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Research article

On the multivalued ρ_*-interpolative contractions in fuzzy metric spaces with application to nonlinear matrix equations

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Abstract: In this study, the subject of the multivalued ρ_*-interpolative Ćirić-Reich-Rus-type fuzzy contractions is introduced and investigated in which the ϑ-comparison functions and the property of the ρ_*-admissibility play an important role. In the first step, the existence of fixed point theorems is proven for such a type of contractions in the context of the complete fuzzy metric spaces. Then, some of the results are extended in the framework of the fuzzy metric spaces equipped with a partial order. In this direction, we give some examples to clarify the obtained results and definitions. Additionally, we demonstrate an application about the solutions of non-linear matrix equations on the basis of fixed points of these new contractions.

Keywords: interpolative contraction; ρ_*-admissible multivalued mapping; fuzzy metric space; partially ordered fuzzy metric space
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

The Banach fixed point theorem has widespread applications in various mathematical fields [1]. It is used to find solutions to ordinary and partial differential equations, optimization problems, and integral equations. Furthermore, its importance extends beyond mathematics such as biology, economics, game theory, and so on. This theorem has been fundamentally extended via various methodological forms. It significantly contributes to the advancement of mathematical theory and applications. In order to increase the usability of the Banach fixed point theorem, a variety of contractive mappings were provided with appropriate conditions. In this direction, it is necessary to present some of the related works.

A large number of researchers have defined contractive-type mappings on complete metric spaces which are generalizations of the standard Banach contraction principle. For instances, Rhoades [2] presented a study that compared the contractions described by important researchers such as Kannan [3], Reich [4], Ciric [5], Rus [6], and Hardy-Rogers [7] with their own names. Recently, Erdal Karapınar's works, interpolative Ćirić-Reich-Rus-type, and Kannan-type contractions, have been the focus of much attention [8–10]. Afterwards, the subject of interpolative contractions entered the literature as a different contraction with important applications of the fixed point theory. Sameul et al. [11, 12] presented new forms in the extended orthogonal S-metric spcaes and controlled orthogonal δ-metric-type spcaes. Devi et al. [13] developed the interpolative cyclic contractions on b-metric spaces. Sezen [14] studied the interpolative best proximity point results under the γ-contractions. In 2025, Jain et al. [15] continued these studies on fuzzy metric spaces and introduced the fuzzy F-interpolative Berinde weak contractions. In the sma eyear, Tomar et al. [16] introduced the ω-interpolative Hardy-Rogers contractions in b-metric spaces, and finally, in another collaboration between Tomar and other colleagues [17] used the geometry of φ-interpolative contractions in neural networks and machine learning models.

Nadler first generalized the Banach fixed point theorem to multivalued fixed point theorems [18]. Afterwards, different versions of multivalued mappings with appropriate conditions were presented, and more general fixed point theorems were obtained for these conractions. Different versions of many contractions for multivalued mappings have been presented in the literature. Of course, along with theses, different fixed point theorems have been proven using multivalued mappings for interpolative contractions (see [19–21]).

On the other hand, after Zadeh [22], who took his first step on the subject of fuzzy sets, these sets became the focus of research in fuzzy metric spaces, which have been extensively studied and have different versions such as [23–25]. Rodríguez-López and Romaguera were the first researchers to introduce the Hausdorff fuzzy metric on the family of the compact nonempty sets in fuzzy metric spaces [26]. Next, the contractions presented in fuzzy metric spaces were generalized for multivalued mappingss in Hausdorff fuzzy metric spaces (see [27]). For other forms in fuzzy metric spaces, the readers can refer to [28–31].

This article demonstrates the multivalued ρ_*-interpolative Ćirić-Reich-Rus-type fuzzy contraction's definition using the ϑ-comparison function. We prove our main theorem in complete fuzzy metric spaces. Additionally, we add fixed point theorems for multivalued ρ_*-interpolative Kannan type contractions, which are the result of our main theorem. We present a few examples to demonstrate the validity of the main theorems. Afterwards, we show that some theorems that previously existed in the literature could be derived from the theorems in our paper. In order to further generalize the theorems in our main theorem, we re-prove them again in partially ordered fuzzy metric spaces defined with a partial order on a metric space. Finally, we present an application of a theorem which shows that the solution of non-linear matrix equations exists.

Regarding the importance of our results, note that every single-valued mapping can be seen as a special case of a multi-valued mapping. While a single-valued mapping has no fixed point, how can a multi-valued mapping have a fixed point? Every single-valued mapping can be seen as a special case of a set-valued mapping, that is, a multi-valued mapping is more general than an self-mapping. Therefore, the results obtained in this study are more general.

2. Some preliminary definitions

Some preliminary definitions are recollected in this section.

Definition 2.1. *[32] A t-norm is a binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *such that*

(1) $*$ *has two properties: commutativity and associativity;*
(2) $*$ *is continuous;*
(3) $*(\tilde{u}, 1) = \tilde{u}$, $\forall\, \tilde{u} \in [0,1]$*; and*
(4) $*(\tilde{u}, \tilde{w}) \leq *(\tilde{a}, \tilde{o})$ *if* $\tilde{u} \leq \tilde{a}$, $\tilde{w} \leq \tilde{o}$, *for all* $\tilde{u}, \tilde{w}, \tilde{a}, \tilde{o} \in [0,1]$.

Two known instances of a continuous t-norm are as follows:

(i) Product operator: $*_p$: $\tilde{u} * \tilde{w} = \tilde{u} \cdot \tilde{w}$; and
(ii) Minimum operator: $*_m$: $\tilde{u} * \tilde{w} = \min\{\tilde{u}, \tilde{w}\}$.

Definition 2.2. *[33] Assume that* $\tilde{G} \neq \emptyset$, $*$ *is a t-norm and* τ *is a fuzzy metric on* $\tilde{G}^2 \times (0, \infty)$. *Then,* $(\tilde{G}, \tau, *)$ *is said to be a fuzzy metric space (FMS) if for all* $\tilde{u}, \tilde{w}, \tilde{y} \in \tilde{G}$ *and* $\check{r}, \tilde{s} > 0$,

(1) $\tau(\tilde{u}, \tilde{w}, \tilde{s}) > 0$*;*
(2) $\tau(\tilde{u}, \tilde{w}, \tilde{s}) = 1$ *if and only if* $\tilde{u} = \tilde{w}$*;*
(3) $\tau(\tilde{u}, \tilde{w}, \tilde{s}) = \tau(\tilde{w}, \tilde{u}, \tilde{s})$*;*
(4) $\tau(\tilde{u}, \tilde{w}, \tilde{s} + \check{r}) \geq \tau(\tilde{u}, \tilde{y}, \tilde{s}) * \tau(\tilde{y}, \tilde{w}, \check{r})$*; and*
(5) $\tau(\tilde{u}, \tilde{w}, \cdot) : (0, \infty) \to [0,1]$ *is continuous.*

Definition 2.3. *[33] Let* $(\tilde{G}, \tau, *)$ *be an FMS and* $\{\tilde{u}_n\}$ *be a sequence in* $\tilde{G}$. *Then,*

(1) $\{\tilde{u}_n\}$ *converges to* $\tilde{u}$ *in* $\tilde{G}$, *illustrated as* $\tilde{u}_n \to \tilde{u}$, *if* $\lim_{n \to +\infty} \tau(\tilde{u}_n, \tilde{u}, \tilde{s}) = 1$ *for all* $\tilde{s} > 0$*;*
(2) $\{\tilde{u}_n\}$ *is Cauchy if for all* $\xi \in (0,1)$ *and* $\tilde{s} > 0$, $\exists\, n_0 \in \mathbb{N}$ *s.t.* $\tau(\tilde{u}_n, \tilde{u}_m, \tilde{s}) \geq 1 - \xi$ *for all* $m > n \geq n_0$*; and*
(3) an FMS is complete if every Cauchy sequence is convergent .

Let $(\tilde{G}, \tau, *)$ be an FMS. Consider the following two classes of sets in the FMS $\tilde{G}$:

$$CL(\tilde{G}) = \{X : X \text{ is a nonempty closed set in } \tilde{G}\},$$

$$K(\tilde{G}) = \{X : X \text{ is a nonempty compact set in } \tilde{G}\}.$$

Let $\mathcal{R} : \tilde{G} \to K(\tilde{G})$ be a multivalued mapping on $\tilde{G}$. A point $\tilde{u} \in \tilde{G}$ is called a fixed point of $\mathcal{R}$ if and only if $\tilde{u} \in \mathcal{R}\tilde{u}$.

Definition 2.4. *[26] A fuzzy metric τ is continuous on $\tilde{G}^2 \times (0, \infty)$ if*

$$\lim_{n\to+\infty} \tau(\tilde{u}_n, \tilde{w}_n, \tilde{s}_n) = \tau(\tilde{u}, \tilde{w}, \tilde{s}),$$

whenever $\{\tau(\tilde{u}_n, \tilde{w}_n, \tilde{s}_n)\}$ is a sequence in $\tilde{G}^2 \times (0, \infty)$, converging to $(\tilde{u}, \tilde{w}, \tilde{s}) \in \tilde{G}^2 \times (0, \infty)$.

