



Research article

Rough statistical convergence for double sequences in L-fuzzy normed spaces

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Abstract: In this paper, we introduce and study the concept of rough statistical convergence for double sequences in L-fuzzy normed spaces. We define the rough statistical limit points of a double sequence and prove that the set is closed and convex. In addition, we define rough statistical cluster points of a double sequence, and we prove that this set is closed. Afterward, we investigate the relationships among rough statistical convergence for double sequences, related notions such as rough statistical boundedness, and rough statistical Cauchy-type conditions.

Keywords: L-fuzzy normed space; statistical convergence; rough convergence; rough statistical limit set

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1. Introduction

The fuzzy set theory proposed by Zadeh in 1965 [39] has provided a powerful mathematical tool for modeling uncertainty in real-world events. This theory was later generalized by various mathematicians, particularly Atanassov [5], who introduced the concept of intuitionistic fuzzy sets. Saadati and Park [31] introduced the concept of intuitionistic fuzzy normed spaces as a generalization of fuzzy metric spaces. While classical fuzzy sets assign membership degrees in the interval $[0,1]$, many real-world situations require more complex structures to express membership or uncertainty. The introduction of L-fuzzy sets [17], in which membership degrees are taken from a complete lattice, has provided additional flexibility and generality. In L-fuzzy normed spaces, a complete lattice is considered instead of the $[0,1]$ unit interval, making it a deep generalization of fuzzy normed spaces. This generalization provides a more flexible and comprehensive framework for examining problems involving uncertainty and ordered structures. In this respect, the study of L-fuzzy normed spaces [34] has yielded new developments in functional analysis, particularly in lattice-valued logic,

fuzzy topology, and the general theory of fuzzy sets. Convergence is one of the fundamental concepts of analysis and functional analysis. Statistical convergence, which is a generalization of convergence in the sense of Cauchy, is based on the concept of natural density of subsets of the natural numbers. Since the notion of statistical convergence was independently introduced by Fast and Steinhaus [15, 35], its applications and various generalizations have been extensively investigated by many researchers, notably Buck [7], Schoenberg [33], Šalát [32], and Fridy [16]. The concept of convergence for double sequences was first introduced by Pringsheim [29]. This notion was later studied in detail by Hardy [18], Móricz [22], and Móricz and Rhoades [24]. Statistical convergence for double sequences was independently extended by Mursaleen and Edely [25] and Móricz [23]. The existence of sequences that are not convergent but can still be related to convergence under certain conditions has led to the emergence of various types of convergence. One of the most important among these is rough convergence, introduced by Phu [27] in finite-dimensional normed spaces as a generalization of Cauchy convergence. Using the notion of natural density, Aytar [6] extended rough convergence to rough statistical convergence and investigated several properties of the set of rough statistical limit points, such as closedness and convexity. Rough statistical convergence, a generalization of statistical and rough convergence, offers an important advancement in convergence theory. This concept was studied in intuitionistic fuzzy normed spaces by Antal et al. [1]. Subsequently, the mentioned concept was extended to different structures, such as probabilistic normed spaces [2] and neutrosophic normed spaces [3]. Similarly, Rashid [30] explored this type of convergence in the context of random 2-normed spaces. For further research, we refer the reader to [12, 13].

The version of rough statistical convergence for double sequences has also been studied in different classes of spaces. Malik and Maity [21] defined this concept for classical normed spaces, while Özcan and Or [26] adapted it for intuitionistic fuzzy normed spaces and Arslan and Dündar [4] for 2-normed spaces. Statistical convergence in L-fuzzy normed spaces was proposed by Yapali [37]. Later, Khan et al. [19] examined ideal convergence in L-fuzzy normed spaces. On the other hand, Aykut et al. [9] explored rough convergence and rough statistical convergence in L-fuzzy normed spaces, introducing new concepts such as rough statistical limit points and rough statistical cluster points for sequences in these spaces.

The literature above demonstrates that rough statistical convergence has been actively investigated for sequences and double sequences in various generalized spaces. However, the specific behavior of rough statistical convergence for double sequences within the rich topological structure of L-fuzzy normed spaces remains relatively unexplored. Motivated by this gap in the literature, this paper investigates rough statistical convergence for double sequences. It examines the topological properties of rough statistical limit points and rough statistical cluster points in these spaces. Furthermore, we propose fundamental theorems that demonstrate the generality of our definitions and results compared to existing work. The paper is organized as follows: Section 2 provides the necessary preliminaries and basic definitions related to L-fuzzy normed spaces. Section 3 introduces the main definitions of rough statistical convergence and the set of rough statistical limit and cluster points. Finally, Section 4 concludes the study with a summary of the findings and suggestions for future research.

2. Preliminaries

In this section, we present some basic definitions and properties needed for the following section. Along this paper, the notations \mathbb{N}, \mathbb{R} represent the set of all natural numbers and real numbers, respectively. For convenience, we recall some definitions as follows.

Definition 2.1. [25] The double natural density of the set $A \subseteq \mathbb{N} \times \mathbb{N}$ is defined by

$$\delta_2(A) = \lim_{m,n \rightarrow \infty} \frac{| \{(j, k) \in A : j \leq m \text{ and } k \leq n\} |}{mn},$$

where $| \{(j, k) \in A : j \leq m \text{ and } k \leq n\} |$ denotes the number of elements of A not exceeding m and n , respectively. It can be observed that if the set A is finite, then $\delta_2(A) = 0$.

Definition 2.2. [25] Let $(\mathbb{X}, \|\cdot\|)$ be a normed linear space. A double sequence (ξ_{nm}) in \mathbb{X} is said to be statistically convergent to $\xi_* \in \mathbb{X}$ if, for all $\varepsilon > 0$, we have

$$\delta_2(\{(n, m) \in \mathbb{N} \times \mathbb{N} : \|\xi_{nm} - \xi_*\| \geq \varepsilon\}) = 0.$$

It is convenient to represent symbolically by $\xi_{nm} \xrightarrow{st} \xi_*$.

