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Research article

Lie group structures and novel soliton solutions of the nonlinear mathematical model

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Abstract: We obtain new computational soliton solutions characterized by topological, rational, exponential, trigonometric, and hyperbolic functions for the modified Zakharov–Kuznetsov equation (mZKE). Utilizing two successful procedures, the extended rational sinh-cosh and Kudryashov expansion methods are applied to derive diverse dynamical wave structures of soliton solutions within the context of evolutionary dynamical structures of solitary wave solutions. It is permissible to choose parameters that show a solution. To improve understanding of the physical phenomena related to these dynamical models in mathematical physics, the empirical demonstration of the physical behavior of these solutions is provided. The technique of symmetry analysis is utilized to examine the governing equation. The implementation of a novel conservation theorem results in the formation of a comprehensive system of one-dimensional subalgebras. The study encompasses the Lie-Bäcklund symmetry generators. Additionally, the methodology for establishing conservation laws for nonlinear partial differential equations is clarified through the presentation of an innovative conservation theorem associated with Lie-Bäcklund symmetries. This strategy is used to come up with the conservation laws that apply to the governing equation.

Keywords: Zakharov–Kuznetsov equation; optimal system; soliton solutions; Lie-Bäcklund symmetries; conservation laws
Mathematics Subject Classification: 35Q53, 65M60

1. Introduction

Nonlinear partial differential equations (PDEs) play a central role in modeling complex phenomena in physics, engineering, and applied sciences. In particular, systems of coupled nonlinear PDEs arise naturally in fluid dynamics, plasma physics, nonlinear optics, and the theory of integrable systems. The mathematical challenges associated with these equations stem from their nonlinear structure and the presence of non-trivial coupling terms, which often prevent the direct application of standard solution techniques [1–3]. These equations are developed by scientists for a variety of reasons, one of which is the study of how waves flow through nonlinear systems. Discovering solutions to nonlinear partial differential equations (NLPDEs), particularly solutions that describe solitary waves or solitons, is one of the most significant challenges that arises in the field of nonlinear phenomena research. For the purpose of explaining complex natural and physical processes, it is essential to have a solid understanding of these solutions. The usage of NLPDEs is a potent tool that is utilized in numerous key scientific domains, such as chemistry, physics, biology, and others, to describe a wide range of events [4–6]. In the 18^{th} and 19^{th} centuries, the PDEs were increasingly popular as a means of providing an explanation for a wide variety of physical events. Many of the systems that exist in the actual world, on the other hand, contain nonlinearities. NLPDEs were developed for the purpose of studying wave propagation in nonlinear systems, which was one of the primary motivations for their creation. In the 20^{th} century, there was a significant amount of progress made in the field of NLPDEs. A number of mathematicians, including Henri Poincaré, conducted research on nonlinear problems. In spite of the fact that general approaches to solving NLPDEs are still difficult to come by, some equations, such as the Korteweg-de Vries equation, have attracted a lot of attention [7]. In order to comprehend complex processes in a wide variety of scientific fields, such as biology, chemistry, and physics, natural language processing and data extraction, NLPDEs have become indispensable [8]. Currently, research on soliton wave treatments is still being conducted [9, 10]. A wide variety of scientific disciplines make use of the phenomenon of nonlinear waves. This spans mathematics, microbiology, and various disciplines of physics, including both asymmetric and nonlinear condensation in optics, fluid dynamics, and molecular physics. Within the fields of applied mathematics, theoretical physics, and a number of engineering subfields, the study of NLPDEs, is exceptionally important. In a recent piece of research [11], a high-dimensional variant of the Zakharov-Kuznetsov equation was investigated. For instance, a three-dimensional isotropic nonlinear equation was used to explain weak nonlinear waves in plasma that had a strong magnetic field and a lossless two-dimensional structure. This equation is supposed to shed light on the manner in which waves travel through a plasma in which the magnetic field is present.

In recent decades, Lie group analysis has emerged as one of the most powerful tools for investigating nonlinear PDEs. By exploiting the invariance properties of differential equations under continuous transformation groups, Lie symmetries provide systematic techniques for symmetry reductions, classification of invariant solutions, construction of optimal systems, and analysis of the underlying algebraic structures. Moreover, Lie symmetries play an essential role in establishing connections between integrability, conservation laws, and invariant solutions [12–14].

Using symmetry approaches in nonlinear PDEs analysis has a long and illustrious history [15–17]. The foundation of continuous symmetries in differential equations was developed by the pioneering studies of Sophus Lie [18]. These concepts were built upon by Bluman and Kumei [19] in subsequent advancements, which led to the invention of viable algorithms for symmetry reductions and similarity solutions. Through the application of Noether's theorem, conservation laws, which are essential structural invariants, have traditionally been produced. However, the introduction of a new theorem [20, 21] has offered a unified framework that can be used for both variational and non-variational PDEs systems. By doing so, it has been possible to investigate a wider range of mathematical and physical models through the application of conservation law analysis. The application of Lie symmetry approaches to nonlinear wave equations, reaction–diffusion systems, and fluid flow models has been the subject of recent research [26,27]. However, there are still a lot of gaps to be explored regarding the symmetry and conservation laws structures for NLPDEs.

Under the influence of this, the purpose of this work is to analyze the soliton solutions and the structure of the Lie group for the modified Zakharov–Kuznetsov equation (mZKE), which is given by [28]:

$$\phi_t + \phi^2\phi_x + \phi_{xxx} + \phi_{xyy} = 0. \tag{1.1}$$

The mZKE has been modified to explain events in discrete electric lattices and nonlinear optics. It describes weakly nonlinear waves, especially ion-acoustic waves in magnetized plasmas [28].

2. The M-truncated derivative

In this section, the fundamental definitions and theorems of the M-truncated derivative are provided [29].

Definition 2.1. *The definition of the new M-truncated derivative of h of order δ is as follows: For $h : [0,\infty) \to \mathbb{R}$,*

$$\mathbb{D}_M^{\delta,\eta}\{(h)(t)\} = \lim_{\epsilon\to 0}\frac{h(t\mathbb{E}_\eta(\epsilon t^{1-\delta})) - h(t)}{\epsilon}, \quad \forall t > 0, \quad 0 < \delta < 1, \ \eta > 0, \tag{2.1}$$

where $\mathbb{E}_\eta(.)$ is a truncated Mittag-Leffler function of one parameter [29].