Definition 2.5. *[26] Let $(\tilde{G}, \tau, *)$ be an FMS. The Hausdorff fuzzy metric induced by a fuzzy metric τ is defined by*

$$\tilde{H}(A, B, \tilde{s}) = \min\left\{\inf_{a\in A}\tau(a, B, \tilde{s}), \inf_{b\in B}\tau(A, b, \tilde{s})\right\}, \qquad A, B \in K(\tilde{G}),$$

for all $\tilde{s} > 0$, where $\tau(\tilde{u}, A, \tilde{s}) = \sup_{a\in A} \tau(\tilde{u}, a, \tilde{s})$ with $\tilde{u} \in \tilde{G}$. Then, $(K(\tilde{G}), \tilde{H}, *)$ is a Hausdorff FMS.

Lemma 2.6. *[26] Let $\tilde{G}$ be an FMS, and let $\tilde{u} \in \tilde{G}$, $B \in K(\tilde{G})$, and $\tilde{s} > 0$ be arbitrary. Then, $\exists\, b_0 \in A$ s.t.*

$$\tau(\tilde{u}, B, \tilde{s}) = \tau(\tilde{u}, b_0, \tilde{s}).$$

Lemma 2.7. *[26] Consider the metric space $(\tilde{G}, d_*)$ and FMS $(\tilde{G}, \tau, *)$. In this case, the Hausdorff fuzzy metric $(\tilde{H}_{\tau_{d_*}}, \cdot)$ induced by the fuzzy metric $(\tau_{d_*}, \cdot)$ coincides with the standard fuzzy metric $(\tau_{\tilde{H}_{d_*}}, \cdot)$ induced by the Hausdorff metric $\tilde{H}_{d_*}$ on $K(\tilde{G})$.*

Now, based on [34], we consider a specific function $\vartheta : [0, 1] \longrightarrow [0, 1]$ so that ϑ is left continuous and nondecreasing; additionally, $\vartheta(\lambda) > \lambda$ for all $\lambda \in (0, 1)$. We denote the family of all such functions ϑ on $[0, 1]$ by Φ. In the next lemma, we review some important properties for ϑ.

Lemma 2.8. *[34] The function ϑ has the following properties:*

(1) $\vartheta(1) = 1$; and
(2) $\lim_{n\to+\infty} \vartheta^n(\lambda) = 1$ for all $\lambda \in (0, 1)$, where $\vartheta^n(\lambda)$ denotes the composition of $\vartheta(\lambda)$ with itself n times.

Definition 2.9. *[35] Consider an FMS $(\tilde{G}, \tau, *)$ and a function $\mathcal{R} : \tilde{G} \to \tilde{G}$. Then, $\mathcal{R}$ is a triangular ρ_*-admissible mapping if $\exists\,\rho_* : \tilde{G} \times \tilde{G} \times (0, \infty) \to (0, \infty)$, provided that*

(1) if $\rho_*(\tilde{u}, \tilde{w}, \tilde{s}) \le 1$, then $\rho_*(\mathcal{R}\tilde{u}, \mathcal{R}\tilde{w}, \tilde{s}) \le 1$ for all $\tilde{u}, \tilde{w} \in \tilde{G}$ and $\tilde{s} > 0$; and
(2) if $\rho_*(\tilde{u}, \tilde{w}, \tilde{s}) \le 1$ and $\rho_*(\tilde{w}, \tilde{y}, \tilde{s}) \le 1$, then $\rho_*(\tilde{u}, \tilde{y}, \tilde{s}) \le 1$ for all $\tilde{u}, \tilde{w}, \tilde{y} \in \tilde{G}$ and $\tilde{s} > 0$.

Lemma 2.10. *[35] Assume that $(\tilde{G}, \tau, *)$ and $\mathcal{R}$ are an FMS and a triangular ρ_*-admissible mapping, respectively. Let $\exists\, \tilde{u}_0 \in X$ s.t. $\rho_*(\tilde{u}_0, \mathcal{R}\tilde{u}_0, \tilde{s}) \le 1$. Then, a sequence $\{\tilde{u}_n\}$, formulated by $\mathcal{R}\tilde{u}_n = \tilde{u}_{n+1}$ for all $n \in \mathbb{N}$, satisfies the following:*

$$\rho_*(\tilde{u}_m, \tilde{u}_n, \tilde{s}) \le 1, \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

Definition 2.11. *[36] Let $(\tilde{G}, \tau, *)$ be an FMS. $\mathcal{R} : \tilde{G} \to \tilde{G}$ is a ρ_*-interpolative Ćirić-Reich-Rus-type fuzzy contraction if there are $\vartheta \in \Phi$, $\rho_* : \tilde{G} \times \tilde{G} \times (0, \infty) \to (0, \infty)$ and positive real numbers γ, β with $\gamma + \beta < 1$ provided that*

$$\rho_*(\tilde{u}, \tilde{w}, \tilde{s})\tau(\mathcal{R}\tilde{u}, \mathcal{R}\tilde{w}, \tilde{s}) \ge \vartheta((\tau(\tilde{u}, \tilde{w}, \tilde{s}))^{\gamma}(\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}))^{\beta}(\tau(\tilde{w}, \mathcal{R}\tilde{w}, \tilde{s}))^{1-\gamma-\beta}),$$

for all $\tilde{u}, \tilde{w} \in \tilde{G} \backslash Fix(\mathcal{R})$.

Definition 2.12. *[36] Let $(\tilde{G}, \tau, *)$ be an FMS. $\mathcal{R} : \tilde{G} \to \tilde{G}$ is a ρ_*-interpolative Kannan-type fuzzy contraction if there are $\vartheta \in \Phi$, $\rho_* : \tilde{G} \times \tilde{G} \times (0, \infty) \to (0, \infty)$ and positive real number γ with $\gamma \in (0, 1)$ provided that*

$$\rho_*(\tilde{u}, \tilde{w}, \tilde{s})\tau(\mathcal{R}\tilde{u}, \mathcal{R}\tilde{w}, \tilde{s}) \ge \vartheta((\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}))^{\gamma}(\tau(\tilde{w}, \mathcal{R}\tilde{w}, \tilde{s}))^{1-\gamma}),$$

for all $\tilde{u}, \tilde{w} \in \tilde{G} \backslash Fix(\mathcal{R})$ and $\tilde{s} > 0$.

3. Main results

The present section aims to introduce two new forms of the ρ_*-interpolative Ćirić-Reich-Rus-type and Kannan-type fuzzy contractions in the multivalued versions and to prove some fixed point theorems in this direction.

Definition 3.1. *Assume that $\tilde{G} \neq \emptyset$, $\mathcal{R} : \tilde{G} \to CL(\tilde{G})$ and $\rho_* : \tilde{G} \times \tilde{G} \times (0,\infty) \to (0,\infty)$. Then, $\mathcal{R}$ is a ρ_*-admissible multivalued mapping if for each $\tilde{u} \in \tilde{G}$ and $\tilde{w} \in \mathcal{R}\tilde{u}$ with $\rho_*(\tilde{u}, \tilde{w}, \tilde{s}) \leq 1$, we have $\rho_*(\tilde{w}, \tilde{y}, \tilde{s}) \leq 1$ for each $\tilde{y} \in \mathcal{R}\tilde{w} \in \mathcal{R}$ and $\tilde{s} > 0$.*

Example 3.2. *Let $\tilde{G} = \mathbb{N} \cup \{0\}$, τ be a fuzzy metric on $\tilde{G}^2 \times (0,\infty)$ given by*

$$\tau(\tilde{u}, \tilde{w}, s) = \frac{\tilde{s}^2}{\tilde{s}^2 + d_*(\tilde{u}, \tilde{w})},$$

for all $\tilde{s} > 0$, and d_* be the usual metric for every $\tilde{u}, \tilde{w} \in \tilde{G}$. Define the mappings $\rho_* : \tilde{G} \times \tilde{G} \times (0,\infty) \to (0,\infty)$ and $\mathcal{R} : \tilde{G} \to CL(\tilde{G})$ by the following:

$$\mathcal{R}\tilde{u} = \begin{cases} \{0, 1\} & , \quad \tilde{u} = 0, 1, \\ \{\tilde{u}, \tilde{u} + 1\} & , \quad \tilde{u} > 1, \end{cases}$$

$$\rho_*(\tilde{u}, \tilde{w}, \tilde{s}) = \begin{cases} \dfrac{1}{4} & , \quad \tilde{u}, \tilde{w} \in \{0, 1\}, \\ e^{-(\tilde{u}+\tilde{w})} & , \quad \tilde{u}, \tilde{w} > 1, \\ 1 & , \quad otherwise. \end{cases}$$

Clearly, $\mathcal{R}$ is ρ_*-admissible.