Definition 2.3. [20] Let $(\mathbb{X}, \|\cdot\|)$ be a normed linear space and r be a non-negative number. A double sequence (ξ_{nm}) in \mathbb{X} is called rough convergent to $\xi_* \in \mathbb{X}$ if, for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $n, m \geq n_\varepsilon$ implies

$$\|\xi_{nm} - \xi_*\| < r + \varepsilon$$

and is denoted by $\xi_{nm} \xrightarrow{r} \xi_*$. The element ξ_* is said to be the rough limit point of the double sequence (ξ_{nm}) , r is called roughness degree. The set

$$r\text{-LIM}_{\xi_{nm}} = \{\xi_* \in \mathbb{X} : \xi_{nm} \xrightarrow{r} \xi_*\}$$

is referred to as the rough limit set of the double sequence (ξ_{nm}) .

Definition 2.4. [21] Let $(\mathbb{X}, \|\cdot\|)$ be a normed linear space and r be a non-negative number. A double sequence (ξ_{nm}) in \mathbb{X} is referred to as rough statistically convergent to $\xi_* \in \mathbb{X}$ if, for all $\varepsilon > 0$,

$$\delta_2(\{(n, m) \in \mathbb{N} \times \mathbb{N} : \|\xi_{nm} - \xi_*\| \geq r + \varepsilon\}) = 0$$

and is written as $\xi_{nm} \xrightarrow{r-st} \xi_*$. Also, the set of all rough statistical limit points of the sequence (ξ_{nm}) is denoted by

$$(r-st)\text{-LIM}_{\xi_{nm}} = \{\xi_* \in \mathbb{X} : \xi_{nm} \xrightarrow{r-st} \xi_*\}.$$

Definition 2.5. [34] Let L be a lattice. A complete lattice is a partially ordered set (L, \leq) in which every subset $S \subseteq L$ has both a supremum ($\sup S \in L$) and an infimum ($\inf S \in L$) within L . It is indicated by $\mathcal{L} = (L, \leq)$. Moreover, the notions $0_{\mathcal{L}}$ and $1_{\mathcal{L}}$ denote $\inf L$ and $\sup L$, respectively.

Definition 2.6. [17] An L -fuzzy set is a mapping $\tau : E \rightarrow L$ for all $e \in E$ and $\tau(e)$ specifies the grade to which any element e in E belongs to the L -fuzzy set τ , where E represents a non-empty universe set.

The symbols \geq , $>$ and $<$ are used with their standard meanings for the given lattice (L, \leq) . The mappings required for L-fuzzy set theory are given in Definitions 2.7 and 2.8 by Cornelis et al. [10].

Definition 2.7. [10] A triangular norm (t-norm) on a complete lattice $\mathcal{L} = (L, \leq)$ is a function $\psi : L \times L \rightarrow L$ with the following assumptions. For every $\rho_1, \rho_2, \rho_3, \rho_4 \in L$,

- 1) $\psi(\rho_1, \rho_2) = \psi(\rho_2, \rho_1)$,
- 2) $\psi(\rho_1, 1_{\mathcal{L}}) = \psi(1_{\mathcal{L}}, \rho_1) = \rho_1$,
- 3) $\psi(\psi(\rho_1, \rho_2), \rho_3) = \psi(\rho_1, \psi(\rho_2, \rho_3))$,
- 4) If $\rho_1 \leq \rho_2$ and $\rho_3 \leq \rho_4$, then $\psi(\rho_1, \rho_3) \leq \psi(\rho_2, \rho_4)$.

Definition 2.8. [10] A negator on $\mathcal{L} = (L, \leq)$ is a function \mathcal{N} from L to L with the following conditions.

- 1) \mathcal{N} is a decreasing function,
- 2) $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$ and $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$.

If, for all $u \in \mathbb{L}$, $\mathcal{N}(\mathcal{N}(u)) = u$, then a negator \mathcal{N} is referred to as involutive. For any $u \in [0, 1]$, the classical negator \mathcal{N}_{cl} on $L = [0, 1]$ is defined as $\mathcal{N}_{cl}(u) = 1 - u$.

An L-fuzzy normed space characterized by Shakeri et al. [34] is presented in the following Definition 2.9.

Definition 2.9. [34] A three tuple (\mathbb{W}, τ, ψ) is said to be an L-fuzzy normed space if \mathbb{W} is real vector space, ψ is a continuous t-norm, and τ is an L-fuzzy set on $\mathbb{W} \times (0, \infty)$ with the following conditions. For every $\rho_1, \rho_2 \in \mathbb{W}$ and $t, s > 0$,

- 1) $\tau(\rho_1, t) > 0_{\mathcal{L}}$,
- 2) $\tau(\rho_1, t) = 1_{\mathcal{L}} \Leftrightarrow \rho_1 = \theta_{\mathbb{W}}$,
- 3) $\tau(c\rho_1, t) = \tau\left(\rho_1, \frac{t}{|c|}\right)$ for $c \neq 0$,
- 4) $\psi(\tau(\rho_1, t), \tau(\rho_2, s)) \leq \tau(\rho_1 + \rho_2, t + s)$,
- 5) $\lim_{t \rightarrow \infty} \tau(\rho_1, t) = 1_{\mathcal{L}}$ and $\lim_{t \rightarrow 0} \tau(\rho_1, t) = 0_{\mathcal{L}}$ for all $\rho_1 \in \mathbb{W} - \{\theta_{\mathbb{W}}\}$,
- 6) The mapping $\tau_{\rho_1} : (0, \infty) \rightarrow L$ defined by $\tau_{\rho_1}(t) = \tau(\rho_1, t)$ is continuous.

In the rest of this study, for the sake of brevity, the term L-fuzzy normed space will be abbreviated as \mathcal{L} -FNS.

Example 2.10. [9] Assume that ψ is a continuous t-norm with

$$\psi(\rho_1, \rho_2) = \rho_1 \rho_2, \quad \forall \rho_1, \rho_2 \in L.$$

Let $(\mathbb{R}, |\cdot|)$ be a normed space and τ be an L-fuzzy set on $\mathbb{R} \times (0, \infty)$ with

$$\tau(\vartheta, t) = \frac{t}{t + |\vartheta|}, \quad \forall t > 0, \vartheta \in \mathbb{R}.$$

Then, (\mathbb{R}, τ, ψ) is an \mathcal{L} -FNS.

Definition 2.11. [34] Let (\mathbb{W}, τ, ψ) be an \mathcal{L} -FNS and \mathcal{N} be a negator on \mathcal{L} . For all $\eta \in \mathbb{W}$, $u > 0$, and $\varepsilon \in L - \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, the set

$$B(\xi, \varepsilon, u) := \{\eta \in \mathbb{W} : \tau(\xi - \eta, u) > \mathcal{N}(\varepsilon)\}$$

is referred to as an open ball with center ξ and radius ε .

