v, w] dx &= \int_{\Omega} \frac{\partial^2 F}{\partial y^2}(x, y, u) G'(u) v G'(u) w q dx \\ &+ \int_{\Omega} \frac{\partial^2 F}{\partial u^2}(x, y, u) v w q dx \\ &+ \int_{\Omega} \frac{\partial^2 F}{\partial y \partial u}(x, y, u) w G'(u) v q dx \\ &+ \int_{\Omega} \frac{\partial^2 F}{\partial y \partial u}(x, y, u) v G'(u) w q dx. \end{aligned} \quad (5.15)$$

Combining (5.14) and (5.15), we get (5.11). The proof is finished.  $\square$

We introduce the following notion of local solutions.

**Definition 5.6.** Let  $1 \leq p \leq \infty$ . We say that  $u \in \mathcal{U}$  is an  $L^p$ -local solution of (1.1) if there exists  $\varepsilon > 0$  such that

$$J(u) \leq J(v) \text{ for every } v \in \mathcal{U} \cap B_{\varepsilon}^p(u), \quad (5.16)$$

where  $B_{\varepsilon}^p(u) := \{u \in L^p(\Omega) : \|v - u\|_{L^p(\Omega)} \leq \varepsilon\}$ . We say that  $u$  is a strict local minimum of (1.1) if the inequality (5.16) is strict whenever  $v \neq u$ .

Let us introduce the Hamiltonian  $H$  of (1.1)–(1.2) given by

$$H : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad H(x, t, \eta, \xi) = L(x, t, \xi) + \eta F(x, t, \xi). \quad (5.17)$$

Note that from Assumptions 2.7 and 2.8,  $H$  is of class  $C^2$  with respect to the last component.

Before going further, we establish the following important result.

**Proposition 5.7.** Let  $s > \frac{1}{2}$ ,  $\tilde{p} > \frac{N}{2s-1}$ , and  $\Omega$  be of class  $C^{1,\sigma}$  for  $0 < \sigma = 2s - \frac{N}{\tilde{p}} - 1 < 1$ . In the case where  $0 < s \leq 1/2$ , assume that  $\tilde{p} < \infty$ ,  $\sigma = 2s - N/\tilde{p}$ , and  $\Omega$  is of class  $C^1$ . Consider a sequence  $(u_k)_{k \geq 1}$  bounded in  $L^{\infty}(\Omega)$  converging strongly to  $u$  in  $L^{\tilde{p}}(\Omega)$ , as  $k \rightarrow \infty$ . If we denote by  $y_k := y(u_k)$  and  $y_u := y(u)$  the states associated with  $u_k$  and  $u$ , respectively, then

$$\lim_{k \rightarrow \infty} (\|y_k - y_u\|_{\mathbb{V}} + \|y_k - y_u\|_{C^{0,\sigma}(\bar{\Omega})}) = 0. \quad (5.18)$$

*Proof.* Let  $(u_k)_{k \geq 1}$  be a bounded sequence in  $L^{\infty}(\Omega)$  such that, as  $k \rightarrow \infty$ ,

$$u_k \rightarrow u \text{ strongly in } L^{\tilde{p}}(\Omega). \quad (5.19)$$

Using (3.4), we can deduce that there is a constant  $M > 0$  such that for all  $k \geq 1$ ,

$$\|y_k\|_{\mathbb{V}} + \|y_k\|_{L^{\infty}(\Omega)} \leq M. \quad (5.20)$$

Moreover,  $y_k - y_u$  satisfies

$$\int_{\Omega} (-\Delta_D)^{\frac{s}{2}}(y_k - y_u)(-\Delta_D)^{\frac{s}{2}}\phi dx = \int_{\Omega} (F(x, y_k, u_k) - F(x, y_u, u))\phi dx, \quad \forall \phi \in \mathbb{V}.$$

Taking  $\phi = y_k - y_u$  as a test function in the latter equation and thanks to the monotonicity of  $F$  and Assumption 2.7, we get

$$\begin{aligned} \|y_k - y_u\|_{\mathbb{V}}^2 &= \int_{\Omega} (F(x, y_k, u_k) - F(x, y_u, u))(y_k - y_u) \, dx \\ &= \int_{\Omega} (F(x, y_k, u_k) - F(x, y_u, u_k))(y_k - y_u) \, dx \\ &\quad + \int_{\Omega} (F(x, y_u, u_k) - F(x, y_u, u))(y_k - y_u) \, dx \\ &\leq \int_{\Omega} (F(x, y_u, u_k) - F(x, y_u, u))(y_k - y_u) \, dx \\ &\leq C_{F,M} \int_{\Omega} |u_k - u| |y_k - y_u| \, dx \\ &\leq C_{F,M} \|u_k - u\|_{L^p(\Omega)} \|y_k - y_u\|_{L^{p'}(\Omega)}, \end{aligned}$$