Example 3.3. *Let $\tilde{G} = \mathbb{N} \cup \{0\}$, τ be a fuzzy metric on $\tilde{G}^2 \times (0,\infty)$ given by*

$$\tau(\tilde{u}, \tilde{w}, s) = \frac{\tilde{s}}{\tilde{s} + d_*(\tilde{u}, \tilde{w})},$$

for all $\tilde{s} > 0$, and d_* be the usual metric for every $\tilde{u}, \tilde{w} \in \tilde{G}$. Define the mappings $\rho_* : \tilde{G} \times \tilde{G} \times (0,\infty) \to (0,\infty)$ and $\mathcal{R} : \tilde{G} \to CL(\tilde{G})$ by the following:

$$\mathcal{R}\tilde{u} = \begin{cases} [2, 2 + \frac{\tilde{u}}{4}] & , \quad \tilde{u} \in [0, 1], \\ [4, 4\tilde{u}] & , \quad \tilde{u} \in (1, \infty), \end{cases}$$

$$\rho_*(\tilde{u}, \tilde{w}, \tilde{s}) = \begin{cases} \dfrac{1}{4} & , \quad \tilde{u}, \tilde{w} \in [0, 1], \\ e^{-(\tilde{u}+\tilde{w})} & , \quad otherwise. \end{cases}$$

Clearly, $\mathcal{R}$ is ρ_*-admissible.

Definition 3.4. *Assume $(\tilde{G}, \tau, *)$ is an FMS. Then, $\mathcal{R} : \tilde{G} \to \tilde{G}$ is a multivalued ρ_*-interpolative Ćirić-Reich-Rus-type fuzzy contraction if there are $\vartheta \in \Phi$, $\rho_* : \tilde{G} \times \tilde{G} \times (0,\infty) \to (0,\infty)$ and positive real numbers γ, β with $\gamma + \beta < 1$ such that*

$$\rho_*(\tilde{u}, \tilde{w}, \tilde{s})\tilde{H}(\mathcal{R}\tilde{u}, \mathcal{R}\tilde{w}, \tilde{s}) \geq \vartheta((\tau(\tilde{u}, \tilde{w}, \tilde{s}))^{\gamma}(\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}))^{\beta}(\tau(\tilde{w}, \mathcal{R}\tilde{w}, \tilde{s}))^{1-\gamma-\beta}), \tag{3.1}$$

for all $\tilde{u}, \tilde{w} \in \tilde{G} \backslash Fix(\mathcal{R})$ and $\tilde{s} > 0$.

Now, the main existence theorem is stated below.

Theorem 3.5. *Assume that $(\tilde{G}, \tau, *)$ and $\mathcal{R} : \tilde{G} \to K(\tilde{G})$ are a complete FMS and a continuous multivalued ρ_*-interpolative Ćirić-Reich-Rus-type fuzzy contraction, respectively. Suppose that*

(1) $\mathcal{R}$ is ρ_*-admissible; and
(2) $\exists\, \tilde{u}_0 \in \tilde{G}$ and $\exists\, \tilde{u}_1 \in \mathcal{R}\tilde{u}_0$ s.t. $\rho(\tilde{u}_0, \tilde{u}_1, \tilde{s}) \leq 1$ for $\tilde{s} > 0$.

Then, $\mathcal{R}$ has a fixed point.

Proof. We assume that $\tilde{u}_0$ is a given point in $\tilde{G}$. As $\mathcal{R}\tilde{u}_0 \in K(\tilde{G})$, we can select $\tilde{u}_1 \in \mathcal{R}\tilde{u}_0$. From the condition (2), we have $\rho_*(\tilde{u}_0, \tilde{u}_1, \tilde{s}) \leq 1$. If $\tilde{u}_1 \in \mathcal{R}\tilde{u}_1$, then $\tilde{u}_1$ is a fixed point of $\mathcal{R}$, and this completes the proof. Let $\tilde{u}_1 \notin \mathcal{R}\tilde{u}_1$; since $\mathcal{R}\tilde{u}_1$ is closed, $\tau(\tilde{u}_1, \mathcal{R}\tilde{u}_1, \tilde{s}) < 1$. Since $\tau(\tilde{u}_1, \mathcal{R}\tilde{u}_1, \tilde{s}) \geq \tilde{H}(\mathcal{R}\tilde{u}_0, \mathcal{R}\tilde{u}_1, \tilde{s})$, from Inequality (3.1), we obtain

$$\begin{aligned}
\tau(\tilde{u}_1, \mathcal{R}\tilde{u}_1, \tilde{s}) &\geq \tilde{H}(\mathcal{R}\tilde{u}_0, \mathcal{R}\tilde{u}_1, \tilde{s}) \\
&\geq \rho_*(\tilde{u}_0, \tilde{u}_1, \tilde{s})\tilde{H}(\mathcal{R}\tilde{u}_0, \mathcal{R}\tilde{u}_1, \tilde{s}) \\
&\geq \vartheta((\tau(\tilde{u}_0, \tilde{u}_1, \tilde{s}))^{\gamma}(\tau(\tilde{u}_1, \mathcal{R}\tilde{u}_1, \tilde{s}))^{\beta}(\tau(\tilde{u}_0, \mathcal{R}\tilde{u}_0, \tilde{s}))^{1-\gamma-\beta}) \\
&= \vartheta((\tau(\tilde{u}_0, \tilde{u}_1, \tilde{s}))^{\gamma}(\tau(\tilde{u}_1, \tilde{u}_2, \tilde{s}))^{\beta}(\tau(\tilde{u}_0, \tilde{u}_1, \tilde{s}))^{1-\gamma-\beta}) \\
&= \vartheta((\tau(\tilde{u}_0, \tilde{u}_1, \tilde{s}))^{1-\beta}(\tau(\tilde{u}_1, \tilde{u}_2, \tilde{s}))^{\beta}) \\
&> (\tau(\tilde{u}_0, \tilde{u}_1, \tilde{s}))^{1-\beta}(\tau(\tilde{u}_1, \tilde{u}_2, \tilde{s}))^{\beta},
\end{aligned} \tag{3.2}$$

for all $\tilde{s} > 0$. Then, if Inequality (3.2) is used, then

$$(\tau(\tilde{u}_1, \tilde{u}_2, \tilde{s}))^{1-\beta} > (\tau(\tilde{u}_0, \tilde{u}_1, \tilde{s}))^{1-\beta}$$

implies that

$$\tau(\tilde{u}_1, \tilde{u}_2, \tilde{s}) > \tau(\tilde{u}_0, \tilde{u}_1, \tilde{s}).$$

On similar lines, we obtain a sequence $\{\tilde{u}_n\}$ such that $\tilde{u}_n \in \mathcal{R}\tilde{u}_{n-1}$, $\tilde{u}_n \neq \tilde{u}_{n-1}$, which implies the following:

$$\rho(\tilde{u}_n, \tilde{u}_{n+1}, \tilde{s}) \leq 1 \quad \text{and} \quad \tau(\tilde{u}_{n+1}, \tilde{u}_n, s) > \tau(\tilde{u}_n, \tilde{u}_{n-1}, \tilde{s}). \tag{3.3}$$

In this case, $\{\tau(\tilde{u}_n, \tilde{u}_{n+1}, \tilde{s})\}$ is a nondecreasing sequence. Eventually, there exists a number $0 \leq \ell \leq 1$ s.t. $\lim_{n\to+\infty} \tau(\tilde{u}_n, \tilde{u}_{n+1}, \tilde{s}) = \ell$. From (3.3), we have the following:

$$\begin{aligned}
(\tau(\tilde{u}_n, \tilde{u}_{n-1}, \tilde{s}))^{1-\beta}(\tau(\tilde{u}_n, \tilde{u}_{n+1}, \tilde{s}))^{\beta} &\geq (\tau(\tilde{u}_n, \tilde{u}_{n-1}, \tilde{s}))^{1-\beta}(\tau(\tilde{u}_n, \tilde{u}_{n-1}, \tilde{s}))^{\beta} \\
&= \tau(\tilde{u}_n, \tilde{u}_{n-1}, \tilde{s}).
\end{aligned} \tag{3.4}$$

Therefore, from (3.2), (3.4), and since ϑ is left continuous and has the nondecreasing property, it leads to the following:

$$\tau(\tilde{u}_n, \tilde{u}_{n+1}, \tilde{s}) \geq \vartheta((\tau(\tilde{u}_n, \tilde{u}_{n-1}, \tilde{s}))^{1-\beta}(\tau(\tilde{u}_n, \tilde{u}_{n+1}, \tilde{s}))^{\beta})$$

$$\geq \vartheta(\tau(\tilde{u}_n, \tilde{u}_{n-1}, \tilde{s})).$$

By repeating this argument, it becomes

$$\begin{aligned}\tau(\tilde{u}_n, \tilde{u}_{n+1}, \tilde{s}) &\geq \vartheta(\tau(\tilde{u}_n, \tilde{u}_{n-1}, \tilde{s})) \geq \vartheta^2(\tau(\tilde{u}_{n-1}, \tilde{u}_{n-2}, \tilde{s})) \\ &\geq \cdots \geq \vartheta^n(\tau(\tilde{u}_0, \tilde{u}_1, \tilde{s})), \end{aligned} \tag{3.5}$$

for all $n \in \mathbb{N}$. As $n \to +\infty$, (3.5) gives the following:

$$\lim_{n\to\infty} \tau(\tilde{u}_n, \tilde{u}_{n+1}, \tilde{s}) = 1. \tag{3.6}$$