Definition 2.12. [38] Let (\mathbb{W}, τ, ψ) be an \mathcal{L} -FNS and \mathcal{N} be a negator on \mathcal{L} . A double sequence (ξ_{jk}) in \mathbb{W} is referred to as convergent to $\xi_0 \in \mathbb{W}$ with respect to fuzzy norm τ if, for all $\varepsilon \in L - \{0_{\mathcal{L}}\}$ and $u > 0$, there exists $n_{\varepsilon} \in \mathbb{N}$ such that

$$\tau(\xi_{jk} - \xi_0, u) > \mathcal{N}(\varepsilon),$$

whenever $n \geq n_{\varepsilon}$. It is convenient to represent symbolically by $\xi_{jk} \xrightarrow{\mathcal{L}} \xi_0$.

Yapali and Gurdal [38] have presented a comprehensive definition of convergence in the Pringsheim sense and of statistical convergence for double sequences.

Definition 2.13. [38] Let (\mathbb{W}, τ, ψ) be an \mathcal{L} -FNS and \mathcal{N} be a negator on \mathcal{L} . A double sequence (ξ_{jk}) in \mathbb{W} is called statistically convergent to $\xi_0 \in \mathbb{W}$ with respect to fuzzy norm τ if, for all $\varepsilon \in L - \{0_{\mathcal{L}}\}$ and $u > 0$, we have

$$\delta_2(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \xi_0, u) \not> \mathcal{N}(\varepsilon)\}) = 0,$$

or equivalently,

$$\lim_{j, k \rightarrow \infty} \frac{1}{jk} |\{j \leq m, k \leq n : \tau(x_{jk} - \xi_0, u) \not> \mathcal{N}(\varepsilon)\}| = 0,$$

whenever $n \geq n_{\varepsilon}$. It is convenient to represent symbolically by $\xi_{jk} \xrightarrow{st\mathcal{L}} \xi_0$.

Lemma 2.14. [38] Let (\mathbb{W}, τ, ψ) be an \mathcal{L} -FNS. Then, the following statements are equivalent: For every $\varepsilon \in L - \{0_{\mathcal{L}}\}$ and $t > 0$,

- (i) $st_{\mathcal{L}}\text{-}\lim_{j, k \rightarrow \infty} \xi_{jk} = \xi_0$,
- (ii) $\delta_2(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \xi_0, u) \not> \mathcal{N}(\varepsilon)\}) = 0$,
- (iii) $\delta_2(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \xi_0, u) > \mathcal{N}(\varepsilon)\}) = 1$,
- (iv) $st_{\mathcal{L}}\text{-}\lim_{j, k \rightarrow \infty} \tau(\xi_{jk} - \xi_0, u) = 1_{\mathcal{L}}$.

3. Main results

This section introduces the concepts of rough convergence and rough statistical convergence for double sequences in \mathcal{L} -FNSs. Then, the relationships among these concepts are examined. In this paper, we consider the elements in the complete lattice $\mathcal{L} = (L, \leq)$ as dense and comparable. Throughout the rest of the paper, unless mentioned otherwise, \mathcal{N} will be shown with a negator on \mathcal{L} .

Definition 3.1. Let (\mathbb{W}, τ, ψ) be an \mathcal{L} -FNS and $r \geq 0$. A double sequence (ξ_{jk}) in \mathbb{W} is said to be rough convergent to $\xi_0 \in \mathbb{W}$ with respect to the fuzzy norm τ if, for all $u > 0$ and $\varepsilon \in L - \{0_{\mathcal{L}}\}$, there exists $n_{\varepsilon} \in \mathbb{N}$ such that

$$\tau(\xi_{jk} - \xi_0; r + u) > \mathcal{N}(\varepsilon) \quad \text{for all } j, k \geq n_{\varepsilon}.$$

In this case, ξ_0 is called a limit point of the double sequence (ξ_{jk}) .

For $r = 0$, the concept of rough convergence for double sequences in \mathcal{L} -FNSs becomes the concept of Pringsheim convergence in \mathcal{L} -FNSs.

The rough limit point of a double sequence may not be unique. Therefore, the set of all the rough limit points for a double sequence (ξ_{jk}) is as follows:

$$r\text{-LIM}_{\xi_{jk}} := \left\{ \xi_0 \in \mathbb{W} : \xi_{jk} \xrightarrow{r\mathcal{L}} \xi_0 \right\}.$$

If $r\text{-LIM}_{\xi_{jk}} \neq \emptyset$, the double sequence (ξ_{jk}) is rough convergent.

Definition 3.2. Let (\mathbb{W}, τ, ψ) be an \mathcal{L} -FNS and $r \geq 0$. A double sequence (ξ_{jk}) in \mathbb{W} is referred to as rough statistically convergent to $\xi_0 \in \mathbb{W}$ with respect to the fuzzy norm τ if, for all $\varepsilon \in L - \{0_L\}$ and $u > 0$,

$$\delta_2(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \xi_0, r + u) \not\prec \mathcal{N}(\varepsilon)\}) = 0.$$

In this case, we show $(r\text{-st})_{\mathcal{L}}\text{-}\lim_{j,k \rightarrow \infty} \xi_{jk} = \xi_0$ or $\xi_{jk} \xrightarrow{(r\text{-st})_{\mathcal{L}}} \xi_0$ as $j, k \rightarrow \infty$.

Remark 3.4. If $r = 0$, then rough statistical convergence coincides with statistical convergence in L -fuzzy normed spaces.

