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Research article

A variant of the Sylvester–Kac matrix which exhibits subsets of integer squared singular values for all orders

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Abstract: In [T. Boros, P. Rozsa, *Linear Algebra Appl.*, **421** (2007), 407–416.], the authors showed that for the Sylvester–Kac matrix of odd order, $2n + 1$ ($n \geq 0$), n of the squared singular values were integers. We present a variation of this matrix for which analogous results are obtained in both the odd- and even-order cases, and we derive explicit formulae for computing the corresponding squared singular values. In the even-order case, we also obtain an explicit form for the determinant. Owing to its simple matrix with a subset of easily calculated singular values makes this matrix a useful test case for numerical software for computing singular values. In addition, report several interesting empirical results regarding the singular values of this variant which we obtained using Maple high precision floating-point arithmetic.

Keywords: Sylvester–Kac matrix; singular values; generating function; fuchsian differential equation; indicial equation; eigenvectors
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1. Introduction

Tridiagonal matrices have been used in many different fields, from engineering and economics to mathematics itself. They arise naturally in system analysis and identification, signal processing, special functions, partial differential equations, and linear algebra (for more details, see [1–4]). Many authors consider certain special cases of tridiagonal matrices which allow associated quantities (for example, spectra, eigenvectors, various factorizations, determinants, inverses, etc.) to be obtained as simple closed forms or with specific qualities (for example, integer values). Such results can, in turn, provide valuable insight into how these approaches may be extended to more complex problems.

The Sylvester matrix of order $n+1$ is the tridiagonal matrix, $\mathcal{A}_n$, with $\{a_{ii}\}_{i=1}^{n} = 0$ and $a_{i,i+1} = a_{n-i+2,n-i+1} = i$, for $i = 1, 2, \ldots, n$, i.e.,

$$\mathcal{A}_n = \begin{pmatrix} 0 & 1 & & & \\ n & 0 & 2 & & \\ & n-1 & \ddots & \ddots & \\ & & \ddots & \ddots & n \\ & & & 1 & 0 \end{pmatrix}.$$

The Sylvester matrix $\mathcal{A}_n$, also known as the Clement or Sylvester–Kac matrix, was first defined by the British mathematician James Joseph Sylvester in his short communication [5], where he conjectured that the determinant of its characteristic matrix $\mathcal{A}_n(x)$ was

$$\det \mathcal{A}_n(x) = \prod_{k=0}^{n} (x + n - 2k). \tag{1.1}$$

This was later proved by both Mazza (see [6, pp. 442]) and, later, by Kac [7], who used generating functions for his proof and also obtained the components of the eigenvectors via polynomial identities. Results concerning the eigenspectrum have been derived, independently rediscovered, and extended by various authors via different approaches [8–10].

Boros and Rózsa [11] showed that, if $n = 2p$, where p is a non-negative integer, and $\{\phi_k\}_{k=1}^{n+1}$ denote the singular values of $\mathcal{A}_n$, then the $(p+1)$ values

$$\phi_k^2 = (n+1)^2 - (2k+1)^2$$

are integers for $0 \le k \le p$. There was no report of a similar result for odd n.

Recently, there has been a revival of interest in the Sylvester–Kac or Clement matrix. While some authors defined certain generalizations of the Sylvester matrix, others considered various analogues of the Sylvester matrix. From these, they then derived various interesting properties such as spectrum, eigenvectors, etc. (for details, we refer to [12–14] and the references therein).

In [15], Chu defined the tridiagonal matrix of order $m+1$ by

$$\Omega_m(x, y) = \left[\varphi_{i,j}(x, y)\right]_{0 \le i,j \le m},$$

where x and y are two free variables, and

$$\varphi_{i,j}(x, y) = \begin{cases} x + 2yj, & i - j = 0, \\ m - j, & i - j = 1, \\ j, & j - i = 1, \\ 0, & |i - j| > 1. \end{cases}$$

He then showed that the eigenvalues of $\Omega_m(x, y)$ were explicitly given by

$$\mathcal{P}_{2n+\delta}(x, y) = \left(\lambda_k := x + (2n+\delta)y + (2k-\delta)\sqrt{1+y^2}\right)_{k=-n}^{n+\delta},$$

where we write $m = \delta + 2n$ with $\delta = 0,\ 1$ depending upon the parity of m.