Consider the opposite to show that $\{\tilde{u}_n\}$ is a Cauchy sequence, that is, assume that the sequence $\{\tilde{u}_n\}$ is not Cauchy. Then, there are $\xi \in (0, 1)$ and $\tilde{s}_0 > 0$ provided that for all $k \in \mathbb{N}$, $\exists\, n_k, m_k \in \mathbb{N}$ with $m_k > n_k \geq k$ and

$$\tau(\tilde{u}_{m_k}, \tilde{u}_{n_k}, \tilde{s}_0) \leq 1 - \xi\ . \tag{3.7}$$

This implies that

$$\tau(\tilde{u}_{m_k}, \tilde{u}_{n_k}, \frac{\tilde{s}_0}{2}) \leq 1 - \xi. \tag{3.8}$$

Let n_k be the least integer exceeding m_k which satisfies Inequality (3.8). Consequently,

$$\tau(\tilde{u}_{m_k-1}, \tilde{u}_{n_k}, \frac{\tilde{s}_0}{2}) > 1 - \xi\ . \tag{3.9}$$

Inequality (3.1) with $\tilde{u} = \bar{a}_{m_k-1}$, $\tilde{w} = \tilde{u}_{n_k-1}$ and $\tilde{s} = \tilde{s}_0$ gives the following:

$$\tau(\tilde{u}_{m_k}, \tilde{u}_{n_k}, \tilde{s}_0) > \vartheta(\tau(\tilde{u}_{m_k-1}, \tilde{u}_{n_{k-1}}, \tilde{s}_0)).$$

As ϑ is nondecreasing, we obtain the following:

$$\tau(\tilde{u}_{m_k}, \tilde{u}_{n_k}, \tilde{s}_0) > \tau(\tilde{u}_{m_k}, \tilde{u}_{n_k}, \tilde{s}_0). \tag{3.10}$$

Now, using (3.7), (3.9), and (3.10), we have the following:

$$\begin{aligned} 1 - \xi\ &\geq \tau(\tilde{u}_{m_k}, \tilde{u}_{n_k}, \tilde{s}_0) \\ &> \tau(\tilde{u}_{m_k-1}, \tilde{u}_{n_{k-1}}, \tilde{s}_0) \\ &\geq \tau(\tilde{u}_{m_k-1}, \tilde{u}_{n_k}, \frac{\tilde{s}_0}{2}) * \tau(\tilde{u}_{n_k}, \tilde{u}_{n_k-1}, \frac{\tilde{s}_0}{2}) \\ &> (1 - \xi\ ) * \tau(\tilde{u}_{n_k}, \tilde{u}_{n_k-1}, \frac{\tilde{s}_0}{2}). \end{aligned} \tag{3.11}$$

By tending $k \to +\infty$ in (3.11), we obtain the following:

$$\lim_{k\to\infty} \tau(\tilde{u}_{m_k}, \tilde{u}_{n_k}, \tilde{s}_0) = 1 - \xi\ . \tag{3.12}$$

Using Inequality (3.1) with $\tilde{u} = \tilde{u}_{m_k-1}$, $\tilde{w} = \tilde{u}_{n_k-1}$, and $\tilde{s} = \tilde{s}_0$, we have the following

$$\begin{aligned}
&\tau(\tilde{u}_{m_k}, \tilde{u}_{n_k}, \tilde{s}_0)\\
&\geq \rho_*(\tilde{u}_{m_{k-1}}, \tilde{u}_{n_{k-1}}, \tilde{s}_0)\tilde{H}(\mathcal{R}\tilde{u}_{m_{k-1}}, \mathcal{R}\tilde{u}_{n_{k-1}}, \tilde{s}_0)\\
&\geq \vartheta((\tau(\tilde{u}_{m_k-1}, \tilde{u}_{n_k-1}, \tilde{s}_0))^{\gamma}(\tau(\tilde{u}_{m_k-1}, \mathcal{R}\tilde{u}_{m_k-1}, \tilde{s}_0))^{\beta}(\tau(\tilde{u}_{n_k-1}, \mathcal{R}\tilde{u}_{n_k-1}, \tilde{s}_0))^{(1-\gamma-\beta)})\\
&= \vartheta((\tau(\tilde{u}_{m_k-1}, \tilde{u}_{n_k-1}, \tilde{s}_0))^{\gamma}(\tau(\tilde{u}_{m_k-1}, \tilde{u}_{m_k}, \tilde{s}_0))^{\beta}(\tau(\tilde{u}_{n_k-1}, \tilde{u}_{n_k}, \tilde{s}_0))^{(1-\gamma-\beta)}).
\end{aligned} \tag{3.13}$$

Tending $k \to +\infty$ in (3.13), and using (3.1), continuity of ϑ, and by (3.6), (3.12), we have the following:

$$(1-\xi) \geq \vartheta((1-\xi)^{\gamma}(1-\xi)^{\beta}(1-\xi\ )^{(1-\gamma-\beta)}).$$

Then, we have

$$(1-\xi) \geq \vartheta((1-\xi)) > (1-\xi),$$

which leads to a contradiction. Hence, $\{\tilde{u}_n\}$ is Cauchy in $\tilde{G}$. Since $\tilde{G}$ is complete, $\exists\, \tilde{u} \in \tilde{G}$ s.t. $\lim_{n\to\infty} \tilde{u}_n = \tilde{u}$. On the other hand, $\mathcal{R}$ is continuous, so

$$\lim_{n\to\infty} \mathcal{R}\tilde{u}_n = \mathcal{R}\tilde{u}.$$

Since the expression

$$\tau(\tilde{u}_n, \mathcal{R}\tilde{u}, \tilde{s}) \geq \tilde{H}(\mathcal{R}\tilde{u}_n, \mathcal{R}\tilde{u}, \tilde{s}) \tag{3.14}$$

is provided, then, as $n \to +\infty$ in (3.14), we obtain $\tau(\tilde{u}_n, \mathcal{R}\tilde{u}, \tilde{s}) = 1$. The compactness of $\mathcal{R}\tilde{u}$ implies that $\tilde{u} \in \mathcal{R}\tilde{u}$. □

In the sequel, we replace the following property instead of the continuity of $\mathcal{R}$:

(A) If $\{\tilde{u}_n\} \subseteq \tilde{G}$ is so that $\rho_*(\tilde{u}_n, \tilde{u}_{n+1}, \tilde{s}) \leq 1$ for all $n \in \mathbb{N}$ and $\tilde{u}_n \to \tilde{u}$ as $n \to \infty$, then $\exists\,\{\tilde{u}_{n_k}\}$ from $\{\tilde{u}_n\}$ s.t. $\rho_*(\tilde{u}_{n_k}, \tilde{u}, \tilde{s}) \leq 1$ for all k.

Theorem 3.6. *Assume that $(\tilde{G}, \tau, *)$ and $\mathcal{R} : \tilde{G} \to K(\tilde{G})$ are a complete FMS and a multivalued ρ_*-interpolative Ćirić-Reich-Rus-type fuzzy contraction, respectively. Suppose that*

(1) condition (A) is satisfied;
(2) $\mathcal{R}$ is ρ_*-admissible; and
(3) $\exists\, \tilde{u}_0 \in \tilde{G}$ and $\exists\, \tilde{u}_1 \in \mathcal{R}\tilde{u}_0$ s.t. $\rho(\tilde{u}_0, \tilde{u}_1, \tilde{s}) \leq 1$ for $\tilde{s} > 0$.

Then, $\mathcal{R}$ has a fixed point.

Proof. Considering the similar steps in the proof of Theorem 3.5, there exists a Cauchy sequence $\{\tilde{u}_n\}$ that converges to some $\tilde{u}$. Assume that $\tilde{u} \notin \mathcal{R}\tilde{u}$. Note that $\tilde{u}_{n(k)} \notin \mathcal{R}\tilde{u}_{n(k)}$ for all $k \geq 0$. According to (A), a subsequence $\{\tilde{u}_{n(k)}\}$ of $\{\tilde{u}_n\}$ exists provided that $\rho_*(\tilde{u}_{n(k)}, \tilde{u}, \tilde{s}) \leq 1$ for all $k \geq 0$. On the other side, $\{\tau(\tilde{u}_{n(k)+1}, \tilde{u}, \tilde{s})\} \to 1$, $\{\tau(\tilde{u}_{n(k)}, \mathcal{R}\tilde{u}_{n(k)}, \tilde{s})\} \to 1$ and $\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}) < 1$. Thus, $\exists\, m \in \mathbb{N}$ such that for all $k \geq m$,

$$\tau(\tilde{u}_{n_k}, \tilde{u}, \tilde{s}) \geq \tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}) \quad \text{and} \quad \tau(\tilde{u}_{n_k}, \mathcal{R}\tilde{u}_{n_k}, \tilde{s}) \geq \tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}).$$

By (3.1), and setting $\tilde{u} = \tilde{u}_{n_k}$ and $\tilde{w} = \tilde{u}$, we obtain the following:

$$\tau(\tilde{u}_{n_k+1}, \mathcal{R}\tilde{u}, \tilde{s}) \geq \rho_*(\tilde{u}_{n_k}, \tilde{u}, \tilde{s})\tau(\mathcal{R}\tilde{u}_n, \mathcal{R}\tilde{u}, \tilde{s})$$

$$\geq \vartheta((\tau(\tilde{u}_{n_k+1}, \tilde{u}, \tilde{s}))^{\gamma}(\tau(\tilde{u}_{n_k}, \mathcal{R}\tilde{u}_{n_k}, \tilde{s}))^{\beta}(\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}))^{1-\gamma-\beta}). \tag{3.15}$$