The rough statistical limit point of a double sequence may not be unique. Hence, the set of rough statistical limit points of a double sequence (ξ_{jk}) is denoted as follows:

$$(r\text{-st})_{\mathcal{L}}\text{-LIM}_{\xi_{jk}} = \left\{ \xi_0 \in \mathbb{W} : \xi_{jk} \xrightarrow{(r\text{-st})_{\mathcal{L}}} \xi_0 \right\}.$$

If (ξ_{jk}) is an unbounded sequence, then $r\text{-LIM}_{\xi_{jk}}$ is an empty set. However, this does not hold in the case of rough statistical convergence. Hence, $(r\text{-st})_{\mathcal{L}}\text{-LIM}_{\xi_{jk}}$ may not be an empty set. This situation is illustrated by Example 3.3 as follows:

Example 3.3. Let $(\mathbb{R}, |\cdot|)$ be a normed space, ψ be a continuous t-norm defined by $\psi(\alpha, \beta) = \alpha\beta$ for $\alpha, \beta \in L$, and τ be an L -fuzzy set on $\mathbb{R} \times (0, \infty)$ defined by $\tau(p, q) = \frac{p}{p+|q|}$ for all $p > 0$ and $q \in \mathbb{R}$. Then, (\mathbb{R}, τ, ψ) is an \mathcal{L} -FNS. Consider a sequence (ξ_{jk}) , for all $j, k \in \mathbb{N}$, such that

$$\xi_{jk} = \begin{cases} (-1)^{j+k}, & j \text{ and } k \text{ are non-squares;} \\ jk, & \text{otherwise.} \end{cases}$$

Then,

$$(r\text{-st})_{\mathcal{L}}\text{-LIM}_{\xi_{jk}} = \begin{cases} \emptyset, & r < 1; \\ [1 - r, r - 1], & \text{otherwise.} \end{cases}$$

Definition 3.4. A double sequence (ξ_{jk}) in \mathcal{L} -FNS (\mathbb{W}, τ, ψ) is called statistically bounded with respect to fuzzy norm τ if, for all $\varepsilon \in L - \{0_L\}$, there exists a real number $C > 0$ such that

$$\delta_2(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk}; C) \not\prec \mathcal{N}(\varepsilon)\}) = 0.$$

Theorem 3.5. A double sequence $\xi = \{\xi_{jk}\}$ in \mathcal{L} -FNS (\mathbb{W}, τ, ψ) is statistically bounded with respect to fuzzy norm τ if and only if there exists an $(r\text{-st})_{\mathcal{L}}\text{-LIM}_{\xi_{jk}} \neq \emptyset$ for $r \geq 0$.

Proof. (\Rightarrow): Let (ξ_{jk}) be a statistically bounded double sequence in the \mathcal{L} -FNS (\mathbb{W}, τ, ψ) . Then, for all $\varepsilon \in L - \{0_L\}$, there exists $C > 0$ such that

$$\delta_2 \left(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk}, C) \not\prec \mathcal{N}(\varepsilon)\} \right) = 0.$$

Let $K = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk}, C) \not\prec \mathcal{N}(\varepsilon)\}$. For $(j, k) \in K^c$, we have

$$\tau(\xi_{jk}, C) > \mathcal{N}(\varepsilon).$$

Moreover, for all $r \geq 0$,

$$\tau(\xi_{jk}, r + C) \geq \psi(\tau(0, r), \tau(\xi_{jk}, C)) = \psi(1_L, \tau(\xi_{jk}, C)) > \mathcal{N}(\varepsilon).$$

Hence,

$$K^c \subset \{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk}, r + C) > \mathcal{N}(\varepsilon)\},$$

and we obtain

$$\delta_2 \left(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk}, r + C) \not\prec \mathcal{N}(\varepsilon)\} \right) = 0.$$

Consequently, $0_L \in (r-st)_{\mathcal{L}}\text{-LIM}_{\xi_{jk}}$, i.e., there exists an $r \geq 0$ such that $(r-st)_{\mathcal{L}}\text{-LIM}_{\xi_{jk}} \neq \emptyset$.

(\Leftarrow): Let $(r-st)_{\mathcal{L}}\text{-LIM}_{\xi_{jk}} \neq \emptyset$ for some $r \geq 0$. Then, there exists $w_0 \in \mathbb{W}$ such that $w_0 \in (r-st)_{\mathcal{L}}\text{-LIM}_{\xi_{jk}} \neq \emptyset$. For all $u > 0$ and $\varepsilon \in L - \{0_L\}$,

$$\delta_2 \left(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - w_0, r + u) \not\prec \mathcal{N}(\varepsilon)\} \right) = 0.$$

Therefore, almost all ξ_{jk} 's are contained in some ball with center w_0 , which implies that the double sequence (ξ_{jk}) is statistically bounded in an \mathcal{L} -FNS. \square

The algebraic conceptualization of rough statistical convergence provides an effective approach for examining the behavior of double sequences in \mathcal{L} -FNSs. We investigate how a double sequence (ξ_{jk}) in \mathbb{W} admits an algebraic characterization of rough statistical convergence within the setting of \mathcal{L} -FNSs.

Theorem 3.6. Let (ξ_{jk}) and (η_{jk}) be two double sequences in \mathcal{L} -FNS (\mathbb{W}, τ, ψ) . Then, for all $r \geq 0$, the following holds:

- (1) If $\xi_{jk} \xrightarrow{(r-st)_{\mathcal{L}}} \xi_0$ and $\alpha \in \mathbb{R}$, then $\alpha \xi_{jk} \xrightarrow{(r-st)_{\mathcal{L}}} \alpha \xi_0$.
- (2) If $\xi_{jk} \xrightarrow{(r-st)_{\mathcal{L}}} \xi_0$ and $\eta_{jk} \xrightarrow{(r-st)_{\mathcal{L}}} \eta_0$, then $\xi_{jk} + \eta_{jk} \xrightarrow{(r-st)_{\mathcal{L}}} \xi_0 + \eta_0$.