Recently, Fonseca and Kılıç [14] introduced a new variant of the Sylvester–Kac matrix, $G_n(x)$, of order $n+1$ and defined as

$$G_n(x) = \begin{pmatrix} x & n & & & & \\ n+3 & x & n-1 & & & \\ & n+4 & x & \ddots & & \\ & & \ddots & \ddots & 2 & \\ & & & 2n+1 & x & 1 \\ & & & & 2n+2 & x \end{pmatrix}. \quad (1.2)$$

They determined its spectrum, $\lambda(G_n(x))$, as

$$\lambda(G_{2n-1}(x)) = \{x \pm 2(2k-1)\}_{k=1}^{n} \text{ and } \lambda(G_{2n}(x)) = \{x \pm 4k\}_{k=0}^{n}.$$

In the rest of this article, we denote the matrix $G_n(0)$ by G_n and we derive explicit formulae for subsets of the singular values of G_n for both *n even* and *n odd*.

Our main results are contained in the following two theorems:

Theorem 1.1. *(Even Parity Case) For even $n = 2m$, the singular values $\{\phi_{2k}\}_{k=0}^{m}$ of G_n are given by*

$$\phi_{2k}^2 = 8(m(2m+3) - k(2k+3)). \quad (1.3)$$

Theorem 1.2. *(Odd Parity Case) For odd $n = 2m+1$, the singular values, $\{\phi_{2k+1}\}_{k=0}^{m}$, of G_n are given by*

$$\phi_{2k+1}^2 = 8((m+1)(2m+3) - (k+1)(2k+1)). \quad (1.4)$$

Now, we can compare the formulae of singular values of the usual Sylvester–Kac matrix $\mathcal{A}_{2n}$ and its variant G_{2n}. We can see that while the Sylvester-Kac matrix $\mathcal{A}_{2n}$ has the following (n+1) singular values

$$\phi_k^2 = (n+1)^2 - (2k+1)^2$$

for $k = 0, ..., n$, we have seen that the variant matrix G_{2n} has the following singular values

$$\phi_k^2 = 8(n(2n+3) - k(2k+3))$$

for $k = 0, ..., n$.

Tables 1 and 2 show the values of $\{\phi_{2k}\}_{k=0}^{m}$ for G_n for $n = 0, 2, 4, 6, 8$, and G_n for $n = 1, 3, 5, 7$ respectively as computed by the Maple procedure, *LinearAlgebra::SingularValues*. The subsets of the integer values predicted by Theorems 1.1 and 1.2 are boxed, and all these values are divisible by 8.

Table 1. The squared singular values of G_n for the first few even values of n.

n	ϕ_0^2	ϕ_1^2	ϕ_2^2	ϕ_3^2	ϕ_4^2	ϕ_5^2	ϕ_6^2	ϕ_7^2	ϕ_8^2
0	0								
2	40	26	0						
4	112	99.55	72	40.45	0				
6	216	204.62	176	145.32	104	56.07	0		
8	352	341.39	312	282.30	240	192.30	136	72.01	0

Table 2. The squared singular values of G_n for the first few odd values of n.

n	ϕ_0^2	ϕ_1^2	ϕ_2^2	ϕ_3^2	ϕ_4^2	ϕ_5^2	ϕ_6^2	ϕ_7^2
1	16	1						
3	72	58.85	32	0.15				
5	160	148.13	120	88.856	48	0.017		
7	280	269.03	240	209.81	168	120.16	64	0.0016

In Figure 1 we show, on the left, the singular values whose squares are integers and, on the right, we display all the squared singular values of G_{50}. The left-hand graph is the quadratic polynomial

$$\phi^2(x) = 8(1326 - 3x - 2x^2)$$

returned from the Maple procedure *CurveFitting::PolynomialFit* as an exact fit (i.e., it is an interpolating polynomial) to the points $\{(k, {\phi_{2k}}^2)\}_{k=0}^{25}$ where $\{{\phi_{2k}}^2\}_{k=0}^{25}$ are the integer subset of the squared singular values computed by the Maple procedure *LinearAlgebra::SingularValues*.