where  $p' = \frac{\tilde{p}}{\tilde{p}-1}$ . Since  $\tilde{p} > \frac{N}{2s-1}$ , we have that  $p' < \frac{N}{N-2s+1} < \frac{N}{N-2s}$ . Therefore, if  $N > 2s$ , we get from (2.3) that  $\mathbb{V} \hookrightarrow L^{\frac{2N}{N-2s}}(\Omega) \hookrightarrow L^{p'}(\Omega)$ . Also if  $N \leq 2s$ , it follows from (2.3) that  $\mathbb{V} \hookrightarrow L^{p'}(\Omega)$ . The same simple calculation works in the case where  $0 < s \leq 1/2$ . Therefore,

$$\|y_k - y_u\|_{\mathbb{V}}^2 \leq C_{F,M} C(p, \Omega) \|u_k - u\|_{L^p(\Omega)} \|y_k - y_u\|_{\mathbb{V}},$$

and so

$$\|y_k - y_u\|_{\mathbb{V}} \leq C_{F,M} C(p, \Omega) \|u_k - u\|_{L^p(\Omega)}. \quad (5.21)$$

Taking the limit, as  $k \rightarrow \infty$ , in (5.21) while using (5.19) yields the strong convergence of the sequence  $(y_k)_k$  to  $y_u$  in  $\mathbb{V}$ . Now, (3.14) along with the boundedness of the sequence  $(u_k)_{k \geq 1}$  in  $L^\infty(\Omega)$  imply that the sequence  $(y_k)_{k \geq 1}$  is bounded in  $C^{0,\sigma}(\bar{\Omega})$ . Therefore, using the compactness of the embedding  $C^{0,\sigma}(\bar{\Omega}) \hookrightarrow C(\bar{\Omega})$  and the latter convergence, we can deduce that  $y_k \rightarrow y_u$  strongly in  $C(\bar{\Omega})$ , as  $k \rightarrow \infty$ . Hence, as  $k \rightarrow \infty$ , we have that

$$y_k \rightarrow y_u \quad \text{strongly in } L^{\tilde{p}}(\Omega). \quad (5.22)$$

On the other hand, recalling again that  $y_k - y_u$  satisfies

$$(-\Delta_D)^s (y_k - y_u) = F(x, y_k, u_k) - F(x, y_u, u), \quad (5.23)$$

we can deduce from (3.8) that

$$\begin{aligned} \|y_k - y_u\|_{C^{0,\sigma}(\bar{\Omega})} &\leq C (\|F(\cdot, y_k(\cdot), u_k(\cdot)) - F(\cdot, y_u(\cdot), u(\cdot))\|_{L^{\tilde{p}}(\Omega)} \\ &\quad + C \|y_k - y_u\|_{\mathbb{V}}). \end{aligned} \quad (5.24)$$

Now, using (2.17) and (5.20), we obtain that

$$\begin{aligned} &\|F(\cdot, y_k(\cdot), u_k(\cdot)) - F(\cdot, y_u(\cdot), u(\cdot))\|_{L^{\tilde{p}}(\Omega)} \\ &\leq C_{F,M} (\|y_k - y_u\|_{L^{\tilde{p}}(\Omega)} + \|u_k - u\|_{L^{\tilde{p}}(\Omega)}). \end{aligned} \quad (5.25)$$

Combining (5.21), (5.24), (5.25), (5.19), and letting  $k \rightarrow \infty$  in (5.21) and (5.24), and using (5.22), we get (5.18). The proof is finished.  $\square$

The following result is crucial for the rest of the paper.

**Theorem 5.8 (First-order necessary optimality conditions).** *Let Assumptions 2.7 and 2.8 hold. Let  $u \in \mathcal{U}$  be an  $L^p$ -local minimum ( $1 \leq p \leq \infty$ ) for (1.1)–(1.2). Then,*

$$J'(u)(v - u) \geq 0 \quad \text{for every } v \in \mathcal{U}, \quad (5.26)$$

and equivalently,

$$\int_{\Omega} \frac{\partial H}{\partial u}(x, y, q, u)(v - u) \, dx \geq 0 \quad \text{for every } v \in \mathcal{U}, \quad (5.27)$$

where  $q$  is the unique weak solution of (5.8) and  $H$  is the Hamiltonian given in (5.17). Moreover, if  $1 \leq p < \infty$ , then

$$\int_{\Omega} H(x, y(x), q(x), u(x)) \, dx = \min_{v \in \mathcal{U}} \int_{\Omega} H(x, y(x), q(x), v(x)) \, dx \quad (5.28)$$

and the pointwise Pontryagin principle

$$H(x, y(x), q(x), u(x)) = \min_{t \in [\alpha, \beta]} H(x, y(x), q(x), t) \quad \text{for a.e. } x \in \Omega \quad (5.29)$$

holds, where  $y$  is the state associated with  $u$ .