Proof. (1) Let (ξ_{jk}) and (η_{jk}) be two double sequences in \mathbb{W} and $r \geq 0$. We assume that $\alpha \neq 0$. In the case $\alpha = 0$, the sequence is identically zero, and hence the assertion holds trivially. Let $\xi_{jk} \xrightarrow{(r-st)_{\mathcal{L}}} \xi_0$. Then, for all $\varepsilon \in L - \{0_L\}$ and $u > 0$,

$$\delta \left(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \xi_0, r + u) \not\prec \mathcal{N}(\varepsilon)\} \right) = 0.$$

Since, for all $r \geq 0$ and $u > 0$ and for special $r_1 = \frac{r}{\alpha} \geq 0$ and $u_1 = \frac{u}{\alpha} > 0$,

$$\delta(K) = \delta \left(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \xi_0, r_1 + u_1) \not\prec \mathcal{N}(\varepsilon)\} \right)$$

$$\begin{aligned}
&= \delta\left(\left\{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau\left(\xi_{jk} - \xi_0, \frac{r+u}{\alpha}\right) \not> \mathcal{N}(\varepsilon)\right\}\right) \\
&= 0
\end{aligned}$$

is provided. Suppose that $(t, s) \in K^c$. Then,

$$\tau\left(\xi_{ts} - \xi_0, \frac{r+u}{\alpha}\right) > \mathcal{N}(\varepsilon).$$

From here,

$$\tau(\alpha\xi_{ts} - \alpha\xi_0, r+u) = \tau\left(\xi_{ts} - \xi_0, \frac{r+u}{\alpha}\right) > \mathcal{N}(\varepsilon)$$

and

$$(t, s) \in \left\{(k, j) \in \mathbb{N} \times \mathbb{N} : \tau(\alpha\xi_{jk} - \alpha\xi_0, r+u) > \mathcal{N}(\varepsilon)\right\}.$$

Hence, $K^c \subset \{n \in \mathbb{N} : \tau(\alpha\xi_{jk} - \alpha\xi_0, r+u) > \mathcal{N}(\varepsilon)\}$ and

$$\delta\left(\{n \in \mathbb{N} : \tau(\alpha\xi_{jk} - \alpha\xi_0, r+u) \not> \mathcal{N}(\varepsilon)\}\right) = 0$$

is obtained. Consequently, $\alpha\xi_{jk} \xrightarrow{(r-st)\mathcal{L}} \alpha\xi_0$.

(2) Let $\xi_{jk} \xrightarrow{(r-st)\mathcal{L}} \xi_0$ and $\eta_{jk} \xrightarrow{(r-st)\mathcal{L}} \eta_0$. Then, for all $u > 0$ and $\varepsilon \in L - \{0_L\}$, $\delta(A) = 0$ and $\delta(B) = 0$, where

$$A = \left\{(k, j) \in \mathbb{N} \times \mathbb{N} : \tau\left(\xi_{jk} - \xi_0, \frac{r+u}{2}\right) \not> \mathcal{N}(\varepsilon)\right\},$$

and

$$B = \left\{(k, j) \in \mathbb{N} \times \mathbb{N} : \tau\left(\eta_{jk} - \eta_0, \frac{r+u}{2}\right) \not> \mathcal{N}(\varepsilon)\right\}.$$

Suppose that $(t, s) \in A^c \cap B^c$. Then,

$$\tau\left(\xi_{ts} - \xi_0, \frac{r+u}{2}\right) > \mathcal{N}(\varepsilon)$$

and

$$\tau\left(\eta_{ts} - \eta_0, \frac{r+u}{2}\right) > \mathcal{N}(\varepsilon).$$

From here,

$$\tau((\xi_{ts} + \eta_{ts}) - (\xi_0 + \eta_0), r+u) \geq \psi\left(\tau\left(\xi_{ts} - \xi_0, \frac{r+u}{2}\right), \tau\left(\eta_{ts} - \eta_0, \frac{r+u}{2}\right)\right) > \psi(\mathcal{N}(\varepsilon), \mathcal{N}(\varepsilon)) > \mathcal{N}(\varepsilon)$$

and

$$(t, s) \in R = \left\{(k, j) \in \mathbb{N} \times \mathbb{N} : \tau((\xi_{jk} + \eta_{jk}) - (\xi_0 + \eta_0), r+u) > \mathcal{N}(\varepsilon)\right\}.$$

Hence, $A^c \cap B^c \subseteq R$. Consequently, $\xi_{jk} + \eta_{jk} \xrightarrow{(r-st)\mathcal{L}} \xi_0 + \eta_0$. \square

The following two theorems establish several topological properties of the set $(r-st)\mathcal{L}\text{-LIM}_{\xi_{jk}}$ associated with a double sequence in an \mathcal{L} -FNS.

Theorem 3.7. Let (ξ_{jk}) be a double sequence in \mathcal{L} -FNS (\mathbb{W}, τ, ψ) and $r \geq 0$. The set $(r-st)\mathcal{L}\text{-LIM}_{\xi_{jk}}$ is a closed set.

Proof. If the set $(r-st)_{\mathcal{L}}\text{-LIM}_{\xi_{jk}} = \emptyset$ is empty, then the theorem holds trivially. Assume that $(r-st)_{\mathcal{L}}\text{-LIM}_{\xi_{jk}} \neq \emptyset$ for some $r > 0$.

Consider a double sequence (x_{pq}) in \mathcal{L} -FNS (\mathbb{W}, τ, ψ) that is convergent to x_{00} . By the definition of convergence in an \mathcal{L} -FNS, for any $\varepsilon \in L - \{0_L\}$ and $u > 0$, there exists $N \in \mathbb{N}$ such that

$$\tau\left(x_{pq} - x_{00}, \frac{u}{2}\right) > \mathcal{N}(\varepsilon) \quad \text{for all } p, q \geq N.$$

We take $x_{p_1q_1} \in (r-st)_{\mathcal{L}}\text{-LIM}_{\xi_{jk}}$ with $p_1, q_1 > N$ and by the definition of rough statistical limit points, we have $\delta_2(U) = 0$, where the set U is defined as

$$U = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \tau\left(\xi_{jk} - x_{p_1q_1}, r + \frac{u}{2}\right) \not> \mathcal{N}(\varepsilon) \right\}.$$

Denote

$$V = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \tau\left(\xi_{jk} - x_{p_1q_1}, r + \frac{u}{2}\right) > \mathcal{N}(\varepsilon) \right\}.$$

Then, $\delta(V) = 1$. For any $(t, s) \in V$, we obtain

$$\tau(\xi_{ts} - x_{00}, r + u) \geq \psi\left(\tau\left(\xi_{ts} - x_{p_1q_1}, r + \frac{u}{2}\right), \tau\left(x_{p_1q_1} - x_{00}, \frac{u}{2}\right)\right) > \mathcal{N}(\varepsilon).$$

Hence,

$$(t, s) \in \mathcal{V} = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - x_{00}, r + u) > \mathcal{N}(\varepsilon) \right\}.$$

Then, $V \subseteq \mathcal{V}$, and hence $\delta(\mathcal{V}) = 1$. Therefore,

$$\delta\left(\left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \xi_0, r + u) \not> \mathcal{N}(\varepsilon) \right\}\right) = 0,$$

which implies that $x_{00} \in (r-st)_{\mathcal{L}}\text{-LIM}_{\xi_{jk}}$. Consequently, $(r-st)_{\mathcal{L}}\text{-LIM}_{\xi_{jk}}$ is closed. \square

Theorem 3.8. Let (ξ_{jk}) be a double sequence in \mathcal{L} -FNS (\mathbb{W}, μ, ψ) and $r \geq 0$. The set $(r-st)_{\mathcal{L}}\text{-LIM}_{\xi_{jk}}$ is a convex set.