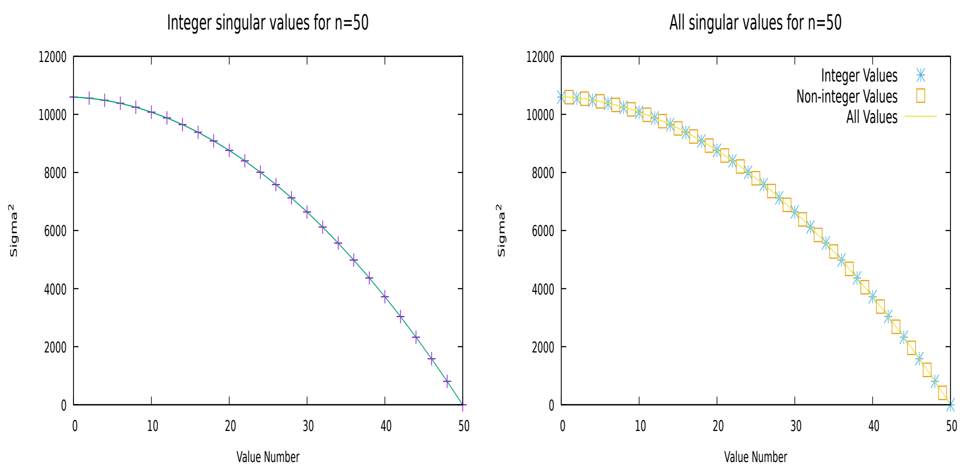

Figure 1. The left-hand plot shows just the integer values of ϕ_i^2 for G_{50}. The right-hand plot displays all the values and illustrates the apparent interlacing of the integer and non-integer squared singular values.

We note that the non-integer ϕ^2 values appear to interlace the integer values. To study this observation further, we utilized Maple to compute the squared singular values of G_n, $n = 4, \ldots, 200$

using integer input and exact arithmetic. This allowed us to isolate the integer subset of values, and we then used high-precision floating-point arithmetic (500 decimal digits) to compute the non-integer values. For all orders, the results exhibited the following interlacing properties:

1) For odd values of n:

$$ip_i > fp_i \wedge ip_i < fp_{i+1}, \quad i = 1, \ldots, (n-1)/2,$$

and

$$ip_{(n+1)/2} > fp_{(n+1)/2}.$$

2) For even values of n:

$$ip_i < fp_i \wedge ip_{i+1} > fp_i, \quad i = 1, \ldots, n/2,$$

where ip and fp are vectors containing the integer and floating point values, respectively, sorted in ascending order. However, we have been unable to prove this property.

In addition, we have noticed that, for increasing values of n, the smallest non-integer values of $\{\phi_k^2\}$ (computed with Maple's high precision arithmetic using 100 decimal digits) became closer to integer values. For example, for G_{51} the two smallest non-integer ϕ^2 values are 8×10^{-29} and $824 + 4 \times 10^{-26}$ and for G_{70} the smallest value is $112 + 0.1 \times 10^{-34}$.

Our variant matrix, $G_n(0)$, has several features that make it useful as a test matrix, for example, testing the accuracy of numerical procedures for computing singular values using floating-point arithmetic. First, the integer elements of $G_n(0)$ can be stored exactly as floating-point numbers for a wide range of values, n; this means that the initial data defining the matrix are free from any representational errors. Second, with a little care, the singular values, $\{\phi_{2k}\}_{k=0}^m$, as defined in Theorems 1.1 and 1.2 can be computed with minimal effects of rounding errors, thus providing very accurate comparison values.

In the proofs of both theorems, we use the following relationship between the eigenvalues of the matrix $X^{\mathrm{T}}X$ and the singular values of the matrix X, where X is a real $(n \times p)$ matrix with $n \geq p$ that states [16, p. 205]:

> *The eigenvalues and eigenvectors of the matrix X^TX are the squares of the singular values and the right singular vectors of X respectively. In addition, when $n > p$ the matrix X^TX has n-p additional zero eigenvalues.*

In Section 2, we prove the even parity case (Theorem 1.1) closely following the method used in [11]. We consider the odd parity case (Theorem 1.2) in Section 3. Here, we show that the subsets of the squared singular values in which we are interested are simply related to the eigenvalues of a particular matrix whose spectrum may be determined by a recurrence technique.

To simplify the proofs of Theorems 1.1 and 1.2, in the following sections, we have changed the indexing of the integer squared singular values from $\{\phi_{2k}\}_{k=0}^m$ to $\{\phi_k\}_{k=0}^m$. The earlier definition was useful for illustrating the interlacing of the values given in Tables 1 and 2.

2. The even parity case

For the case G_{2m} with the integer m, we follow the approach used in [11, