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Research article

Boundedness and asymptotic behavior of a prey-taxis model with modified Leslie-Gower and hunting cooperation

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Abstract: This work investigated the dynamics of a prey-taxis model with modified Leslie-Gower and hunting cooperation. Using energy estimates combined with a newly developed weighted technique, we proved that solutions remain globally bounded in both two or higher dimensions, provided the prey-taxis coefficient is sufficiently small. Furthermore, by constructing a suitable Lyapunov functional, we analyzed the large-time behavior of solutions under some conditions.

Keywords: Leslie-Gower; hunting cooperation; boundedness; large time behavior
Mathematics Subject Classification: 35A01, 35B35, 35K57, 92C17

1. Introduction

The crucial role of the complex dynamic relationship between predators and prey in maintaining ecological balance has been widely recognized, making this relationship an important research topic in the field of biological mathematics. Since the pioneering contributions of Lotka and Volterra, scholars have introduced various functional response functions to describe this dynamic interaction, including Holling type [11, 21], ratio-dependent type [31], Beddington-DeAngelis type [7, 9], Sigmoid type [12], and so on.

Leslie and Gower [13, 22] considered that there is a positive proportional relationship between the environmental carrying capacity of the predator and its preferred prey. Subsequently, they proposed a Leslie-Gower functional response function. However, in real ecosystems, predators typically do not rely on a single food source. When the population of their preferred prey is insufficient to support the survival of the predator population, predators will switch to other food sources, yet their growth remains constrained. Aziz-Alaoui et al. [5] argued that the denominator of the Leslie-Gower term should be augmented by a constant to better reflect the interaction between predators and prey. Thus, a modified Leslie-Gower predator-prey model is formulated as follows:

$$\begin{cases} \frac{du}{dt} = \sigma u \left(1 - \frac{u}{\kappa}\right) - G(u,v)v, \\ \frac{dv}{dt} = v\left(r - \frac{sv}{m+u}\right), \end{cases} \tag{1.1}$$

where $u(x,t)$ is the prey population density and $v(x,t)$ is the predator population density. The prey grows logistically as $\sigma u\left(1 - \frac{u}{\kappa}\right)$ with $\sigma, \kappa > 0$. $G(u,v)$ is known as the functional response function. $r > 0$ means the intrinsic growth rate of the predator. The term $\frac{sv}{m+u}$ with $s, m > 0$ is called the modified Leslie-Gower functional response term. When $G(u,v) = u$, Saha et al. [6] utilized geometric singular perturbation theory and blow-up techniques to investigate the dynamics behavior of (1.1), including Hopf bifurcations, canard orbits, bistability, multiple relaxation oscillations, and so on. Khofifah and Savitri [20] analyzed the local stability of four equilibrium points for (1.1). If $G(u,v) = \frac{u}{\mu+u}$, the existence and global attractivity of positive periodic solutions for (1.1) were proved by Zhu and Wang [32]. Aziz-Alaoui and Okiye [4] demonstrated the boundedness and global stability of the solution for (1.1). For other functional response functions, scholars have extensively studied the properties of the solution for (1.1), including boundedness, stability, Hopf bifurcation, and pattern formation [16, 17, 23, 24].

The predation rate of a population is not only closely related to the environmental carrying capacity, but also associated with the survival pattern of other populations in the ecosystem. For instance, some social predators including gray wolves, common chimpanzees, and several corvid species achieve significantly higher hunting success when individuals align their moves, especially under low-resource or severe-weather conditions that would outcompete solitary foragers [8, 14, 25]. These cooperative strategies contribute to increase short-term energy intake, and strengthen group stability and cross-generation information flow, thereby enhancing population persistence across variable environments. To simplify the description of cooperative hunting behavior in a population, Alves and Hilker [1] introduced the term $G(u,v) = (1 + av)u$ with $a > 0$ as the cooperation coefficient, and they investigated the equilibrium stability in the phase plane and bifurcation diagrams. After some time, González-Olivares and Rojas-Palma [12] studied the cooperative Leslie-Gower model:

$$\begin{cases} \frac{du}{dt} = \sigma u\left(1 - \frac{u}{\kappa}\right)(u - c) - (1 + av)uv, \\ \frac{dv}{dt} = v\left(r - \frac{sv}{m+u}\right), \end{cases} \tag{1.2}$$

proved the existence and stability of its nonnegative equilibria, and identified the Allee term $\sigma u(1 - u/\kappa)(u - c)$ with Allee parameter $c > 0$.

As predators and prey undergo random motion in space, predators tend to accumulate in prey-rich patches. This behavior was first suggested by Keller and Segel and later cast into a partial differential equation (PDE) framework by Kareiva and Odell [19] through the following model:

$$\begin{cases} u_t = d_1\Delta u + \sigma u(1 - \frac{u}{\kappa}) - G(u,v)v, \\ v_t = d_2\Delta v - \chi\nabla\cdot(v\nabla u) - \beta G(u,v)v - sv, \end{cases} \tag{1.3}$$

where $d_i > 0$ $(i = 1, 2)$ are the diffusion coefficients of the prey and predator, respectively. The term $-\chi\nabla\cdot(v\nabla u)$ with $\chi > 0$ represents prey-taxis, i.e., the advective flux of predators toward prey-rich patches. The parameter $\beta > 0$ gives the conversion efficiency from prey to predator biomass, and $s > 0$

is the background death rate of predators. The system has been studied extensively, with results on boundedness, stability, pattern formation, and global bifurcations [15, 18, 29].

Based on (1.2) and (1.3), we extend model (1.2) by incorporating random dispersal of both prey and predators, prey-taxis, and the classic logistic term. Among other things, we propose the following modified Leslie-Gower prey-taxis model with cooperative hunting:

$$\begin{cases} u_t = d_1\Delta u + \sigma u\left(1 - \frac{u}{\kappa}\right) - (1 + av)uv, & x \in \Omega, t > 0, \\ v_t = d_2\Delta v - \chi\nabla\cdot(v\nabla u) + v\left(r - \frac{sv}{m+u}\right), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

with bounded smooth domain $\Omega \subset \mathbb{R}^n (n \geq 2)$. We assume the initial data satisfies

$$(u_0, v_0) \in W^{1,\infty}(\Omega) \times C^0\left(\overline{\Omega}\right) \text{ with } u_0, v_0 \geq 0 \text{ in } \overline{\Omega}. \tag{1.5}$$

The primary findings are outlined as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a smooth bounded domain. For any (u_0, v_0) fulfilling (1.5), (1.4) possesses a nonnegative classical solution $(u, v) \in \left[C^0\left(\bar{\Omega} \times [0,\infty)\right) \cap C^{2,1}(\Omega \times (0,\infty))\right]^2$ and then there exists $\chi^* = \dfrac{\pi}{2\beta_2 K_1} \in (0, +\infty]$ such that for all $\chi \in (0, \chi^*)$,*

$$\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq K,$$

in which constant $K > 0$ is independent of t. Moreover, $0 < u \leq K_1 := \max\{\kappa, \|u_0\|_{L^\infty(\Omega)}\}$. In particular, when $n = 2$, we have $\chi^* = +\infty$.

Remark 1.1. *To facilitate the proof of the long time behavior of the coexistence equilibrium in (1.4), we have*

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leq M(1 + s^{-2})^{16} := K_0,$$

where $M = M(d_1, d_2, \sigma, \kappa, a, \chi, r, m, |\Omega|, \|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^\infty(\Omega)}) > 0$.

We next check that model (1.4) admits four possible homogeneous equilibria:
- Trivial equilibrium $(0, 0)$.
- Prey-only equilibrium $(\kappa, 0)$.
- Predator-only equilibrium $(0, v^*) = (0, \frac{rm}{s})$.
- Coexisting equilibrium (u_*, v_*), where

$$(u_*, v_*) = \left(\frac{\sqrt{4a\sigma\kappa r^2 s^2(m+\kappa) + s^2(\kappa r + \sigma s)^2} - 2a\kappa r^2 m - \kappa r s - \sigma s^2}{2a\kappa r^2}, \frac{r}{s}(m + u_*)\right)$$

with $s > \frac{rm\left(1+\sqrt{1+4a\sigma}\right)}{2\sigma}$.

By utilizing the method outlined in [10], we can demonstrate that $(0, 0)$ and $(\kappa, 0)$ are unstable.