Proof. Let $y_1, y_2 \in (r-st)_{\mathcal{L}}\text{-LIM}_{\xi_{jk}}$. To prove the convexity of the set $(r-st)_{\mathcal{L}}\text{-LIM}_{\xi}$, we must show that the convex combination $y_{\lambda} = (1 - \lambda)y_1 + \lambda y_2$ belongs to $(r-st)_{\mathcal{L}}\text{-LIM}_{\xi}$ for all $\lambda \in (0, 1)$. For all $\varepsilon \in L - \{0_L\}$ and $u > 0$, we define the following sets of indices:

$$U = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \tau\left(\xi_{jk} - y_1, \frac{r + u}{2(1 - \lambda)}\right) \geq \mathcal{N}(\varepsilon) \right\}$$

and

$$V = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \tau\left(\xi_{jk} - y_2, \frac{r + u}{2\lambda}\right) \geq \mathcal{N}(\varepsilon) \right\}.$$

Since $y_1, y_2 \in (r-st)_{\mathcal{L}}\text{-LIM}_{\xi}$, we have $\delta_2(U) = 0$ and $\delta_2(V) = 0$. Let $(j, k) \in U^c \cap V^c$. Then,

$$\tau\left(\xi_{jk} - y_1, \frac{r + u}{2(1 - \lambda)}\right) > \mathcal{N}(\varepsilon), \quad \tau\left(\xi_{jk} - y_2, \frac{r + u}{2\lambda}\right) > \mathcal{N}(\varepsilon).$$

Using the convexity property of the L-fuzzy norm, we obtain

$$\tau(\xi_{jk} - y_{\lambda}, r + u) = \tau\left((1 - \lambda)(\xi_{jk} - y_1) + \lambda(\xi_{jk} - y_2), r + u\right)$$

$$\begin{aligned} &\geq \phi\left(\tau\left(\xi_{jk} - y_1, \frac{r+u}{2(1-\lambda)}\right), \tau\left(\xi_{jk} - y_2, \frac{r+u}{2\lambda}\right)\right) \\ &> \mathcal{N}(\varepsilon). \end{aligned}$$

Define

$$B = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \xi_\lambda, r+u) \not> \mathcal{N}(\varepsilon)\}.$$

Since $U^c \cap V^c \subseteq B^c$, we obtain $\delta(B) = 0$. Therefore,

$$y_\lambda \in (r\text{-}st)_{\mathcal{L}\text{-LIM}}_{\xi_{jk}}.$$

Hence, $(r\text{-}st)_{\mathcal{L}\text{-LIM}}_{\xi_{jk}}$ is a convex set. \square

Theorem 3.9. Let (ξ_{jk}) be a double sequence in \mathcal{L} -FNS (\mathbb{W}, τ, ψ) and $r \geq 0$. If there exists a double sequence (η_{jk}) in \mathbb{W} such that $\eta_{jk} \xrightarrow{st_{\mathcal{L}}} \xi_0$ and

$$\tau(\xi_{jk} - \eta_{jk}, r) > \mathcal{N}(\varepsilon) \quad \text{for all } \varepsilon \in L - \{0_L\},$$

then $\xi_{jk} \xrightarrow{(r\text{-}st)_{\mathcal{L}}} \xi_0$.

Proof. Let (ξ_{jk}) be a double sequence in (\mathbb{W}, τ, ψ) , and let $r \geq 0$. Assume that there exists a double sequence (η_{jk}) such that $\eta_{jk} \xrightarrow{st_{\mathcal{L}}} \xi_0$. Let $\varepsilon \in L - \{0_L\}$ and $u > 0$ be arbitrary. We define

$$A = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\eta_{jk} - \xi_0, u) \not> \mathcal{N}(\varepsilon)\},$$

and

$$B = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \eta_{jk}, r) \not> \mathcal{N}(\varepsilon)\}.$$

Since $\eta_{jk} \xrightarrow{st_{\mathcal{L}}} \xi_0$, we have $\delta_2(A) = 0$. By assumption, $\delta_2(B) = 0$. Hence, $\delta_2(A^c \cap B^c) = 1$. For any $(j, k) \in A^c \cap B^c$, using the triangular property of the L-fuzzy norm, we obtain

$$\tau(\xi_{jk} - \xi_0, r+u) \geq \psi\left(\tau(\xi_{jk} - \eta_{jk}, r), \tau(\eta_{jk} - \xi_0, u)\right) > \mathcal{N}(\varepsilon).$$

Define

$$C = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \xi_0, r+u) \not> \mathcal{N}(\varepsilon)\}.$$

Then, $C \subseteq A \cup B$, which implies

$$\delta_2(C) \leq \delta_2(A) + \delta_2(B) = 0.$$

Therefore, $\xi_{jk} \xrightarrow{(r\text{-}st)_{\mathcal{L}}} \xi_0$. \square

Theorem 3.10. Let (ξ_{jk}) be a double sequence in \mathcal{L} -FNS (\mathbb{W}, τ, ψ) , and let $r \geq 0$. Then, the set $(r\text{-}st)_{\mathcal{L}\text{-LIM}}_{\xi_{jk}}$ cannot contain two distinct points ξ_1 and ξ_2 such that

$$\tau(\xi_1 - \xi_2, mr) \not> \mathcal{N}(\varepsilon)$$

for some $\varepsilon \in L - \{0_L\}$ and $m > 2$.

