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Research article

Fuzzy GE-norms and fuzzy normed GE-algebras

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Abstract: In this paper, we introduce the notion of a fuzzy GE-norm on a GE-algebra $(\mathbf{X}, *, 1_{\mathbf{X}})$ as a scale-dependent fuzzy analogue of the GE-norm introduced in [2]. We established fundamental structural properties of fuzzy GE-norms, derived several identities and generalizations of classical norm inequalities, and developed examples demonstrating the variety of fuzzy behaviors that arise in GE-algebraic settings. Particular attention was given to the interaction between fuzzy GE-norms and GE-morphisms. We showed that a fuzzy GE-norm is not automatically preserved under mappings induced by a GE-morphism. Positive preservation results were obtained when the morphism is injective or surjective and satisfies natural compatibility requirements. A central result of the paper is a complete characterization of fuzzy normability: a GE-algebra admits a fuzzy GE-norm if and only if its induced order is transitive. This theorem identifies transitivity as the precise structural requirement for norm-like behavior in GE-algebras. Additional results include uniqueness of fuzzy limits in commutative GE-algebras, continuity of left and right translations, fuzzy convergence criteria, and Lipschitz-type estimates for the rational fuzzy GE-norm.

Keywords: GE-algebra; fuzzy GE-norm; transitivity; GE-morphism; fuzzy convergence
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1. Introduction

GE-algebras, introduced in [1], provide an algebraic framework that extends the behavior of implication-like operations and order-theoretic structures. In a recent work [2], GE-norms were introduced as real-valued functions satisfying a triangle-type inequality compatible with the

GE-operation. These norms equip GE-algebras with a geometric flavor and enable a metric-like analysis of algebraic structure.

The purpose of the present paper is to develop a fuzzy analogue of GE-norms. Motivated by classical fuzzy normed spaces [3], we define a *fuzzy GE-norm* as a mapping $\pi : \mathbf{X} \times (0, \infty) \to [0, 1]$ depending on a positive scale parameter and satisfying fuzzy versions of the zero property, monotonicity, large-scale limit behavior, and the GE-triangle inequality. This framework naturally captures degrees of proximity in GE-algebras and extends the deterministic structure of GE-norms to a richer fuzzy setting.

We begin by establishing fundamental properties of fuzzy GE-norms, proving that they induce a nontrivial topology compatible with GE-operations and that they satisfy several identities analogous to those in the crisp GE-norm theory of [2]. We then investigate the behavior of fuzzy GE-norms under GE-morphisms, extending earlier studies of morphisms in GE-algebras such as [4]. Examples demonstrate that transferring a fuzzy GE-norm along a GE-morphism does not, in general, preserve fuzzy normhood. Nevertheless, when the morphism is injective or surjective, appropriate compatibility conditions ensure that the transferred mapping retains the structure of a fuzzy GE-norm.

The main theorem of the paper provides a complete characterization of fuzzy normability: A GE-algebra carries a fuzzy GE-norm if and only if its induced order is transitive. This result identifies transitivity as the precise structural requirement for fuzzy norm behavior. As a consequence, we show that every transitive GE-algebra admits simple explicit fuzzy GE-norms, while nontransitive GE-algebras admit none.

Further sections develop the theory of fuzzy convergence, uniqueness of fuzzy limits in commutative GE-algebras, behavior under GE-morphisms, fuzzy continuity of left and right translations, and stability results such as Lipschitz-type estimates for the rational model. Taken together, these results show that fuzzy GE-norms form a natural and robust extension of crisp GE-norms and open a pathway to fuzzy geometric analysis in GE-algebras.

2. Preliminaries

Definition 2.1 ([1]). A *GE-algebra* is a nonempty set $\mathbf{X}$ with a constant 1 and a binary operation "$*$" satisfying the following axioms:

(GE1) $\omega_1 * \omega_1 = 1$,

(GE2) $1 * \omega_1 = \omega_1$,

(GE3) $\omega_1 * (\omega_2 * \omega_3) = \omega_1 * (\omega_2 * (\omega_1 * \omega_3))$

for all $\omega_1, \omega_2, \omega_3 \in \mathbf{X}$.

In a GE-algebra $\mathbf{X}$, a binary relation "$\leq_{\mathbf{X}}$" is defined by

$$(\forall \omega_1, \omega_2 \in \mathbf{X}) \, (\omega_1 \leq_{\mathbf{X}} \omega_2 \;\Leftrightarrow\; \omega_1 * \omega_2 = 1) \,. \tag{2.1}$$

Definition 2.2 ([1]). A GE-algebra $\mathbf{X}$ is called

- *transitive* if it satisfies:

$$(\forall \omega_1, \omega_2, \omega_3 \in \mathbf{X}) \, (\omega_1 * \omega_2 \leq_{\mathbf{X}} (\omega_3 * \omega_1) * (\omega_3 * \omega_2)) \,. \tag{2.2}$$

- *commutative* if it satisfies:

$$(\forall \omega_1, \omega_2 \in \mathbf{X})\,((\omega_1 * \omega_2) * \omega_2 = (\omega_2 * \omega_1) * \omega_1)\,. \tag{2.3}$$

Proposition 2.3 ([1]). *Every GE-algebra* $\mathbf{X}$ *satisfies the following properties:*

$$\omega_1 * 1 = 1, \tag{2.4}$$
$$\omega_1 * (\omega_1 * \omega_2) = \omega_1 * \omega_2, \tag{2.5}$$
$$\omega_1 \leq_{\mathbf{X}} \omega_2 * \omega_1, \tag{2.6}$$
$$\omega_1 * (\omega_2 * \omega_3) \leq_{\mathbf{X}} \omega_2 * (\omega_1 * \omega_3), \tag{2.7}$$
$$1 \leq_{\mathbf{X}} \omega_1 \;\Rightarrow\; \omega_1 = 1, \tag{2.8}$$
$$\omega_1 \leq_{\mathbf{X}} (\omega_2 * \omega_1) * \omega_1, \tag{2.9}$$
$$\omega_1 \leq_{\mathbf{X}} (\omega_1 * \omega_2) * \omega_2, \tag{2.10}$$
$$\omega_1 \leq_{\mathbf{X}} \omega_2 * \omega_3 \;\Leftrightarrow\; \omega_2 \leq_{\mathbf{X}} \omega_1 * \omega_3, \tag{2.11}$$

for all $\omega_1, \omega_2, \omega_3 \in \mathbf{X}$.

Definition 2.4 ([4]). Let $(\mathbf{X}, *_{\mathbf{X}}, 1_{\mathbf{X}})$ and $(\mathbf{Y}, *_{\mathbf{Y}}, 1_{\mathbf{Y}})$ be GE-algebras. A mapping $\varphi : \mathbf{X} \to \mathbf{Y}$ is called a *GE-morphism* if it satisfies:

$$(\forall \upsilon_1, \upsilon_2 \in \mathbf{X})(\varphi(\upsilon_1 *_{\mathbf{X}} \upsilon_2) = \varphi(\upsilon_1) *_{\mathbf{Y}} \varphi(\upsilon_2)). \tag{2.12}$$

Definition 2.5 ([2]). A *GE-norm* on $\mathbf{X} := (\mathbf{X}, *, 1_{\mathbf{X}})$ is defined to be a mapping $\|\cdot\| : \mathbf{X} \to \mathbb{R}$ that satisfies:

$$(\forall \upsilon_1 \in \mathbf{X})\,(\|\upsilon_1\| \geq 0)\,, \tag{2.13}$$
$$(\forall \upsilon_1 \in \mathbf{X})\,(\|\upsilon_1\| = 0 \;\Leftrightarrow\; \upsilon_1 = 1_{\mathbf{X}})\,, \tag{2.14}$$
$$(\forall \upsilon_1, \upsilon_2, \upsilon_3 \in \mathbf{X})\,(\|\upsilon_1 * \upsilon_3\| \leq \|\upsilon_1 * \upsilon_2\| + \|\upsilon_2 * \upsilon_3\|)\,. \tag{2.15}$$

A *normed GE-algebra* is a GE-algebra $\mathbf{X} := (\mathbf{X}, *, 1_{\mathbf{X}})$ equipped with a GE-norm $\|\cdot\| : \mathbf{X} \to \mathbb{R}$ and it is denoted by $(\mathbf{X}, \|\cdot\|)$.

Given a GE-algebra $\mathbf{X} := (\mathbf{X}, *, 1_{\mathbf{X}})$, if there exists a function $\|\cdot\|$ mapping elements of $\mathbf{X}$ to non-negative real numbers satisfying the conditions (2.14) and (2.15), then $(\mathbf{X}, \|\cdot\|)$ is a normed GE-algebra.

3. Fuzzy GE-norms and fuzzy normed GE-algebras

Let $(\mathbf{X}, *, 1_{\mathbf{X}})$ be a GE-algebra and suppose $\|\cdot\| : \mathbf{X} \to \mathbb{R}_{\geq 0}$ is a GE-norm as in Section 2. We now introduce a fuzzy analogue of a GE-norm that uses a membership degree depending on a positive scale parameter.

3.1. Basic properties of fuzzy GE-norms

Definition 3.1 (Fuzzy GE-norm). Let $(\mathbf{X}, *, 1_{\mathbf{X}})$ be a GE-algebra. A mapping $\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1]$ is called a *fuzzy GE-norm* on $\mathbf{X}$ if, for all $\xi, \sigma, \tau \in \mathbf{X}$ and all $t_1, t_2 > 0$, the following conditions hold:

1) **(FN1) Zero property:**

$$\pi_{\mathbf{X}}(\xi, t_2) = 1 \quad \text{for all } t_2 > 0 \quad \Longleftrightarrow \quad \xi = 1_{\mathbf{X}}.$$

2) **(FN2) Monotonicity in scale:**

$$0 < t_1 \le t_2 \quad \Longrightarrow \quad \pi_{\mathbf{X}}(\xi, t_1) \le \pi_{\mathbf{X}}(\xi, t_2).$$

3) **(FN3) Limit at large scales (vanishing norm analogue):**

$$\lim_{t_2 \to \infty} \pi_{\mathbf{X}}(\xi, t_2) = 1 \qquad \text{for every } \xi \in \mathbf{X}.$$

4) **(FN4) Triangle inequality (fuzzy analogue of** (2.15)**):** For all $t_1, t_2 > 0$,

$$\pi_{\mathbf{X}}(\xi * \tau,\ t_1 + t_2) \ \ge\ \min\{\pi_{\mathbf{X}}(\xi * \sigma,\ t_1),\ \pi_{\mathbf{X}}(\sigma * \tau,\ t_2)\}.$$

Remark 3.2 (Necessity of (FN3))**.** Axiom (FN3) cannot be omitted from the definition of a fuzzy GE-norm. Indeed, there exist mappings satisfying (FN1), (FN2), and (FN4) but failing (FN3), which induces a pathological fuzzy topology.