Theorem 1.2. *Assume that the conditions in Theorem 1.2 are satisfied. Then (1) If the model parameters satisfy*

$$s \leq \frac{rm^2}{\sigma(m + K_1)}$$

and

$$\chi^2 < \frac{2sd_1d_2}{\kappa m^2},$$

the solution (u, v) *converges exponentially to* $(0, v^*)$ *in the manner described below:*

$$\|u\|_{L^\infty(\Omega)} + \|v - v^*\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t}, \quad t > 0,$$

where the constant $C > 0$ *and* $\lambda > 0$*.*

(2) If the model parameters satisfy

$$s > \max\left\{\frac{rm\left(1 + \sqrt{1 + 4a\sigma}\right)}{2\sigma}, \frac{\kappa(m + K_1)}{\sigma}\left(a^2K_0^2 + a^2v_*^2 + \frac{1}{2} + \frac{r^2}{4m^2}\right)\right\}$$

and

$$\chi^2 < \frac{4d_1d_2u_*}{K_1^2v_*},$$

the solution (u, v) *converges exponentially to* (u_*, v_*) *in the manner described below:*

$$\|u - u_*\|_{L^\infty(\Omega)} + \|v - v_*\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t}, \quad t > 0,$$

where the constant $C > 0$ *and* $\lambda > 0$*.*

The rest of the paper is structured as follows. In Section 2, we derive the local existence result for model (1.4) and provide some preliminary lemmas. Section 3 focuses on proving Theorem 1.2, that is, the global boundedness of classical solutions to (1.4) when $n \geq 2$. Finally, we investigate the large time behavior of solutions, and demonstrate Theorem 1.2 in Section 4.

2. Preliminaries

This section introduces several lemmas to be employed in the subsequent proof of Theorem 1.1. First, the local existence of solutions to model (1.4) is presented. Since this lemma can be directly derived from Amann's theorem [2, 3], the proof is omitted.

Lemma 2.1. *Let* $\Omega \subset \mathbb{R}^n (n \geq 2)$ *be a smooth bounded domain, and* (u_0, v_0) *satisfies (1.5). There exists constant* $T_{max} \in (0, \infty]$ *guaranteeing that (1.4) possesses a nonnegative classical solution*

$$(u, v) \in \left[C^0\left(\bar{\Omega} \times [0, T_{max})\right) \cap C^{2,1}\left(\Omega \times (0, T_{max})\right)\right]^2.$$

In addition, if $T_{max} < \infty$*, then*

$$\limsup_{t \nearrow T_{max}} \left\{\|u(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}\right\} = \infty.$$

We first note that the boundedness of u follows from a comparison principle. The inequality of (2.1) is a direct consequence of the results established in [18].

Lemma 2.2. *Suppose that the conditions in Lemma 2.1 are fulfilled. There exists a constant* $K_1 := \max\{\kappa, \|u_0\|_{L^\infty(\Omega)}\} > 0$ *such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq K_1, \quad t \in (0, T_{max}). \tag{2.1}$$

Now we establish a boundedness criterion.

Lemma 2.3. *[30] Suppose that the assumptions in Theorem 1.2 hold and* (u, v) *is the solution of the system (1.4). If there exists* $p > \frac{3n}{2}$ $(n \geq 2)$ *and a generic constant* $C > 0$ *such that*

$$\sup_{t\in(0,T_{\max})} \|v(\cdot,t)\|_{L^p(\Omega)} \leq C, \tag{2.2}$$

then it holds that

$$\sup_{t\in(0,T_{\max})} \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq C. \tag{2.3}$$

Remark 2.1. *The model considered in [30] is not exactly the same as the one studied here; thus, it would be helpful to clarify that Lemma 2.3 follows by an argument similar to that in Lemma 2.6 of [30].*

Lemma 2.4. *Let* $T > 0$ *and* $\phi : [0, T) \to [0, \infty)$ *be absolutely continuous. There exist constants* $k_1 > 0$ *and* $k_2 > 0$ *satisfying*

$$\phi'(t) + k_1\phi(t) \leq k_2, \quad t \in (0, T_{max}).$$

Then

$$\phi(t) \leq \max\left\{\phi(0), \frac{k_2}{k_1}\right\}.$$

Lemma 2.5. *[28] Let* $T_{max} > 0$ *and* $\tau \in (0, T_{max})$*. Suppose that the nonnegative function* $\psi(t) \in L^1_{loc}([0, T_{max}))$ *satisfies*

$$\int_t^{t+\tau} \psi(s)\,ds \leq k_1, \quad t \in [0, T_{max} - \tau),$$

and $\phi : [0, T_{max}) \to [0, \infty)$ *is absolutely continuous. Then there exists two constants* $k_1 > 0$ *and* $k_2 > 0$ *ensuring that*

$$\phi'(t) + k_2\phi(t) \leq \psi(t),$$

$$\phi(t) \leq \max\left\{\phi(0) + k_1, \frac{k_1}{k_2\tau} + 2k_1\right\}, \quad t \in (0, T_{max}).$$

3. Proof of Theorem 1.1 for $n = 2$ and $n \geq 3$

This section is dedicated to proving Theorem 1.2. As the large time behavior of solutions in the next section requires us to clarify how the upper bounds of $\|v(\cdot,t)\|_{L^\infty(\Omega)}$ depend on the model parameters, we shall preserve the dependencies of corresponding estimates on the model parameters for subsequent utilization. We first prove the boundedness of $\|v\|_{L^1(\Omega)}$.

Lemma 3.1. *Provided that the conditions in Lemma 2.1 are fulfilled, let* (u, v) *be the solution of (1.4). There exists constant* $K_2 = M_1(1 + s^{-2}) > 0$ *such that*

$$\|v(\cdot, t)\|_{L^1(\Omega)} \leq K_2, \quad t \in (0, T_{max}) \tag{3.1}$$

and

$$\int_t^{t+\tau} \int_\Omega v^2 \leq K_2, \quad t \in (0, T_{max} - \tau), \tag{3.2}$$

where the constant $M_1 = M_1(\sigma, \kappa, r, m, |\Omega|, \|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^\infty(\Omega)}) > 0$ *and* $0 < \tau < \min\left\{1, \frac{1}{2}T_{max}\right\}$.

Proof. Using the second equation of (1.4) and Young's inequality, we derive

$$\begin{aligned}\frac{d}{dt}\int_\Omega v + \int_\Omega v &= (r+1)\int_\Omega v - s\int_\Omega \frac{v^2}{m+u} \\ &\leq -\frac{s}{2(m+K_1)}\int_\Omega v^2 + \frac{(r+1)^2(m+K_1)|\Omega|}{2s}.\end{aligned}$$

It follows from Lemma 2.4 that

$$\int_\Omega v \leq \max\left\{\|v_0\|_{L^\infty(\Omega)}, \frac{(r+1)^2(m+K_1)|\Omega|}{2s}\right\} \leq C_1(1 + s^{-1}),$$

where $C_1 > 0$ is independent of s. Then integrating the aforementioned equation over $(t, t+\tau)$ yields

$$\int_t^{t+\tau}\int_\Omega v^2 \leq \frac{(r+1)^2(m+K_1)^2|\Omega|\tau}{s^2} + \frac{2(m+K_1)}{s}\int_\Omega v \leq C_2(1 + s^{-2}).$$

For all $t \in (0, T_{max} - \tau)$ and $C_2 > 0$ independent of s, let $M_1 = \max\{C_1, C_2\}$, which gives (3.1) by Lemma 2.4. □

Lemma 3.2. *Provided that the conditions in Lemma 2.1 are fulfilled, let* (u, v) *be the solution of (1.4). There exists constant* $K_3 = M_2(1 + s^{-2})^{\frac{1}{2}} > 0$ *ensuring that*

$$\|\nabla u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \leq K_3, \quad t \in (0, T_{max}), \tag{3.3}$$

and there exists a constant $K_4 = M_3(1 + s^{-2})^2 > 0$ *such that*

$$\int_t^{t+\tau}\int_\Omega |\Delta u|^2 + \int_t^{t+\tau}\int_\Omega v^3 \leq K_4, \quad t \in (0, T_{max} - \tau), \tag{3.4}$$

where the constants

$$M_2 = M_2(d_1, \sigma, \kappa, a, \chi, r, m, |\Omega|, \|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^\infty(\Omega)}) > 0$$

and

$$M_3 = M_3(d_1, \sigma, \kappa, a, \chi, r, m, |\Omega|, \|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^\infty(\Omega)}) > 0.$$