Proof. Assume, on the contrary, that there exist $\xi_1, \xi_2 \in (r - st)_{\mathcal{L}} - \text{LIM}_{\xi_{jk}}$ such that

$$\tau(\xi_1 - \xi_2, mr) \not> \mathcal{N}(\varepsilon)$$

for some $\varepsilon \in L - \{0_L\}$ and $m > 2$. Since $\xi_1, \xi_2 \in (r - st)_{\mathcal{L}} - \text{LIM}_{\xi}$, we define the sets

$$U = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \tau\left(\xi_{jk} - y, r + \frac{u}{2}\right) \not> \mathcal{N}(\varepsilon) \right\},$$

and

$$V = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \tau\left(\xi_{jk} - z, r + \frac{u}{2}\right) \not> \mathcal{N}(\varepsilon) \right\}.$$

such that $\delta_2(U) = 0$ and $\delta_2(V) = 0$. Let $(j, k) \in U^c \cap V^c$, and by the triangular property of the L-fuzzy norm,

$$\tau\left(\xi_{jk} - \xi_1, r + \frac{u}{2}\right) > \mathcal{N}(\varepsilon) \quad \text{and} \quad \tau\left(\xi_{jk} - \xi_2, r + \frac{u}{2}\right) > \mathcal{N}(\varepsilon).$$

Then,

$$\tau(\xi_1 - \xi_2, 2r + u) \geq \psi\left(\tau\left(\xi_1 - \xi_{jk}, r + \frac{u}{2}\right), \tau\left(\xi_{jk} - \xi_2, r + \frac{u}{2}\right)\right) > \mathcal{N}(\varepsilon).$$

Therefore,

$$\tau(\xi_1 - \xi_2, 2r + u) > \mathcal{N}(\varepsilon).$$

Since $m > 2$, we have $mr > 2r$, which implies $mr - 2r > 0$. We choose a real number u such that $0 < u \leq mr - 2r$. Since the fuzzy norm $\tau(x, t)$ is non-decreasing with respect to t , this implies

$$\tau(\xi_1 - \xi_2, mr) \geq \tau(\xi_1 - \xi_2, 2r + u) > \mathcal{N}(\varepsilon).$$

This result directly contradicts the hypothesis. Therefore, there does not exist $\xi_1, \xi_2 \in (r - st)_{\mathcal{L}} - \text{LIM}_{\xi_{jk}}$ such that $\tau(\xi_1 - \xi_2, mr) \not> \mathcal{N}(\varepsilon)$ for some $\varepsilon \in L - \{0_L\}$ and $m > 2$. \square

Following the examination of the topological and geometric properties of rough statistical limit points, we introduce the concept of rough statistical cluster points for double sequences in \mathcal{L} -FNSs and investigate some of its fundamental properties.

Definition 3.11. Let (\mathbb{W}, τ, ψ) be an \mathcal{L} -FNS, (ξ_{jk}) be a double sequence in \mathbb{W} , and $r \geq 0$. An element $\xi_0 \in \mathbb{W}$ is called a rough statistical cluster point of the double sequence (ξ_{jk}) with respect to the fuzzy norm τ if, for all $\varepsilon \in L - \{0_L\}$ and $u > 0$,

$$\delta_2\left(\left\{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \xi_0, r + u) > \mathcal{N}(\varepsilon)\right\}\right) > 0.$$

In this context, ξ_0 is referred to as a rough statistical cluster point for the double sequence (ξ_{jk}) with respect to the L-fuzzy norm τ . The set of all rough statistical cluster points of the double sequence (ξ_{jk}) is denoted by $(r-st)_{\mathcal{L}} - \Gamma_{\xi_{jk}}$. In the special case where $r = 0$, this set corresponds to the statistical cluster points with respect to the fuzzy norm τ in an \mathcal{L} -FNS, denoted by $(st)_{\mathcal{L}} - \Gamma_{\xi_{jk}}$.

Theorem 3.12. Let (ξ_{jk}) be a double sequence in \mathcal{L} -FNS (\mathbb{W}, τ, ψ) , and let $r \geq 0$. The set of rough statistical limit points, denoted by $(r-st)_{\mathcal{L}} - \Gamma_{\xi_{jk}}$, is a closed set.

Proof. If the set $(r-st)_{\mathcal{L}\text{-}\Gamma_{\xi_{jk}}}$ is empty, the theorem holds trivially. Thus, assume that $(r-st)_{\mathcal{L}\text{-}\Gamma_{\xi_{jk}}} \neq \emptyset$. Let $\eta^* \in \overline{(r-st)_{\mathcal{L}\text{-}\Gamma_{\xi_{jk}}}}$. By the definition of the closure, there exists a sequence (η_{jk}) in $(r-st)_{\mathcal{L}\text{-}\Gamma_{\xi_{jk}}}$ such that $\eta_{jk} \xrightarrow{\mathcal{L}} \eta^*$. To prove the theorem, it suffices to show that $\eta^* \in (r-st)_{\mathcal{L}\text{-}\Gamma_{\xi_{jk}}}$. By the definition of convergence of a double sequence in an \mathcal{L} -FNS, $\varepsilon \in L - \{0_L\}$ and $u > 0$, and there exists an integer $N_\varepsilon \in \mathbb{N}$ such that

$$\tau\left(\eta_{jk} - \eta^*, \frac{u}{2}\right) > \mathcal{N}(\varepsilon).$$

Let $j_0, k_0 \in \mathbb{N}$ such that $j_0, k_0 > N_\varepsilon$. Then, $\tau\left(\eta_{j_0 k_0} - \eta^*, \frac{u}{2}\right) > \mathcal{N}(\varepsilon)$. Since (η_{jk}) is a double sequence in $(r-st)_{\mathcal{L}\text{-}\Gamma_{\xi_{jk}}}$, $\eta_{j_0 k_0} \in (r-st)_{\mathcal{L}\text{-}\Gamma_{\xi_{jk}}}$. Hence, $\delta_2(U) = 0$ such that

$$U = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \tau\left(\xi_{jk} - \eta_{j_0 k_0}, r + \frac{u}{2}\right) > \mathcal{N}(\varepsilon) \right\}.$$

We consider $(t, s) \in U$.

$$\tau\left(\xi_{ts} - \eta_{j_0 k_0}, r + \frac{u}{2}\right) > \mathcal{N}(\varepsilon).$$

Hence,

$$\begin{aligned} \tau(\xi_{ts} - \eta^*, r + u) &\geq \psi\left(\tau\left(\xi_{ts} - \eta_{j_0 k_0}, r + \frac{u}{2}\right), \tau\left(\eta_{j_0 k_0} - \eta^*, \frac{u}{2}\right)\right) \\ &\Rightarrow \mu(\xi_{jk} - \eta^*, r + u) > \mathcal{N}(\varepsilon). \end{aligned}$$