*Proof.* The proofs of (5.26) and (5.27) are classical and follow from the convexity of  $\mathcal{U}$ . The proofs of (5.28) and (5.29) are a simple adaptation of the arguments of [16, Theorem 4.1]. We omit the details for the sake of brevity.  $\square$

**Lemma 5.9.** *Let  $u \in L^\infty(\Omega)$ . Let Assumptions 2.7 and 2.8 hold. Let  $s > \frac{1}{2}$ ,  $\tilde{p} > \frac{N}{2s-1}$ , and  $\Omega$  be of class  $C^{1,\sigma}$  for  $0 < \sigma = 2s - \frac{N}{\tilde{p}} - 1 < 1$ . In the case where  $0 < s \leq 1/2$ , assume that  $\tilde{p} < \infty$ ,  $\sigma = 2s - N/\tilde{p}$ , and  $\Omega$  is of class  $C^1$ . Then the linear mapping  $v \mapsto J'(u)v$  can be extended to a linear continuous mapping  $J'(u) : L^2(\Omega) \rightarrow \mathbb{R}$  given by (5.10). Moreover, the bilinear mapping  $(v, w) \mapsto J''(u)[v, w]$  can be extended to a bilinear continuous mapping  $J''(u) : L^2(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R}$  given by (5.11). In addition, there is a constant  $C = C(N, s, \Omega, \alpha, \beta) > 0$  such that for all  $v, w \in L^2(\Omega)$ ,*

$$|J'(u)v| \leq C\|v\|_{L^2(\Omega)} \quad (5.30)$$

and

$$|J''(u)[v, w]| \leq C\|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}. \quad (5.31)$$

*Proof.* Let  $u \in L^\infty(\Omega)$  and  $v \in L^2(\Omega)$ . From (5.10), we have that

$$J'(u)v = \int_{\Omega} \left( \frac{\partial L}{\partial u}(x, y, u) + \frac{\partial F}{\partial u}(x, y, u)q \right) v \, dx,$$

where  $q$  is the weak solution of (5.8). Using (3.5), (2.17), (2.21), and (5.9), we have that there is a constant  $C > 0$  independent of  $v$  such that (5.30) holds. Thus, the mapping  $v \mapsto J'(u)v$  is linear and continuous on  $L^2(\Omega)$ . Next, let  $v, w \in L^2(\Omega)$ . Using Cauchy-Schwarz inequality, (2.17), (2.21), (5.7), and (5.9), we get from (5.11) that there is a constant  $C > 0$  such that (5.31) holds. Hence, the mapping  $(v, w) \mapsto J''(u)[v, w]$  is a bilinear continuous mapping on  $L^2(\Omega) \times L^2(\Omega)$ .  $\square$

## 6. Second-order optimality conditions

The main concern of this section is to derive the second-order necessary and sufficient conditions of optimality for the control problems (1.1)–(1.2) and (1.1)–(1.4).

### 6.1. Second-order necessary optimality conditions

Since the cost functional  $J$  associated with the optimization problem (1.1)–(1.2) or (1.1)–(1.4) is non-convex, we have that the first-order optimality conditions given in Theorem 5.8 are necessary but not sufficient for optimality.

We first introduce the cone of critical directions associated with a control  $u \in \mathcal{U}$  defined by

$$C_u := \left\{ v \in L^2(\Omega) : J'(u)v = 0 \text{ and } v \text{ fulfills (6.2)} \right\}, \quad (6.1)$$

that is, for a.e  $x \in \Omega$ ,

$$\begin{cases} v(x) \geq 0 & \text{if } u(x) = \alpha, \\ v(x) \leq 0 & \text{if } u(x) = \beta, \\ 0 & \text{if } d(x) \neq 0, \end{cases} \quad (6.2)$$

where

$$d(x) := \frac{\partial L}{\partial u}(x, y, u) + \frac{\partial F}{\partial u}(x, y, u)q.$$

In the rest of the paper, we will adopt the notation  $J''(u)v^2 := J''(u)[v, v]$ . The following result presents the second-order necessary conditions for optimality. The proof is classical and can be found in [38, p. 246].

**Theorem 6.1 (Second-order necessary optimality conditions).** *Let  $u \in \mathcal{U}$  be an  $L^p$ -local solution of (1.1) with  $1 \leq p \leq \infty$ . Then,  $J''(u)v^2 \geq 0$  for all  $v \in C_u$ .*

### 6.2. Second-order sufficient optimality conditions for $L^\infty$ -local solutions

In this section, as in the local case [16], we impose the following suitable structural assumption on  $\frac{\partial H}{\partial u}(x, y, q, u)$ , where  $u \in \mathcal{U}$  is a control satisfying (5.27), and  $y, q$  are the associated state and adjoint state, respectively.