Proof. Multiplying the u-equation in (1.4) by $\frac{\Delta u}{u}$ and integrating over Ω yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} + d_1\int_\Omega \frac{|\Delta u|^2}{u} + \frac{\sigma}{\kappa}\int_\Omega |\nabla u|^2 \\
&= \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}u_t + \int_\Omega v\Delta u - 2a\int_\Omega v\nabla u\cdot\nabla v \\
&\le \frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u + \frac{\sigma}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega v\Delta u - 2a\int_\Omega v\nabla u\cdot\nabla v.
\end{aligned} \tag{3.5}$$

Then

$$\frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u = d_1\int_\Omega \frac{|\Delta u|^2}{u} - d_1\int_\Omega \frac{|D^2u|^2}{u} + \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}ds + d_1\int_\Omega \frac{|\nabla u|^4}{u^3} - d_1\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u.$$

Utilizing

$$\int_\Omega u|D^2\ln u|^2 = \int_\Omega \frac{|D^2u|^2}{u} - \int_\Omega \frac{|\nabla u|^4}{u^3} + \int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u,$$

it follows that

$$\frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u = \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}ds + 2d_1\int_\Omega \frac{|\Delta u|^2}{u} - d_1\int_\Omega u|D^2\ln u|^2. \tag{3.6}$$

Plugging (3.6) into (3.5) gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} + d_1\int_\Omega u|D^2\ln u|^2 + \frac{3\sigma}{2\kappa}\int_\Omega |\nabla u|^2 \\
&\le \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}ds + \frac{\sigma}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega v\Delta u - 2a\int_\Omega v\nabla u\cdot\nabla v.
\end{aligned} \tag{3.7}$$

From (1.4), multiplying the v-equation in (1.4) by $\frac{a}{\chi}$ and integrating over Ω yields

$$\frac{a}{\chi}\frac{d}{dt}\int_\Omega v^2 + \frac{2ad_2}{\chi}\int_\Omega |\nabla v|^2 = \frac{2ar}{\chi}\int_\Omega v^2 - \frac{2as}{\chi}\int_\Omega \frac{v^3}{m+u} + 2a\int_\Omega v\nabla u\cdot\nabla v. \tag{3.8}$$

Adding (3.7) and (3.8) yields

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \frac{a}{\chi}\int_\Omega v^2\right) + d_1\int_\Omega u|D^2\ln u|^2 + \frac{3\sigma}{2\kappa}\int_\Omega |\nabla u|^2 + \frac{2as}{\chi(m+K_1)}\int_\Omega v^3 \\
&\le \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}ds + \frac{\sigma}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega v\Delta u + \frac{2ar}{\chi}\int_\Omega v^2.
\end{aligned} \tag{3.9}$$

Using $\int_\Omega u|D^2\ln u|^2 \ge C_1\left(\int_\Omega \frac{|D^2u|^2}{u} + \int_\Omega \frac{|\nabla u|^4}{u^3}\right)$ with the constant $C_1 > 0$ independent of s, which substituted into (3.9), yields

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \frac{a}{\chi}\int_\Omega v^2\right) + d_1C_1\left(\int_\Omega \frac{|D^2u|^2}{u} + \int_\Omega \frac{|\nabla u|^4}{u^3}\right) + \frac{2as}{\chi(m+K_1)}\int_\Omega v^3 \\
&\le \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}ds + \frac{\sigma}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega v\Delta u + \frac{2ar}{\chi}\int_\Omega v^2.
\end{aligned} \tag{3.10}$$

According to the trace inequality [27], Cauchy-Schwarz inequality, and $\frac{\partial|\nabla u|^2}{\partial\nu} \le l|\nabla u|^2$ with $l > 0$, it holds that

$$\frac{d_1}{2}\int_{\partial\Omega}\frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}ds \le ld_1\int_{\partial\Omega}\frac{|\nabla u|^2}{u}ds \le \frac{d_1C_1}{2}\int_\Omega\left(\frac{|D^2u|^2}{u}+\frac{|\nabla u|^4}{u^3}\right)+C_2\int_\Omega\frac{|\nabla u|^2}{u}, \tag{3.11}$$

where C_2 is a positive constant independent of s. It can be inferred from Hölder's inequality that

$$\left(\frac{\sigma+1+2C_2}{2}\right)\int_\Omega\frac{|\nabla u|^2}{u} \le C_3\left(\int_\Omega\frac{|\nabla u|^4}{u^3}\right)^{\frac{1}{2}}\left(\int_\Omega u\right)^{\frac{1}{2}} \le \frac{d_1C_1}{4}\int_\Omega\frac{|\nabla u|^4}{u^3}+C_4 \tag{3.12}$$

with $C_3 = \frac{\sigma+1+2C_2}{2}$ and $C_4 = \frac{K_1C_3{}^2}{d_1C_1}$ independent of s. Utilizing Young's inequality and $|\Delta u| \le \sqrt{2}|D^2u|$, we have

$$\int_\Omega v\Delta u \le \frac{d_1C_1}{8}\int_\Omega\frac{|\Delta u|^2}{u}+\frac{2}{d_1C_1}\int_\Omega uv^2 \le \frac{d_1C_1}{4}\int_\Omega\frac{|D^2u|^2}{u}+\frac{2K_1}{d_1C_1}\int_\Omega v^2. \tag{3.13}$$

Plugging (3.11)–(3.13) into (3.10), it follows that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_\Omega\frac{|\nabla u|^2}{u}+\frac{a}{\chi}\int_\Omega v^2\right)+\left(\frac{1}{2}\int_\Omega\frac{|\nabla u|^2}{u}+\frac{a}{\chi}\int_\Omega v^2\right)\\&\quad+\frac{d_1C_1}{8}\left(\int_\Omega\frac{|\Delta u|^2}{u}+\int_\Omega\frac{|\nabla u|^4}{u^3}\right)+\frac{2as}{\chi(m+K_1)}\int_\Omega v^3 \le \left(\frac{2ar}{\chi}+\frac{2K_1}{d_1C_1}\right)\int_\Omega v^2+C_4,\end{aligned}$$

which gives (3.3) and (3.4) by applying Lemma 2.5 and (3.2). □

Lemma 3.3. *Provided that the conditions in Lemma 2.1 are fulfilled and assuming that n=2, Let* (u, v) *be the solution of (1.4). There exists constant* $K_5 = M_4(1+s^{-2})^{\frac{4}{3}}$ *guaranteeing that*

$$\|v\|_{L^3(\Omega)} \le K_5 \quad t \in (0, T_{max}) \tag{3.14}$$

with $M_4 = M_4(d_1, d_2, \sigma, \kappa, a, \chi, r, m, |\Omega|, \|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^\infty(\Omega)}) > 0$.

Proof. According to (1.4), we end up with

$$\begin{aligned}\frac{1}{3}\frac{d}{dt}\int_\Omega v^3+\int_\Omega v^3 &= -2d_2\int_\Omega v|\nabla v|^2+2\chi\int_\Omega v^2\nabla u\cdot\nabla v+(r+1)\int_\Omega v^3-s\int_\Omega\frac{v^4}{m+u}\\&\le -d_2\int_\Omega v|\nabla v|^2+\frac{\chi^2}{d_2}\left(\int_\Omega v^6\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla u|^4\right)^{\frac{1}{2}}+(r+1)\int_\Omega v^3.\end{aligned} \tag{3.15}$$

Utilizing the Gagliardo-Nirenberg inequality, it holds that

$$\|v^{\frac{3}{2}}\|^2_{L^4(\Omega)} \le C_1\left(\|\nabla v^{\frac{3}{2}}\|_{L^2(\Omega)}\|v^{\frac{3}{2}}\|_{L^2(\Omega)}+\|v^{\frac{3}{2}}\|^2_{L^2(\Omega)}\right) \tag{3.16}$$

and

$$\|\nabla u\|^2_{L^4(\Omega)} \le C_2\left(\|\Delta u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}+\|\nabla u\|^2_{L^2(\Omega)}\right) \le C_2\left(K_3\|\Delta u\|_{L^2(\Omega)}+K_3^2\right), \tag{3.17}$$

in which the positive constants C_1, C_2 are independent of s. Thanks to (3.16) and (3.17), one gets