Example 3.3. Let $(\mathbf{X}, *, 1_{\mathbf{X}})$ be a GE-algebra with at least two elements. Define a mapping $\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1]$ by

$$\pi_{\mathbf{X}}(\xi, t_2) = \begin{cases} 1, & \xi = 1_{\mathbf{X}}, \\ \frac{1}{2}, & \xi \ne 1_{\mathbf{X}}, \end{cases} \qquad t_2 > 0.$$

Proof. We verify the axioms one by one. **(FN1) Zero property:** If $\xi = 1_{\mathbf{X}}$, then $\pi_{\mathbf{X}}(\xi, t_2) = 1$ for all $t_2 > 0$. Conversely, if $\pi_{\mathbf{X}}(\xi, t_2) = 1$ for all $t_2 > 0$, then by definition, $\xi = 1_{\mathbf{X}}$. Hence (FN1) holds. **(FN2) Monotonicity in scale:** For any $\xi \in \mathbf{X}$ and $0 < t_1 \le t_2$, we have $\pi_{\mathbf{X}}(\xi, t_1) = \pi_{\mathbf{X}}(\xi, t_2)$. Thus $\pi_{\mathbf{X}}(\xi, t_1) \le \pi_{\mathbf{X}}(\xi, t_2)$, and (FN2) is satisfied. **(FN4) Triangle inequality:** Let $\xi, \sigma, \tau \in \mathbf{X}$ and $t_1, t_2 > 0$. If either $\pi_{\mathbf{X}}(\xi * \sigma, t_1) = \frac{1}{2}$ or $\pi_{\mathbf{X}}(\sigma * \tau, t_2) = \frac{1}{2}$, then

$$\min\{\pi_{\mathbf{X}}(\xi * \sigma, t_1), \pi_{\mathbf{X}}(\sigma * \tau, t_2)\} = \tfrac{1}{2},$$

while $\pi_{\mathbf{X}}(\xi * \tau, t_1 + t_2) \ge \frac{1}{2}$ by definition. If both values are equal to 1, then $\xi * \sigma = 1_{\mathbf{X}}$ and $\sigma * \tau = 1_{\mathbf{X}}$, and hence $\xi * \tau = 1_{\mathbf{X}}$, so that $\pi_{\mathbf{X}}(\xi * \tau, t_1 + t_2) = 1$. Therefore (FN4) holds in all cases. **Failure of (FN3):** For any $\xi \ne 1_{\mathbf{X}}$,

$$\lim_{t_2 \to \infty} \pi_{\mathbf{X}}(\xi, t_2) = \tfrac{1}{2} \ne 1.$$

Thus axiom (FN3) is violated. □

Remark 3.4. For any $0 < \varepsilon < \frac{1}{2}$ and any $t_2 > 0$, we have

$$U(\xi_0, t_2, \varepsilon) = \{\xi \in \mathbf{X} : \pi_{\mathbf{X}}(\xi * \xi_0, t_2) > 1 - \varepsilon\} = \{\xi \in \mathbf{X} : \xi * \xi_0 = 1_{\mathbf{X}}\} = \{\xi \in \mathbf{X} : \xi \le_X \xi_0\}.$$

Thus the fuzzy neighborhood is independent of the scale parameter t_2. Consequently, fuzzy balls do not expand as the scale increases, and the induced fuzzy topology is degenerate.

Definition 3.5. A pair $(\mathbf{X}, \pi_{\mathbf{X}})$, where $\pi_{\mathbf{X}}$ is a fuzzy GE-norm, is called a *fuzzy normed GE-algebra*.

We give two fundamental examples of fuzzy GE-norms on a GE-algebra $(\mathbf{X}, *, 1_{\mathbf{X}})$ equipped with a GE-norm $\|\cdot\|$ (Definition 3.1). In both examples, $t_2 > 0$ denotes the positive scale parameter.

Example 3.6 (Rational (canonical) model)**.** Define

$$\pi_{\mathbf{X}}(\xi, t_2) \ := \ \frac{t_2}{t_2 + \|\xi\|} \qquad (\xi \in \mathbf{X},\ t_2 > 0).$$

We verify (FN1)–(FN4).

(FN1): If $\xi = 1_{\mathbf{X}}$, then $\|\xi\| = \|1_{\mathbf{X}}\| = 0$ (Def. 3.1), hence for every $t_2 > 0$,

$$\pi_{\mathbf{X}}(1_{\mathbf{X}}, t_2) = \frac{t_2}{t_2 + 0} = 1.$$

Conversely, if $\pi_{\mathbf{X}}(\xi, t_2) = 1$ for all $t_2 > 0$, then for each t_2, we have $t_2/(t_2 + \|\xi\|) = 1$, which forces $\|\xi\| = 0$. By the crisp GE zero property (Def. 3.1), $\|\xi\| = 0 \iff \xi = 1_{\mathbf{X}}$. Thus (FN1) holds.

(FN2): Fix $\xi \in \mathbf{X}$ and $0 < t_1 \leq t_2$. Then

$$\pi_{\mathbf{X}}(\xi, t_1) = \frac{t_1}{t_1 + \|\xi\|} \leq \frac{t_2}{t_2 + \|\xi\|} = \pi_{\mathbf{X}}(\xi, t_2),$$

since the function $t_3 \mapsto \dfrac{t_3}{t_3 + \alpha}$ is increasing for $t_3 > 0$ when $\alpha \geq 0$. Thus (FN2) holds.

(FN3): For fixed ξ, $\lim\limits_{t_2 \to \infty} \dfrac{t_2}{t_2 + \|\xi\|} = 1$, so (FN3) holds.

(FN4): We must show for all $\xi, \sigma, \tau \in \mathbf{X}$ and $t_1, t_2 > 0$:

$$\pi_{\mathbf{X}}(\xi * \tau, t_1 + t_2) \ \geq \ \min\{\pi_{\mathbf{X}}(\xi * \sigma, t_1),\ \pi_{\mathbf{X}}(\sigma * \tau, t_2)\}.$$

Set

$$\mathbf{A} := \pi_{\mathbf{X}}(\xi * \sigma, t_1) = \frac{t_1}{t_1 + \|\xi * \sigma\|}, \qquad \mathbf{B} := \pi_{\mathbf{X}}(\sigma * \tau, t_2) = \frac{t_2}{t_2 + \|\sigma * \tau\|},$$

and

$$\mathbf{C} = \pi_{\mathbf{X}}(\xi * \tau, t_1 + t_2) = \frac{t_1 + t_2}{t_1 + t_2 + \|\xi * \tau\|}.$$

Using the GE–norm triangle inequality (2.15), $\|\xi * \tau\| \leq \|\xi * \sigma\| + \|\sigma * \tau\|$, and we obtain the lower bound

$$\mathbf{C} \geq \frac{t_1 + t_2}{t_1 + t_2 + \|\xi * \sigma\| + \|\sigma * \tau\|}.$$

Thus, it suffices to prove

$$\frac{t_1 + t_2}{t_1 + t_2 + t_3 + t_4} \ \geq \ \min\Big\{\frac{t_1}{t_1 + t_3},\ \frac{t_2}{t_2 + t_4}\Big\},$$

for arbitrary nonnegative t_3, t_4 (take $t_3 = \|\xi * \sigma\|$, $t_4 = \|\sigma * \tau\|$). Set $\mathbf{A} = \frac{t_1}{t_1 + t_3}$, $\mathbf{B} = \frac{t_2}{t_2 + t_4}$, and $\mathbf{C} = \frac{t_1 + t_2}{t_1 + t_2 + t_3 + t_4}$, and argue by two cases:
Case 1: $\mathbf{A} \le \mathbf{B}$ (equivalently $t_2 t_3 \ge t_1 t_4$). Compute

$$\mathbf{C} - \mathbf{A} = \frac{t_1 + t_2}{t_1 + t_2 + t_3 + t_4} - \frac{t_1}{t_1 + t_3} = \frac{t_2 t_3 - t_1 t_4}{(t_1 + t_2 + t_3 + t_4)(t_1 + t_3)} \ge 0,$$

so $\mathbf{C} \ge \mathbf{A} = \min\{\mathbf{A}, \mathbf{B}\}$.
Case 2: $\mathbf{B} \le \mathbf{A}$ (equivalently $t_2 t_3 \le t_1 t_4$). Compute

$$\mathbf{C} - \mathbf{B} = \frac{t_1 + t_2}{t_1 + t_2 + t_3 + t_4} - \frac{t_2}{t_2 + t_4} = \frac{t_1 t_4 - t_2 t_3}{(t_1 + t_2 + t_3 + t_4)(t_2 + t_4)} \ge 0,$$

so $\mathbf{C} \ge \mathbf{B} = \min\{\mathbf{A}, \mathbf{B}\}$.

In either case, $\mathbf{C} \ge \min\{\mathbf{A}, \mathbf{B}\}$, which proves (FN4) for the rational model.

Example 3.7. Define the mapping $\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1]$ by

$$\pi_{\mathbf{X}}(\xi, t_2) := \begin{cases} 1, & \text{if } \|\xi\| < t_2, \\ 0, & \text{if } \|\xi\| \ge t_2. \end{cases}$$

Then, $\pi_{\mathbf{X}}$ is a fuzzy GE-norm in the sense of (FN1)-(FN4):

- (FN1): If $\xi = 1_{\mathbf{X}}$, then $\|\xi\| = 0 < t_2$, so $\pi_{\mathbf{X}}(\xi, t_2) = 1$ for every $t_2 > 0$. Conversely, if $\pi_{\mathbf{X}}(\xi, t_2) = 1$ for all $t_2 > 0$, then in particular, $\pi_{\mathbf{X}}(\xi, 1) = 1$, so $\|\xi\| < 1$. Repeating with arbitrarily small scales shows $\|\xi\| = 0$ and hence $\xi = 1_{\mathbf{X}}$.
- (FN2): If $t_1 \le t_2$ and $\pi_{\mathbf{X}}(\xi, t_1) = 1$, then $\|\xi\| < t_1 \le t_2$, so $\pi_{\mathbf{X}}(\xi, t_2) = 1$; if $\pi_{\mathbf{X}}(\xi, t_1) = 0$, then monotonicity of the predicate also gives $\pi_{\mathbf{X}}(\xi, t_2) \ge 0$.
- (FN3): For fixed ξ, if $t_2 \to \infty$, then eventually $t_2 > \|\xi\|$, so $\pi_{\mathbf{X}}(\xi, t_2) = 1$ and hence the limit is 1.
- (FN4): For ξ, σ, τ, and $t_1, t_2 > 0$, if either $\pi_{\mathbf{X}}(\xi * \sigma, t_1) = 0$ or $\pi_{\mathbf{X}}(\sigma * \tau, t_2) = 0$, the right-hand side is 0 and the inequality is trivial. If both are 1, then $\|\xi * \sigma\| < t_1$ and $\|\sigma * \tau\| < t_2$, so by (2.15), $\|\xi * \tau\| \le \|\xi * \sigma\| + \|\sigma * \tau\| < t_1 + t_2$, hence $\pi_{\mathbf{X}}(\xi * \tau, t_1 + t_2) = 1$. Therefore (FN4) holds.

Example 3.8. Let $\mathbf{X} = \{\ell_1, \ell_a\}$ be a set with the following Cayley table:

$*$	ℓ_1	ℓ_a
ℓ_1	ℓ_1	ℓ_a
ℓ_a	ℓ_1	ℓ_1

Then, $(\mathbf{X}, *, \ell_1)$ is a GE-algebra. Define $\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1]$ by $\pi_{\mathbf{X}}(\ell_1, t_2) = 1$ and $\pi_{\mathbf{X}}(\ell_a, t_2) = \frac{t_2}{t_2+1}$ for all $t_2 > 0$. Then $\pi_{\mathbf{X}}$ is a fuzzy GE-norm on $\mathbf{X}$, so $(\mathbf{X}, \pi_{\mathbf{X}})$ is a fuzzy normed GE-algebra.

Proposition 3.9. *Every fuzzy normed GE-algebra $(\mathbf{X}, \pi_{\mathbf{X}})$ satisfies for all $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathbf{X}$ and $t_1, t_2, t_4 > 0$:*

(1.1) $\pi_{\mathbf{X}}(1_{\mathbf{X}}, t_2) = 1$,
(1.2) $\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_1, t_2) = 1$,

(1.3) $\pi_{\mathbf{X}}(\upsilon_1, t_1 + t_2) \geq \min\{\pi_{\mathbf{X}}(\upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 * \upsilon_1, t_2)\}$,
(1.4) $\pi_{\mathbf{X}}(\upsilon_1 * (\upsilon_1 * \upsilon_2), t_2) = \pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_2)$,
(1.5) $\pi_{\mathbf{X}}(\upsilon_1 * (\upsilon_2 * \upsilon_3), t_1 + t_2) \geq \min\{\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 * (\upsilon_1 * \upsilon_3), t_2)\}$,
(1.6) $\pi_{\mathbf{X}}(\upsilon_1 * (\upsilon_2 * \upsilon_3), t_1 + t_2 + t_4) \geq \min\{\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 * \upsilon_1, t_2), \pi_{\mathbf{X}}(\upsilon_1 * \upsilon_3, t_4)\}$,
(1.7) $\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1 + t_2) \geq \pi_{\mathbf{X}}(\upsilon_2, t_2)$.