**Assumption 6.2.** There exist three constants  $K > 0$ ,  $\varepsilon_0 > 0$ , and  $\gamma \in (0, +\infty]$  such that  $\forall \varepsilon \leq \varepsilon_0$ ,

$$\left\{ x \in \Omega : \left| \frac{\partial L}{\partial u}(x, y(x), u(x)) + q(x) \frac{\partial F}{\partial u}(x, y(x), u(x)) \right| < \varepsilon \right\} \leq K\varepsilon^\gamma. \quad (6.3)$$

**Theorem 6.3.** *Let Assumptions 2.7, 2.8, and 6.2 hold. Let  $u \in \mathcal{U}$  be a control satisfying (5.27). Then, there is a constant  $\kappa > 0$  such that*

$$J'(u)(v - u) \geq \kappa \|v - u\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \quad \forall v \in \mathcal{U}, \quad (6.4)$$

where

$$\kappa = \frac{1}{2[4 \max(|\alpha|, |\beta|)K]^\frac{1}{\gamma}}. \quad (6.5)$$

In addition, if  $\gamma = +\infty$ , then there exists  $\eta > 0$  such that

$$J(v) \geq J(u) + \frac{\kappa}{2} \|v - u\|_{L^1(\Omega)} \quad \forall v \in \mathcal{U} \cap B_\eta^\infty(u), \quad (6.6)$$

where  $B_\eta^\infty(u)$  is the open ball in  $L^\infty(\Omega)$  with center  $u$  and radius  $\eta$ . So,  $u$  is strictly locally optimal in the sense of  $L^\infty(\Omega)$ .

*Proof.* The proof is an adaptation of [16, Theorem 5.2]. We proceed in two steps.

**Step 1.** We prove first that there is a constant  $\kappa > 0$  such that (6.4) holds. Indeed, from (5.27), we have that for a.e  $x \in \Omega$ ,

$$\left( \frac{\partial L}{\partial u}(x, y(x), u(x)) + q(x) \frac{\partial F}{\partial u}(x, y(x), u(x)) \right) (v(x) - u(x)) \geq 0,$$

for every  $v \in \mathcal{U}$ . Now, let  $\varepsilon > 0$  and define the set

$$\Omega_\varepsilon := \left\{ x \in \Omega : \left| \frac{\partial L}{\partial u}(x, y(x), u(x)) + q(x) \frac{\partial F}{\partial u}(x, y(x), u(x)) \right| \geq \varepsilon \right\}.$$

It follows from (6.3) that

$$|\Omega \setminus \Omega_\varepsilon| \leq K\varepsilon^\gamma. \quad (6.7)$$

Hence,

$$\begin{aligned} & J'(u)(v - u) \\ &= \int_{\Omega} \left( \frac{\partial L}{\partial u}(x, y(x), u(x)) + q(x) \frac{\partial F}{\partial u}(x, y(x), u(x)) \right) (v(x) - u(x)) \, dx \\ &\geq \int_{\Omega} \left| \frac{\partial L}{\partial u}(x, y(x), u(x)) + q(x) \frac{\partial F}{\partial u}(x, y(x), u(x)) \right| |v(x) - u(x)| \, dx \\ &\geq \varepsilon \|v - u\|_{L^1(\Omega_\varepsilon)} = \varepsilon \|v - u\|_{L^1(\Omega)} - \varepsilon \|v - u\|_{L^1(\Omega \setminus \Omega_\varepsilon)}. \end{aligned}$$

Thanks to (6.7), we have that

$$\|v - u\|_{L^1(\Omega \setminus \Omega_\varepsilon)} = \int_{\Omega \setminus \Omega_\varepsilon} |v(x) - u(x)| \, dx \leq 2 \max(|\alpha|, |\beta|) K\varepsilon^\gamma. \quad (6.8)$$

Combining (6.8)–(6.8), we can deduce that

$$J'(u)(v - u) \geq \varepsilon \|v - u\|_{L^1(\Omega)} - 2\varepsilon \max(|\alpha|, |\beta|) K\varepsilon^\gamma. \quad (6.9)$$

Taking

$$\varepsilon := \frac{\|v - u\|_{L^1(\Omega)}^{\frac{1}{\gamma}}}{[4 \max(|\alpha|, |\beta|) K]^{\frac{1}{\gamma}}},$$

it follows from (6.9) that

$$J'(u)(v - u) \geq \frac{\|v - u\|_{L^1(\Omega)}^{1 + \frac{1}{\gamma}}}{2[4 \max(|\alpha|, |\beta|) K]^{\frac{1}{\gamma}}}.$$

Therefore, (6.4) holds with  $\kappa$  given by (6.5).