$$\begin{aligned}\frac{\chi^2}{d_2}\left(\int_\Omega v^6\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^4\right)^{\frac{1}{2}} &\leq \frac{\chi^2 C_1 C_2}{d_2}\left(\|\nabla v^{\frac{3}{2}}\|_{L^2(\Omega)}\|v^{\frac{3}{2}}\|_{L^2(\Omega)} + \|v^{\frac{3}{2}}\|^2_{L^2(\Omega)}\right)\left(K_3\|\Delta u\|_{L^2(\Omega)} + K_3^2\right)\\ &\leq \frac{\chi^2 C_1 C_2 K_3}{d_2}\|\nabla v^{\frac{3}{2}}\|_{L^2(\Omega)}\|v^{\frac{3}{2}}\|_{L^2(\Omega)}\|\Delta u\|_{L^2(\Omega)} + \frac{\chi^2 C_1 C_2 K_3^2}{d_2}\|v^{\frac{3}{2}}\|^2_{L^2(\Omega)}\\ &\quad + \frac{\chi^2 C_1 C_2 K_3^2}{d_2}\|\nabla v^{\frac{3}{2}}\|_{L^2(\Omega)}\|v^{\frac{3}{2}}\|_{L^2(\Omega)} + \frac{\chi^2 C_1 C_2 K_3}{d_2}\|v^{\frac{3}{2}}\|^2_{L^2(\Omega)}\|\Delta u\|_{L^2(\Omega)}\\ &\leq d_2\int_\Omega v|\nabla v|^2 + C_3\|\Delta u\|^2_{L^2(\Omega)}\|v\|^3_{L^3(\Omega)} + C_4\|v\|^3_{L^3(\Omega)},\end{aligned} \tag{3.18}$$

where

$$C_3 := \frac{9\chi^4 C_1^2 C_2^2 K_3^2}{4d_2^3} \quad and \quad C_4 := \frac{9\chi^4 C_1^2 C_2^2 K_3^4}{8d_2^3} + \frac{\chi^2 C_1 C_2 K_3^2}{d_2} + \frac{2d_2}{9}$$

dependent on s. Substituting (3.18) into (3.15), it is not difficult to get

$$\frac{d}{dt}\|v\|^3_{L^3(\Omega)} \leq C_3\|\Delta u\|^2_{L^2(\Omega)}\|v\|^3_{L^3(\Omega)} + (C_4 + r + 1)\|v\|^3_{L^3(\Omega)},$$

which gives (3.14) by (3.4) and Gronwall's inequality. □

Then, we prove that $\|v\|_{L^4(\Omega)}$ is bounded.

Lemma 3.4. *Provided the conditions in Lemma 2.1 are fulfilled, let (u, v) be the solution of (1.4). There exists constant $K_6 = M_5(1 + s^{-2})^{\frac{9}{2}}$ guaranteeing that*

$$\|v\|_{L^4(\Omega)} + \|\nabla u\|_{L^4(\Omega)} \leq K_6, \quad t \in (0, T_{max}), \tag{3.19}$$

with $M_5 = M_5(d_1, d_2, \sigma, \kappa, a, \chi, r, m, |\Omega|, \|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^\infty(\Omega)}) > 0$.

Proof. Utilizing (1.4), we conclude

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_\Omega v^4 + \int_\Omega v^4 &\leq -3d_2\int_\Omega v^2|\nabla v|^2 + 3\chi\int_\Omega v^3\nabla u\cdot\nabla v + (r+1)\int_\Omega v^4 - s\int_\Omega \frac{v^5}{m+u}\\ &\leq -2d_2\int_\Omega v^2|\nabla v|^2 + \frac{9\chi^2}{4d_2}\int_\Omega v^4|\nabla u|^2 + (r+1)\int_\Omega v^4 - \frac{s}{m+K_1}\int_\Omega v^5.\end{aligned} \tag{3.20}$$

Applying (1.4) and $\nabla u\cdot\nabla\Delta u = \frac{1}{2}\Delta|\nabla u|^2 - |D^2u|^2$ yields

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla u|^4 + \frac{d_1}{2}\int_\Omega |\nabla|\nabla u|^2|^2 + d_1\int_\Omega |\nabla u|^2|D^2u|^2 \qquad (3.21)\\ &\leq \frac{d_1}{2}\int_{\partial\Omega}|\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}ds + K_1\int_\Omega v\left(|\nabla u|^2|\Delta u| + |\nabla|\nabla u|^2||\nabla u|\right)\\ &\quad + aK_1\int_\Omega v^2\left(|\nabla u|^2|\Delta u| + |\nabla|\nabla u|^2||\nabla u|\right) + \sigma\int_\Omega |\nabla u|^4\\ &\leq \frac{d_1}{2}\int_{\partial\Omega}|\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}ds + \sigma\int_\Omega|\nabla u|^4 + \left(\sqrt{2}+2\right)K_1\int_\Omega v|\nabla u|^2|D^2u|\end{aligned}$$

$$+\left(\sqrt{2}+2\right)aK_1\int_\Omega v^2|\nabla u|^2|D^2u|$$
$$\le \frac{d_1}{2}\int_{\partial\Omega}|\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}ds+\frac{d_1}{2}\int_\Omega|\nabla u|^2|D^2u|^2+C_1\int_\Omega|\nabla u|^4+r\int_\Omega v^4+C_2\int_\Omega v^4|\nabla u|^2,$$

where

$$C_1=\frac{\left(\sqrt{2}+2\right)^4K_1^4}{4d_1^2r}+\sigma \ \ and \ \ C_2=\frac{\left(\sqrt{2}+2\right)^2a^2K_1^2}{d_1}$$

independent of s. Utilizing the trace inequality and $\frac{\partial|\nabla u|^2}{\partial\nu}\le l|\nabla u|^2$, with $l>0$, it holds that

$$\frac{d_1}{2}\int_{\partial\Omega}|\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}ds\le\frac{ld_1}{2}\||\nabla u|^2\|^2_{L^2(\partial\Omega)}\le\frac{d_1}{8}\int_\Omega|\nabla|\nabla u|^2|^2+C_3\int_\Omega|\nabla u|^4, \tag{3.22}$$

in which the positive constant C_3 is independent of s. Thanks to the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}(1+C_1+C_3)\int_\Omega|\nabla u|^4&=(1+C_1+C_3)\||\nabla u|^2\|^2_{L^2(\Omega)}\\&\le C_4\|\nabla|\nabla u|^2\|_{L^2(\Omega)}\||\nabla u|^2\|_{L^1(\Omega)}+C_4\||\nabla u|^2\|^2_{L^1(\Omega)}\\&\le\frac{d_1}{8}\int_\Omega|\nabla|\nabla u|^2|^2+C_4\end{aligned} \tag{3.23}$$

with $C_4:=\frac{(2+d_2)C_5^2K_3^4}{d_2}$ and $C_5>0$ independent of s. Combining (3.20)–(3.23), one gets

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\left(\int_\Omega v^4+\int_\Omega|\nabla u|^4\right)+\int_\Omega v^4+\int_\Omega|\nabla u|^4+2d_2\int_\Omega v^2|\nabla v|^2+\frac{d_1}{4}\int_\Omega|\nabla|\nabla u|^2|^2\\&\le C_6\left(\int_\Omega v^6\right)^{\frac{2}{3}}\left(\int_\Omega|\nabla u|^6\right)^{\frac{1}{3}}-\frac{s}{2(m+K_1)}\int_\Omega v^5+C_7,\end{aligned} \tag{3.24}$$

where

$$C_6:=\left(\frac{9\chi^2}{4d_2}+\frac{\left(\sqrt{2}+2\right)^2a^2K_1^2}{d_1}\right)$$

is independent of s, while $C_7=C_5+\frac{8^4(m+K_1)^4(2r+1)^5}{5^5s^4}$ is dependent on s. Then

$$\begin{aligned}C_6\|v^2\|^2_{L^3(\Omega)}\|\nabla u\|^2_{L^6(\Omega)}&\le C_6\|v^2\|^2_{L^3(\Omega)}\left(C_8\|\nabla|\nabla u|^2\|^{\frac{2}{3}}_{L^2(\Omega)}\||\nabla u|^2\|^{\frac{1}{3}}_{L^1(\Omega)}+C_8\||\nabla u|^2\|_{L^1(\Omega)}\right)\\&\le C_6C_8K_3^{\frac{2}{3}}\|v^2\|^2_{L^3(\Omega)}\|\nabla|\nabla u|^2\|^{\frac{2}{3}}_{L^2(\Omega)}+C_6C_8K_3^2\|v^2\|^2_{L^3(\Omega)}\\&\le\frac{d_1}{4}\|\nabla|\nabla u|^2\|^2_{L^2(\Omega)}+C_9\|v^2\|^3_{L^3(\Omega)}+C_{10}\end{aligned} \tag{3.25}$$

with $C_9:=\frac{8C_6^{\frac{3}{2}}C_8^{\frac{3}{2}}K_3}{3\sqrt{3}d_1^{\frac{1}{2}}}$ and $C_{10}:=\frac{d_1K_3^4}{4}$ dependent on s and $C_{11}>0$ independent of s. Utilizing the