Proof. (1.1) It comes directly from the zero properties.

(1.2) It comes directly from (GE1) and (1.1).

(1.3) It is obtained by putting $\upsilon_1 = 1$ in the triangle inequality and using (GE2).

(1.4) It comes directly from (2.5).

(1.5) For every $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathbf{X}$ and $t_1, t_2, t_4 > 0$, we have

$$\begin{aligned}\pi_{\mathbf{X}}(\upsilon_1 * (\upsilon_2 * \upsilon_3), t_1 + t_2) &\overset{(GE3)}{=} \pi_{\mathbf{X}}(\upsilon_1 * (\upsilon_2 * (\upsilon_1 * \upsilon_3)), t_1 + t_2)\\ &\overset{(FN4)}{\geq} \min\{\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 * (\upsilon_2 * (\upsilon_1 * \upsilon_3)), t_2)\}\\ &\overset{(2.5)}{=} \min\{\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 * (\upsilon_1 * \upsilon_3), t_2)\}.\end{aligned}$$

(1.6) For every $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathbf{X}$ and $t_1, t_2, t_4 > 0$, we have

$$\pi_{\mathbf{X}}(\upsilon_1 * (\upsilon_2 * \upsilon_3), t_1 + t_2 + t_4) \geq \min\{\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 * (\upsilon_1 * \upsilon_3), t_2 + t_4)\} \tag{3.1}$$

by (1.5), and

$$\begin{aligned}\pi_{\mathbf{X}}(\upsilon_2 * (\upsilon_1 * \upsilon_3), t_2 + t_4) &\geq \min\{\pi_{\mathbf{X}}(\upsilon_2 * \upsilon_1, t_2), \pi_{\mathbf{X}}(\upsilon_1 * (\upsilon_1 * \upsilon_3), t_4)\}\\ &= \min\{\pi_{\mathbf{X}}(\upsilon_2 * \upsilon_1, t_2), \pi_{\mathbf{X}}(\upsilon_1 * \upsilon_3, t_4)\}\end{aligned} \tag{3.2}$$

by the triangle inequality and (2.5). The combination of (3.1) and (3.2) induces

$$\pi_{\mathbf{X}}(\upsilon_1 * (\upsilon_2 * \upsilon_3), t_1 + t_2 + t_4) \geq \min\{\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 * \upsilon_1, t_2), \pi_{\mathbf{X}}(\upsilon_1 * \upsilon_3, t_4)\}.$$

(1.7) For every $\upsilon_1, \upsilon_2 \in \mathbf{X}$ and $t_1, t_2 > 0$, we have

$$\begin{aligned}\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1 + t_2) &\geq \min\{\pi_{\mathbf{X}}(\upsilon_1 * 1_{\mathbf{X}}, t_1), \pi_{\mathbf{X}}(1_{\mathbf{X}} * \upsilon_2, t_2)\}\\ &= \min\{\pi_{\mathbf{X}}(1_{\mathbf{X}}, t_1), \pi_{\mathbf{X}}(\upsilon_2, t_2)\}\\ &= \min\{1, \pi_{\mathbf{X}}(\upsilon_2, t_2)\} = \pi_{\mathbf{X}}(\upsilon_2, t_2)\end{aligned}$$

by the triangle inequality, (2.4), (GE2), and the zero property. □

3.2. Morphisms and fuzzy norm transfer

Let $(\mathbf{X}, *_{\mathbf{X}}, 1_{\mathbf{X}})$ and $(\mathbf{Y}, *_{\mathbf{Y}}, 1_{\mathbf{Y}})$ be GE-algebras, and let $\varphi : \mathbf{X} \to \mathbf{Y}$ be a GE-morphism. Consider two mappings $\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1]$ and $\pi_{\mathbf{Y}} : \mathbf{Y} \times (0, \infty) \to [0, 1]$ such that $\pi_{\mathbf{X}} = \pi_{\mathbf{Y}} \circ \varphi$. We now ask the question: If $\pi_{\mathbf{X}}$ is a fuzzy GE-norm on $\mathbf{X}$, then is $\pi_{\mathbf{Y}}$ a fuzzy GE-norm on $\mathbf{Y}$?

The example below gives a negative answer to this question.

Example 3.10. (i) Let $\mathbf{X} = \{\ell_1\}$ be the GE-algebra with constant ℓ_1 and binary operation $*$ defined by $\ell_1 * \ell_1 = \ell_1$. Let $\mathbf{Y} = \{\ell_0, \ell_1\}$ be the GE-algebra with constant ℓ_1 and binary operation $*$ defined by $\ell_0 * \ell_0 = \ell_1$, $\ell_0 * \ell_1 = \ell_1$, $\ell_1 * \ell_0 = \ell_0$, and $\ell_1 * \ell_1 = \ell_1$. Define the GE-morphism $\varphi : \mathbf{X} \to \mathbf{Y}$ by $\varphi(\ell_1) = \ell_1$; and define

$$\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1],\ (\ell_1, t_2) \mapsto 1.$$

Define a map $\pi_{\mathbf{Y}} : \mathbf{Y} \times (0, \infty) \to [0, 1]$ by $\pi_{\mathbf{Y}}(\ell_0, t) = 1 = \pi_{\mathbf{Y}}(\ell_1, t_2)$ for all $t_2 > 0$. Then $\pi_{\mathbf{X}}$ is a fuzzy GE-norm on $\mathbf{X}$ and $\pi_{\mathbf{X}} = \pi_{\mathbf{Y}} \circ \varphi$. But $\pi_{\mathbf{Y}}$ is not a fuzzy GE-norm on $\mathbf{Y}$ because $\pi_{\mathbf{Y}}(\ell_0, t_2) = 1$ for all $t_2 > 0$ even though $\ell_0 \neq \ell_1$.

(ii) Consider two sets $\mathbf{X} = \{1_{\mathbf{X}}, a_{\mathbf{X}}, b_{\mathbf{X}}, c_{\mathbf{X}}, d_{\mathbf{X}}\}$ and $\mathbf{Y} = \{1_{\mathbf{Y}}, 2_{\mathbf{Y}}, 3_{\mathbf{Y}}, 4_{\mathbf{Y}}, 5_{\mathbf{Y}}\}$ with the following Cayley tables:

$*_{\mathbf{X}}$	$1_{\mathbf{X}}$	$a_{\mathbf{X}}$	$b_{\mathbf{X}}$	$c_{\mathbf{X}}$	$d_{\mathbf{X}}$
$1_{\mathbf{X}}$	$1_{\mathbf{X}}$	$a_{\mathbf{X}}$	$b_{\mathbf{X}}$	$c_{\mathbf{X}}$	$d_{\mathbf{X}}$
$a_{\mathbf{X}}$	$1_{\mathbf{X}}$	$1_{\mathbf{X}}$	$c_{\mathbf{X}}$	$c_{\mathbf{X}}$	$c_{\mathbf{X}}$
$b_{\mathbf{X}}$	$1_{\mathbf{X}}$	$a_{\mathbf{X}}$	$1_{\mathbf{X}}$	$d_{\mathbf{X}}$	$d_{\mathbf{X}}$
$c_{\mathbf{X}}$	$1_{\mathbf{X}}$	$a_{\mathbf{X}}$	$1_{\mathbf{X}}$	$1_{\mathbf{X}}$	$1_{\mathbf{X}}$
$d_{\mathbf{X}}$	$1_{\mathbf{X}}$	$a_{\mathbf{X}}$	$1_{\mathbf{X}}$	$1_{\mathbf{X}}$	$1_{\mathbf{X}}$

$*_{\mathbf{Y}}$	$1_{\mathbf{Y}}$	$2_{\mathbf{Y}}$	$3_{\mathbf{Y}}$	$4_{\mathbf{Y}}$	$5_{\mathbf{Y}}$
$1_{\mathbf{Y}}$	$1_{\mathbf{Y}}$	$2_{\mathbf{Y}}$	$3_{\mathbf{Y}}$	$4_{\mathbf{Y}}$	$5_{\mathbf{Y}}$
$2_{\mathbf{Y}}$	$1_{\mathbf{Y}}$	$1_{\mathbf{Y}}$	$1_{\mathbf{Y}}$	$1_{\mathbf{Y}}$	$1_{\mathbf{Y}}$
$3_{\mathbf{Y}}$	$1_{\mathbf{Y}}$	$2_{\mathbf{Y}}$	$1_{\mathbf{Y}}$	$4_{\mathbf{Y}}$	$2_{\mathbf{Y}}$
$4_{\mathbf{Y}}$	$1_{\mathbf{Y}}$	$5_{\mathbf{Y}}$	$3_{\mathbf{Y}}$	$1_{\mathbf{Y}}$	$5_{\mathbf{Y}}$
$5_{\mathbf{Y}}$	$1_{\mathbf{Y}}$	$3_{\mathbf{Y}}$	$4_{\mathbf{Y}}$	$1_{\mathbf{Y}}$	$1_{\mathbf{Y}}$

Then, $(\mathbf{X}, *_{\mathbf{Y}}, 1_{\mathbf{X}})$ and $(\mathbf{Y}, *_{\mathbf{Y}}, 1_{\mathbf{Y}})$ are GE-algebras. Define a mapping $\varphi : \mathbf{X} \to \mathbf{Y}$ by

$$\varphi(\upsilon_1) = \begin{cases} 1_{\mathbf{Y}}, & \text{if } \upsilon_1 \in \{1_{\mathbf{X}}, b_{\mathbf{X}}, c_{\mathbf{X}}, d_{\mathbf{X}}\}, \\ 2_{\mathbf{Y}}, & \text{if } \upsilon_1 = a_{\mathbf{X}}. \end{cases}$$

It is routine to check that φ is a GE-morphism. Define a GE-norm $\|\cdot\|$ on $\mathbf{X}$ by

$$\|1\| = 0, \qquad \|\upsilon_1\| = 1 \quad \text{for } \upsilon_1 \in \{a_{\mathbf{X}}, b_{\mathbf{X}}, c_{\mathbf{X}}, d_{\mathbf{X}}\}.$$

This is a GE-norm on $\mathbf{X}$. Now define the standard "rational" fuzzy GE-norm on $\mathbf{X}$ by

$$\pi_{\mathbf{X}}(\upsilon_1, t_2) := \frac{t_2}{t_2 + \|\upsilon_1\|}, \qquad \upsilon_1 \in \mathbf{X},\ t_2 > 0.$$

Hence

$$\pi_{\mathbf{X}}(1_{\mathbf{X}}, t_2) = 1, \qquad \pi_{\mathbf{X}}(\upsilon_1, t_2) = \frac{t_2}{t_2 + 1} \text{ for } \upsilon_1 \neq 1_{\mathbf{X}}.$$

Define a map $\pi_{\mathbf{Y}} : \mathbf{Y} \times (0, \infty) \to [0, 1]$ by

$$\pi_{\mathbf{Y}}(1_{\mathbf{Y}}, t_2) = 1 = \pi_{\mathbf{Y}}(3_{\mathbf{Y}}, t_2) = \pi_{Y}(4_{\mathbf{Y}}, t_2) = \pi_{\mathbf{Y}}(5_{\mathbf{Y}}, t_2) \text{ and } \pi_{\mathbf{Y}}(2_{\mathbf{Y}}, t_2) = \pi_{\mathbf{X}}(a_{\mathbf{X}}, t_2)$$

for all $t_2 > 0$. Then $\pi_{\mathbf{X}}$ is a fuzzy GE-norm on $\mathbf{X}$ and $\pi_{\mathbf{X}} = \pi_{\mathbf{Y}} \circ \varphi$. But $\pi_{\mathbf{Y}}$ is not a fuzzy GE-norm on $\mathbf{Y}$ because $\pi_{\mathbf{Y}}(3_{\mathbf{Y}}, t_2) = 1$ for all $t_2 > 0$ even though $3_{\mathbf{Y}} \neq 1_{\mathbf{Y}}$. Hence π_{Y} is not a fuzzy GE-norm on $\mathbf{Y}$.