**Step 2.** We prove (6.6). Let  $u \in \mathcal{U} \cap B_\eta^\infty(u)$ , where  $\eta$  will be fixed later. Using Proposition 5.10, we have that  $J$  is of class  $C^2$ . Therefore, the second-order Taylor expansion of  $J$  at  $u$  gives

$$J(v) = J(u) + J'(u)(v - u) + \frac{1}{2}J''(u + \theta(v - u))(v - u)^2,$$

where  $\theta \in (0, 1)$ . Using (5.31) and (6.4) with  $\gamma = +\infty$ , we obtain that

$$J(v) \geq J(u) + \kappa\|v - u\|_{L^1(\Omega)} - C\|v - u\|_{L^2(\Omega)}^2,$$

where the constant  $C$  is independent of  $u$  and  $v$ . Observe that

$$\|v - u\|_{L^2(\Omega)}^2 \leq \|v - u\|_{L^\infty} \|v - u\|_{L^1(\Omega)} \leq \eta \|v - u\|_{L^1(\Omega)}.$$

Therefore,

$$J(v) \geq J(u) + (\kappa - \eta C)\|v - u\|_{L^1(\Omega)}.$$

Taking  $\eta = \frac{\kappa}{2C}$  leads us to (6.6). This completes the proof.  $\square$

### 6.3. Second-order sufficient optimality conditions for $L^2(\Omega)$ -local solutions

The main concern of the present section is to find sufficient conditions for  $L^2(\Omega)$ -local optimality of a feasible control  $\bar{u} \in \mathcal{U}$  that satisfies the first-order necessary optimality conditions. To do this, we need the following result.

**Proposition 6.4.** *Let Assumptions 2.7 and 2.8 hold. Let  $s > \frac{1}{2}$ ,  $\tilde{p} > \frac{N}{2s-1}$ , and  $\Omega$  be of class  $C^{1,\sigma}$  for  $0 < \sigma = 2s - \frac{N}{\tilde{p}} - 1 < 1$ . In the case where  $0 < s \leq 1/2$ , assume that  $\tilde{p} < \infty$ ,  $\sigma = 2s - N/\tilde{p}$ , and  $\Omega$  is of class  $C^1$ . Then, there are positive constants  $C$  and  $C_{\tilde{p}}$  such that for all  $u, \bar{u} \in \mathcal{U}$ , the following estimates hold:*

$$\|y - \bar{y}\|_{L^2(\Omega)} \leq C\|u - \bar{u}\|_{L^1(\Omega)} \quad \text{if } N < 4s \tag{6.10}$$

$$\|y - \bar{y}\|_{L^\infty(\Omega)} \leq C_{\tilde{p}}\|u - \bar{u}\|_{L^{\tilde{p}}(\Omega)} \tag{6.11}$$

$$\|q - \bar{q}\|_{L^\infty(\Omega)} \leq C_{\tilde{p}}\|u - \bar{u}\|_{L^{\tilde{p}}(\Omega)} \tag{6.12}$$

$$\|G'(u)v\|_{L^2(\Omega)} \leq C\|v\|_{L^1(\Omega)}, \quad \forall v \in L^\infty(\Omega) \text{ and } N < 4s, \tag{6.13}$$

where  $y$  and  $\bar{y}$ ,  $q$  and  $\bar{q}$  are the states and adjoint states associated to the controls  $u$  and  $\bar{u}$ , respectively.

*Proof.* It follows from (3.15) that there is a constant  $M > 0$  such that

$$\|y\|_{C(\bar{\Omega})} + \|u\|_{L^\infty(\Omega)} \leq M \quad \forall u \in \mathcal{U}. \tag{6.14}$$

Subtracting the equations satisfied by  $y$  and  $\bar{y}$ , and using the mean value theorem we get

$$\begin{aligned} (-\Delta_D)^s(y - \bar{y}) &= \left(F(x, y, \bar{u}) - F(x, \bar{y}, \bar{u})\right) + \left(F(x, y, u) - F(x, y, \bar{u})\right) \\ &= \frac{\partial F}{\partial y}(x, \bar{y} + \theta(y - \bar{y}), \bar{u})(y - \bar{y}) + \left(F(x, y, u) - F(x, y, \bar{u})\right). \end{aligned} \tag{6.15}$$

Since  $N < 4s$ , we have that  $2 < N/(N - 2s)$ . Thus, using Theorem 2.3, and Assumption 2.7, we can deduce that there is a constant  $C_1 > 0$  such that

$$\|y - \bar{y}\|_{L^2(\Omega)} \leq C_1 \|F(x, y, u) - F(x, y, \bar{u})\|_{L^1(\Omega)} \leq CC_{F,M} \|u - \bar{u}\|_{L^1(\Omega)}$$

and we have shown (6.10).