Theorem 2.2. *Let $0 < \delta \le 1$, $\eta > 0$, $q,\ r \in \mathbb{R}$, and let $g,\ h$ be δ-differentiable at a point $t > 0$. Then,*

(1) $\mathbb{D}_M^{\delta,\eta}\{(qg + rh)(t)\} = q\mathbb{D}_M^{\delta,\eta}\{g(t)\} + r\mathbb{D}_M^{\delta,\eta}\{h(t)\}$.
(2) $\mathbb{D}_M^{\delta,\eta}\{(g\ .\ h)(t)\} = g(t)\mathbb{D}_M^{\delta,\eta}\{h(t)\} + h(t)\mathbb{D}_M^{\delta,\eta}\{g(t)\}$.
(3) $\mathbb{D}_M^{\delta,\eta}\{\frac{g}{h}(t)\} = \frac{h(t)\mathbb{D}_M^{\delta,\eta}\{g(t)\} - g(t)\mathbb{D}_M^{\delta,\eta}\{h(t)\}}{[h(t)]^2}$.
(4) $\mathbb{D}_M^{\delta,\eta}\{c\} = 0$, *where $g(t) = c$ is a constant.*
(5) $\mathbb{D}_M^{\delta,\eta}\{g(t)\} = \frac{t^{1-\delta}}{\Gamma(\eta+1)}\frac{dg(t)}{dt}$, *if g is differentiable.*

3. Analysis of the methods

The extended rational sine-cosine and sinh-cosh methods are analyzed in this section [30].

Take into account the form, in general, for the M-truncated partial differential equation along with travelling wave

$$P(\mathbb{D}_{M,x}^{\delta,\eta}\Theta,\ \Theta^2\mathbb{D}_{M,x}^{2\delta,\eta}\Theta,\ \mathbb{D}_{M,t}^{\delta,\eta}\Theta,\ \mathbb{D}_{M,t}^{\delta,\eta}\mathbb{D}_{M,x}^{\delta,\eta}\Theta,\ \ldots) = 0, \tag{3.1}$$

$$\Theta(x,t) = \Theta(\xi), \quad \xi = \frac{\Gamma(\eta+1)}{\delta}\mu(x^\delta - \rho t^\delta), \tag{3.2}$$

respectively, where $\mathbb{D}_{M,*}^{\delta,\eta}\Theta$ is the δ derivative with respect to $*$ (x or t), $\Theta(x,t)$ is the unknown function, μ is a non-zero constant, and ρ is the speed of the wave.

Putting Eq (3.2) into Eq (3.1) gives

$$D(\Theta,\ \Theta',\ \Theta'',\ \Theta^2\Theta',\ \ldots) = 0. \tag{3.3}$$

In this case, Θ represents the unknown function of η, and the superscript shows the derivative of Θ with respect to η.

3.1. Extended rational sine-cosine

This section lays out the fundamental procedures for the extended rational sine-cosine.

Step I: The following solutions to Eq (3.3) are assumed to be valid:

$$\Theta(\xi) = \frac{\Upsilon_0 \sin(\mu\xi)}{\Upsilon_2 + \Upsilon_1 \cos(\mu\xi)}, \quad \cos(\mu\xi) \neq -\frac{\Upsilon_2}{\Upsilon_1}, \tag{3.4}$$

or

$$\Theta(\xi) = \frac{\Upsilon_0 \cos(\mu\xi)}{\Upsilon_2 + \Upsilon_1 \sin(\mu\xi}, \quad \sin(\mu\xi) \neq -\frac{\Upsilon_2}{\Upsilon_1}, \tag{3.5}$$

where μ is the wave number, and $\Upsilon_j,\ (j = 0, 1, 2)$ unknown parameters.

Step II: By substituting Eq (3.5) or Eq (3.4) into Eq (3.3), the unknown parameters can be derived. The result is a polynomial with roots in either $\cos(\mu\xi)$ or $\sin(\mu\xi)$. After setting each sum equal to zero, a set of algebraic equations is obtained by collecting the coefficients of $\cos(\mu\xi)$ or $\sin(\mu\xi)$ to the same power.

Step III: Equations (3.4) or (3.5) can be used to guarantee the solutions of Eq (3.1) by replacing the values of the unknown parameters.

3.2. Extended rational sinh-cosh

Here we provide the broad strokes of the extended rational sinh-cosh.

Step I: The following solutions to Eq (3.3) are assumed to be valid:

$$\Theta(\xi) = \frac{\Upsilon_0 sinh(\mu\xi)}{\Upsilon_2 + \Upsilon_1 \cosh(\mu\xi)}, \quad \cosh(\mu\xi) \neq -\frac{\Upsilon_2}{\Upsilon_1}, \tag{3.6}$$

or

$$\Theta(\xi) = \frac{\Upsilon_0 cosh(\mu\xi)}{\Upsilon_2 + \Upsilon_1 \sinh(\mu\xi)}, \quad \sinh(\mu\xi) \neq -\frac{\Upsilon_2}{\Upsilon_1}, \tag{3.7}$$

where $\Upsilon_j,\ (j = o, 1, 2)$ are unknown parameters and μ is the wave number.

Step II: By substituting Eq (3.6) or Eq (3.7) into Eq (3.3), the unknown parameters can be derived. In terms of powers of $cosh(\mu\xi)$ or $sinh(\mu\xi)$, this produces a polynomial. After setting each sum equal to zero, what results from raising the coefficients of $cosh(\mu\xi)$ or $sinh(\mu\xi)$ to the same power are a collection of algebraic equations.

Step III: By plugging the unknown parameter values into either Eq (3.6) or Eq (3.7), the solutions of Eq (3.1) are guaranteed.

3.3. Analysis of the KEM

Here is the description of the Kudryashov expansion method (KEM) [31]. Consider Eqs (3.1) and (3.2).

Inserting Eq (3.1) into Eq (3.2), we get (3.3).

Implementing Eq (3.3) using the text function as a test case, we get

$$K(\xi) = \sum_{r=0}^{\theta} A_r \Theta^r(\xi), \quad A_\theta \neq 0. \tag{3.8}$$

We get the integer value of Θ by balancing the highest-power nonlinear and highest derivative terms.

The function $\Theta(\xi)$ solves:

$$\Theta^{'} = \delta\Theta(\Theta - 1). \tag{3.9}$$

Employing the separable approach on (3.9), provides

$$\Theta(\xi) = \frac{1}{1 + \eta e^{\delta\xi}}. \tag{3.10}$$

When a fixed value of θ is inputted into (3.8) with the potential derivatives and (3.9), a polynomial in powers of θ is generated. If we add the coefficients of like powers of Θ and set each sum to zero, we can build a collection of algebraic equations. To find the answers to the questions that were asked, one needs to solve the system of algebraic equations.