Gagliardo-Nirenberg inequality, we deduce

$$\begin{aligned} C_9\|v^2\|^3_{L^3(\Omega)} &\le C_9C_{11}\left(\|\nabla v^2\|^{\frac{3}{2}}_{L^2(\Omega)}\|v^2\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\Omega)} + \|v^2\|^2_{L^{\frac{3}{2}}(\Omega)}\right) \\ &\le C_9C_{11}K_5\|\nabla v^2\|^{\frac{3}{2}}_{L^2(\Omega)} + C_9C_{11}K_5^{\frac{4}{3}} \\ &\le d_2\int_\Omega v^2|\nabla v|^2 + C_{12}, \end{aligned} \tag{3.26}$$

in which $C_{12} := \frac{27C_9^4C_{11}^4K_5^{12}}{4d_2^3} + C_9C_{11}K_5^6$ is dependent on s. Inserting (3.25) and (3.26) into (3.24), one can find a constant $C_{13} = C_7 + C_{10} + C_{12}$ dependent on s such that

$$\frac{1}{4}\frac{d}{dt}\left(\int_\Omega v^4 + \int_\Omega |\nabla u|^4\right) + \int_\Omega v^4 + \int_\Omega |\nabla u|^4 \le C_{13},$$

which gives (3.19) with Lemma 2.4. □

Focusing on dimensions $n \ge 3$, we next give a proof of Theorem 1.2 via weighted L^p-estimates on v with $p > 1$. For this purpose, we fix $p > 1$ and introduce a collection of positive constants that are determined by K_1 in (2.1) and the chosen exponent p.

$$\begin{cases} \beta_1 := \dfrac{d_1p^2 - d_1p}{pd_1^2 + pd_2^2 + 2d_1d_2}, & \beta_2 := \dfrac{1}{2d_1}\left[p\left(p + \dfrac{2d_1}{d_2}\right)\right]^{1/2}, \\ \beta_3 := \dfrac{2d_1d_2(p-1)}{pd_1^2 + pd_2^2 + 2d_1d_2}, & \chi_* := \dfrac{\pi}{2\beta_2K_1}. \end{cases} \tag{3.27}$$

We then construct the following function, which we will use later.

Lemma 3.5. *Fix $p > 1$ and let the parameters $\beta_1, \beta_2, \beta_3$ and the threshold χ_* be as specified in* (3.27). *Throughout, we restrict χ to the interval $(0, \chi_*)$. Then for all $z \in [0, K_1]$, the function*

$$\varphi(z) := e^{\beta_1\chi z}[\cos(\chi\beta_2 z)]^{-\beta_3} \tag{3.28}$$

satisfies

$$1 \le \varphi(z) \le \varphi(K_1) < +\infty, \tag{3.29}$$

$$0 < \chi\beta_1 \le \frac{\varphi'(z)}{\varphi(z)} \le \chi[\beta_1 + \beta_2\beta_3\tan(\chi\beta_2K_1)] =: k_{\chi,p} < +\infty, \tag{3.30}$$

$$\frac{d_1}{p}\varphi''(z) - \chi\varphi'(z) = \frac{[(\chi(p-1)\varphi(z) - (d_1+d_2)\varphi'(z))]^2}{2d_2(p-1)\varphi(z)}. \tag{3.31}$$

Proof. Obviously, (3.27) and $\chi \in (0, \chi_*)$ imply

$$0 < \chi\beta_2 z \le \chi\beta_2K_1 < \frac{\pi}{2} \quad \text{for all } z \in [0, K_1].$$

As a consequence, it follows from (3.27) and (3.28) that $\varphi(z)$ is increasing on $[0, K_1]$, and (3.29) holds. By straightforward computation, one has

$$\frac{\varphi'(z)}{\varphi(z)} = \chi[\beta_1 + \beta_2\beta_3\tan(\chi\beta_2 z)] \quad \text{for all } z \in [0, K_1],$$

and then (3.30) is obvious. We next prove (3.31). By tedious calculations, we arrive at

$$
\begin{aligned}
&2d_2(p-1)\varphi(z)\left(\frac{d_1}{p}\varphi''(z)-\chi\varphi'(z)\right)-[\chi(p-1)\varphi(z)-(d_1+d_2)\varphi'(z)]^2\\
&=-\frac{1}{p}\left(\frac{\varphi(z)}{\cos(\chi\beta_2 z)}\right)^2\Big\{\beta_1^2\chi^2(p(d_1^2+d_2^2)+2d_1d_2)-2d_1d_2\beta_2^2\beta_3(p-1)\chi^2\\
&\quad+(p-1)p\chi[(p-1)\chi-2d_1\beta_1\chi]\cos^2(\chi\beta_2 z)\\
&\quad+\beta_2\beta_3\chi\sin(\chi\beta_2 z)[2(\beta_1\chi(p(d_1^2+d_2^2)+2d_1d_2)-d_1p(p-1)\chi)\cos(\chi\beta_2 z)\\
&\quad+(\beta_3(p(d_1^2+d_2^2)+2d_1d_2)-2d_1d_2(p-1))\beta_2\chi\sin(\chi\beta_2 z)]\Big\}.
\end{aligned}
\tag{3.32}
$$

Substituting the result of (3.27) into (3.32) yields (3.31). The proof is complete. □

With the auxiliary function established above, we can now employ it as a weight to establish a uniform estimate for $\|v(\cdot,t)\|_{L^p(\Omega)}$.

Lemma 3.6. *Let $p > 1$ and assume the hypotheses of Theorem 1.2. For the solution (u, v) of system (1.4), take $\beta_1, \beta_2, \beta_3, \chi_*$ from (3.27), fix any $\chi \in (0, \chi_p)$, and let $k_{\chi,p}$ be defined by (3.30). Then there exists a time-independent constant $C(p) > 0$ such that*

$$\|v(\cdot,t)\|_{L^p(\Omega)} \le C(p) \quad \textit{for all } t \in (0, T_{\max}).$$

Proof. Define $\varphi(z)$ through (3.28), and then (2.1) together with (3.29) ensures that for every $z \in [0, K_1]$,

$$1 \le \varphi(u(x,t)) \le \varphi(K_1) \quad \text{for all } (x,t) \in \Omega \times (0, T_{\max}). \tag{3.33}$$

Integrating by parts yields

$$
\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega v^p\varphi(u) &= \int_\Omega v^{p-1}\varphi(u)\left[d_2\Delta v-\chi\nabla\cdot(v\nabla u)+\ v\left(r-\frac{sv}{m+u}\right)\right]\\
&\quad+\frac{1}{p}\int_\Omega v^p\varphi'(u)\left[d_1\Delta u+\left(\sigma u\left(1-\frac{u}{\kappa}\right)-(1+av)uv\right)\right]\\
&=-d_2(p-1)\int_\Omega v^{p-2}\varphi(u)|\nabla v|^2-\int_\Omega\left(\frac{d_1}{p}\varphi''(u)-\chi\varphi'(u)\right)v^p|\nabla u|^2\\
&\quad+\int_\Omega(\chi(p-1)\varphi(u)-(d_1+d_2)\varphi'(u)]\,v^{p-1}\nabla v\cdot\nabla u\\
&\quad+\int_\Omega v^{p-1}\varphi(u)\left(\left(\sigma u\left(1-\frac{u}{\kappa}\right)-(1+av)uv\right)\frac{v\varphi'(u)}{p\varphi(u)}+v(r-\frac{sv}{m+u})\right)
\end{aligned}
\tag{3.34}
$$

for all $t \in (0, T_{\max})$. Therefore,

$$I_1 := \int_\Omega [\chi(p-1)\varphi(u)-(d_1+d_2)\varphi'(u)]\,v^{p-1}\nabla v\cdot\nabla u$$

and

$$I_2 := \int_\Omega v^{p-1}\varphi(u)\left(\left(\sigma u\left(1-\frac{u}{\kappa}\right)-(1+av)uv\right)\frac{v\varphi'(u)}{p\varphi(u)}+v(r-\frac{sv}{m+u})\right).$$

Using Young's inequality, we obtain

$$I_1 \le \frac{d_2(p-1)}{2}\int_\Omega v^{p-2}\varphi(u)|\nabla v|^2 + \int_\Omega \frac{[\chi(p-1)\varphi(u)-(d_1+d_2)\varphi'(u)]^2}{2d_2(p-1)\varphi(u)} v^p|\nabla u|^2. \tag{3.35}$$