Let V be the set of indices defined by

$$V = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \eta^*, r + u) > \mathcal{N}(\varepsilon) \right\}.$$

From the implication above, it is evident that $U \subseteq V$. Due to the monotonicity of the double asymptotic density, we have

$$\delta_2(V) \geq \delta_2(U) > 0.$$

Since $\delta_2(V) > 0$ holds for any arbitrary ε and u , we conclude that η^* satisfies the condition to be a rough statistical cluster point. Thus, $\eta^* \in (r-st)_{\mathcal{L}\text{-}\Gamma_{\xi_{jk}}}$, which implies that the set is closed. \square

Theorem 3.13. Let (ξ_{jk}) be a double sequence in \mathcal{L} -FNS (\mathbb{W}, τ, ψ) , $r \geq 0$, and $\eta \in (st)_{\mathcal{L}\text{-}\Gamma_{\xi_{jk}}}$. If an element $\xi \in \mathbb{W}$ satisfies the condition

$$\tau(\xi - \eta, r) > \mathcal{N}(\varepsilon)$$

for an arbitrary $\varepsilon \in L - \{0_L\}$, then $\xi \in (r-st)_{\mathcal{L}\text{-}\Gamma_{\xi_{jk}}}$.

Proof. Let $\eta \in (st)_{\mathcal{L}\text{-}\Gamma_{\xi_{jk}}}$. Then, for all $u > 0$ and $\varepsilon \in L - \{0_L\}$, we define

$$U = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \eta, u) > \mathcal{N}(\varepsilon) \right\}$$

such that $\delta_2(U) > 0$. For $(t, s) \in U$, we have $\tau(\xi_{ts} - \eta, u) > \mathcal{N}(\varepsilon)$. Using the triangle inequality of the \mathcal{L} -fuzzy norm

$$\tau(\xi_{ts} - \xi, r + u) \geq \psi(\tau(\xi_{ts} - \eta, u), \tau(\eta - \xi, r)).$$

From the hypothesis, we know that $\tau(\xi - \eta, r) > \mathcal{N}(\varepsilon)$. Using the symmetry of the norm, $\tau(\eta - \xi, r) > \mathcal{N}(\varepsilon)$. Based on the properties of the t-norm in \mathcal{L} -FNSs (specifically that $\psi(a, b) > \mathcal{N}(\varepsilon)$ whenever $a, b > \mathcal{N}(\varepsilon)$), we obtain

$$\tau(\xi_{ts} - \xi, r + u) > \mathcal{N}(\varepsilon).$$

We choose the set

$$V = \{(t, s) \in \mathbb{N} \times \mathbb{N} : \mu(\xi_{ts} - \xi, r + u) > \mathcal{N}(\varepsilon)\}.$$

Then, $U \subseteq V$. Due to the monotonicity of the double asymptotic density, it follows that

$$\delta_2(V) \geq \delta_2(U) > 0.$$

Consequently, $\xi \in (r-st)_{\mathcal{L}}\Gamma_{\xi_{jk}}$. □

Theorem 3.14. Let (ξ_{jk}) be a double sequence in \mathcal{L} -FNS (\mathbb{W}, τ, ψ) , and let $r \geq 0$. If $\xi_{jk} \xrightarrow{st_{\mathcal{L}}} \xi$, then

$$\overline{B(\xi, \varepsilon, r)} \subset (r-st)_{\mathcal{L}}\text{-LIM}_{\xi_{jk}}, \quad \forall \varepsilon \in L \setminus \{0_L\}.$$

Proof. Since (ξ_{jk}) is statistically convergent to $\xi \in \mathbb{W}$ with respect to fuzzy norm τ , then for every $\varepsilon \in L \setminus \{0_L\}$ and $u > 0$, we have $\delta_2(A) = 0$ such that

$$A = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \tau(\xi_{jk} - \xi, u) \not> \mathcal{N}(\varepsilon)\}.$$

Let $\eta \in \overline{B(\xi, \varepsilon, r)}$, i.e., $\tau(\xi - \eta, r) > \mathcal{N}(\varepsilon)$. For $(t, s) \in A^c$, we have $\tau(\xi_{ts} - \xi, r + u) > \mathcal{N}(\varepsilon)$. By the triangle inequality,

$$\tau(\xi_{ts} - \eta, r + u) \geq \psi(\tau(\xi_{ts} - \xi, r + u), \tau(\xi - \eta, r)).$$

Thus, $\tau(\xi_{ts} - \eta, r + u) > \mathcal{N}(\varepsilon)$ for all $(j, k) \in A^c$. Therefore,

$$\{(j, k) : \tau(\xi_{jk} - \eta, r + u) \not> \mathcal{N}(\varepsilon)\} \subset A,$$

and since $\delta_2(A) = 0$, we have

$$\delta_2(\{(j, k) : \tau(\xi_{jk} - \eta, r + u) \not> \mathcal{N}(\varepsilon)\}) = 0.$$

Consequently, $\eta \in (r-st)_{\mathcal{L}}\text{-LIM}_{\xi_{jk}}$. □

4. Conclusions

This study establishes a comprehensive framework for the rough convergence and statistical convergence in L-fuzzy normed spaces. It introduces rough statistical convergence for double sequences and investigates their fundamental properties. These investigations clarify the relationships among rough statistical convergence and boundedness. Additionally, the paper defines rough statistical limit points and rough statistical cluster points and establishes their relationship. As the results obtained here are more general than those for fuzzy normed spaces, some findings parallel classical ones, yet the proofs follow a different approach. Beyond its theoretical contributions, this study addresses a gap in the literature by precisely characterizing convergence in fuzzy settings. We believe this paper leaves sufficient scope for future work. On the other hand, we are believed to apply to addressing complex problems across diverse real-world domains, including stability analysis in fuzzy proportional-integral-derivative control [11] and structural design processes in topology optimization [36].

Author contributions

Aykut Or: Investigation, validation, writing–original draft preparation, conceptualization, methodology, writing–reviewing and editing; Ahmet Çakı: Investigation, validation, writing–original draft preparation, conceptualization, methodology, writing–reviewing and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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