Once again using Eq (6.15), the  $L^\infty(\Omega)$ -estimates for the linear system given in Theorem 2.3 and Assumption 2.7, we obtain that there is a constant  $C(\tilde{p}) > 0$  such that

$$\|y - \bar{y}\|_{L^\infty(\Omega)} \leq C(\tilde{p}) \|F(x, y, u) - F(x, y, \bar{u})\|_{L^{\tilde{p}}(\Omega)} \leq C(\tilde{p})C_{F,M} \|u - \bar{u}\|_{L^{\tilde{p}}(\Omega)}$$

and this gives (6.11).

To prove (6.12), we subtract the equations satisfied by  $q$  and  $\bar{q}$  to get

$$\begin{aligned} (-\Delta_D)^s(q - \bar{q}) &= \frac{\partial F}{\partial y}(x, y, u)q - \frac{\partial F}{\partial y}(x, \bar{y}, \bar{u})\bar{q} + \frac{\partial L}{\partial y}(x, y, u) - \frac{\partial L}{\partial y}(x, \bar{y}, \bar{u}) \\ &= \frac{\partial F}{\partial y}(x, y, u)(q - \bar{q}) + \left[ \frac{\partial F}{\partial y}(x, y, u) - \frac{\partial F}{\partial y}(x, \bar{y}, \bar{u}) \right] q \\ &\quad + \left[ \frac{\partial L}{\partial y}(x, y, u) - \frac{\partial L}{\partial y}(x, \bar{y}, \bar{u}) \right]. \end{aligned}$$

Using the  $L^\infty(\Omega)$ -estimates given in Theorem 2.3, Assumptions 2.7 and 2.8, and (6.11), we can deduce that there is a constant  $C(\tilde{p}) > 0$  such that

$$\begin{aligned} \|q - \bar{q}\|_{L^\infty(\Omega)} &\leq C(\tilde{p}) [C_{F,N}M \|y - \bar{y}\|_{L^{\tilde{p}}(\Omega)} + \|u - \bar{u}\|_{L^{\tilde{p}}(\Omega)}] \\ &\quad + C(\tilde{p})C_{L,M} \|y - \bar{y}\|_{L^{\tilde{p}}(\Omega)} \\ &\leq C(\tilde{p}) \|u - \bar{u}\|_{L^{\tilde{p}}(\Omega)} \end{aligned}$$

and we have shown (6.12). Finally, (6.13) is a direct consequence of Assumption 2.7, Theorem 2.3, and (5.5). The proof is finished.  $\square$

In the rest of the paper, we tacitly assume the following.

**Assumption 6.5.** There exists  $\nu > 0$  such that

$$\frac{\partial^2 H}{\partial u^2}(x, y(x), q(x), u(x)) \geq \nu \text{ for a.e. } x \in \Omega, \forall u(x) \in [\alpha, \beta], \quad (6.16)$$

where  $y$  and  $q$  are the state and adjoint state associated with  $u$ , respectively, and  $H$  is the Hamiltonian given in (5.17).

**Remark 6.6.** The second Fréchet derivative of  $J$  can be rewritten in terms of the Hamiltonian function  $H$  defined in (5.17) as follows:

$$\begin{aligned} J''(u)v^2 &= \int_{\Omega} \frac{\partial^2 H}{\partial y^2}(x, y, q, u)(G'(u)v)^2 dx \\ &\quad + 2 \int_{\Omega} \frac{\partial^2 H}{\partial y \partial u}(x, y, q, u)(vG'(u)v) dx \\ &\quad + \int_{\Omega} \frac{\partial^2 H}{\partial u^2}(x, y, q, u)v^2 dx. \end{aligned} \quad (6.17)$$

**Theorem 6.7.** Let  $u \in \mathcal{U}$  satisfy the first-order optimality condition (5.26) and  $y, q$  be the associated state and adjoint state, respectively. Also, let Assumptions 2.7 and 2.8 hold and  $N < 4s$ . We assume that the structural assumption (6.3) holds with  $\gamma > 1$ . Then, there exists  $\varepsilon > 0$  such that

$$J(v) \geq J(u) + \frac{\kappa}{2} \|v - u\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} + \frac{\nu}{8} \|v - u\|_{L^2(\Omega)}^2 \quad \forall v \in \mathcal{U} \cap \bar{B}_\varepsilon^2(u), \quad (6.18)$$

where  $\bar{B}_\varepsilon^2(u)$  is the closed ball in  $L^2(\Omega)$  with center  $u$  and radius  $\varepsilon$ , and  $\kappa$  is given in (6.5).