For the term I_2, it follows from (3.30) that

$$\begin{aligned} I_2 &= \int_\Omega v^{p-1}\varphi(u)\left[v\left(r-\frac{sv}{m+u}\right)+\frac{\varphi'(u)}{p\varphi(u)}v\left(\sigma u\left(1-\frac{u}{\kappa}\right)-(1+av)uv\right)\right] \\ &\le \int_\Omega v^p\varphi(u)\left[r+\frac{\sigma k_{\chi,p}}{p}u\right] \quad \text{for all } t\in(0,T_{\max}), \end{aligned}$$

which means

$$I_2 \le 2\max\left\{r, \frac{\sigma k_{\chi,p}}{p}K_1\right\}\int_\Omega v^p\varphi(u) \quad \text{for all } t\in(0,T_{\max}). \tag{3.36}$$

The combination of (3.31) and (3.34)–(3.36) yields

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v^p\varphi(u)+\frac{1}{p}\int_\Omega v^p\varphi(u)+\frac{d_1(p-1)}{2}\int_\Omega v^{p-2}|\nabla v|^2\varphi(u) \le C_{14}\int_\Omega v^p\varphi(u), \tag{3.37}$$

where $C_{14} := 2\max\left\{r, \frac{\sigma k_{\chi,p}}{p}K_1\right\}$ is independent of s. The last term on the right-hand side of the preceding inequality can be estimated by invoking (3.33) together with the Gagliardo-Nirenberg inequality. This yields a positive constant C_{15} such that

$$\begin{aligned} C_{14}\int_\Omega v^p\varphi(u) &\le C_{14}\,\varphi(K_1)\|v^{\frac{p}{2}}\|^2_{L^2(\Omega)} \\ &\le C_{14}\,\varphi(K_1)C_{15}\Big(\|\nabla v^{\frac{p}{2}}\|^{2\theta}_{L^2(\Omega)}\,\|v^{\frac{p}{2}}\|^{2(1-\theta)}_{L^{\frac{2}{p}}(\Omega)}+\|v^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\Big) \end{aligned} \tag{3.38}$$

with $\theta = \frac{\frac{p}{2}-\frac{1}{2}}{\frac{p}{2}-\frac{1}{2}+\frac{1}{n}} \in (0,1)$. Since $0 < 2\theta < 2$, by Young's inequality, (3.1), and (3.33), we obtain from (3.38) that

$$\begin{aligned} C_{14}\int_\Omega v^p\varphi(u) &\le C_{14}\varphi(K_1)C_{15}(K_2+1)^p\left(\|\nabla v^{\frac{p}{2}}\|^{2\theta}_{L^2(\Omega)}+1\right) \\ &\le C_{14}\varphi(K_1)C_{15}(K_2+1)^p\Bigg[\frac{d_2(p-1)}{p^2C_2\varphi(K_1)C_{15}(K_2+1)^p}\|\nabla v^{\frac{p}{2}}\|^2_{L^2(\Omega)} \\ &\quad +\left(\frac{d_2(p-1)}{p^2C_2\varphi(K_1)C_{15}(K_2+1)^p}\right)^{-\frac{\theta}{1-\theta}}+1\Bigg] \\ &\le \frac{d_2(p-1)}{4}\int_\Omega v^{p-2}|\nabla v|^2\varphi(u)+C_{16}. \end{aligned} \tag{3.39}$$

Substituting (3.39) into (3.37), one has

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v^p\varphi(u)+\frac{1}{p}\int_\Omega v^p\varphi(u)+\frac{d_2(p-1)}{4}\int_\Omega v^{p-2}|\nabla v|^2\varphi(u) \le C_{16},$$

in which $C_{16} > 0$ is independent of s. Upon solving the foregoing inequality and invoking (3.33), the Sobolev embedding $W^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$ (with $q > n$), and the assumption $v_0 \in W^{1,q}(\Omega)$, we obtain positive constants C_{17} and C_{18} independent of s such that

$$\|v(\cdot,t)\|^p_{L^p(\Omega)} \le \int_\Omega v^p\varphi(u) \le \max\left\{\varphi(K_1)\|v_0\|^p_{L^p(\Omega)}, pC_{16}\right\} \le C_{17}\left(\|v_0\|^p_{W^{1,q}(\Omega)} + 1\right) \le C_{18}$$

for all $t \in (0, T_{\max})$. This completes the proof. □

Proof of Theorem 1.1. The combination of Lemmas 2.2, 2.3, 3.4, and 3.6, yields a positive constant K such that

$$\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \le K, \quad t \in (0, T_{max}),$$

which together with the extension criterion in Lemma 2.1 proves Theorem 1.1. □

4. Large time behavior

Before studying the large time behavior of the solution for (1.4), we need a standard parabolic regularity lemma.

Lemma 4.1. *[26] There exist constants $\theta \in (0,1)$ and $C > 0$ satisfying*

$$\|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} + \|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} \le C, \quad t \ge 1.$$

In particular, one can find $C > 0$ such that

$$\|\nabla u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \le C, \quad t > 1.$$

4.1. The predator-only steady state: $(0, v^*)$

The objective of this subsection is to prove that the solution of (1.4) converges to $(0, v^*)$ under certain conditions. To achieve this, we construct the Lyapunov functional

$$\mathcal{F}_1(t) := \int_\Omega\left(v - v^* - v^*\ln\frac{v}{v^*}\right) + \frac{sv^*}{m^2}\int_\Omega u + \frac{sv^*}{\kappa m^2}\int_\Omega u^2.$$

Lemma 4.2. *Let (u,v) be a nonnegative solution for (1.4). Provided that the parameter satisfies $s < \frac{rm^2}{\sigma(m+K_1)}$ and $\chi^2 < \frac{2sd_1d_2}{\kappa m^2}$, it holds that*

$$\|u\|_{L^\infty(\Omega)} + \|v - v^*\|_{L^\infty(\Omega)} \le K_8 e^{-\delta_1 t}, \quad t > 0,$$

in which the constant $K_8 > 0$ and $\delta_1 > 0$.

Proof. From (1.4), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega\left(v - v^* - v^*\ln\frac{v}{v^*}\right) &= \int_\Omega\left(1 - \frac{v^*}{v}\right)\left(d_2\Delta v - \chi\nabla\cdot(v\nabla u) + v\left(r - \frac{sv}{m+u}\right)\right)\\ &\le -d_2v^*\int_\Omega\frac{|\nabla v|^2}{v^2} + \chi v^*\int_\Omega\frac{\nabla u\cdot\nabla v}{v} - s\int_\Omega\frac{1}{m+u}(v-v^*)^2 + \frac{sv^*}{m^2}\int_\Omega uv - \frac{sv^{*2}}{m}\int_\Omega\frac{u}{m+u}.\end{aligned} \tag{4.1}$$

Utilizing the first equation in (1.4), we have

$$\frac{sv^*}{m^2}\frac{d}{dt}\int_\Omega u = \frac{s\sigma v^*}{m^2}\int_\Omega u - \frac{s\sigma v^*}{\kappa m^2}\int_\Omega u^2 - \frac{sv^*}{m^2}\int_\Omega uv - \frac{asv^*}{m^2}\int_\Omega uv^2 \tag{4.2}$$

and

$$\frac{sv^*}{4\kappa m^2}\frac{d}{dt}\int_\Omega u^2 = -\frac{sd_1v^*}{2\kappa m^2}\int_\Omega |\nabla u|^2 + \frac{s\sigma v^*}{2\kappa m^2}\int_\Omega u^2 - \frac{s\sigma v^*}{2\kappa^2 m^2}\int_\Omega u^3 - \frac{sv^*}{2\kappa m^2}\int_\Omega (1+av)u^2 v. \tag{4.3}$$

Combining (4.1)–(4.3) yields

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) &\le -d_2 v^*\int_\Omega \frac{|\nabla v|^2}{v^2} + \chi v^*\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - \frac{sd_1v^*}{2\kappa m^2}\int_\Omega|\nabla u|^2 - s\int_\Omega \frac{(v-v^*)^2}{m+u}\\ &\quad - \frac{sv^*}{m}\int_\Omega\left(\frac{v^*}{m+u}-\frac{\sigma}{m}\right)u - \frac{s\sigma v^*}{2\kappa m^2}\int_\Omega u^2\\ &\le -\left(\frac{sd_1v^*}{2\kappa m^2}-\frac{\chi^2 v^*}{4d_2}\right)\int_\Omega|\nabla u|^2 - s\int_\Omega\frac{(v-v^*)^2}{m+u} - \frac{sv^*}{m}\int_\Omega\left(\frac{v^*}{m+u}-\frac{\sigma}{m}\right)u - \frac{s\sigma v^*}{2\kappa m^2}\int_\Omega u^2.\end{aligned} \tag{4.4}$$