Since ϑ is nondecreasing, from (3.15), we have the following

$$\begin{aligned}\tau(\tilde{u}_{n+1}, \mathcal{R}\tilde{u}, \tilde{s}) &\geq \vartheta((\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}))^{\gamma}(\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}))^{\beta}(\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}))^{1-\gamma-\beta}) \\ &= \vartheta(\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s})).\end{aligned}$$

Letting $k \to +\infty$,

$$\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}) \geq \vartheta(\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s})) > \tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}),$$

which is a contradiction. Thus, $\tilde{u} \in \mathcal{R}\tilde{u}$. □

Example 3.7. *Let $\tilde{G} = \{0, 1, 2\}$, τ be a fuzzy metric on $\tilde{G}^2 \times (0, \infty)$ given by the following:*

$$\tau(\tilde{u}, \tilde{w}, s) = \frac{\tilde{s}^2}{\tilde{s}^2 + d_*(\tilde{u}, \tilde{w})}, \quad \tilde{s} > 0,$$

where d_* is the usual metric for all $\tilde{u}, \tilde{w} \in \tilde{G}$. Define the mappings $\rho_* : \tilde{G} \times \tilde{G} \times (0, \infty) \to (0, \infty)$ and $\mathcal{R} : \tilde{G} \to CL(\tilde{G})$ by

$$\mathcal{R}\tilde{u} = \begin{cases} \{0\} & , \quad \tilde{u} = 1, \\ \{0, 1\} & , \quad \tilde{u} \neq 1, \end{cases}$$

and

$$\rho_*(\tilde{u}, \tilde{w}, \tilde{s}) = \begin{cases} 1 & , \quad \tilde{u} \in \{0, 1\} \text{ and } \tilde{w} \in \{2\} \text{ or } \tilde{u}, \tilde{w} \in \{0\}, \\ \left(\dfrac{\tilde{s}^2}{\tilde{s}^2 + 1}\right)^{\frac{3}{10}} & , \quad \text{otherwise}, \end{cases}$$

respectively. It is easy to show this fact that $\mathcal{R}$ is ρ_*-admissible. Moreover, let $\vartheta : [0, 1] \to [0, 1]$ be defined by $\vartheta(\lambda) = \lambda^{\frac{4}{5}}$. Let $\tilde{u}, \tilde{w} \in \tilde{G} \backslash Fix(\mathcal{R})$ and $\gamma = \dfrac{1}{2}$; then, the following cases are distinguished.
(a) If $\tilde{u} = \tilde{w} = 0$, then we get the following:

$$\tilde{H}(\mathcal{R}0, \mathcal{R}0, \tilde{s}) = 1,$$

$$d_*(\tilde{u}, \mathcal{R}\tilde{u}) = d_*(0, \mathcal{R}0) = \frac{\tilde{s}^2}{\tilde{s}^2 + 0} = 1,$$

$$d_*(\tilde{w}, \mathcal{R}\tilde{w}) = d_*(0, \mathcal{R}0) = \frac{\tilde{s}^2}{\tilde{s}^2 + 0} = 1.$$

Then, we have the following:

$$\begin{aligned}\tilde{H}(\mathcal{R}0, \mathcal{R}0, \tilde{s}) = \frac{\tilde{s}^2}{\tilde{s}^2 + 0} &= \sqrt{\frac{\tilde{s}^2}{\tilde{s}^2 + 0}}\sqrt{\frac{\tilde{s}^2}{\tilde{s}^2 + 0}} \\ &= \tau(0, \mathcal{R}0, \tilde{s})^{\frac{1}{2}}\tau(0, \mathcal{R}0, \tilde{s})^{1-\frac{1}{2}}.\end{aligned}$$

By applying function ϑ to the right-hand side of the above quation, we have the following:

$$\tilde{H}(\mathcal{R}0, \mathcal{R}0, \tilde{s}) \geq \vartheta(\tau(0, \mathcal{R}0, \tilde{s})^{\frac{1}{2}} \tau(0, \mathcal{R}0, \tilde{s})^{1-\frac{1}{2}}).$$

For $\rho_*(0, 0, \tilde{s}) = 1$, *we can write the following:*

$$\begin{aligned} 1 &= \rho_*(0, 0, \tilde{s})\tilde{H}(\mathcal{R}0, \mathcal{R}0, \tilde{s}) \\ &\geq \vartheta(\tau(0, \mathcal{R}0, \tilde{s})^{\frac{1}{2}} \tau(0, \mathcal{R}0, \tilde{s})^{1-\frac{1}{2}}) \\ &= 1. \end{aligned}$$

b) If $\tilde{u} = \tilde{w} = 1$, *then we get the following:*

$$\tilde{H}(\mathcal{R}1, \mathcal{R}1, \tilde{s}) = 1,$$

$$d_*(\tilde{u}, \mathcal{R}\tilde{u}) = d_*(1, \mathcal{R}1) = \frac{\tilde{s}^2}{\tilde{s}^2 + 1},$$

$$d_*(\tilde{w}, \mathcal{R}\tilde{w}) = d_*(1, \mathcal{R}1) = \frac{\tilde{s}^2}{\tilde{s}^2 + 1}.$$

Then, we have the following:

$$\begin{aligned} \tilde{H}(\mathcal{R}1, \mathcal{R}1, \tilde{s}) &= \frac{\tilde{s}^2}{\tilde{s}^2 + 0} > \frac{\tilde{s}^2}{\tilde{s}^2 + 1} = \sqrt{\frac{\tilde{s}^2}{\tilde{s}^2 + 1}} \sqrt{\frac{\tilde{s}^2}{\tilde{s}^2 + 1}} \\ &= \tau(1, \mathcal{R}1, \tilde{s})^{\frac{1}{2}} \tau(1, \mathcal{R}1, \tilde{s})^{1-\frac{1}{2}}. \end{aligned}$$

By applying the function ϑ to the right-hand side of the above equations, we have the following:

$$\tilde{H}(\mathcal{R}1, \mathcal{R}1, \tilde{s}) > \vartheta(\tau(1, \mathcal{R}1, \tilde{s})^{\frac{1}{2}} \tau(1, \mathcal{R}1, \tilde{s})^{1-\frac{1}{2}}).$$

For $\rho_*(1, 1, \tilde{s}) = \left(\frac{\tilde{s}^2}{\tilde{s}^2 + 1}\right)^{\frac{3}{10}}$, *we can write the following:*

$$\begin{aligned} \left(\frac{\tilde{s}^2}{\tilde{s}^2 + 1}\right)^{\frac{3}{10}} &= \rho_*(1, 1, \tilde{s})\tilde{H}(\mathcal{R}1, \mathcal{R}1, \tilde{s}) \\ &\geq \vartheta(\tau(1, \mathcal{R}1, \tilde{s})^{\frac{1}{2}} \tau(1, \mathcal{R}1, \tilde{s})^{1-\frac{1}{2}}) \\ &= \left(\frac{\tilde{s}^2}{\tilde{s}^2 + 1}\right)^{\frac{4}{5}}. \end{aligned}$$

For $\rho_*(1, 1, \tilde{s}) = 1$, *we can write the following:*

$$\left(\frac{\tilde{s}^2}{\tilde{s}^2 + 1}\right)^{\frac{3}{10}} = \rho_*(1, 1, \tilde{s})\tilde{H}(\mathcal{R}1, \mathcal{R}1, \tilde{s})$$

$$\geq \vartheta(\tau(1,\mathcal{R}1,\tilde{s})^{\frac{1}{2}}\tau(1,\mathcal{R}1,\tilde{s})^{1-\frac{1}{2}})$$

$$= \left(\frac{\tilde{s}^2}{\tilde{s}^2+1}\right)^{\frac{4}{10}}.$$

(c)If $\tilde{u} = \omega = 2$, then it can be shown in a similar method as in case (b).
(d) If $\tilde{u} = 0$, $\tilde{w} = 1$ or $\tilde{u} = 1$, $\tilde{w} = 0$, then we obtain the following:

$$\tilde{H}(\mathcal{R}0,\mathcal{R}1,\tilde{s}) = \min\Big\{\inf_{\tilde{u}\in\{0,1\}} M(\tilde{u},\{0\},\tilde{s}), \inf_{\tilde{w}\in\{0\}} M(\{0,1\},\tilde{w},\tilde{s})\Big\} = 1,$$

$$d_*(\tilde{u},\mathcal{R}\tilde{u}) = d_*(0,\mathcal{R}0) = \frac{\tilde{s}^2}{\tilde{s}^2+0} = 1,$$

$$d_*(\tilde{w},\mathcal{R}\tilde{w}) = d_*(1,\mathcal{R}1) = \frac{\tilde{s}^2}{\tilde{s}^2+1}.$$

Then, we have the following:

$$\tilde{H}(\mathcal{R}1,\mathcal{R}1,\tilde{s}) = \frac{\tilde{s}^2}{\tilde{s}^2+0} > 1\cdot\sqrt{\frac{\tilde{s}^2}{\tilde{s}^2+1}}$$

$$= \tau(0,\mathcal{R}0,\tilde{s})^{\frac{1}{2}}\tau(1,\mathcal{R}1,\tilde{s})^{1-\frac{1}{2}}.$$