By strengthening the condition of GE-morphism φ, we provide a positive answer to the above question in the following theorem.

Theorem 3.11. *Let* $(\mathbf{X}, *_{\mathbf{X}}, 1_{\mathbf{X}})$ *and* $(\mathbf{Y}, *_{\mathbf{Y}}, 1_{\mathbf{Y}})$ *be GE-algebras, and let* $\varphi : \mathbf{X} \to \mathbf{Y}$ *be an onto GE-morphism. Consider two mappings* $\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1]$ *and* $\pi_{\mathbf{Y}} : \mathbf{Y} \times (0, \infty) \to [0, 1]$ *such that* $\pi_{\mathbf{X}} = \pi_{\mathbf{Y}} \circ \varphi$. *If* $\pi_{\mathbf{X}}$ *is a fuzzy GE-norm on* $\mathbf{X}$, *then* $\pi_{\mathbf{Y}}$ *is a fuzzy GE-norm on* $\mathbf{Y}$.

Proof. Suppose $\pi_{\mathbf{Y}}(\upsilon_2, t_2) = 1$ for all $t_2 > 0$. Since φ is onto, there exists $\upsilon_1 \in \mathbf{X}$ such that $\varphi(\upsilon_1) = \upsilon_2$. Hence

$$\pi_{\mathbf{X}}(\upsilon_1, t_2) = \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2) = \pi_{\mathbf{Y}}(\upsilon_2, t_2) = 1$$

for all $t_2 > 0$. The zero property of $\pi_{\mathbf{X}}$ forces $\upsilon_1 = 1_{\mathbf{X}}$, so $\upsilon_2 = \varphi(\upsilon_1) = \varphi(1_{\mathbf{X}}) = 1_{\mathbf{Y}}$. Conversely, assume $\upsilon_2 = 1_{\mathbf{Y}}$. Then

$$\pi_{\mathbf{Y}}(\upsilon_2, t_2) = \pi_{\mathbf{Y}}(1_{\mathbf{X}}, t_2) = \pi_{\mathbf{Y}}(\varphi(1_{\mathbf{X}}), t_2) = \pi_{\mathbf{X}}(1_{\mathbf{X}}, t_2) = 1$$

for all $t_2 > 0$ by the zero property of $\pi_{\mathbf{X}}$. This shows that $\pi_{\mathbf{Y}}$ satisfies the zero property. Let $\upsilon_2 \in \mathbf{X}$ and $0 < t_1 \leq t_2$. Then $\varphi(\upsilon_1) = \upsilon_2$ for some $\upsilon_1 \in \mathbf{X}$ since φ is onto. Thus

$$\pi_{\mathbf{Y}}(\upsilon_2, t_1) = \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_1) = \pi_{\mathbf{X}}(\upsilon_1, t_1) \leq \pi_{\mathbf{X}}(\upsilon_1, t_2) = \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2) = \pi_{\mathbf{Y}}(\upsilon_2, t_2),$$

so the monotonicity of $\pi_{\mathbf{Y}}$ is valid. In order to confirm that the limit at large scales of $\pi_{\mathbf{Y}}$, for every $\upsilon_2 \in \mathbf{Y}$, choose $\upsilon_1 \in \mathbf{X}$ such that $\varphi(\upsilon_1) = \upsilon_2$. Then

$$\lim_{t_2 \to \infty} \pi_{\mathbf{Y}}(\upsilon_2, t_2) = \lim_{t_2 \to \infty} \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2) = \lim_{t_2 \to \infty} \pi_{\mathbf{X}}(\upsilon_1, t_2) = 1$$

by the limit property of $\pi_{\mathbf{X}}$. Finally, we will check the triangle inequality for $\pi_{\mathbf{Y}}$. Let $\jmath_1, \jmath_2, \jmath_3 \in \mathbf{Y}$ and $t_1, t_2 > 0$. Then there exist $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathbf{X}$ such that $\varphi(\upsilon_1) = \jmath_1$, $\varphi(\upsilon_2) = \jmath_2$, and $\varphi(\upsilon_3) = \jmath_3$. Since φ is a GE-morphism, we have $\jmath_1 * \jmath_3 = \varphi(\upsilon_1) * \varphi(\upsilon_3) = \varphi(\upsilon_1 * \upsilon_3)$, $\jmath_1 * \jmath_2 = \varphi(\upsilon_1) * \varphi(\upsilon_2) = \varphi(\upsilon_1 * \upsilon_2)$, and $\jmath_2 * \jmath_3 = \varphi(\upsilon_2) * \varphi(\upsilon_3) = \varphi(\upsilon_2 * \upsilon_3)$. It follows from the triangle inequality for $\pi_{\mathbf{X}}$ that

$$\begin{aligned}
\pi_{\mathbf{Y}}(\jmath_1 * \jmath_3, t_1 + t_2) &= \pi_{\mathbf{Y}}(\varphi(\upsilon_1 * \upsilon_3), t_1 + t_2) = \pi_{\mathbf{X}}(\upsilon_1 * \upsilon_3, t_1 + t_2) \\
&\geq \min\{\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 * \upsilon_3, t_2)\} \\
&= \min\{\pi_{\mathbf{Y}}(\varphi(\upsilon_1 * \upsilon_2), t_1), \pi_{\mathbf{Y}}(\varphi(\upsilon_2 * \upsilon_3), t_1)\} \\
&= \min\{\pi_{\mathbf{Y}}(\jmath_1 * \jmath_2, t_1), \pi_{\mathbf{Y}}(\jmath_2 * \jmath_3, t_2)\}.
\end{aligned}$$

Hence $\pi_{\mathbf{Y}}$ is a fuzzy GE-norm on $\mathbf{Y}$. □

Let $(\mathbf{X}, *_{\mathbf{X}}, 1_{\mathbf{X}})$ and $(\mathbf{Y}, *_{\mathbf{Y}}, 1_{\mathbf{Y}})$ be GE-algebras, and let $\varphi : \mathbf{X} \to \mathbf{Y}$ be a GE-morphism. Consider two mappings $\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1]$ and $\pi_{\mathbf{Y}} : \mathbf{Y} \times (0, \infty) \to [0, 1]$ such that $\pi_{\mathbf{X}} = \pi_{\mathbf{Y}} \circ \varphi$. We now ask the question: If $\pi_{\mathbf{Y}}$ is a fuzzy GE-norm on $\mathbf{Y}$, then is $\pi_{\mathbf{X}}$ a fuzzy GE-norm on $\mathbf{X}$?

The example below gives a negative answer to this question.

Example 3.12. Let $\mathbf{X} = \mathbf{Y} = \{\ell_0, \ell_1\}$ be a set with the following Cayley table:

$*$	ℓ_0	ℓ_1
ℓ_0	ℓ_1	ℓ_1
ℓ_1	ℓ_0	ℓ_1

Then, $(\mathbf{X}, *, \ell_1)$ and $(\mathbf{Y}, *, \ell_1)$ are GE-algebras. Note that the mapping $\varphi : \mathbf{X} \to \mathbf{Y}$ by $\varphi(\ell_0) = \ell_1 = \varphi(\ell_1)$ is a GE-morphism. Define $\pi_{\mathbf{Y}} : \mathbf{Y} \times (0, \infty) \to [0, 1]$ by $\pi_{\mathbf{Y}}(\ell_0, t_2) = 1 - e^{-t_2}$ and $\pi_{\mathbf{Y}}(\ell_1, t_2) = 1$ for all $t_2 > 0$. By definition, $\pi_{\mathbf{Y}}$ satisfies the zero property. If $0 < t_1 \leq t_2$, then $\pi_{\mathbf{Y}}(\ell_1, t_1) = 1 = \pi_{\mathbf{Y}}(\ell_1, t_2)$ and

$$\pi_{\mathbf{Y}}(\ell_0, t_1) = 1 - e^{-t_1} \leq 1 - e^{-t_2} = \pi_{\mathbf{Y}}(\ell_0, t_2).$$

We have $\lim_{t_2 \to \infty} \pi_{\mathbf{Y}}(\ell_1, t_2) = 1$ and $\lim_{t_2 \to \infty} \pi_{\mathbf{Y}}(\ell_0, t_2) = \lim_{t_2 \to \infty} (1 - e^{-t_2}) = 1$. Hence $\pi_{\mathbf{Y}}$ satisfies the limit at large scales. In order to check the triangle inequality of $\pi_{\mathbf{Y}}$, let $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathbf{Y}$ and $t_1, t_2 > 0$. Note that $\upsilon_1 * \upsilon_3 = \ell_0$ only when $(\upsilon_1, \upsilon_3) = (\ell_1, \ell_0)$; otherwise $\upsilon_1 * \upsilon_3 = \ell_1$. If $(\upsilon_1, \upsilon_3) \in (\mathbf{Y} \times \mathbf{Y}) \setminus \{(\ell_1, \ell_0)\}$, then

$$\pi_{\mathbf{Y}}(\upsilon_1 * \upsilon_3, t_1 + t_2) = \pi_{\mathbf{Y}}(\ell_1, t_1 + t_2) = 1 \geq \min\{\pi_{\mathbf{Y}}(\upsilon_1 * \upsilon_2, t_1), \pi_{\mathbf{Y}}(\upsilon_2 * \upsilon_3, t_2)\}.$$

Let $(\upsilon_1, \upsilon_3) = (\ell_1, \ell_0)$. Then $\pi_{\mathbf{Y}}(\ell_1 * \ell_0, t_1 + t_2) = \pi_{\mathbf{Y}}(\ell_0, t_1 + t_2) = 1 - e^{-(t_1 + t_2)}$. If $\upsilon_2 = \ell_1$, then

$$\begin{aligned}
&\min\{\pi_{\mathbf{Y}}(\ell_1 * \ell_1, t_1), \pi_{\mathbf{Y}}(\ell_1 * \ell_0, t_2)\} = \min\{\pi_{\mathbf{Y}}(\ell_1, t_1), \pi_{\mathbf{Y}}(\ell_0, t_2)\} \\
&= \min\{1, 1 - e^{-t_2}\} = 1 - e^{-t_2} \leq 1 - e^{-(t_1 + t_2)} \\
&= \pi_{\mathbf{Y}}(\ell_1 * \ell_0, t_1 + t_2).
\end{aligned}$$

If $\upsilon_2 = \ell_0$, then

$$\begin{aligned}
&\min\{\pi_{\mathbf{Y}}(\ell_1 * \ell_0, t_1), \pi_{\mathbf{Y}}(\ell_0 * \ell_0, t_2)\} = \min\{\pi_{\mathbf{Y}}(\ell_0, t_1), \pi_{\mathbf{Y}}(\ell_1, t_2)\} \\
&= \min\{1 - e^{-t_1}, 1\} = 1 - e^{-t_1} \leq 1 - e^{-(t_1 + t_2)} \\
&= \pi_{\mathbf{Y}}(\ell_1 * \ell_0, t_1 + t_2).
\end{aligned}$$

Hence, $\pi_{\mathbf{Y}}$ satisfies the triangle inequality, so $\pi_{\mathbf{Y}}$ is a fuzzy GE-norm on $\mathbf{Y}$. Since $\pi_{\mathbf{X}}(\upsilon_1, t_2) = \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2) = \pi_{\mathbf{Y}}(\ell_1, t_2) = 1$ for all $\upsilon_1 \in \mathbf{X}$ and $t_2 > 0$, we have $\pi_{\mathbf{X}}(\ell_0, t_2) = 1$ for all $t_2 > 0$, but $\ell_0 \neq \ell_1$. Therefore $\pi_{\mathbf{X}}$ does not satisfies the zero property, so it is not a fuzzy GE-norm on $\mathbf{X}$.

By strengthening the condition of GE-morphism φ, we provide a positive answer to the above question in the following theorem.

Theorem 3.13. *Let* $(\mathbf{X}, *_{\mathbf{X}}, 1_{\mathbf{X}})$ *and* $(\mathbf{Y}, *_{\mathbf{Y}}, 1_{\mathbf{Y}})$ *be GE-algebras, and let* $\varphi : \mathbf{X} \to \mathbf{Y}$ *be a GE-morphism. Consider two mappings* $\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1]$ *and* $\pi_{\mathbf{Y}} : \mathbf{Y} \times (0, \infty) \to [0, 1]$ *such that* $\pi_{\mathbf{X}} = \pi_{\mathbf{Y}} \circ \varphi$, *i.e.,* $\pi_{\mathbf{X}}(\upsilon_1, t_2) = \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2)$ *for all* $t_2 > 0$. *If* φ *satisfies*

$$(\forall \upsilon_1 \in \mathbf{X})(\varphi(\upsilon_1) = 1_{\mathbf{Y}} \ \Rightarrow \ \upsilon_1 = 1_{\mathbf{X}}) \tag{3.3}$$

and $\pi_{\mathbf{Y}}$ *is a fuzzy GE-norm on* $\mathbf{Y}$, *then* $\pi_{\mathbf{X}}$ *is a fuzzy GE-norm on* $\mathbf{X}$.

Proof. (FN1) Zero property: Let $\upsilon_1 \in \mathbf{X}$. If $\pi_{\mathbf{X}}(\upsilon_1, t_2) = 1$ for all $t_2 > 0$, then $\pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2) = 1$ for all $t_2 > 0$, so $\varphi(\upsilon_1) = 1_{\mathbf{Y}}$ by the zero property of $\pi_{\mathbf{Y}}$. It follows from (3.3) that $\upsilon_1 = 1_{\mathbf{X}}$. Suppose $\upsilon_1 = 1_{\mathbf{X}}$. Then $\pi_{\mathbf{X}}(1_{\mathbf{X}}, t_2) = \pi_{\mathbf{Y}}(\varphi(1_{\mathbf{X}}), t_2) = \pi_{\mathbf{Y}}(1_{\mathbf{Y}}, t_2) = 1$ for all $t_2 > 0$ by the zero property of $\pi_{\mathbf{Y}}$.

(FN2) Monotonicity in scale: For every $\upsilon_1 \in \mathbf{X}$ and $0 < t_1 \leq t_2$, we have

$$\pi_{\mathbf{X}}(\upsilon_1, t_1) = \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_1) \leq \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2) = \pi_{\mathbf{X}}(\upsilon_1, t_2).$$

(FN3) Limit at large scales: For every $\upsilon_1 \in \mathbf{X}$ and $t_2 > 0$, we have

$$\lim_{t_2\to\infty} \pi_{\mathbf{X}}(\upsilon_1, t_2) = \lim_{t_2\to\infty} \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2) = 1.$$

(FN4) Triangle inequality: For every $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathbf{X}$ and $t_1, t_2 > 0$, we have

$$\begin{aligned}\pi_{\mathbf{X}}(\upsilon_1 *_{\mathbf{X}} \upsilon_3, t_1 + t_2) &= \pi_{\mathbf{Y}}(\varphi(\upsilon_1 *_{\mathbf{X}} \upsilon_3), t_1 + t_2) = \pi_{\mathbf{Y}}(\varphi(\upsilon_1) *_{\mathbf{Y}} \varphi(\upsilon_3), t_1 + t_2)\\ &\geq \min\{\pi_{\mathbf{Y}}(\varphi(\upsilon_1) *_{\mathbf{Y}} \varphi(\upsilon_2), t_1), \pi_{\mathbf{Y}}(\varphi(\upsilon_2) *_{\mathbf{Y}} \varphi(\upsilon_3), t_2)\}\\ &= \min\{\pi_{\mathbf{Y}}(\varphi(\upsilon_1 *_{\mathbf{X}} \upsilon_2), t_1), \pi_{\mathbf{Y}}(\varphi(\upsilon_2 *_{\mathbf{X}} \upsilon_3), t_2)\}\\ &= \min\{\pi_{\mathbf{X}}(\upsilon_1 *_{\mathbf{X}} \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 *_{\mathbf{X}} \upsilon_3, t_2)\}.\end{aligned}$$

Consequently, $\pi_{\mathbf{X}}$ is a fuzzy GE-norm on $\mathbf{X}$. □

Remark 3.14. Let $(\mathbf{X}, *_{\mathbf{X}}, 1_{\mathbf{X}})$ and $(\mathbf{Y}, *_{\mathbf{Y}}, 1_{\mathbf{Y}})$ be GE-algebras, and let $\varphi : \mathbf{X} \to \mathbf{Y}$ be a GE-morphism. Consider two mappings $\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1]$ and $\pi_{\mathbf{Y}} : \mathbf{Y} \times (0, \infty) \to [0, 1]$ such that $\pi_{\mathbf{X}} = \pi_{\mathbf{Y}} \circ \varphi$, i.e., $\pi_{\mathbf{X}}(\upsilon_1, t_2) = \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2)$ for all $\upsilon_1 \in \mathbf{X}$ and $t_2 > 0$. Suppose $\pi_{\mathbf{Y}}$ is a fuzzy GE-norm on $\mathbf{Y}$. Through Example 3.12, we found that $\pi_{\mathbf{X}}$ is not a fuzzy GE-norm on $\mathbf{X}$. But we know that $\pi_{\mathbf{X}}$ satisfies the triangle inequality, the limit at large scales, and the monotonicity in scale by the following calculations.

(FN4) Triangle inequality: Let $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathbf{X}$ and $t_1, t_2 > 0$. Then

$$\begin{aligned}\pi_{\mathbf{X}}(\upsilon_1 *_{\mathbf{X}} \upsilon_3, t_1 + t_2) &= \pi_{\mathbf{Y}}(\varphi(\upsilon_1 *_{\mathbf{X}} \upsilon_3), t_1 + t_2) = \pi_{\mathbf{Y}}(\varphi(\upsilon_1) *_{\mathbf{Y}} \varphi(\upsilon_3), t_1 + t_2)\\ &\geq \min\{\pi_{\mathbf{Y}}(\varphi(\upsilon_1) *_{\mathbf{Y}} \varphi(\upsilon_2), t_1), \pi_{\mathbf{Y}}(\varphi(\upsilon_2) *_{\mathbf{Y}} \varphi(\upsilon_3), t_2)\}\\ &= \min\{\pi_{\mathbf{Y}}(\varphi(\upsilon_1 *_{\mathbf{X}} \upsilon_2), t_1), \pi_{\mathbf{Y}}(\varphi(\upsilon_2 *_{\mathbf{X}} \upsilon_3), t_2)\}\\ &= \min\{\pi_{\mathbf{X}}(\upsilon_1 *_{\mathbf{X}} \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 *_{\mathbf{X}} \upsilon_3, t_2)\}\end{aligned}$$

(FN3) Limit at large scales: For every $\upsilon_1 \in \mathbf{X}$ and $t_2 > 0$, we have

$$\lim_{t_2\to\infty} \pi_{\mathbf{X}}(\upsilon_1, t_2) = \lim_{t_2\to\infty} \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2) = 1.$$

(FN2) Monotonicity in scale: For every $\upsilon_1 \in \mathbf{X}$ and $0 < t_1 \leq t_2$, we have

$$\pi_{\mathbf{X}}(\upsilon_1, t_1) = \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_1) \leq \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2) = \pi_{\mathbf{X}}(\upsilon_1, t_2).$$

Theorem 3.15. *Let $\varphi : \mathbf{X} \to \mathbf{Y}$ be a GE-morphism of GE-algebras $(\mathbf{X}, *_{\mathbf{X}}, 1_{\mathbf{X}})$ and $(\mathbf{Y}, *_{\mathbf{Y}}, 1_{\mathbf{Y}})$. Consider two mappings $\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1]$ and $\pi_{\mathbf{Y}} : \mathbf{Y} \times (0, \infty) \to [0, 1]$ such that $\pi_{\mathbf{X}} = \pi_{\mathbf{Y}} \circ \varphi$, i.e., $\pi_{\mathbf{X}}(\upsilon_1, t_2) = \pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2)$ for all $\upsilon_1 \in \mathbf{X}$ and $t_2 > 0$. If φ is injective and $\pi_{\mathbf{Y}}$ is a fuzzy GE-norm on $\mathbf{Y}$, then $\pi_{\mathbf{X}}$ is a fuzzy GE-norm on $\mathbf{X}$.*

Proof. Suppose φ is injective and $\pi_{\mathbf{Y}}$ is a fuzzy GE-norm on $\mathbf{Y}$. Through the Remark 3.14, we find that $\pi_{\mathbf{X}}$ satisfies the triangle inequality, limit at large scales, and monotonicity in scale, which are independent of the injectivity of φ. So we need to verify the zero property of $\pi_{\mathbf{X}}$ using the injectivity of φ. Let $\upsilon_1 \in \mathbf{X}$ and suppose $\pi_{\mathbf{X}}(\upsilon_1, t_2) = 1$ for all $t_2 > 0$. Then $\pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2) = 1$ for all $t_2 > 0$, so $\varphi(\upsilon_1) = 1_{\mathbf{Y}}$ by the zero property of $\pi_{\mathbf{Y}}$. Since φ is injective and $\varphi(1_{\mathbf{X}}) = 1_{\mathbf{Y}}$, it follows that $\upsilon_1 = 1_{\mathbf{X}}$. Conversely, if $\upsilon_1 = 1_{\mathbf{X}}$, then $\pi_{\mathbf{X}}(\upsilon_1, t_2) = \pi_{\mathbf{X}}(1_{\mathbf{X}}, t_2) = \pi_{\mathbf{Y}}(\varphi(1_{\mathbf{X}}), t_2) = \pi_{\mathbf{Y}}(1_{\mathbf{Y}}, t_2) = 1$ for all $t_2 > 0$. Hence $\pi_{\mathbf{X}}$ is a fuzzy GE-norm on $\mathbf{X}$. □

Theorem 3.16. *A GE-algebra* $\mathbf{X} := (\mathbf{X}, *, 1_{\mathbf{X}})$ *admits a fuzzy GE-norm if and only if its induced order* $\leq_{\mathbf{X}}$ *is transitive.*

Proof. Assume that $\mathbf{X} := (\mathbf{X}, *, 1_{\mathbf{X}})$ admits a fuzzy GE-norm $\pi_{\mathbf{X}}$. Suppose the induced order $\leq_{\mathbf{X}}$ is not transitive. Then there exist $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathbf{X}$ such that $\upsilon_1 \leq_{\mathbf{X}} \upsilon_2$ and $\upsilon_2 \leq_{\mathbf{X}} \upsilon_3$ but $\upsilon_1 \nleq_{\mathbf{X}} \upsilon_3$, that is, $\upsilon_1 * \upsilon_2 = 1_{\mathbf{X}}$, $\upsilon_2 * \upsilon_3 = 1_{\mathbf{X}}$, and $\upsilon_1 * \upsilon_3 \neq 1_{\mathbf{X}}$. Using the triangle inequality for $\pi_{\mathbf{X}}$ induces

$$\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_3, t_1 + t_2) \geq \min\{\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 * \upsilon_3, t_2)\} = \min\{\pi_{\mathbf{X}}(1_{\mathbf{X}}, t_1), \pi_{\mathbf{X}}(1_{\mathbf{X}}, t_2)\} = 1$$

for all $t_1, t_2 > 0$. Hence $\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_3, \upsilon_4) = 1$ for all $\upsilon_4 := t_1 + t_2 > 0$, so $\upsilon_1 * \upsilon_3 = 1_{\mathbf{X}}$ by the zero property for $\pi_{\mathbf{X}}$. This is a contradiction, so $\leq_{\mathbf{X}}$ is transitive.