4. Applications

In this section, we present the application of the extended rational sine-cosine/sinh-cosh method to the local M-truncated mZKE given as

$$\mathbb{D}_{M,t}^{\delta,\eta}\phi + \phi^2\mathbb{D}_{M,x}^{\delta,\eta}\phi + \mathbb{D}_{M,x}^{3\delta,\eta}\phi + \mathbb{D}_{M,x}^{\delta,\eta}\mathbb{D}_{M,y}^{2\delta,\eta}\phi = 0. \tag{4.1}$$

Take into account the M-truncated wave transformation:

$$\phi = \Theta(\xi),\ \xi = \frac{\Gamma(\eta+1)\left(x^\delta + \alpha y^\delta - \rho t^\delta\right)}{\delta}. \tag{4.2}$$

Substituting Eq (4.2) into Eq (4.1), gives

$$3(1+\alpha^2)\Theta^{''} + \Theta^3 - 3\rho\Theta = 0. \tag{4.3}$$

Balancing $\Theta^{''}$ and Θ^3 in Eq (4.3), gives $m = 1$.

Assume that Eq (4.3) possess the solution of the form

$$\Theta(\xi) = \frac{\Upsilon_0 \sin(\mu\xi)}{\Upsilon_2 + \Upsilon_1 cos(\mu\xi)}. \tag{4.4}$$

When Eq (4.4) is substituted into Eq (4.3), a polynomial in trigonometric functions is produced. You get a set of algebraic equations by adding together all the trigonometric function coefficients that have the same power of $\cos(\mu\xi)$. The unknown parameters are given the following values after solving the system of algebraic equations:

Set-1: For

$$\Upsilon_0 = \Upsilon_1 \sqrt{-3\rho},\ \Upsilon_2 = -\Upsilon_1,\ \mu = -\frac{\sqrt{2\rho}}{\sqrt{\alpha^2+1}},$$

we attain

$$\phi_{1.1}(x,y,t) = -\frac{\sqrt{3}\Upsilon_1 \sqrt{-\rho}\sin\left(\frac{\sqrt{2}\sqrt{\rho}\Gamma(\eta+1)\left(-\rho t^\delta + x^\delta + \alpha y^\delta\right)}{\delta\sqrt{\alpha^2+1}}\right)}{\Upsilon_1\cos\left(\frac{\sqrt{2}\sqrt{\rho}\Gamma(\eta+1)\left(-\rho t^\delta + x^\delta + \alpha y^\delta\right)}{\delta\sqrt{\alpha^2+1}}\right) - \Upsilon_1}. \tag{4.5}$$

Set-2: When

$$\Upsilon_0 = \Upsilon_1 \sqrt{-3\rho},\ \Upsilon_2 = 0,\ \mu = \frac{\sqrt{\rho}}{\sqrt{2\alpha^2+2}},$$

we get

$$\phi_{1.2}(x,y,t) = \sqrt{-3\rho}\tan\left(\frac{\sqrt{\rho}\Gamma(\eta+1)\left(-\rho t^\delta + x^\delta + \alpha y^\delta\right)}{\delta\sqrt{2\alpha^2+2}}\right). \tag{4.6}$$

Assume that Eq (4.3) has the solution of the form

$$\Theta(\xi) = \frac{\Upsilon_0\cos(\mu\xi)}{\Upsilon_2 + \Upsilon_1\sin(\mu\xi)}. \tag{4.7}$$

A polynomial in trigonometric functions is obtained by substituting Eq (4.7) into Eq (4.3). Equations in algebra are obtained by adding up the coefficients of all the trigonometric functions that are raised to the same power as $\sin(\mu\xi)$. The unknown parameters are given the following values after solving the system of algebraic equations:

Set-1: For

$$\Upsilon_0 = \Upsilon_1\left(-\sqrt{-3\rho}\right),\ \Upsilon_2 = \Upsilon_1,\ \mu = -\frac{\sqrt{2\rho}}{\sqrt{\alpha^2+1}},$$

we attain

$$\phi_{2.1}(x,y,t) = -\frac{\sqrt{3}\Upsilon_1 \sqrt{-\rho}\cos\left(\frac{\sqrt{2}\sqrt{\rho}\Gamma(\eta+1)\left(-\rho t^\delta + x^\delta + \alpha y^\delta\right)}{\sqrt{\alpha^2+1}\delta}\right)}{\Upsilon_1 - \Upsilon_1\sin\left(\frac{\sqrt{2}\sqrt{\rho}\Gamma(\eta+1)\left(-\rho t^\delta + x^\delta + \alpha y^\delta\right)}{\sqrt{\alpha^2+1}\delta}\right)}. \tag{4.8}$$

Set-2: When

$$\Upsilon_0 = \Upsilon_1\left(-\sqrt{-3\rho}\right),\ \Upsilon_2 = 0,\ \mu = \frac{\sqrt{\rho}}{\sqrt{2\alpha^2+2}},$$

we get

$$\phi_{2.2}(x,y,t) = -\sqrt{3}\sqrt{-\rho}\cot\left(\frac{\sqrt{\rho}\Gamma(\eta+1)\left(-\rho t^{\delta}+x^{\delta}+\alpha y^{\delta}\right)}{\sqrt{2\alpha^2+2}\delta}\right). \tag{4.9}$$

Assume that Eq (4.3) has the solution of the form

$$\Theta(\eta) = \frac{\Upsilon_0 sinh(\mu\eta)}{\Upsilon_2 + \Upsilon_1 cosh(\mu\eta)}. \tag{4.10}$$

A polynomial in trigonometric functions is obtained by substituting Eq (4.10) into Eq (4.3). One can obtain a set of algebraic equations by adding up the coefficients of all the trigonometric functions that are powers of $\cosh(\mu\xi)$. The following values for the unknown parameters are obtained by solving the system of algebraic equations:

Set-1: When

$$\Upsilon_0 = \Upsilon_1\sqrt{3\rho},\ \Upsilon_2 = -\Upsilon_1,\ \mu = \frac{\sqrt{2\rho}}{\sqrt{-\alpha^2-1}},$$

we get

$$\phi_{3.1}(x,y,t) = \frac{\sqrt{3}\Upsilon_1\sqrt{\rho}\sinh\left(\frac{\sqrt{2}\sqrt{\rho}\Gamma(\eta+1)\left(-\rho t^{\delta}+x^{\delta}+\alpha y^{\delta}\right)}{\sqrt{-\alpha^2-1}\delta}\right)}{\Upsilon_1\cosh\left(\frac{\sqrt{2}\sqrt{\rho}\Gamma(\eta+1)\left(-\rho t^{\delta}+x^{\delta}+\alpha y^{\delta}\right)}{\sqrt{-\alpha^2-1}\delta}\right)-\Upsilon_1}. \tag{4.11}$$

Set-2: When

$$\Upsilon_0 = \Upsilon_1\left(-\sqrt{3\rho}\right),\ \Upsilon_2 = 0,\ \mu = \sqrt{-\frac{\rho}{2\left(\alpha^2+1\right)}},$$

we get

$$\phi_{3.2}(x,y,t) = -\sqrt{3}\sqrt{\rho}\tanh\left(\frac{\sqrt{-\frac{\rho}{\alpha^2+1}}\Gamma(\eta+1)\left(-\rho t^{\delta}+x^{\delta}+\alpha y^{\delta}\right)}{\delta\sqrt{2}}\right). \tag{4.12}$$

Assume that Eq (4.3) has a solution of the form

$$\Theta(\xi) = \frac{\Upsilon_0\cosh(\mu\xi)}{\Upsilon_2 + \Upsilon_1\sinh(\mu\xi)}. \tag{4.13}$$

When Eq (4.13) is substituted into Eq (4.3), a polynomial in trigonometric functions is produced. A set of algebraic equations is obtained by adding up the coefficients of all the trigonometric functions that are raised to the same power as $\sinh(\mu\xi)$. The unknown parameters are given the following values after solving the system of algebraic equations:

Set-1: For

$$\Upsilon_0 = -\sqrt{3}\Upsilon_1\sqrt{\rho},\ \Upsilon_2 = -i\Upsilon_1,\ \mu = \frac{\sqrt{2}\sqrt{\rho}}{\sqrt{-\alpha^2-1}},$$

we attain

$$\phi_{4.1}(x,y,t) = -\frac{\sqrt{3}\Upsilon_1\sqrt{\rho}\cosh\left(\frac{\sqrt{2}\sqrt{\rho}\Gamma(\eta+1)\left(-\rho t^\delta+x^\delta+\alpha y^\delta\right)}{\delta\sqrt{-\alpha^2-1}}\right)}{\Upsilon_1\sinh\left(\frac{\sqrt{2}\sqrt{\rho}\Gamma(\eta+1)\left(-\rho t^\delta+x^\delta+\alpha y^\delta\right)}{\delta\sqrt{-\alpha^2-1}}\right)-i\Upsilon_1}. \tag{4.14}$$

Set-2: When

$$\Upsilon_0 = \sqrt{3}\Upsilon_1\sqrt{\rho},\ \Upsilon_2 = 0,\ \mu = \frac{\sqrt{\rho}}{\sqrt{2}\sqrt{-\alpha^2-1}},$$

we get

$$\phi_{4.2}(x,y,t) = \sqrt{3}\sqrt{\rho}\coth\left(\frac{\sqrt{\rho}\Gamma(\eta+1)\left(-\rho t^\delta+x^\delta+\alpha y^\delta\right)}{\sqrt{2}\sqrt{-\alpha^2-1\delta}}\right). \tag{4.15}$$

4.1. Solutions via KEM

Here, we acquire the analytical solutions of (4.1).

Consider the following wave transformation:

$$\phi = \Theta(\xi),\ \xi = \frac{\mu\Gamma(\eta+1)\left(x^\delta+\alpha y^\delta-\rho t^\delta\right)}{\delta}. \tag{4.16}$$

Plugging (4.16) into (4.1), provides

$$3\mu^2(1+\alpha^2)\Theta'' + \Theta^3 - 3\rho\Theta = 0. \tag{4.17}$$

Balancing the terms Θ'' and Θ^3 provide $\theta = 1$.

For $\theta = 1$, the trial series solution of Eq (4.17) receives the following shape:

$$K(\xi) = A_0 + A_1\phi. \tag{4.18}$$

One can get a polynomial in powers of Θ by plugging in (4.18), its second derivative, and (3.9) into (4.17). If we add the coefficients of like powers of Θ and set each sum to zero, we can build a collection of algebraic equations. Here are the solutions to the equations that were investigated once the system of algebraic equations was solved:

(I): When

$$A_0 = -\sqrt{3}\sqrt{\rho},\ A_1 = 2\sqrt{3}\sqrt{\rho},\ \mu = -\frac{\sqrt{2}\sqrt{\rho}}{\sqrt{-\alpha^2\delta^2-\delta^2}},$$

we have

$$\phi^I(x,y,t) = \frac{2\sqrt{3}\sqrt{\rho}}{\eta\exp\left(-\frac{\sqrt{2}\sqrt{\rho}\Gamma(\eta+1)\left(-\rho t^\delta+x^\delta+\alpha y^\delta\right)}{\sqrt{-\alpha^2\delta^2-\delta^2}}\right)+1} - \sqrt{3}\sqrt{\rho}. \tag{4.19}$$

(II): When

$$A_0 = -i\sqrt{\frac{3}{2}}\sqrt{\alpha^2+1}\delta\mu,\ A_1 = i\sqrt{6}\sqrt{\alpha^2+1}\delta\mu,\ \rho = -\frac{1}{2}\left(\alpha^2+1\right)\delta^2\mu^2,$$

we have

$$\phi^{II}(x,y,t) = \frac{i\sqrt{6}\sqrt{\alpha^2+1}\delta\mu}{\eta\exp\left(\mu\Gamma(\eta+1)\left(\frac{1}{2}\left(\alpha^2+1\right)\delta^2\mu^2t^\delta + x^\delta + \alpha y^\delta\right)\right)+1} - i\sqrt{\frac{3}{2}}\sqrt{\alpha^2+1}\delta\mu. \tag{4.20}$$

5. Lie group structures and conservation laws

We work with evolutionary vector fields:

$$X_Q \ = \ Q\frac{\partial}{\partial u},$$

where $Q = Q(x, t, y, u, u_x, u_{xx}, \ldots)$. The Lie bracket of two evolutionary fields is again evolutionary:

$$[X_Q, X_R] = X_{[Q,R]_E},$$

where the evolutionary Lie bracket is

$$[Q,R]_E \ = \ \mathcal{L}_Q(R) - \mathcal{L}_R(Q),$$

and $\mathcal{L}_Q$ denotes the Fréchet derivative or linearization of Q:

$$\mathcal{L}_Q(\phi) \ = \ \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} Q(u+\varepsilon\phi).$$

Equivalently, if Q depends on finitely many u-derivatives,

$$\mathcal{L}_Q(\phi) \ = \ \sum_J \frac{\partial Q}{\partial u_J} D_J(\phi),$$

where J denotes a multi-index of derivatives and D_J the corresponding total derivative.

We will compute the three linearizations needed and then the three brackets:

$$[Q_1, Q_2]_E, [Q_1, Q_3]_E, [Q_2, Q_3]_E.$$

5.1. Linearizations

For easier computation, consider $a(y) := F_{34}(y),\ b(y) := F_{36}(y),$ *and* $c(y) := F_{37}(y)$.

(i) Linearization of $Q_1 = a(y)u_x$

The only dependence of Q_1 on u is via u_x. Therefore,

$$\mathcal{L}_{Q_1}(\phi) = a(y)\,D_x(\phi).$$

That is, the Fréchet derivative of Q_1 is the operator $a(y)D_x$.

(ii) Linearization of $Q_2 = b(y)(u^2u_x + u_{xxx})$

Differentiate termwise:

$$\begin{aligned}\mathcal{L}_{Q_2}(\phi) &= b(y)(\mathcal{L}_{u^2u_x}(\phi) + \mathcal{L}_{u_{xxx}}(\phi))\\ &= b(y)(2u\phi \cdot u_x + u^2D_x(\phi) + D_x^3(\phi)).\end{aligned}$$

Thus, $\mathcal{L}_{Q_2}$ acts on ϕ by

$$\mathcal{L}_{Q_2}(\phi) = b(y)(2uu_x\,\phi + u^2D_x(\phi) + D_x^3(\phi)).$$

(iii) Linearization of $Q_3 = c(y)(3tu^2u_x + 3tu_{xxx} - u_{xx} - u)$

Similarly,

$$\mathcal{L}_{Q_3}(\phi) = c(y)\Big(3t(2uu_x\,\phi + u^2D_x(\phi)) + 3tD_x^3(\phi) - D_x^2(\phi) - \phi\Big).$$

5.2. Lie brackets of the characteristics

By definition,

$$[Q_i, Q_j]_E = \mathcal{L}_{Q_i}(Q_j) - \mathcal{L}_{Q_j}(Q_i).$$

We compute each bracket using the linearizations above. We keep the result in a compact operator form to summarize the expressions that are with D_x, D_x^2.

(A) $[Q_1, Q_2]_E$

Use $\mathcal{L}_{Q_1}(\cdot) = aD_x(\cdot)$ and $\mathcal{L}_{Q_2}(\cdot) = b(2uu_x(\cdot) + u^2D_x(\cdot) + D_x^3(\cdot))$.

$$\begin{aligned}\mathcal{L}_{Q_1}(Q_2) &= a(y)\,D_x\Big(b(y)(u^2u_x + u_{xxx})\Big)\\ &= ab\,D_x(u^2u_x + u_{xxx}) + ab'\,u_y\text{-terms},\end{aligned}$$

but since $b = b(y)$ depends only on y, $D_x(b) = 0$. So,

$$\mathcal{L}_{Q_1}(Q_2) = ab\,D_x(u^2u_x + u_{xxx}).$$

Next,

$$\begin{aligned}\mathcal{L}_{Q_2}(Q_1) &= b(y)\Big(2uu_x\,Q_1 + u^2D_x(Q_1) + D_x^3(Q_1)\Big)\\ &= b\Big(2uu_x \cdot au_x + u^2D_x(au_x) + D_x^3(au_x)\Big).\end{aligned}$$

We use $D_x(au_x) = a\,u_{xx} + a'u_{xy}$ and so on; recall $a' = da/dy$ and $D_x(a) = 0$ since a depends only on y (but $D_x(au_x)$ still produces $a'u_{xy}$ because D_x hits u_x producing u_{xx} and acts on the y-dependence only through mixed derivatives of u). Keeping this structure, we get

$$[Q_1, Q_2]_E = ab\,D_x(u^2u_x + u_{xxx}) - b\Big(2uu_x \cdot au_x + u^2D_x(au_x) + D_x^3(au_x)\Big).$$

(B) $[Q_1, Q_3]_E$

Similarly,

$$\mathcal{L}_{Q_1}(Q_3) = a(y)D_x\Big\{c(y)(3tu^2u_x + 3tu_{xxx} - u_{xx} - u)\Big\} = ac\, D_x(\cdots),$$

since c depends only on y. Also

$$\mathcal{L}_{Q_3}(Q_1) = c(y)\Big(3t(2uu_x\, Q_1 + u^2D_x(Q_1)) + 3tD_x^3(Q_1) - D_x^2(Q_1) - Q_1\Big).$$

Therefore,

$$[Q_1, Q_3]_E = ac\, D_x(3tu^2u_x + 3tu_{xxx} - u_{xx} - u) - c\Big(3t(2uu_x\, au_x + u^2D_x(au_x)) + 3tD_x^3(au_x) - D_x^2(au_x) - au_x\Big).$$

(C) $[Q_2, Q_3]_E$

Finally,

$$\mathcal{L}_{Q_2}(Q_3) = b(y)\Big(2uu_x\, Q_3 + u^2D_x(Q_3) + D_x^3(Q_3)\Big),$$
$$\mathcal{L}_{Q_3}(Q_2) = c(y)\Big(3t(2uu_x\, Q_2 + u^2D_x(Q_2)) + 3tD_x^3(Q_2) - D_x^2(Q_2) - Q_2\Big).$$

So,

$$[Q_2, Q_3]_E = b\Big(2uu_x\, Q_3 + u^2D_x(Q_3) + D_x^3(Q_3)\Big) - c\Big(3t(2uu_x\, Q_2 + u^2D_x(Q_2)) + 3tD_x^3(Q_2) - D_x^2(Q_2) - Q_2\Big).$$

This completes the algebraic expressions of the three commutators.

5.3. On closure, abelianity, and algebraic structure

- Since each commutator is again an evolutionary characteristic built from u and its derivatives multiplied by functions $a(y), b(y), c(y)$, and their derivatives, the space spanned by $\{Q_1, Q_2, Q_3\}$ does not close in general to a finite 3-dimensional Lie algebra unless the coefficient functions a, b, c are chosen specially. In the generic case (arbitrary a, b, c), the algebra generated is infinite-dimensional.
- The algebra is not generally abelian. The brackets above are nonzero in general.
- If one imposes simplifying assumptions, for example,. a, b, c are constants, and moreover, there are certain relations among them, the algebra may reduce to a finite algebra; to classify that case one must substitute those choices and simplify the expressions.

5.4. Adjoint action and adjoint table

For vector fields X_Q and X_R, the adjoint action is

$$\mathrm{Ad}_{\exp(\varepsilon X_Q)}(X_R) = X_R - \varepsilon[X_Q, X_R] + \tfrac{\varepsilon^2}{2}[X_Q, [X_Q, X_R]] - \cdots .$$

We present the adjoint action truncated at first order, which is the standard to build optimal systems. Using $[X_{Q_i}, X_{Q_j}] = X_{[Q_i,Q_j]_E}$, we get the first-order adjoint formulas:

$$\mathrm{Ad}_{\exp(\varepsilon X_{Q_1})}(X_{Q_1}) = X_{Q_1},$$
$$\mathrm{Ad}_{\exp(\varepsilon X_{Q_1})}(X_{Q_2}) = X_{Q_2} - \varepsilon\, X_{[Q_1,Q_2]_E} + O(\varepsilon^2),$$
$$\mathrm{Ad}_{\exp(\varepsilon X_{Q_1})}(X_{Q_3}) = X_{Q_3} - \varepsilon\, X_{[Q_1,Q_3]_E} + O(\varepsilon^2),$$

and similarly with $1 \leftrightarrow 2$, $1 \leftrightarrow 3$.