Then, thanks to $s \le \frac{rm^2}{\sigma(m+K_1)}$ and $\chi^2 < \frac{2sd_1d_2}{\kappa m^2}$, it holds that

$$\frac{d}{dt}\mathcal{F}_1(t) \le -c_1\left(\int_\Omega u + \int_\Omega u^2 + \int_\Omega (v-v^*)^2\right), \tag{4.5}$$

in which constant $c_1 > 0$. Integrating (4.5) over $[1, +\infty)$, we have

$$\int_1^\infty\int_\Omega u + \int_1^\infty\int_\Omega u^2 + \int_1^\infty\int_\Omega (v-v^*)^2 \le c_2, \tag{4.6}$$

which together with Lemma 4.1 yields

$$\int_\Omega u + \int_\Omega u^2 + \int_\Omega (v-v^*)^2 \to 0 \quad \text{as } t\to\infty. \tag{4.7}$$

We utilize the Gagliardo-Nirenberg inequality and Lemma 4.1 to obtain

$$\|u\|_{L^\infty(\Omega)} \le c_4\|u\|_{W^{1,\infty}(\Omega)}^{\frac{2}{N+2}}\|u\|_{L^2(\Omega)}^{\frac{2}{N+2}} \tag{4.8}$$

and

$$\|v-v^*\|_{L^\infty(\Omega)} \le c_3\|v-v^*\|_{W^{1,\infty}(\Omega)}^{\frac{2}{N+2}}\|v-v^*\|_{L^2(\Omega)}^{\frac{2}{N+2}}, \tag{4.9}$$

in which the constant $c_3 > 0$ and $c_4 > 0$. From L'Hoptial's rule, for any $s_0 > 0$, it holds that

$$\lim_{s\to s_0}\frac{s-s_0-s_0\ln\frac{s}{s_0}}{(s-s_0)^2} = \lim_{s\to s_0}\frac{1-\frac{s_0}{s}}{2\,(s-s_0)} = \lim_{s\to s_0}\frac{1}{2s} = \frac{1}{2s_0},$$

which gives a constant $\varepsilon > 0$ such that

$$\frac{1}{4s_0}(s-s_0)^2 \le s - s_0 - s_0\ln\frac{s}{s_0} \le \frac{1}{s_0}(s-s_0)^2 \quad \text{for all } |s-s_0|\le\varepsilon. \tag{4.10}$$

Applying (4.8) and (4.9), there exists $t_0 > 0$ ensuring that

$$\|u\|_{L^\infty(\Omega)} + \|v - v^*\|_{L^\infty(\Omega)} \le \varepsilon, \quad t \ge t_0.$$

Therefore, by (4.10), we get

$$\frac{1}{4v^*}\int_\Omega (v - v^*)^2 \le \int_\Omega \left(v - v^* - v^* \ln \frac{v}{v^*}\right) \le \frac{1}{v^*}\int_\Omega (v - v^*)^2, \quad t \ge t_0. \tag{4.11}$$

Thanks to (4.11), we are guaranteed that

$$\mathcal{F}_1(t) \le \frac{1}{c_5}\left(\int_\Omega u + \int_\Omega u^2 + \int_\Omega (v - v^*)^2\right), \quad t \ge t_0, \tag{4.12}$$

in which constant $c_5 > 0$. Plugging (4.12) into (4.4), it follows that

$$\frac{d}{dt}\mathcal{F}_1(t) \le -\left(\int_\Omega u + \int_\Omega u^2 + \int_\Omega (v - v^*)^2\right) \le -c_5\mathcal{F}_1(t)$$

for all $t \ge t_0$. Therefore, we obtain

$$\mathcal{F}_1(t) \le c_6 e^{-\delta_1 t}, \quad t \ge t_0,$$

with $c_6 > 0$ and $\delta_1 > 0$. □

4.2. The coexistence steady state: (u_*, v_*)

The objective of this subsection is to prove that the solution of (1.4) converges to (u_*, v_*) under certain conditions. To achieve this, we construct the Lyapunov functional

$$\mathcal{F}_2(t) := \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) + \int_\Omega \left(v - v_* - v_* \ln \frac{v}{v_*}\right).$$

Lemma 4.3. *Let (u, v) be a global bounded solution for (1.4). If the parameters satisfy*

$$s > \max\left\{\frac{rm\left(1 + \sqrt{1 + 4a\sigma}\right)}{2\sigma}, \frac{\kappa(m + K_1)}{\sigma}\left(a^2K_0^2 + a^2v_*^2 + \frac{1}{2} + \frac{r^2}{4m^2}\right)\right\} \tag{4.13}$$

and

$$\chi^2 < \frac{4d_1d_2u_*}{K_1^2 v_*}, \tag{4.14}$$

it holds that

$$\|u - u_*\|_{L^\infty(\Omega)} + \|v - v_*\|_{L^\infty(\Omega)} \le K_9 e^{-\delta_2 t}, \quad t > 0,$$

in which constant $K_9 > 0$ and $\delta_2 > 0$.

Proof. Using $\sigma\left(1-\frac{u_*}{\kappa}\right)-(1+av_*)\,v_*=0$, one gets

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(u-u_*-u_*\ln\frac{u}{u_*}\right)\\
&=\int_\Omega\left(1-\frac{u_*}{u}\right)\left(d_1\Delta u+u\sigma\left(1-\frac{u}{\kappa}\right)-(1+av)uv\right)\\
&=-d_1u_*\int_\Omega\frac{|\nabla u|^2}{u^2}+\int_\Omega(u-u_*)\left(\sigma\left(1-\frac{u}{\kappa}\right)-(1+av)\,v\right)\\
&=-d_1u_*\int_\Omega\frac{|\nabla u|^2}{u^2}-\frac{\sigma}{\kappa}\int_\Omega(u-u_*)^2-\int_\Omega(u-u_*)\,(v-v_*)-a\int_\Omega(u-u_*)\left(v^2-v_*^2\right).
\end{aligned}\tag{4.15}$$

Similarly, utilizing $r-\frac{sv_*}{m+u_*}=0$, we end up with

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(v-v_*-v_*\ln\frac{v}{v_*}\right)=\int_\Omega\left(1-\frac{v_*}{v}\right)\left(d_2\Delta v-\chi\nabla\cdot(v\nabla u)+v\left(r-\frac{sv}{m+u}\right)\right)\\
&=-d_2v_*\int_\Omega\frac{|\nabla v|^2}{v^2}+\chi v_*\int_\Omega\frac{\nabla u\cdot\nabla v}{v}-s\int_\Omega\frac{1}{m+u}(v-v_*)^2+r\int_\Omega\frac{1}{m+u}(u-u_*)(v-v_*).
\end{aligned}\tag{4.16}$$

Combining (4.15) and (4.16), we are guaranteed that

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_2(t)\le&-d_1u_*\int_\Omega\frac{|\nabla u|^2}{u^2}-d_2v_*\int_\Omega\frac{|\nabla v|^2}{v^2}+\chi v_*\int_\Omega\frac{\nabla u\cdot\nabla v}{v}-\frac{\sigma}{\kappa}\int_\Omega(u-u_*)^2\\
&-s\int_\Omega\frac{1}{m+u}(v-v_*)^2-\int_\Omega\left(1+a(v+v_*)-\frac{r}{m+u}\right)(u-u_*)(v-v_*)\\
:=&-XQX^T-YPY^T,
\end{aligned}$$

where $X=\left(\frac{\nabla u}{u},\frac{\nabla v}{v}\right)$, $Y=(u-u_*,v-v_*)$, and we have the matrices defined by

$$Q=\begin{pmatrix}d_1u_* & -\frac{\chi v_*u}{2}\\ -\frac{\chi v_*u}{2} & d_2v_*\end{pmatrix}$$

and

$$P=\begin{pmatrix}\frac{\sigma}{\kappa} & \frac{1+a(v+v_*)-\frac{r}{m+u}}{2}\\ \frac{1+a(v+v_*)-\frac{r}{m+u}}{2} & \frac{s}{m+u}\end{pmatrix}.$$