Conversely, suppose the induced order $\leq_{\mathbf{X}}$ is transitive. Define a mapping

$$\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1], \qquad (\upsilon_1, t_2) \mapsto \begin{cases} 1, & \text{if } \upsilon_1 = 1_{\mathbf{X}}, \\ 1 - e^{-t_2}, & \text{if } \upsilon_1 \neq 1_{\mathbf{X}}. \end{cases}$$

(FN1): Suppose $\upsilon_1 = 1_{\mathbf{X}}$. Then, by definition,

$$\pi_{\mathbf{X}}(\upsilon_1, t_2) = \pi_{\mathbf{X}}(1_{\mathbf{X}}, t_2) = 1 \quad \text{for all } t_2 > 0.$$

Conversely, assume that $\pi_{\mathbf{X}}(\upsilon_1, t_2) = 1$ for all $t_2 > 0$. If $\upsilon_1 \neq 1_{\mathbf{X}}$, then

$$\pi_{\mathbf{X}}(\upsilon_1, t_2) = 1 - e^{-t_2} < 1 \quad \text{for every } t_2 > 0,$$

which is a contradiction. Hence $\upsilon_1 = 1_{\mathbf{X}}$. Therefore, axiom (FN1) holds. **(FN2):** Let $\upsilon_1 \in \mathbf{X}$ and let $0 < t_1 \leq t_2$. If $\upsilon_1 = 1_{\mathbf{X}}$, then

$$\pi_{\mathbf{X}}(\upsilon_1, t_1) = \pi_{\mathbf{X}}(\upsilon_1, t_2) = 1.$$

If $\upsilon_1 \neq 1_{\mathbf{X}}$, then

$$\pi_{\mathbf{X}}(\upsilon_1, t_1) = 1 - e^{-t_1} \leq 1 - e^{-t_2} = \pi_{\mathbf{X}}(\upsilon_1, t_2),$$

since the function $t_2 \mapsto 1 - e^{-t_2}$ is increasing on $(0, \infty)$. Thus axiom (FN2) is satisfied. **(FN3):** Let $\upsilon_1 \in \mathbf{X}$. If $\upsilon_1 = 1_{\mathbf{X}}$, then $\pi_{\mathbf{X}}(\upsilon_1, t_2) = 1$ for all $t_2 > 0$, and hence

$$\lim_{t_2 \to \infty} \pi_{\mathbf{X}}(\upsilon_1, t_2) = 1.$$

If $\upsilon_1 \neq 1_{\mathbf{X}}$, then

$$\lim_{t_2 \to \infty} \pi_{\mathbf{X}}(\upsilon_1, t_2) = \lim_{t_2 \to \infty} (1 - e^{-t_2}) = 1.$$

Therefore, axiom (FN3) holds for all $\upsilon_1 \in \mathbf{X}$. **(FN4):** We need to show that

$$\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_3, t_1 + t_2) \geq \min\{\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 * \upsilon_3, t_2)\} \tag{3.4}$$

for all $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathbf{X}$ and all $t_1, t_2 > 0$.

If $\upsilon_1 * \upsilon_3 = 1_{\mathbf{X}}$, then $\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_3, t_1 + t_2) = 1$, and (3.4) holds trivially. Suppose $\upsilon_1 * \upsilon_3 \neq 1_{\mathbf{X}}$. Then

$$\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_3, t_1 + t_2) = 1 - e^{-(t_1+t_2)}.$$

Since the induced order $\leq_{\mathbf{X}}$ is transitive, there exists no $\upsilon_2 \in \mathbf{X}$ such that $\upsilon_1 \leq_{\mathbf{X}} \upsilon_2$ and $\upsilon_2 \leq_{\mathbf{X}} \upsilon_3$. Hence, for every $\upsilon_2 \in \mathbf{X}$, either $\upsilon_1 * \upsilon_2 \neq 1_{\mathbf{X}}$ or $\upsilon_2 * \upsilon_3 \neq 1_{\mathbf{X}}$. Without loss of generality, assume $t_1 \leq t_2$. Then

$$\min\{\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 * \upsilon_3, t_2)\} \leq 1 - e^{-t_2}.$$

Therefore,

$$\begin{aligned}\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_3, t_1 + t_2) &= 1 - e^{-(t_1+t_2)} = 1 - e^{-t_1}e^{-t_2} > 1 - e^{-t_2} \\ &= \min\{\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 * \upsilon_3, t_2)\},\end{aligned}$$

which proves (3.4). This completes the proof. □

3.3. Fuzzy convergence and continuity

Definition 3.17 (Fuzzy convergence)**.** A sequence $(\xi_n) \subset \mathbf{X}$ *converges* to $\xi \in \mathbf{X}$ (write $\xi_n \to_{\pi_{\mathbf{X}}} \xi$) if for every $t_2 > 0$ and every $\varepsilon \in (0, 1)$, there exists N such that for all $n \geq N$,

$$\pi_{\mathbf{X}}(\xi_n * \xi, t_2) > 1 - \varepsilon \quad \text{and} \quad \pi_{\mathbf{X}}(\xi * \xi_n, t_2) > 1 - \varepsilon.$$

Theorem 3.18 (Uniqueness of fuzzy limits)**.** *Let* $(\mathbf{X}, *, 1_{\mathbf{X}})$ *be a* commutative *GE-algebra equipped with a fuzzy GE-norm* $\pi_{\mathbf{X}}$ *satisfying (FN1)-(FN4). If a sequence* $(\xi_n) \subset \mathbf{X}$ *converges (in the fuzzy sense) to both* ξ *and* η*, then* $\xi = \eta$*.*

Proof. Fix an arbitrary scale $t_2 > 0$ and an arbitrary tolerance $\varepsilon \in (0, 1)$. Since $\xi_n \to_{\pi_{\mathbf{X}}} \xi$ and $\xi_n \to_{\pi_{\mathbf{X}}} \eta$, by the definition of fuzzy convergence, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have the four inequalities

$$\pi_{\mathbf{X}}(\xi_n * \xi, t_2) > 1 - \varepsilon, \qquad \pi_{\mathbf{X}}(\xi * \xi_n, t_2) > 1 - \varepsilon,$$

and

$$\pi_{\mathbf{X}}(\xi_n * \eta, t_2) > 1 - \varepsilon, \qquad \pi_{\mathbf{X}}(\eta * \xi_n, t_2) > 1 - \varepsilon.$$

Applying (FN4) to the pair (ξ, η) with intermediate ξ_n yields

$$\pi_{\mathbf{X}}(\xi * \eta, 2t_2) \; \geq \; \min\{\pi_{\mathbf{X}}(\xi * \xi_n, t_2), \; \pi_{\mathbf{X}}(\xi_n * \eta, t_2)\}.$$

For $n \geq N$, the right-hand side exceeds $1 - \varepsilon$, hence $\pi_{\mathbf{X}}(\xi * \eta, 2t_2) > 1 - \varepsilon$. Letting $\varepsilon \to 0$ gives $\pi_{\mathbf{X}}(\xi * \eta, 2t_2) = 1$ for all $t_2 > 0$. By (FN1), we therefore obtain $\xi * \eta = 1_{\mathbf{X}}$. Repeating the same argument with the roles of ξ and η swapped yields $\eta * \xi = 1_{\mathbf{X}}$. Since $\mathbf{X}$ is commutative, $\xi = \eta$. This completes the proof. □

Definition 3.19 (Fuzzy Cauchy)**.** The sequence (ξ_n) is *fuzzy-Cauchy* if for every $t_2 > 0$ and every $\varepsilon \in (0, 1)$ there exists N such that for all $m, n \geq N$,

$$\pi_{\mathbf{X}}(\xi_n * \xi_m, t_2) > 1 - \varepsilon \quad \text{and} \quad \pi_{\mathbf{X}}(\xi_m * \xi_n, t_2) > 1 - \varepsilon.$$

Theorem 3.20 (Preservation under GE-morphisms)**.** *Let* $\varphi : \mathbf{X} \to \mathbf{Y}$ *be a GE-morphism between GE-algebras. Suppose* $\pi_{\mathbf{X}}$ *and* $\pi_{\mathbf{Y}}$ *are fuzzy GE-norms with compatibility*

$$\pi_{\mathbf{Y}}(\varphi(\xi), t_2) = \pi_{\mathbf{X}}(\xi, t_2) \quad \textit{for all } \xi \in \mathbf{X}, \; t_2 > 0.$$

Then, $\xi_n \to_{\pi_{\mathbf{X}}} \xi$ *iff* $\varphi(\xi_n) \to_{\pi_{\mathbf{Y}}} \varphi(\xi)$*.*

Proof. Assume $\varphi : \mathbf{X} \to \mathbf{Y}$ is a GE-morphism and that the fuzzy GE-norms $\pi_{\mathbf{X}}$ on $\mathbf{X}$ and $\pi_{\mathbf{Y}}$ on $\mathbf{Y}$ satisfy the compatibility condition

$$\pi_{\mathbf{Y}}(\varphi(\xi), t_2) = \pi_{\mathbf{X}}(\xi, t_2) \qquad \text{for all } \xi \in \mathbf{X},\ t_2 > 0.$$

Suppose $\xi_n \to_{\pi_{\mathbf{X}}} \xi$ in $\mathbf{X}$.

Fix an arbitrary $t_2 > 0$ and $\varepsilon \in (0, 1)$. By the definition of fuzzy convergence $\xi_n \to_{\pi_{\mathbf{X}}} \xi$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\pi_{\mathbf{X}}(\xi_n * \xi, t_2) > 1 - \varepsilon \qquad \text{and} \qquad \pi_{\mathbf{X}}(\xi * \xi_n, t_2) > 1 - \varepsilon.$$

Because φ is a GE-morphism, we have the identities

$$\varphi(\xi_n) * \varphi(\xi) = \varphi(\xi_n * \xi) \quad \text{and} \quad \varphi(\xi) * \varphi(\xi_n) = \varphi(\xi * \xi_n)$$

for every n. Applying the compatibility of the fuzzy norms, for $n \geq N$, we obtain

$$\pi_{\mathbf{Y}}\big(\varphi(\xi_n) * \varphi(\xi), t_2\big) = \pi_{\mathbf{Y}}\big(\varphi(\xi_n * \xi), t_2\big) = \pi_{\mathbf{X}}(\xi_n * \xi, t_2) > 1 - \varepsilon,$$

and similarly

$$\pi_{\mathbf{Y}}(\varphi(\xi) * \varphi(\xi_n), t_2) = \pi_{\mathbf{X}}(\xi * \xi_n, t_2) > 1 - \varepsilon.$$

Since $t_2 > 0$ and $\varepsilon \in (0, 1)$ were arbitrary, we have $\varphi(\xi_n) \to_{\pi_{\mathbf{Y}}} \varphi(\xi)$ in $\mathbf{Y}$. Conversely, suppose $\varphi(\xi_n) \to_{\pi_{\mathbf{Y}}} \varphi(\xi)$.