*Proof.* Let  $v \in \mathcal{U} \cap \bar{B}_\varepsilon^2(u)$ , where  $\varepsilon$  will be chosen later. We set  $v_\theta := u + \theta(v - u)$ ,  $\theta \in (0, 1)$ . Using the Taylor expansion and (6.4), we write

$$\begin{aligned} J(v) &= J(u) + J'(u)(v - u) + \frac{1}{2} J''(v_\theta)(v - u)^2 \\ &\geq J(u) + \kappa \|v - u\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} + \frac{1}{2} J''(v_\theta)(v - u)^2. \end{aligned} \quad (6.19)$$

We have that,

$$\begin{aligned} J''(v_\theta)(v - u)^2 &= \int_{\Omega} \frac{\partial^2 H}{\partial y^2}(x, y_\theta, q_\theta, v_\theta) (G'(v_\theta)(v - u))^2 dx \\ &\quad + 2 \int_{\Omega} \frac{\partial^2 H}{\partial y \partial u}(x, y_\theta, q_\theta, v_\theta) (v - u) G'(v_\theta)(v - u) dx \\ &\quad + \int_{\Omega} \frac{\partial^2 H}{\partial u^2}(x, y_\theta, q_\theta, v_\theta) (v - u)^2 dx, \end{aligned} \quad (6.20)$$

where  $y_\theta$  and  $q_\theta$  are the state and adjoint state associated with  $v_\theta$ , respectively. We note that from (3.15) and (5.9), we have that there exists  $M > 0$  such that

$$\|y_\theta\|_{C(\bar{\Omega})} + \|q_\theta\|_{C(\bar{\Omega})} \leq M. \quad (6.21)$$

Using Young's inequality, (2.17), (2.21), (6.13), and (6.21), we obtain that there is a constant  $C > 0$  such that

$$\left| \int_{\Omega} \frac{\partial^2 H}{\partial y^2}(x, y_\theta, q_\theta, v_\theta) (G'(v_\theta)(v - u))^2 dx \right| \leq C \|v - u\|_{L^1(\Omega)}^2, \quad (6.22)$$

and

$$\begin{aligned} &2 \left| \int_{\Omega} \frac{\partial^2 H}{\partial y \partial u}(x, y_\theta, q_\theta, v_\theta) (v - u) G'(v_\theta)(v - u) dx \right| \\ &\leq C \int_{\Omega} |(v - u) G'(v_\theta)(v - u)| dx \\ &\leq \frac{\nu}{4} \|v - u\|_{L^2(\Omega)}^2 + C \|v - u\|_{L^1(\Omega)}^2. \end{aligned}$$

For the last term of (6.20), we write

$$\begin{aligned} & \int_{\Omega} \frac{\partial^2 H}{\partial u^2}(x, y_{\theta}, q_{\theta}, v_{\theta})(v - u)^2 \, dx \\ &= \int_{\Omega} \left[ \frac{\partial^2 L}{\partial u^2}(x, y_{\theta}, v_{\theta}) - \frac{\partial^2 L}{\partial u^2}(x, y, v_{\theta}) \right] (v - u)^2 \, dx \\ &+ \int_{\Omega} q \left[ \frac{\partial^2 F}{\partial u^2}(x, y_{\theta}, v_{\theta}) - \frac{\partial^2 F}{\partial u^2}(x, y, v_{\theta}) \right] (v - u)^2 \, dx \\ &+ \int_{\Omega} (q_{\theta} - q) \frac{\partial^2 F}{\partial u^2}(x, y_{\theta}, v_{\theta})(v - u)^2 \, dx \\ &+ \int_{\Omega} \frac{\partial^2 H}{\partial u^2}(x, y, q, v_{\theta})(v - u)^2 \, dx. \end{aligned}$$

Using (2.17), (2.18), (2.22), (6.21), and (6.12) with  $\tilde{p} = 2$ , we get

$$\left| \int_{\Omega} \left[ \frac{\partial^2 L}{\partial u^2}(x, y_{\theta}, v_{\theta}) - \frac{\partial^2 L}{\partial u^2}(x, y, v_{\theta}) \right] (v - u)^2 \, dx \right| \leq \varepsilon \|v - u\|_{L^2(\Omega)}^2, \quad (6.23)$$

$$\left| \int_{\Omega} q \left[ \frac{\partial^2 F}{\partial u^2}(x, y_{\theta}, v_{\theta}) - \frac{\partial^2 F}{\partial u^2}(x, y, v_{\theta}) \right] (v - u)^2 \, dx \right| \leq \varepsilon M \|v - u\|_{L^2(\Omega)}^2, \quad (6.24)$$

$$\left| \int_{\Omega} (q_{\theta} - q) \frac{\partial^2 F}{\partial u^2}(x, y_{\theta}, v_{\theta})(v - u)^2 \, dx \right| \leq C_{F,M} \|v - u\|_{L^2(\Omega)}^3. \quad (6.25)$$

Choosing  $\varepsilon > 0$  sufficiently small such that  $\|v - u\|_{L^2(\Omega)} < \varepsilon$  and  $\varepsilon(1 + M + C_{F,M}) \leq \frac{\nu}{2}$ , we can deduce from (6.23), (6.25), and (6.16) that

$$\begin{aligned} & \int_{\Omega} \frac{\partial^2 H}{\partial u^2}(x, y_{\theta}, q_{\theta}, v_{\theta})(v - u)^2 \, dx \\ & \geq -\frac{\nu}{2} \|v - u\|_{L^2(\Omega)}^2 + \int_{\Omega} \frac{\partial^2 H}{\partial u^2}(x, y, q, v_{\theta})(v - u)^2 \, dx \\ & \geq -\frac{\nu}{2} \|v - u\|_{L^2(\Omega)}^2 + \nu \|v - u\|_{L^2(\Omega)}^2 \\ & = \frac{\nu}{2} \|v - u\|_{L^2(\Omega)}^2. \end{aligned}$$

Combining (6.20), (6.22), (6.23), and (6.26), we can deduce that

$$J''(v_{\theta})(v - u)^2 \geq \frac{\nu}{4} \|v - u\|_{L^2(\Omega)}^2 - 2C \|v - u\|_{L^1(\Omega)}^2. \quad (6.26)$$

Hence, (6.19) leads us to

$$J(v) \geq J(u) + \kappa \|v - u\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} - C \|v - u\|_{L^1(\Omega)}^2 + \frac{\nu}{8} \|v - u\|_{L^2(\Omega)}^2. \quad (6.27)$$

Now, since  $\gamma > 1$ , we have that

$$\kappa \|v - u\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} - C \|v - u\|_{L^1(\Omega)}^2 = \|v - u\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \left[ \kappa - C \|v - u\|_{L^1(\Omega)}^{1-\frac{1}{\gamma}} \right]. \quad (6.28)$$

So, if we choose  $\varepsilon > 0$  such that

$$C \|v - u\|_{L^1(\Omega)}^{1-\frac{1}{\gamma}} \leq C |\Omega|^{\frac{1}{2}-\frac{1}{2\gamma}} \|v - u\|_{L^2(\Omega)}^{1-\frac{1}{\gamma}} \leq C |\Omega|^{\frac{1}{2}-\frac{1}{2\gamma}} \varepsilon^{1-\frac{1}{\gamma}} \leq \frac{\kappa}{2},$$

we can deduce (6.18) from (6.28). The proof is finished.  $\square$

Now, we can state the last result of the paper. Its proof is classical and can be done by following the proof of [16, Theorem 6.1] with minor modifications.

**Theorem 6.8.** *Let  $u \in \mathcal{U}$  satisfy the first-order optimality condition (5.26) and  $y, q$  be the associated state and adjoint state, respectively. Also, let Assumptions 2.7 and 2.8 hold. We assume that*

$$J''(u)v^2 > 0 \quad \forall v \in C_u \setminus \{0\}. \quad (6.29)$$

*Then, there exist  $\varepsilon > 0$  and  $\delta > 0$  such that*

$$J(v) \geq J(u) + \frac{\delta}{2} \|v - u\|_{L^2(\Omega)}^2 \quad \forall v \in \mathcal{U} \cap \bar{B}_\varepsilon^2(u), \quad (6.30)$$

*where  $\bar{B}_\varepsilon^2(u)$  is the closed ball in  $L^2(\Omega)$  with center  $u$  and radius  $\varepsilon$ .*

## 7. Conclusions

We studied optimal control problems for semilinear fractional elliptic equations involving either the spectral fractional Dirichlet Laplacian or the integral fractional Laplacian with homogeneous exterior condition. In contrast with the standard setting in which the control enters the state equation linearly, we allowed a nonlinear dependence on the control, which leads to a nonconvex problem.

Under suitable assumptions on functions involved, we proved existence and regularity results for the state equation. These estimates were then used, together with some structural conditions and a measurable selection argument, to establish existence of optimal controls. We also derived first-order necessary optimality conditions and a pointwise Pontryagin-type minimum principle. Finally, we obtained second-order conditions for optimality under additional assumptions. Extensions to more general constraints and time-dependent fractional models will be addressed in future work.

## Author contributions

Cyrille Kenne: Conceptualization, methodology, formal analysis, investigation, writing – original draft, writing – review and editing; Gisèle Mophou: Conceptualization, methodology, formal analysis, investigation, writing – original draft, writing – review and editing; Mahamadi Warma: Conceptualization, methodology, formal analysis, investigation, writing – original draft, writing – review and editing. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

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