If (4.13) and (4.14) hold, we check

$$|Q|=d_1d_2u_*v_*-\frac{\chi^2v_*^2u^2}{4}>d_1d_2u_*v_*-\frac{\chi^2v_*^2K_1^2}{4}>0$$

and

$$\begin{aligned}
|P|&=\frac{\sigma s}{\kappa(m+u)}-\frac{1}{4}\left(1+a(v+v_*)-\frac{r}{m+u}\right)^2\\
&>\frac{\sigma s}{\kappa(m+K_1)}-\frac{1}{2}-a^2v^2-a^2v_*^2-\frac{r^2}{4m^2}\\
&>0,
\end{aligned}$$

which means that the matrices Q and P are positive definite. Consequently, for any u, v, there exists positive constants $c_i\,(i = 1, 2)$ such that

$$\frac{d}{dt}\mathcal{F}_2\,(t) \leq -c_1 \int_\Omega \left(\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2}\right) - c_2 \int_\Omega \left((u - u_*)^2 + (v - v_*)^2\right).$$

Then, similar to Lemma 4.2, we prove the solution (u, v) has exponential convergence to (u_*, v_*) as $t \to \infty$ in $L^\infty(\Omega)$. □

Proof of Theorem 1.2. To sum up, combining Lemmas 4.2 and 4.3, we prove Theorem 1.2. □

Author contributions

Weibo Lin: Formal analysis, Investigation, Validation, Writing-original draft; Lu Xu: Conceptualization, Methodology, Writing-review and editing, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. M. Alves, F. Hilker, Hunting cooperation and Allee effects in predators, *J. Theor. Biol.*, **419** (2017), 13–22. https://doi.org/10.1016/j.jtbi.2017.02.002
2. H. Amann, Dynamic theory of quasilinear parabolic equations. II. Reaction-diffusion systems, *Differ. Integr. Equ.*, **3** (1990), 13–75.
3. H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, In: *Function Spaces, Differential Operators and Nonlinear Analysis*, **133** (1993), 9–126. https://doi.org/10.1007/978-3-663-11336-2-1
4. M. A. Aziz-Alaoui, M. D. Okiye, Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes, *Appl. Math. Lett.*, **16** (2003), 1069–1075. https://doi.org/10.1016/S0893-9659(03)90096-6

5. M. A. Aziz-Alaoui, M. Cadivel, A. F. Nindjin, Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay, *Nonl. Anal. Real World Appl.*, **7** (2006), 1104–1118. https://doi.org/10.1016/j.nonrwa.2005.10.003
6. M. Banerjee, P. J. Pal, T. Saha, Slow-fast analysis of a modified Leslie-Gower model with Holling type I functional response, *Nonlinear Dyn.*, **108** (2022), 4531–4555. https://doi.org/10.1007/s11071-022-07370-1
7. J. R. Beddington, Mutual interference between parasites or predators and its effect on searching efficiency, *J. Anim. Ecol.*, **44** (1975), 331–340. https://doi.org/10.2307/3866
8. N. M. Creel, S. Creel, Communal hunting and pack size in African wild dogs, Lycaon pictus, *Anim. Behav.*, **50** (1995), 1325–1339. https://doi.org/10.1016/0003-3472(95)80048-4
9. D. DeAngelis, R. A. Goldstein, R. V. O. Neill, A model for trophic interaction, *Ecology*, **56** (1975), 881–892. https://doi.org/10.2307/1936298
10. E. X. DeJesus, C. Kaufman, Routh-Hurwitz criterion in the examination of eigenvalues of a system of nonlinear ordinary differential equations, *Phys. Rev. A*, **35** (1987), 5288–5290. https://doi.org/10.1103/PhysRevA.35.5288
11. P. Gai, H. Zhang, Qualitative analysis of a prey-predator system with Holling I functional response, *J. Jilin Univ. Sci.*, **44** (2006), 373–376.
12. E. González-Olivares, A. Rojas-Palma, Influence of the collaboration among predators and the weak Allee effect on prey in a modified Leslie-Gower predation model, In: *Mathematical Methods for Engineering Applications. ICMASE 2022*, Springer, **414** (2023), 147–164. https://doi.org/10.1007/978-3-031-21700-5-15
13. J. C. Gower, P. H. Leslie, The properties of a stochastic model for the predator-prey type of interaction between two species, *Biometrika*, **47** (1960), 219–234. https://doi.org/10.2307/2333294
14. D. P. Hector, Cooperative hunting and its relationship to foraging success and prey size in an avian predator, *Ethology*, **73** (1986), 247–257. https://doi.org/10.1111/j.1439-0310.1986.tb00915.x
15. T. Hillen, J. M. Lee, M. A. Lewis, Continuous traveling waves for prey-taxis, *Bull. Math. Biol.*, **70** (2008), 654–676. https://doi.org/10.1007/s11538-007-9271-4
16. C. Ji, D. Jiang, N. Shi, Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation, *J. Math. Anal. Appl.*, **359** (2009), 482–498. https://doi.org/10.1016/j.jmaa.2009.05.039
17. C. Ji, D. Jiang, N. Shi, A note on a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation, *J. Math. Anal. Appl.*, **377** (2011), 435–440. https://doi.org/10.1016/j.jmaa.2010.11.008
18. H. Y. Jin, Z. A. Wang, Global stability of prey-taxis systems, *J. Differ. Equ.*, **262** (2017), 1257–1290. https://doi.org/10.1016/j.jde.2016.10.010
19. P. Kareiva, G. Odell, Swarms of predators exhibit "preytaxis" if individual predators use area-restricted search, *Am. Nat.*, **130** (1987), 233–270.
20. R. F. Khofifah, D. Savitri, Modified Leslie-Gower model with Holling type I functional responses and cannibalism in prey, *SITEKIN: J. Sains, Teknologi Industri*, **21** (2023), 58–64. http://doi.org/10.24014/sitekin.v21i1.24529

21. R. Kohno, R. Miyazaki, J. Sugie, On a predator-prey system of Holling type, *Proc. Am. Math. Soc.*, **125** (1997), 2041–2050. https://doi.org/10.1090/S0002-9939-97-03901-4

22. P. H. Leslie, Some further notes on the use of matrices in population mathematics, *Biometrika*, **35** (1948), 213–245. https://doi.org/10.2307/2332342

23. Y. Li, D. Xiao, Bifurcations of a predator-prey system of Holling and Leslie types, *Chaos Solitons Fract.*, **34** (2007), 606–620. https://doi.org/10.1016/j.chaos.2006.03.068

24. M. Liu, D. S. Xu, X. F. Xu, Analysis of a stochastic predator-prey system with modified Leslie-Gower and Holling-type IV schemes, *Phys. A*, **537** (2020), 122761. https://doi.org/10.1016/j.physa.2019.122761

25. L. D. Mech, P. A. Schmidt, Wolf pack size and food acquisition, *Amer. Nat.*, **150** (1997), 513–517.

26. M. Porzio, V. Vespri, Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations, *J. Differ. Equ.*, **103** (1993), 146–178. https://doi.org/10.1006/jdeq.1993.1045

27. P. Quittner, P. Souplet, *Superlinear Parabolic Problems: Blow-Up, Global Existence and Steady States*, Basel/Boston/Berlin: Birkhäuser Advanced Texts, 2007.

28. C. Stinner, C. Surulescu, M. Winkler, Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion, *SIAM J. Math. Anal.*, **46** (2014), 1969–2007. https://doi.org/10.1137/13094058X

29. J. P. Shi, B. Y. Wu, S. N. Wu, Global existence of solutions and uniform persistence of a diffusive predator-prey model with prey-taxis, *J. Differ. Equ.*, **260** (2016), 5847–5874. https://doi.org/10.1016/j.jde.2015.12.024

30. W. R. Tao, Z. A. Wang, Global well-posedness and Turing-Hopf bifurcation of prey-taxis systems with hunting cooperation, *Eur. J. Appl. Math.*, **36** (2025), 1121–1147. https://doi.org/10.1017/S0956792525000026

31. M. Wang, Stationary patterns for a prey-predator model with prey-dependent and ratio-dependent functional responses and diffusion, *Phys. D: Nonlinear Phenom.*, **196** (2004), 172–192. https://doi.org/10.1016/j.physd.2004.05.007

32. K. Wang, Y. Zhu, Existence and global attractivity of positive periodic solutions for a predator-prey model with modified Leslie-Gower and Holling-type II schemes, *J. Math. Anal. Appl.*, **384** (2011), 400–408. https://doi.org/10.1016/j.jmaa.2011.05.081