By applying the function ϑ to the right-hand side of the above equation, we have the following:

$$\tilde{H}(\mathcal{R}0,\mathcal{R}1,\tilde{s}) > \vartheta(\tau(0,\mathcal{R}0,\tilde{s})^{\frac{1}{2}}\tau(1,\mathcal{R}1,\tilde{s})^{1-\frac{1}{2}}).$$

For $\rho_*(0,1,\tilde{s}) = \left(\frac{\tilde{s}^2}{\tilde{s}^2+1}\right)^{\frac{3}{10}}$, we can write the following:

$$\left(\frac{\tilde{s}^2}{\tilde{s}^2+1}\right)^{\frac{3}{10}} = \rho_*(0,1,\tilde{s})\tilde{H}(\mathcal{R}0,\mathcal{R}1,\tilde{s})$$

$$\geq \vartheta(\tau(0,\mathcal{R}0,\tilde{s})^{\frac{1}{2}}\tau(1,\mathcal{R}1,\tilde{s})^{1-\frac{1}{2}})$$

$$= \left(\frac{\tilde{s}^2}{\tilde{s}^2+1}\right)^{\frac{4}{10}}.$$

(e) If $\tilde{u} = 0$, $\tilde{w} = 2$ or $\tilde{u} = 2, \tilde{w} = 0$ then, we obtain the following:

$$\tilde{H}(\mathcal{R}0,\mathcal{R}2,\tilde{s}) = \min\Big\{\inf_{\tilde{u}\in\{0,1\}} M(\tilde{u},\{0,1\},\tilde{s}), \inf_{\tilde{w}\in\{0,1\}} M(\{0,1\},\tilde{w},\tilde{s})\Big\} = 1,$$

$$\rho_*(0,2,\tilde{s}) = 1.$$

Thus, the inequality holds in this case.
(f) If $\tilde{u} = 1, \tilde{w} = 2$ *or* $\tilde{u} = 2, \tilde{w} = 1$*, then we get the following:*

$$\tilde{H}(\mathcal{R}1, \mathcal{R}2, \tilde{s}) = \min\Big\{ \inf_{\tilde{u}\in\{0\}} M(\tilde{u}, \{0,1\}, \tilde{s}), \inf_{\tilde{w}\in\{0,1\}} M(\{0\}, \tilde{w}, \tilde{s})\Big\} = 1,$$

$$\rho_*(1, 2, \tilde{s}) = 1.$$

Hence, the desired inequality is true in this case.

It is notable that $\mathcal{R}$ *is not continuous. We establish that (A) is satisfied. Some* $\tilde{u}_0 \in \tilde{G}$ *exists provided that* $\rho_*(\tilde{u}_0, \mathcal{R}\tilde{u}_0, \tilde{s}) \leq 1$*. Indeed, for* $\tilde{u}_0 = 0$*, we have* $\rho_*(0, P0, \tilde{s}) \leq 1$*. Finally,* $\{\tilde{u}_n\}$ *is a sequence in* $\tilde{G}$ *with* $\rho_*(\tilde{u}_n, \tilde{u}_{n+1}, \tilde{s}) \leq 1$ *for all* $n \in \mathbb{N}$ *and* $\tilde{u}_n \to \tilde{u}$ *as* $n \to +\infty$*. We know that there is only one convergent sequence in* $\tilde{G}$ *and it is the constant sequence* $\{\tilde{u}_n\}$*, by assuming* $\tilde{u}_n = a$ *with* $a \in \tilde{G}$*; therefore,* $\rho_*(\tilde{u}_n, \tilde{u}, \tilde{s}) \leq 1$*. Consequently, these items show that Theorem 3.5 is fulfilled.*

In the following, we introduce the notion of the multivalued ρ_* -interpolative Kannan-type fuzzy contraction.

Definition 3.8. *Assume that* $(\tilde{G}, \tau, *)$ *is an FMS.* $\mathcal{R} : \tilde{G} \to \tilde{G}$ *is a multivalued* ρ_**-interpolative Kannan-type fuzzy contraction if there are* $\vartheta \in \Phi$*,* $\rho_* : \tilde{G} \times \tilde{G} \times (0, \infty) \to (0, \infty)$ *and a positive real number* γ *with* $\gamma \in (0, 1)$ *provided that*

$$\rho_*(\tilde{u}, \tilde{w}, \tilde{s})\tilde{H}(\mathcal{R}\tilde{u}, \mathcal{R}\tilde{w}, \tilde{s}) \geq \vartheta((\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}))^{\gamma}(\tau(\tilde{w}, \mathcal{R}\tilde{w}, \tilde{s}))^{1-\gamma}), \tag{3.16}$$

for all $\tilde{u}, \tilde{w} \in \tilde{G}\backslash Fix(\mathcal{R})$*, and* $\tilde{s} > 0$*.*

Theorem 3.9. *Assume that* $(\tilde{G}, \tau, *)$ *and* $\mathcal{R} : \tilde{G} \to K(\tilde{G})$ *are a complete FMS and a continuous multivalued* ρ_**-interpolative Kannan-type fuzzy contraction, respectively; additionally, suppose that*

(1) $\mathcal{R}$ *is* ρ_**-admissible; and*
(2) $\exists\, \tilde{u}_0 \in \tilde{G}$ *and* $\exists\, \tilde{u}_1 \in \mathcal{R}\tilde{u}_0$ *s.t.* $\rho(\tilde{u}_0, \tilde{u}_1, \tilde{s}) \leq 1$ *for* $\tilde{s} > 0$*.*

Then, $\mathcal{R}$ *has a fixed point.*

Proof. The proof of this theorem can be established similar to the proof of Theorem 3.5. □

If we use the condition (A) above, instead of the continuity of $\mathcal{R}$, we get the following theorem.

Theorem 3.10. *Assume that* $(\tilde{G}, \tau, *)$ *and* $\mathcal{R} : \tilde{G} \to K(\tilde{G})$ *are a complete FMS and a multivalued* ρ_**-interpolative Kannan-type fuzzy contraction, respectively; additionally, suppose that*

(1) condition (A) is satisfied;
(2) $\mathcal{R}$ *is* ρ_**-admissible; and*
(3) $\exists\, \tilde{u}_0 \in \tilde{G}$ *and* $\exists\, \tilde{u}_1 \in \mathcal{R}\tilde{u}_0$ *s.t.* $\rho(\tilde{u}_0, \tilde{u}_1, \tilde{s}) \leq 1$ *for* $\tilde{s} > 0$*.*

Then, $\mathcal{R}$ *has a fixed point.*

Proof. The proof of this theorem can be established similar to the proof of Theorem 3.6. □

The following corollaries are easily proven based on the conclusions of the above theorems.

Corollary 3.11. *Suppose that $(\tilde{G}, \tau, *)$ is a complete FMS, $\mathcal{R} : \tilde{G} \to K(\tilde{G})$ is an arbitrary mapping, and $\vartheta \in \Phi$; for positive real numbers γ, β with $\gamma + \beta < 1$, the inequality*

$$\tilde{H}(\mathcal{R}\tilde{u}, \mathcal{R}\tilde{w}, \tilde{s}) \geq \vartheta((\tau(\tilde{u}, \tilde{w}, \tilde{s}))^{\gamma}(\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}))^{\beta}(\tau(\tilde{w}, \mathcal{R}\tilde{w}, \tilde{s}))^{1-\gamma-\beta})$$

holds for every $\tilde{u}, \tilde{w} \in \tilde{G} \backslash Fix(\mathcal{R})$ and $\tilde{s} > 0$. Then, $\mathcal{R}$ has a fixed point.

Corollary 3.12. *Suppose that $(\tilde{G}, \tau, *)$ is a complete FMS, $\mathcal{R} : \tilde{G} \to K(\tilde{G})$ is an arbitrary mapping, and $\vartheta \in \Phi$; for positive real number γ with $\gamma \in (0, 1)$, the inequality*

$$\tilde{H}(\mathcal{R}\tilde{u}, \mathcal{R}\tilde{w}, \tilde{s}) \geq \vartheta((\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}))^{\gamma}(\tau(\tilde{w}, \mathcal{R}\tilde{w}, \tilde{s}))^{1-\gamma})$$

holds for every $\tilde{u}, \tilde{w} \in \tilde{G} \backslash Fix(\mathcal{R})$ and $\tilde{s} > 0$. Then, $\mathcal{R}$ has a fixed point.

3.1. Reduction to single-valued cases

In the present part of the paper, some reductions of the above results in the multivalued version to the single-valued mappings are provided.

Theorem 3.13. *Assume $(\tilde{G}, \tau, *)$ and $\mathcal{R} : \tilde{G} \to \tilde{G}$ are a complete FMS and an ρ_*-interpolative Ćirić-Reich-Rus-type fuzzy contraction, respectively; additionally, assume that*

(1) either $\mathcal{R}$ is continuous or condition (A) holds;
(2) $\mathcal{R}$ is ρ_*-admissible; and
(3) $\exists\, \tilde{u}_0 \in \tilde{G}$ s.t. $\rho_*(\tilde{u}_0, \mathcal{R}\tilde{u}_0, \tilde{s}) \leq 1$ for $\tilde{s} > 0$.