Again fix arbitrary $t_2 > 0$ and $\varepsilon \in (0, 1)$. From $\varphi(\xi_n) \to_{\pi_{\mathbf{Y}}} \varphi(\xi)$ there exists N such that for all $n \geq N$,

$$\pi_{\mathbf{Y}}(\varphi(\xi_n) * \varphi(\xi), t_2) > 1 - \varepsilon \quad \text{and} \quad \pi_{\mathbf{Y}}(\varphi(\xi) * \varphi(\xi_n), t_2) > 1 - \varepsilon.$$

We use the GE-morphism identity $\varphi(\xi_n) * \varphi(\xi) = \varphi(\xi_n * \xi)$ and the compatibility $\pi_{\mathbf{Y}}(\varphi(\upsilon_1), t_2) = \pi_{\mathbf{X}}(\upsilon_1, t_2)$ to obtain, for $n \geq N$,

$$\pi_{\mathbf{X}}(\xi_n * \xi, t_2) = \pi_{\mathbf{Y}}\big(\varphi(\xi_n * \xi), t_2\big) = \pi_{\mathbf{Y}}\big(\varphi(\xi_n) * \varphi(\xi), t_2\big) > 1 - \varepsilon,$$

and similarly $\pi_{\mathbf{X}}(\xi * \xi_n, t_2) > 1 - \varepsilon$. This shows $\xi_n \to_{\pi_{\mathbf{X}}} \xi$, which completes the proof. □

Definition 3.21. A fuzzy GE-norm $\pi : \mathbf{X} \times (0, \infty) \to [0, 1]$ is said to be *order-preserving* if for all $\upsilon_1, \upsilon_2 \in \mathbf{X}$ and all $t_2 > 0$,

$$\upsilon_1 \leq_X \upsilon_2 \quad \Longrightarrow \quad \pi(\upsilon_1, t_2) \leq \pi(\upsilon_2, t_2).$$

Proposition 3.22. *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed GE-algebra such that*

$$\upsilon_1 \leq_X \upsilon_2 \quad \Longrightarrow \quad \|\upsilon_1\| \geq \|\upsilon_2\|.$$

Define $\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1]$ *by*

$$\pi_{\mathbf{X}}(\upsilon_1, t_2) = \frac{t_2}{t_2 + \|\upsilon_1\|}.$$

Then $\pi_{\mathbf{X}}$ *is an order-preserving fuzzy GE-norm.*

Proof. By Example 3.6, $\pi_{\mathbf{X}}$ is a fuzzy GE-norm. Let $\upsilon_1, \upsilon_2 \in \mathbf{X}$ with $\upsilon_1 \leq_X \upsilon_2$ and let $t_2 > 0$. By assumption, $\|\upsilon_1\| \geq \|\upsilon_2\|$. Then, we obtain

$$\pi_{\mathbf{X}}(\upsilon_1, t_2) = \frac{t_2}{t_2 + \|\upsilon_1\|} \leq \frac{t_2}{t_2 + \|\upsilon_2\|} = \pi_{\mathbf{X}}(\upsilon_2, t_2).$$

Hence $\pi_{\mathbf{X}}$ is order-preserving. □

Theorem 3.23. *Let $(\mathbf{X}, *, 1_{\mathbf{X}})$ be a transitive GE-algebra and let $\pi_{\mathbf{X}}$ be a fuzzy GE-norm. Assume*

(1) $\xi_n \leq_{\mathbf{X}} \xi_{n+1}$ for every n,
(2) for all $\wp_1, \wp_2, \wp_3 \in X$, $\wp_1 \leq_{\mathbf{X}} \wp_2 \Rightarrow \wp_3 * \wp_1 \leq_{\mathbf{X}} \wp_3 * \wp_2$,
(3) $\pi_{\mathbf{X}}$ is order–preserving: $\wp_1 \leq_{\mathbf{X}} \wp_2 \Rightarrow \pi_{\mathbf{X}}(\wp_1, t_2) \leq \pi_{\mathbf{X}}(\wp_2, t_2)$ for all $t_2 > 0$.

Then for each fixed $t_2 > 0$, the sequence $n \mapsto \pi_{\mathbf{X}}(\xi * \xi_n, t_2)$ is non-decreasing. Moreover, if $\xi_n \to_{\pi_{\mathbf{X}}} \xi$ (in the two-sided sense), then $\pi_{\mathbf{X}}(\xi * \xi_n, t_2) \to 1$ as $n \to \infty$.

Proof. Fix an arbitrary $t_2 > 0$. For each n, we have $\xi_n \leq_{\mathbf{X}} \xi_{n+1}$ by (1). Applying left-translation with the fixed element ξ (hypothesis (2)) yields

$$\xi * \xi_n \leq_{\mathbf{X}} \xi * \xi_{n+1}.$$

Now we apply the order-preservation of $\pi_{\mathbf{X}}$ (hypothesis (3)) to obtain

$$\pi_{\mathbf{X}}(\xi * \xi_n, t_2) \leq \pi_{\mathbf{X}}(\xi * \xi_{n+1}, t_2).$$

Since n was arbitrary, the sequence $n \mapsto \pi_{\mathbf{X}}(\xi * \xi_n, t_2)$ is non-decreasing.
First note that for every n and fixed t_2, we have $0 \leq \pi_{\mathbf{X}}(\xi * \xi_n, t_2) \leq 1$, so the non-decreasing sequence is bounded above by 1 and therefore has a limit $L \in [0, 1]$.
Now assume $\xi_n \to_{\pi_{\mathbf{X}}} \xi$ in the (two-sided) fuzzy sense. By the definition of convergence, for every $\varepsilon \in (0, 1)$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\pi_{\mathbf{X}}(\xi * \xi_n, t_2) > 1 - \varepsilon.$$

Because the sequence is non-decreasing, for every $m \geq N$, we have $\pi_{\mathbf{X}}(\xi * \xi_N, t_2) \leq \pi_{\mathbf{X}}(\xi * \xi_m, t_2)$, hence all tail values $m \geq N$ exceed $1 - \varepsilon$. Therefore, the limit satisfies $L \geq 1 - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude $L = 1$, i.e.,

$$\lim_{n \to \infty} \pi_{\mathbf{X}}(\xi * \xi_n, t_2) = 1.$$

This proves both claims. □

Theorem 3.24. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed GE-algebra and let the fuzzy GE-norm be $\pi_{\mathbf{X}}(\upsilon_1, t_2) = \dfrac{t_2}{t_2 + \|\upsilon_1\|}$. If $\xi_n \to_{\pi_{\mathbf{X}}} \xi$, then for every $t_2 > 0$, there exists N such that*

$$\pi_{\mathbf{X}}(\xi_n,\ t_2 + \|\xi\|) > \tfrac{1}{2} \qquad \forall n \geq N.$$

Proof. Fix $t_2 > 0$ and set $\varepsilon = \frac{1}{2}$. By convergence, there exists N with $\pi_{\mathbf{X}}(\xi * \xi_n, t_2) > \frac{1}{2}$ for all $n \geq N$. Hence for such n,

$$\frac{t_2}{t_2 + \|\xi * \xi_n\|} > \frac{1}{2} \quad \Longrightarrow \quad \|\xi * \xi_n\| < t_2.$$

Using the GE-norm inequality $\|\xi_n\| \leq \|\xi * \xi_n\| + \|\xi\|$, we obtain $\|\xi_n\| < t_2 + \|\xi\|$ for all $n \geq N$. Therefore

$$\pi_{\mathbf{X}}(\xi_n,\ t_2 + \|\xi\|) = \frac{t_2 + \|\xi\|}{t_2 + \|\xi\| + \|\xi_n\|} > \frac{t_2 + \|\xi\|}{2(t_2 + \|\xi\|)} = \frac{1}{2},$$

which proves the claim. □

Theorem 3.25. *Let* $(\mathbf{X}, *, 1_{\mathbf{X}})$ *be a GE-algebra equipped with a fuzzy GE-norm* $\pi_{\mathbf{X}}$. *If* $\xi_n \to_{\pi_{\mathbf{X}}} \xi$ *and* $\tau_n \to_{\pi_{\mathbf{X}}} \tau$, *then*

$$\xi_n * \tau_n \to_{\pi_{\mathbf{X}}} \xi * \tau.$$

Proof. Let $t_2 > 0$ and $\varepsilon \in (0, 1)$ be arbitrary. We show that there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\pi_{\mathbf{X}}\big((\xi_n * \tau_n) * (\xi * \tau),\ t_2\big) > 1 - \varepsilon \quad \text{and} \quad \pi_{\mathbf{X}}\big((\xi * \tau) * (\xi_n * \tau_n),\ t_2\big) > 1 - \varepsilon.$$

Since $\xi_n \to_{\pi_{\mathbf{X}}} \xi$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$\pi_{\mathbf{X}}(\xi_n * \xi, t_2/2) > 1 - \varepsilon \quad \text{and} \quad \pi_{\mathbf{X}}(\xi * \xi_n, t_2/2) > 1 - \varepsilon.$$

Similarly, since $\tau_n \to_{\pi_{\mathbf{X}}} \tau$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$\pi_{\mathbf{X}}(\tau_n * \tau, t_2/2) > 1 - \varepsilon \quad \text{and} \quad \pi_{\mathbf{X}}(\tau * \tau_n, t_2/2) > 1 - \varepsilon.$$

Let $N = \max\{N_1, N_2\}$ and fix $n \geq N$. Applying axiom (FN4) with scale parameters $t_2/2$ and $t_2/2$, we obtain

$$\pi_{\mathbf{X}}\big((\xi_n * \tau_n) * (\xi * \tau),\ t_2\big) \geq \min\{\pi_{\mathbf{X}}(\xi_n * \xi, t_2/2), \pi_{\mathbf{X}}(\tau_n * \tau, t_2/2)\} > 1 - \varepsilon.$$

Similarly, another application of (FN4) yields

$$\pi_{\mathbf{X}}\big((\xi * \tau) * (\xi_n * \tau_n),\ t_2\big) \geq \min\{\pi_{\mathbf{X}}(\xi * \xi_n, t_2/2), \pi_{\mathbf{X}}(\tau * \tau_n, t_2/2)\} > 1 - \varepsilon.$$

Since $t_2 > 0$ and $\varepsilon \in (0, 1)$ were arbitrary, it follows that

$$\xi_n * \tau_n \to_{\pi_{\mathbf{X}}} \xi * \tau.$$

This completes the proof. □

Theorem 3.26. *Let* $(\mathbf{X}, *, 1_{\mathbf{X}})$ *be a transitive GE-algebra and let* $\pi_{\mathbf{X}}$ *be a fuzzy GE-norm that is order-preserving:*

$$\wp_1 \leq_{\mathbf{X}} \wp_2 \quad \Longrightarrow \quad \pi_{\mathbf{X}}(\wp_1, t_2) \leq \pi_{\mathbf{X}}(\wp_2, t_2) \quad \textit{for all } t_2 > 0.$$

Fix $\upsilon_1 \in \mathbf{X}$ *and define the left and right translation maps*

$$L_{\upsilon_1}(\xi) := \upsilon_1 * \xi, \qquad R_{\upsilon_1}(\xi) := \xi * \upsilon_1 \qquad (\xi \in \mathbf{X}).$$

If $\xi_n \to_{\pi_{\mathbf{X}}} \xi$, *then*

$$L_{\upsilon_1}(\xi_n) \to_{\pi_{\mathbf{X}}} L_{\upsilon_1}(\xi) \qquad \textit{and} \qquad R_{\upsilon_1}(\xi_n) \to_{\pi_{\mathbf{X}}} R_{\upsilon_1}(\xi).$$

In other words, both L_{υ_1} *and* R_{υ_1} *are fuzzy-continuous.*

Proof. Fix $t_2 > 0$ and $\varepsilon \in (0, 1)$. Since $\xi_n \to_{\pi_{\mathbf{X}}} \xi$, choose N such that for all $n \geq N$,

$$\pi_{\mathbf{X}}(\xi_n * \xi, t_2) > 1 - \varepsilon \qquad \text{and} \qquad \pi_{\mathbf{X}}(\xi * \xi_n, t_2) > 1 - \varepsilon.$$

Since **X** is transitive, we have $(\xi_n * \xi) \leq_{\mathbf{X}} (\upsilon_1 * \xi_n) * (\upsilon_1 * \xi)$ and $(\xi * \xi_n) \leq_{\mathbf{X}} (\upsilon_1 * \xi) * (\upsilon_1 * \xi_n)$. Since $\pi_{\mathbf{X}}$ is order-preserving, we have

$$\pi_{\mathbf{X}}\big((\upsilon_1 * \xi_n) * (\upsilon_1 * \xi), t_2\big) \;\geq\; \pi_{\mathbf{X}}(\xi_n * \xi, t_2) > 1 - \varepsilon$$

and

$$\pi_{\mathbf{X}}\big((\upsilon_1 * \xi) * (\upsilon_1 * \xi_n), t_2\big) \;\geq\; \pi_{\mathbf{X}}(\xi * \xi_n, t_2) > 1 - \varepsilon.$$

Since **X** is transitive, we have $(\xi * \xi_n) \leq_{\mathbf{X}} (\xi_n * \upsilon_1) * (\xi * \upsilon_1)$ and $(\xi_n * \xi) \leq_{\mathbf{X}} (\xi * \upsilon_1) * (\xi_n * \upsilon_1)$. Since $\pi_{\mathbf{X}}$ is order-preserving, we have

$$\pi_{\mathbf{X}}\big((\xi_n * \upsilon_1) * (\xi * \upsilon_1), t_2\big) \;\geq\; \pi_{\mathbf{X}}(\xi * \xi_n, t_2) > 1 - \varepsilon$$

and

$$\pi_{\mathbf{X}}\big((\xi * \upsilon_1) * (\xi_n * \upsilon_1), t_2\big) \;\geq\; \pi_{\mathbf{X}}(\xi_n * \xi, t_2) > 1 - \varepsilon.$$