Thus we can record the adjoint table (first order) symbolically, (see, Table 1):

Table 1. Adjoint table.

$\mathrm{Ad}_{\exp(\varepsilon X_{Q_i})}(X_{Q_j})$	X_{Q_1}	X_{Q_2}	X_{Q_3}
$\exp(\varepsilon X_{Q_1})$	X_{Q_1}	$X_{Q_2} - \varepsilon X_{[Q_1,Q_2]_E}$	$X_{Q_3} - \varepsilon X_{[Q_1,Q_3]_E}$
$\exp(\varepsilon X_{Q_2})$	$X_{Q_1} - \varepsilon X_{[Q_2,Q_1]_E}$	X_{Q_2}	$X_{Q_3} - \varepsilon X_{[Q_2,Q_3]_E}$
$\exp(\varepsilon X_{Q_3})$	$X_{Q_1} - \varepsilon X_{[Q_3,Q_1]_E}$	$X_{Q_2} - \varepsilon X_{[Q_3,Q_2]_E}$	X_{Q_3}

5.5. Classification and one-dimensional optimal system

A full classification of one-dimensional subalgebras for the infinite-dimensional algebra cannot be performed without specifying functional spaces for a, b, c or imposing constraints on them as constants. Here, we use the standard method used in applications; this yields a manageable optimal system for the finite subalgebra spanned by the three types Q_1, Q_2, Q_3.

Let a general one-dimensional subalgebra be generated by

$$V = \alpha X_{Q_1} + \beta X_{Q_2} + \gamma X_{Q_3}, \qquad (\alpha, \beta, \gamma) \in \mathbb{R}^3,$$

where coefficients multiply the whole characteristics (that is αX_{Q_1} means characteristic $\alpha a(y) u_x$). Two generators V and $\tilde{V}$ are equivalent if there exists g in the adjoint group such that $\tilde{V} = \mathrm{Ad}_g(V)$.

Using the adjoint actions in subsection 5.4, we may simplify V by choosing appropriate adjoint transformations to normalize coefficients. Thus:

(1) If $\beta \neq 0$, we use scaling to set $\beta = 1$. Then, we apply $\mathrm{Ad}_{\exp(\varepsilon X_{Q_1})}$ (and/or $\mathrm{Ad}_{\exp(\delta X_{Q_3})}$) to eliminate α and γ by solving for ε, δ from

$$\alpha - \varepsilon \langle [Q_1, Q_2]_E, \cdot \rangle - \delta \langle [Q_3, Q_2]_E, \cdot \rangle = 0.$$

(2) If $\beta = 0$ but $\gamma \neq 0$, we scale to $\gamma = 1$, and use the adjoint of X_{Q_1} to eliminate α, if possible.
(3) If $\beta = \gamma = 0$ and $\alpha \neq 0$, we scale to $\alpha = 1$.

Since the commutators are complicated differential expressions, one cannot always eliminate both extra coefficients for arbitrary a, b, c. However, the following set of representatives is a one-dimensional optimal system for the span of $\{X_{Q_1}, X_{Q_2}, X_{Q_3}\}$:

$$\{ \langle X_{Q_1} \rangle,\ \langle X_{Q_2} \rangle,\ \langle X_{Q_3} \rangle,\ \langle X_{Q_1} + X_{Q_2} \rangle,\ \langle X_{Q_1} + X_{Q_3} \rangle,\ \langle X_{Q_2} + X_{Q_3} \rangle,\ \langle X_{Q_1} + X_{Q_2} + X_{Q_3} \rangle \}.$$

These seven representatives cover the generic possibilities encountered when reducing with one generator. For special choices of a, b, c, some of the representatives may collapse or be further simplified.

5.6. Conservation laws via the new conservation theorem

According to the new conservation theorem [22–25], each Lie-Backlund symmetry Q yields a conservation law of Eq (4.1) through the formal Lagrangian:

$$\mathcal{L} = v\left(u_t + u^2u_x + u_{xxx} + u_{xyy}\right),$$

where v is the adjoint variable. The conserved vector $C = (C^t, C^x, C^y)$ is obtained from

$$C^i = W\left(\frac{\partial \mathcal{L}}{\partial u_i} - D_j\frac{\partial \mathcal{L}}{\partial u_{ij}} + D_jD_k\frac{\partial \mathcal{L}}{\partial u_{ijk}} - \cdots\right), \tag{5.1}$$

with characteristic $W = Q$.

We consider the PDE:

$$F[u] \equiv u_t + u^2u_x + u_{xxx} + u_{xyy} = 0. \tag{5.2}$$

Using new conservation theorem with the formal Lagrangian,

$$L = v\,F[u],$$

and noting that the equation is strictly self-adjoint ($v \equiv 1$), a conservation law corresponding to any symmetry with evolutionary characteristic Q is

$$\begin{aligned} C^t &= Q, \\ C^x &= Qu^2 + D_x^2(Q) + D_y^2(Q), \\ C^y &= 2D_xD_y(Q), \end{aligned} \tag{5.3}$$

with conservation equation

$$D_tC^t + D_xC^x + D_yC^y = Q\,F[u].$$

1) Symmetry with $Q_1 = F_{34}(y)\,u_x$

$$\begin{aligned} C_1^t &= F_{34}(y)\,u_x, \\ C_1^x &= F_{34}(y)\,u^2u_x + F_{34}(y)\,u_{xxx} + F_{34}''(y)\,u_x + 2F_{34}'(y)\,u_{xy} + F_{34}(y)\,u_{xyy}, \\ C_1^y &= 2\Big(F_{34}'(y)\,u_{xx} + F_{34}(y)\,u_{xxy}\Big). \end{aligned} \tag{5.4}$$

2) Symmetry with $Q_2 = F_{36}(y)\,(u^2u_x + u_{xxx})$

$$\begin{aligned} C_2^t &= F_{36}(y)\,(u^2u_x + u_{xxx}), \\ C_2^x &= Q_2u^2 + D_x^2(Q_2) + D_y^2(Q_2), \\ C_2^y &= 2D_xD_y(Q_2), \end{aligned} \tag{5.5}$$

where

$$
\begin{aligned}
D_x^2(Q_2) &= F_{36}(y)\Big(2u_x^3 + 6uu_xu_{xx} + u^2u_{xxx} + u_{xxxxx}\Big),\\
D_y(Q_2) &= F'_{36}(y)(u^2u_x + u_{xxx}) + F_{36}(y)(2uu_yu_x + u^2u_{xy} + u_{xxxy}),\\
D_y^2(Q_2) &= F''_{36}(y)(u^2u_x + u_{xxx})\\
&\quad + 2F'_{36}(y)(2uu_yu_x + u^2u_{xy} + u_{xxxy})\\
&\quad + F_{36}(y)(2u_y^2u_x + 2uu_{yy}u_x + 4uu_yu_{xy} + u^2u_{xyy} + u_{xxxyy}),\\
D_xD_y(Q_2) &= F'_{36}(y)(2uu_yu_x + u^2u_{xy} + u_{xxxy})\\
&\quad + F_{36}(y)(2u_yu_x^2 + 2uu_{xy}u_x + 2uu_yu_{xx} + u^2u_{xxy} + u_{xxxxy}).
\end{aligned}
\tag{5.6}
$$