Then, $\mathcal{R}$ has a fixed point.

Proof. We know that $\{\tilde{u}\} \in K(\tilde{G})$ for every $\tilde{u} \in \tilde{G}$. We define a mapping $\mathfrak{I} : \tilde{G} \to K(\tilde{G})$ as $\mathfrak{I}\tilde{u} = \{\mathcal{R}\tilde{u}\}$. In this case, all conditions of this theorem clearly reduce to the hypothses of Theorem 3.5 and Theorem 3.6; thus, the conclusions of Theorem 3.5 and Theorem 3.6 imply the existence of $\tilde{u} \in \tilde{G}$ so that $\{\tilde{u}\} = \{\mathfrak{I}\tilde{u}\}$. By the definition of $\mathfrak{I}$, we have $\mathfrak{I}\tilde{u} = \{\mathcal{R}\tilde{u}\}$. Hence, $\tilde{u} = \mathcal{R}\tilde{u}$, meaning that $\tilde{u}$ is a fixed point of $\mathcal{R}$. □

Theorem 3.14. *Assume $(\tilde{G}, \tau, *)$ and $\mathcal{R} : \tilde{G} \to \tilde{G}$ are a complete FMS and an ρ_*-interpolative Kannan-type fuzzy contraction, respectively; additionally, assume that*

(1) either $\mathcal{R}$ is continuous or condition (A) holds;
(2) $\mathcal{R}$ is ρ_*-admissible; and
(3) $\exists\, \tilde{u}_0 \in \tilde{G}$ s.t. $\rho_*(\tilde{u}_0, \mathcal{R}\tilde{u}_0, \tilde{s}) \leq 1$ for $\tilde{s} > 0$.

Then, $\mathcal{R}$ has a fixed point.

Proof. Following the similar arguments in the proof of Theorem 3.13, and in view of the conclusions of Theorem 3.9 and Theorem 3.10, we get the desired result. □

Remark 3.15. *In Theorems 3.13 and 3.14, considering $\rho_*(\tilde{u}, \tilde{w}, \tilde{s}) = 1$, we have Corollary 1 and Corollary 2 in the paper published by Sezen et al. [36] in 2025.*

3.2. Some other results on partially ordered FMS

In this part, we have some other fixed point theorems for multivalued ordered interpolative Ćirić-Reich-Rus-type and Kannan-type fuzzy contractions on partially ordered FMSs. Let us present the condition that we will use it in the following propositions.

(B) If $\{\tilde{u}_n\} \subseteq \tilde{G}$ so that $\tilde{u}_n \preceq \tilde{u}_{n+1}$, for all $n \in \mathbb{N}$ and $\tilde{u}_n \to \tilde{u}$ as $n \to \infty$, then a subsequence like $\{\tilde{u}_{n_k}\}$ of $\{\tilde{u}_n\}$ exists provided that $\tilde{u}_{n_k} \preceq \tilde{u}$ for every k.

Proposition 3.16. *Assume that $(\tilde{G}, \tau, *, \preceq)$ is a partially ordered FMS and $\mathcal{R} : \tilde{G} \to K(\tilde{G})$ be an arbitrary mapping. Let there exist $\vartheta \in \Phi$, $\rho_* : \tilde{G} \times \tilde{G} \times (0, \infty) \to (0, \infty)$ and positive real numbers γ, β with $\gamma + \beta < 1$ provided that*

$$\rho_*(\tilde{u}, \tilde{w}, \tilde{s})\tilde{H}(\mathcal{R}\tilde{u}, \mathcal{R}\tilde{w}, \tilde{s}) \geq \vartheta((\tau(\tilde{u}, \tilde{w}, \tilde{s}))^{\gamma}(\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}))^{\beta}(\tau(\tilde{w}, \mathcal{R}\tilde{w}, \tilde{s}))^{1-\gamma-\beta}),$$

for all $\tilde{u}, \tilde{w} \in \tilde{G} \backslash Fix(\mathcal{R}))$ with $\tilde{u} \preceq \tilde{w}$ and $\tilde{s} > 0$. Additionally, suppose that

(1) $\mathcal{R}$ is increasing w.r.t. $\preceq$;
(2) $\exists\, \tilde{u}_0 \in \tilde{G}$ s.t. $\tilde{u}_0 \preceq \mathcal{R}\tilde{u}_0$; and
(3) either $\mathcal{R}$ is continuous or condition (B) is true.

Then, some $\tilde{u}$ exists s.t. $\tilde{u} \in \mathcal{R}\tilde{u}$.

Proof. In Theorems 3.5 and 3.6, if ρ_* is defined

$$\rho_*(\tilde{u}, \tilde{w}, \tilde{s}) = \begin{cases} 1 & \text{if} \quad (\tilde{u} \preceq \tilde{w}) \text{ or } (\tilde{w} \preceq \tilde{u}), \\ 0 \quad , & \qquad \text{otherwise,} \end{cases}$$

then the conclusion is immediately established and this completes the proof. □

Proposition 3.17. *Assume that $(\tilde{G}, \tau, *, \preceq)$ and $\mathcal{R} : \tilde{G} \to K(\tilde{G})$ are a partially ordered FMS and an arbitrary mapping, respectively. Assume that there are $\vartheta \in \Phi$, $\rho_* : \tilde{G} \times \tilde{G} \times (0, \infty) \to (0, \infty)$ and positive real number γ with $\gamma \in (0, 1)$ provided that*

$$\rho_*(\tilde{u}, \tilde{w}, \tilde{s})\tilde{H}(\mathcal{R}\tilde{u}, \mathcal{R}\tilde{w}, \tilde{s}) \geq \vartheta((\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s})))^{\gamma}(\tau(\tilde{w}, \mathcal{R}\tilde{w}, \tilde{s}))^{1-\gamma}))$$

for all $\tilde{u}, \tilde{w} \in \tilde{G} \backslash Fix(\mathcal{R}))$ with $\tilde{u} \preceq \tilde{w}$ and $\tilde{s} > 0$. Additionally, suppose that

(1) $\mathcal{R}$ is increasing with respect to $\preceq$;
(2) there is $\tilde{u}_0 \in \tilde{G}$ such that $\tilde{u}_0 \preceq \mathcal{R}\tilde{u}_0$; and
(3) either $\mathcal{R}$ is continuous or condition (B) is true.

Then, some $\tilde{u}$ exists s.t. $\tilde{u} \in \mathcal{R}\tilde{u}$.

Proof. In Theorems 3.9 and 3.10, if ρ_* is defined as

$$\rho_*(\tilde{u}, \tilde{w}, \tilde{s}) = \begin{cases} 1 & \text{if} \quad (\tilde{u} \preceq \tilde{w}) \text{ or } (\tilde{w} \preceq \tilde{u}), \\ 0 \quad , & \qquad \text{otherwise,} \end{cases}$$

then the conclusion is immediately established and this completes the proof. □

Remark 3.18. *In Theorems 3.13 and 3.14, considering $\rho_*(\tilde{u}, \tilde{w}, \tilde{s}) = 1$, we obtain the conclusions of Corollary 5 and Corollary 6 established in the paper published by Sezen et al. [36] in 2025.*

4. Application to nonlinear matrix equations

This section presents an application of our main results in the field of the nonlinear matrix equations. The family of all $n \times n$ Hermitian matrices on $\mathbb{C}$ is denoted by $H(n)$. Moreover, the family of all $n \times n$ positive semi-definite matrices is denoted by $K(n) \subset H(n)$, and $P(n) \subset K(n)$ specifies the family of all $n \times n$ positive definite matrices. For the family of all $n \times n$ matrices on $\mathbb{C}$, we use the symbol $M(n)$.

To indicate that a matrix $\tilde{u} \in P(n)$ is positive definite, the relation $\tilde{u} > 0$ is used. Similarly, for a positive semi-definite matrix, we use $\tilde{u} \geq 0$. Accordingly, $\tilde{u}_1 > \tilde{u}_2$ and $\tilde{u}_1 \geq \tilde{u}_2$ give $\tilde{u}_1 - \tilde{u}_2 > 0$ and $\tilde{u}_1 - \tilde{u}_2 \geq 0$, respectively. The symbol $\|\cdot\|$ is the spectral norm of the matrix $\tilde{u}$, defined by the following:

$$\|\tilde{u}\| = \sqrt{\lambda^{+}\tilde{u}^{*}\tilde{u}},$$

where λ is the largest eigenvalue of $\tilde{u}^{*}\tilde{u}$. Morever,

$$\|\tilde{u}\|_{tr} = \sum_{i=1}^{m} s_i(\tilde{u}),$$

so that $s_i(\tilde{u})$ (for $1 \leq i \leq m$) shows the singular values of $\tilde{u} \in K(n)$. In this case, $(H(n), \|\cdot\|_{tr})$ will be a complete metric space [37–39]. On $H(n)$, we define a binary relation $\leq$ as follows:

$$\tilde{u} \leq \tilde{w} \quad \text{if and only if} \quad \tilde{w} \leq \tilde{u},$$

for every $\tilde{u},\tilde{w} \in H(n)$.

These two lemmas are required in the next discussions.