This shows

$$L_{\upsilon_1}(\xi_n) = \upsilon_1 * \xi_n \to_{\pi_{\mathbf{X}}} \upsilon_1 * \xi = L_{\upsilon_1}(\xi) \text{ and } R_{\upsilon_1}(\xi_n) = \xi_n * \upsilon_1 \to_{\pi_{\mathbf{X}}} \xi * \upsilon_1 = R_{\upsilon_1}(\xi).$$

This proves that both L_{υ_1} and R_{υ_1} are fuzzy-continuous. □

3.4. Metric and topological aspects

Lemma 3.27. *Let $\pi_{\mathbf{X}}$ be the crisp fuzzy GE-norm defined by*

$$\pi_{\mathbf{X}}(\xi, t_2) = \begin{cases} 1, & \|\xi\| < t_2, \\ 0, & \|\xi\| \geq t_2. \end{cases}$$

Let $\delta(\xi, \sigma) = \|\xi * \sigma\|$ be the quasi-metric induced by the GE-norm. Then the topology generated by the fuzzy neighborhoods

$$U(\xi_0, t_2, 1) = \{\xi : \pi_{\mathbf{X}}(\xi * \xi_0, t_2) = 1\}$$

coincides with the topology generated by the δ-balls

$$B(\xi_0; t_2) = \{\xi : \delta(\xi, \xi_0) < t_2\}.$$

Proof. For any $\xi_0 \in \mathbf{X}$ and $t_2 > 0$, we compute directly

$$U(\xi_0, t_2, 1) = \{\xi : \pi_{\mathbf{X}}(\xi * \xi_0, t_2) = 1\} = \{\xi : \|\xi * \xi_0\| < t_2\} = B(\xi_0; t_2).$$

Thus, every fuzzy neighborhood of level 1 is exactly a δ-ball. Conversely, every δ-ball $B(\xi_0; t_2)$ is obtained as $U(\xi_0, t_2, 1)$.

Since the family of sets $\{U(\xi_0, t_2, 1) : t_2 > 0\}$ forms a neighborhood base in the fuzzy topology, and the family $\{B(\xi_0; t_2) : t_2 > 0\}$ forms a neighborhood base in the quasi-metric topology, the two topologies have identical bases at each point.

Hence, the fuzzy topology induced by the crisp fuzzy GE-norm $\pi_{\mathbf{X}}$ is precisely the quasi-metric topology induced by δ. □

Theorem 3.28. *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed GE-algebra and let the rational fuzzy GE-norm be*

$$\pi_{\mathbf{X}}(\wp_1, t_2) = \frac{t_2}{t_2 + \|\wp_1\|} \qquad (\wp_1 \in \mathbf{X},\ t_2 > 0).$$

Then, for all $\wp_1, \wp_2 \in \mathbf{X}$ *and* $t_2 > 0$*,*

$$\left|\pi_{\mathbf{X}}(\wp_1, t_2) - \pi_{\mathbf{X}}(\wp_2, t_2)\right| \leq \frac{t_2}{(t_2 + \|\wp_1\|)(t_2 + \|\wp_2\|)} \|\wp_1 * \wp_2\|.$$

Proof. Compute the difference directly:

$$\left|\pi_{\mathbf{X}}(\wp_1, t_2) - \pi_{\mathbf{X}}(\wp_2, t_2)\right| = \left|\frac{t_2}{t_2 + \|\wp_1\|} - \frac{t_2}{t_2 + \|\wp_2\|}\right| = \frac{t_2 \left|\|\wp_2\| - \|\wp_1\|\right|}{(t_2 + \|\wp_1\|)(t_2 + \|\wp_2\|)}.$$

Thus it suffices to estimate $|\|\wp_2\| - \|\wp_1\||$. From the GE-norm triangle inequality, we get

$$\|\wp_2\| = \|1 * \wp_2\| \leq \|1 * \wp_1\| + \|\wp_1 * \wp_2\| = \|\wp_1\| + \|\wp_1 * \wp_2\|,$$

which gives

$$|\|\wp_2\| - \|\wp_1\|| \leq \|\wp_1 * \wp_2\|.$$

Substitute this into the earlier display to obtain

$$\left|\pi_{\mathbf{X}}(\wp_1, t_2) - \pi_{\mathbf{X}}(\wp_2, t_2)\right| \leq \frac{t_2 \|\wp_1 * \wp_2\|}{(t_2 + \|\wp_1\|)(t + \|\wp_2\|)},$$

which is the asserted bound. □

Theorem 3.29. *Let* $\varphi : \mathbf{X} \to \mathbf{Y}$ *be a GE-morphism between fuzzy normed commutative GE-algebras* $(\mathbf{X}, \pi_{\mathbf{X}})$ *and* $(\mathbf{Y}, \pi_{\mathbf{Y}})$ *such that* $\pi_{\mathbf{Y}}(\varphi(\wp_1 *_{\mathbf{X}} \wp_2), t_2) = \pi_{\mathbf{X}}(\wp_1 *_{\mathbf{X}} \wp_2, t_2)$ *for all* $\wp_1, \wp_2 \in \mathbf{X}$ *and* $t_2 > 0$*. Then* φ *is injective.*

Proof. Assume $\varphi : \mathbf{X} \to \mathbf{Y}$ is a GE-morphism satisfying $\pi_{\mathbf{Y}}(\varphi(\wp_1 * \wp_2), t_2) = \pi_{\mathbf{X}}(\wp_1 * \wp_2, t_2)$ for all $\wp_1, \wp_2 \in \mathbf{X}$ and $t_2 > 0$. Suppose $\varphi(\wp_1) = \varphi(\wp_2)$. Then

$$\pi_{\mathbf{X}}(\wp_1 *_{\mathbf{X}} \wp_2, t_2) = \pi_{\mathbf{Y}}(\varphi(\wp_1 *_{\mathbf{X}} \wp_2), t_2) = \pi_{\mathbf{Y}}(\varphi(\wp_1) *_{\mathbf{Y}} \varphi(\wp_2), t_2) = \pi_{\mathbf{Y}}(1_{\mathbf{Y}}, t_2) = 1$$

for all $t_2 > 0$. By (FN1), we get $\wp_1 *_{\mathbf{X}} \wp_2 = 1$, so $\wp_1 \leq_{\mathbf{X}} \wp_2$. Similarly $\wp_2 \leq_{\mathbf{X}} \wp_1$, and since $\mathbf{X}$ is a commutative GE-algebra, this yields $\wp_1 = \wp_2$. Thus φ is injective. □

Theorem 3.30. *Let* $(\mathbf{X}, *, 1)$ *be a GE-algebra and let* $\pi_{\mathbf{X}} : \mathbf{X} \times (0, \infty) \to [0, 1]$ *be a fuzzy GE-norm. For* $t_2 > 0$*, define a mapping* $d_{t_2} : \mathbf{X} \times \mathbf{X} \to [0, 1]$ *by*

$$d_{t_2}(\upsilon_1, \upsilon_2) = 1 - \pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_2).$$

Then the following assertions hold:

(i) $d_{t_2}(\upsilon_1, \upsilon_1) = 0$ *for all* $\upsilon_1 \in \mathbf{X}$*;*
(ii) $d_{t_2}(\upsilon_1, \upsilon_2) \geq 0$ *for all* $\upsilon_1, \upsilon_2 \in \mathbf{X}$*;*

(iii) for all $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathbf{X}$ *and all* $t_1, t_2 > 0$,

$$d_{t_1+t_2}(\upsilon_1, \upsilon_3) \le d_{t_1}(\upsilon_1, \upsilon_2) + d_{t_2}(\upsilon_2, \upsilon_3).$$

Proof. **(i).** Let $\upsilon_1 \in \mathbf{X}$ be arbitrary. By axiom (GE1), $\upsilon_1 * \upsilon_1 = 1$. Using Proposition 3.9(1.1), we obtain

$$\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_1, t_2) = \pi_{\mathbf{X}}(1, t_2) = 1 \quad \text{for all } t_2 > 0.$$

Hence

$$d_{t_2}(\upsilon_1, \upsilon_1) = 1 - \pi_{\mathbf{X}}(\upsilon_1 * \upsilon_1, t_2) = 0.$$

(ii). For any $\upsilon_1, \upsilon_2 \in \mathbf{X}$ and $t_2 > 0$, we have $\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_2) \in [0, 1]$. Therefore,

$$d_{t_2}(\upsilon_1, \upsilon_2) = 1 - \pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_2) \ge 0.$$

(iii). Let $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathbf{X}$ and let $t_1, t_2 > 0$. By (FN4),

$$\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_3, t_1 + t_2) \ge \min\{\pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1), \pi_{\mathbf{X}}(\upsilon_2 * \upsilon_3, t_2)\}.$$

Subtracting both sides from 1, we obtain

$$1 - \pi_{\mathbf{X}}(\upsilon_1 * \upsilon_3, t_1 + t_2) \le \max\{1 - \pi_{\mathbf{X}}(\upsilon_1 * \upsilon_2, t_1), 1 - \pi_{\mathbf{X}}(\upsilon_2 * \upsilon_3, t_2)\}.$$

Using the elementary inequality $\max\{a, b\} \le a + b$ for all $a, b \ge 0$, it follows that

$$d_{t_1+t_2}(\upsilon_1, \upsilon_3) \le d_{t_1}(\upsilon_1, \upsilon_2) + d_{t_2}(\upsilon_2, \upsilon_3).$$

This completes the proof. □

4. Conclusions

We introduced and developed the theory of fuzzy GE-norms on GE-algebras, establishing a fuzzy analogue of the geometric structure previously studied for crisp GE-norms. Our investigation revealed several structural and foundational results: fuzzy GE-norms satisfy a collection of identities that generalize those of normed GE-algebras, interact predictably with GE-operations, and induce a meaningful fuzzy topology compatible with quasi-metric structures.

A major contribution of this work is the complete characterization of fuzzy normability: A GE-algebra admits a fuzzy GE-norm precisely when its induced order is transitive. This criterion not only clarifies when fuzzy norms exist but also explains why non-transitive GE-algebras cannot support norm-like behavior. Additional results show how fuzzy norms behave under GE-morphisms, including both positive preservation theorems and counterexamples demonstrating the limitations of morphic transfer.

Finally, fuzzy convergence, uniqueness of fuzzy limits, and continuity of translations were established, revealing that fuzzy normed GE–algebras exhibit rich analytic behavior analogous to fuzzy normed linear spaces. These results lay foundational groundwork for future investigations in fuzzy geometric methods, fixed point theory, and applications of GE–algebras in approximation, decision theory, and algebraic logic.

Author contributions

Amal S. Alali: Conceptualization, methodology, validation, writing–original draft preparation, writing review and editing, supervision, funding acquisition; Ravi Kumar Bandaru: Conceptualization, methodology, validation, writing–original draft preparation, writing review and editing, supervision; Seok-Zun Song: writing–original draft preparation, writing review and editing, validation; Young Bae Jun: Conceptualization, methodology, validation, writing–original draft preparation, writing review and editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. R. Bandaru, A. B. Saeid, Y. B. Jun, On GE-algebras, *Bull. Sect. Log.*, **50** (2021), 81–96. https://doi.org/10.18778/0138-0680.2020.20
2. R. Bandaru, Y. B. Jun, GE-algebras with norms, *Int. J. Maps Math.*, **8** (2025), 681–701.
3. T. Bag, S. K. Samanta, Finite dimensional fuzzy normed linear spaces, *Ann. Fuzzy Math. Inform.*, 2013. Print Version.
4. A. Rezaei, R. Bandaru, A. B. Saeid, Y. B. Jun, Prominent GE-filters and GE-morphisms in GE-algebras, *Afr. Mat.*, **32** (2021), 1121–1136. https://doi.org/10.1007/s13370-021-00886-6