3) Symmetry with $Q_3 = F_{37}(y)\,(3tu^2u_x + 3tu_{xxx} - u_{xx} - u)$

$$
\begin{aligned}
C_3^t &= F_{37}(y)\,(3tu^2u_x + 3tu_{xxx} - u_{xx} - u),\\
C_3^x &= Q_3u^2 + D_x^2(Q_3) + D_y^2(Q_3),\\
C_3^y &= 2D_xD_y(Q_3).
\end{aligned}
\tag{5.7}
$$

Here only F_{37} depends on y, so

$$D_yF_{37} = F'_{37}(y), \qquad D_tF_{37} = D_xF_{37} = 0.$$

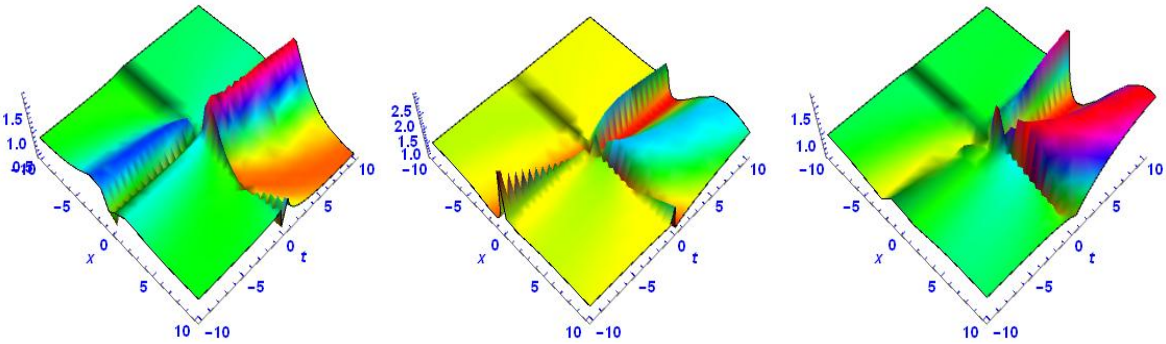
Thus, for each symmetry, the conservation law takes the form

$$D_tC_i^t + D_xC_i^x + D_yC_i^y = 0, \qquad i = 1, 2, 3,$$

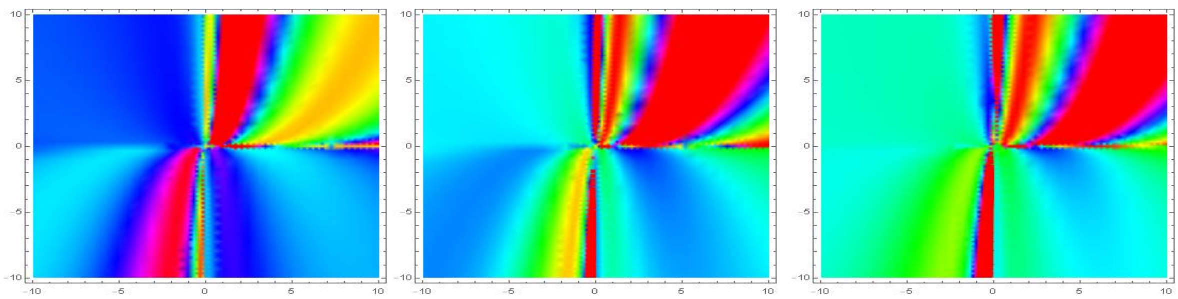
on the solution set of the PDE.

6. Results and discussion

In this section, we interpret the reported results of this study. Using the extended rational sine-cosine/sinh-cosh and Kudryashov expansion methods, we successfully reported topological, rational, exponential, trigonometric, and hyperbolic function solutions to the M-truncated modified Zakharov–Kuznetsov equation. Hyperbolic functions are frequently employed in physics and engineering to explain hanging chain structures, satellite orbits, and wave propagation. They parameterise Lorentz transformations in special relativity and describe the rate at which velocity is added. Bulut et al. [28] studied the modified Zakharov–Kuznetsov equation and reported some important hyperbolic function solutions. Jhangeer et al. [32] constructed the travelling wave patterns of the studied equation. Saha et al. [33] presented the exact solutions and multistability of the studied equation. Eslami et al. constructed some exact travelling wave solutions for the modified Zakharov–Kuznetsov equation using the Riccati equation method [34]. Al-Amin *et al.* revealed some important solitary wave solutions to the modified Zakharov–Kuznetsov equation using the enhanced auxiliary equation method [35]. This study presented the solutions in their rational form alongside nonclassical derivatives. When appropriate parameter values of the parameters involved in the solutions and the fractional value of δ are chosen, the 3D and density graphs are plotted to provide a clear and comprehensive explanation of the physical characteristics of the constructed soliton solutions. The perspective view of the periodic wave solution, Eq (4.11), can be seen in the 3D graphs which appear in Figure 1 at (a) $\delta = 0.1$, (b) $\delta = 0.8$, and (c) $\delta = 1$ for the fractional parameter values. The perspective view of the topological soliton, Eq (4.12), can be seen in the 3D graphs which appear in Figure 2 at (a) $\delta = 0.5$, (b) $\delta = 0.9$, and (c) $\delta = 1$ for the fractional parameter values. The perspective view of the singular soliton, Eq (4.15), can be seen in the 3D graphs which appear in Figure 3 at (a) $\delta = 0.6$, (b) $\delta = 0.7$, and (c) $\delta = 1$ for the fractional parameter values. An alternative to 3D visualizations are density graphs. The plots of density in Figures 1–3 show how the waves propagate both steadily and erratically.



(a) **(b)** **(c)**



(d) **(e)** **(f)**

Figure 1. The 3D and density profiles of (4.11).