Lemma 4.1. *[40] Let $\tilde{u} \geq \tilde{w}$ and $\tilde{w} \geq \tilde{u}$ be two $n \times n$ matrices. Then,*

$$0 \leq tr(\tilde{u}\tilde{w}) \leq \|\tilde{u}\|\, tr(\tilde{w}).$$

Lemma 4.2. *[41] Let $\tilde{u} \in H(n)$ be so that $\tilde{u} \prec \tilde{\imath}_n$. Then, $\|\tilde{u}\| < 1$.*

As the main purpose, we will investigate the existence and uniqueness of the solution in relation to the given nonlinear matrix equation

$$\tilde{u} = \tilde{E} + \sum_{i=1}^{m} \tilde{a}_i^{*}\mathcal{R}(\tilde{u})\tilde{a}_i, \tag{4.1}$$

so that $\tilde{E}$ is assumed as a Hermitian positive definite matrix, $\tilde{a}_i^{*}$ specifies the conjugate transpose of the $n \times n$ matrix $\tilde{a}_i$, and $\mathcal{R}$ introduces an order-preserving continuous mapping from the family of all Hermitian matrices $H(n)$ to the family of all positive definite matrices $P(n)$ provided that $\mathcal{R}(0) = 0$.

Theorem 4.3. *Consider the nonlinear matrix equation formulated as (4.1) and let*

(1) $\exists\, \tilde{E} \in P(n)$ *s.t.* $\sum_{i=1}^{m} \tilde{a}_i^{*}\mathcal{R}(\tilde{u})\tilde{a}_i > 0$;

(2) for all $\tilde{u}, \tilde{w} \in P(n)$ *with* $\tilde{u} \preceq \tilde{w}$*, we have*

$$\sum_{i=1}^{m} \tilde{a}_i^* \mathcal{R}(\tilde{u}) \tilde{a}_i \neq \sum_{i=1}^{m} \tilde{a}_i^* \mathcal{R}(\tilde{w}) \tilde{a}_i;$$

and

(3) $\exists\, \delta \in (0,1)$ *for which* $\sum_{i=1}^{m} \tilde{a}_i^* \tilde{a}_i \prec \delta \tilde{\imath}_n$ *and*

$$|tr(\mathcal{R}(\tilde{u}) - \mathcal{R}(\tilde{w}))| \leq \frac{1}{\delta}(\gamma\, |tr(\tilde{u} - \tilde{w})| + \beta\, |tr(\tilde{u} - \mathcal{R}\tilde{u})| + (1 - \gamma - \beta)\, |tr(\tilde{w} - \mathcal{R}\tilde{w})|),$$

for all positive real numbers γ, β *with* $\gamma + \beta < 1$.

Then, the matrix equation (4.1) admits a unique solution in $P(n)$*.*

Proof. Let $\tilde{G} = P(n)$. Define $\mathcal{R} : P(n) \to P(n)$ as following:

$$\mathcal{R}(\tilde{u}) = \tilde{E} + \sum_{i=1}^{m} \tilde{a}_i^* \mathcal{R}(\tilde{u}) \tilde{a}_i,$$

for each $\tilde{u} \in P(n)$. Note that the fixed point of $\mathcal{R}$ is a solution of the matrix equation (4.1). Define a fuzzy metric with $\tau : P(n) \times P(n) \times (0, \infty) \to [0, 1]$ by the following:

$$\tau(\mathcal{R}\tilde{u}, \mathcal{R}\tilde{w}, \tilde{s}) = e^{-\frac{d_*(\tilde{u} - \tilde{w})}{\tilde{s}}},$$

where $d_*(\tilde{u}, \tilde{w}) = \|\tilde{u} - \tilde{w}\|_1 = \|\tilde{u} - \tilde{w}\|_{tr} = \|\tilde{u} - \tilde{w}\|^2 + \|\tilde{u}\|$. Let $\tilde{u}, \tilde{w} \in P(n)$ with $\tilde{u} \preceq \tilde{w}$. Then, $\mathcal{R}\tilde{u} \preceq \mathcal{R}\tilde{w}$. Therefore, for $d_*(\tilde{u}, \tilde{w}) > 0$ and by using the hypotheses (1)-(3), we have the following:

$$\begin{aligned}
d_*(\mathcal{R}\tilde{u}, \mathcal{R}\tilde{w}) &= \|\mathcal{R}\tilde{u} - \mathcal{R}\tilde{w}\|_{tr} = tr(\mathcal{R}\tilde{u} - \mathcal{R}\tilde{w}) \\
&= tr(\sum_{i=1}^{m} \tilde{a}_i^* (\mathcal{R}(\tilde{u}) - \mathcal{R}(\tilde{w})) \tilde{a}_i) \\
&= \sum_{i=1}^{m} tr(\tilde{a}_i^* \tilde{a}_i (\mathcal{R}(\tilde{u}) - \mathcal{R}(\tilde{w}))) \\
&= tr\left(\left(\sum_{i=1}^{m} \tilde{a}_i^* \tilde{a}_i \right) (\mathcal{R}(\tilde{u}) - \mathcal{R}(\tilde{w})) \right) \\
&\leq \left\| \sum_{i=1}^{m} \tilde{a}_i^* \tilde{a}_i \right\| \|\mathcal{R}(\tilde{u}) - \mathcal{R}(\tilde{w})\|_{tr} \\
&\leq \left\| \sum_{i=1}^{m} \tilde{a}_i^* \tilde{a}_i \right\| \frac{1}{\delta} (\gamma\, \|\tilde{u} - \tilde{w}\|_{tr} + \beta\, \|\tilde{u} - \mathcal{R}\tilde{u}\|_{tr} + (1 - \gamma - \beta)\, \|\tilde{w} - \mathcal{R}\tilde{w}\|_{tr})
\end{aligned}$$

$$< \gamma \|\tilde{u} - \tilde{w}\|_{tr} + \beta \|\tilde{u} - \mathcal{R}\tilde{u}\|_{tr} + (1 - \gamma - \beta) \|\tilde{w} - \mathcal{R}\tilde{w}\|_{tr} .$$

These imply that

$$\|\mathcal{R}\tilde{u} - \mathcal{R}\tilde{w}\|_{tr} < \gamma \|\tilde{u} - \tilde{w}\|_{tr} + \beta \|\tilde{u} - \mathcal{R}\tilde{u}\|_{tr} + (1 - \gamma - \beta) \|\tilde{w} - \mathcal{R}\tilde{w}\|_{tr} .$$

Consequently, we obtain the following:

$$e^{-\frac{\|\mathcal{R}\tilde{u} - \mathcal{R}\tilde{w}\|_{tr}}{\tilde{s}}} > e^{-\frac{\gamma \|\tilde{u} - \tilde{w}\|_{tr}}{\tilde{s}}} e^{-\frac{\beta \|\tilde{u} - \mathcal{R}\tilde{u}\|_{tr}}{\tilde{s}}} e^{-\frac{(1 - \gamma - \beta) \|\tilde{u} - \mathcal{R}\tilde{u}\|_{tr}}{\tilde{s}}} .$$

From the property of ϑ, we obtain the following:

$$\tau(\mathcal{R}\tilde{u}, \mathcal{R}\tilde{w}, \tilde{s}) \geq \vartheta((\tau(\tilde{u}, \tilde{w}, \tilde{s}))^{\gamma}(\tau(\tilde{u}, \mathcal{R}\tilde{u}, \tilde{s}))^{\beta}(\tau(\tilde{w}, \mathcal{R}\tilde{w}, \tilde{s}))^{1-\gamma-\beta}).$$

Thus, we have a result of Corollary 3.16 which considers $\rho_*(\tilde{u}, \tilde{w}, \tilde{s}) = 1$. Therefore, $\mathcal{R}$ has a fixed point $\tilde{u} \in P(n)$ so that $\mathcal{R}\tilde{u} = \tilde{u}$, that is, the nonlinear matix equation (4.1) admits a solution in $P(n)$. □

5. Conclusions

This article covered a variety of results related to the existence of fixed points for multivalued ρ_*-interpolative Ćirić-Reich-Rus-type fuzzy contractions in complete fuzzy metric spaces by using concepts of ρ_*-admissibility and the ϑ-comparison function. We presented a few examples to support the definitions and main results we introduced. As a consequence of the our main theorem, we proved the fixed point theorem of multivalued ρ_*-interpolative Kannan-type fuzzy contractions and presented some results related to Kannan-type fuzzy contractions. Additionally, we proved some fixed point results for multivalued ordered ρ_*-interpolative Ćirić-Reich-Rus-type and Kannan-type fuzzy contractions on a partially ordered fuzzy metric space. As an application, we showed solution for non-linear matrix equations. For the next studies, one can extend these results in other types of fuzzy metric spaces such as the multiplicative fuzzy metric spaces.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Authors' contributions

Conceptualization: M.S.S., S.E.; Methodology: M.S.S., M.J.M., S.E.; Software: M.S.S., S.E.; Validation: S.E., J.T.; Formal analysis: M.J.M., J.T.; Investigation: S.E., J.T.; Writing-original draft preparation: M.S.S., M.J.M.; Writing-review and editing: S.E., J.T.; Supervision: J.T.; Project administration: J.T. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflict of interest.

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