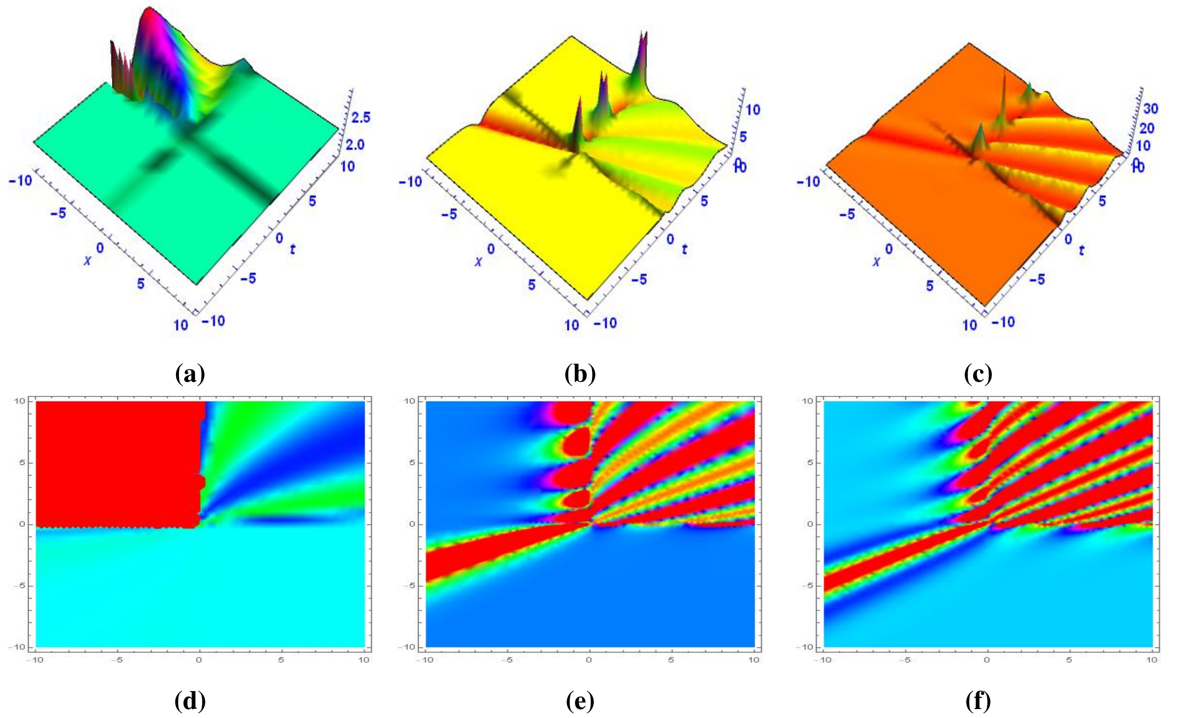


Figure 2. The 3D and density profiles of (4.12).

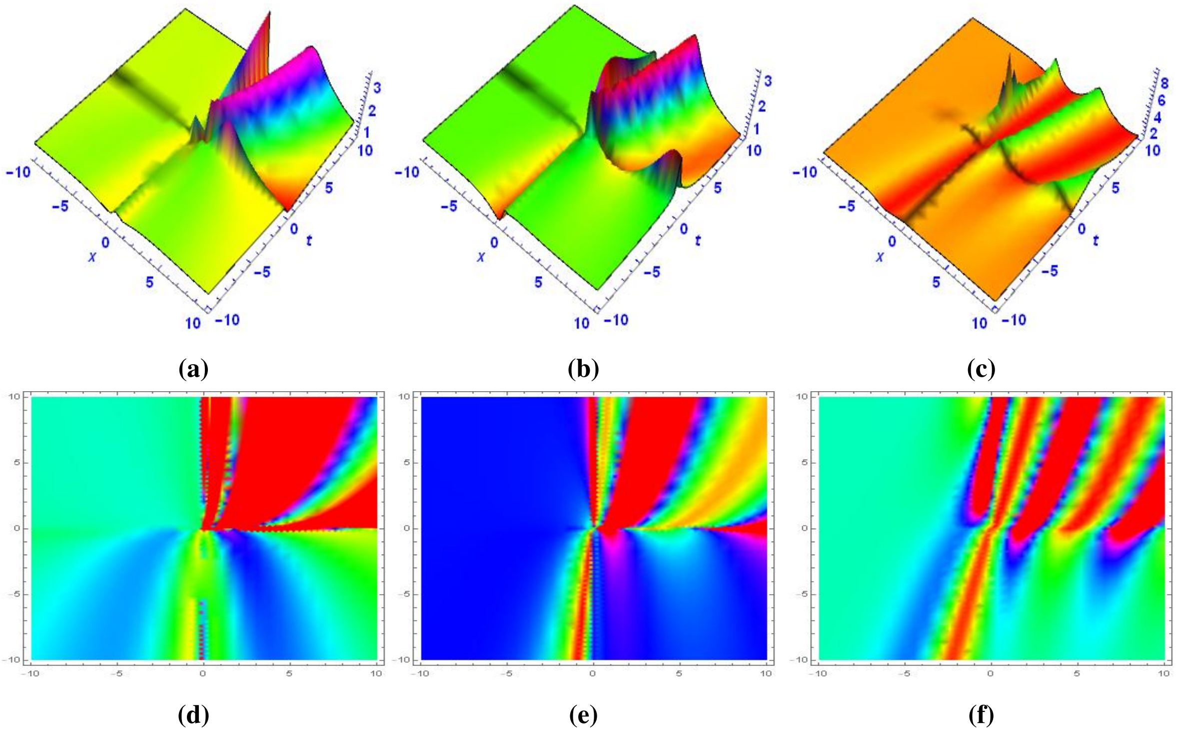


Figure 3. The 3D and density profiles of (4.15).

7. Conclusions

In this paper, the mZKE has been investigated. The Lie symmetry structure, conservation laws, and new soliton solutions of the equation have been investigated. Following the examination of the Lie point symmetries, we proceeded to construct the Lie algebra that corresponds to them. The use of the commutator table, the adjoint representation, and the classification of the most efficient system of subalgebras have all contributed to the accomplishment of the task of examining the structure of algebra. Conservation laws have been derived in a systematic manner by the application of the conservation theorem. Symmetries are connected to physically significant invariants through the use of these principles. New solutions for soliton problems have been discovered through the utilization of two efficient integration methods. Findings that are based on symmetry give a substantial framework for the governing equation. This framework is provided by the obtained results. Having this knowledge makes it easier to comprehend nonlinearly connected partial differential equations and to comprehend how these equations could be utilized in examination. It can be stated that there are a lot of gaps to explore with regard to future work in this area. That is to say, future research on symmetry and conservation laws will increasingly concentrate on bridging the gap between quantum mechanics and general relativity, investigating hidden symmetries in high-energy physics, and applying Lie symmetry approaches to complicated nonlinear systems. These are the three main areas of attention. The application of Noether-based approaches to study non-local conservation laws in advanced material science and nanotechnology are two key research fields. The relevance of broken or emergent symmetries in explaining dark matter and dark energy is another important research direction.

Use of Generative-AI tools declaration

The author declares that he/she has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that there is no conflict of interest associated with this publication.

Data availability statement

All datasets utilized in this study are fully presented within the body of the manuscript.

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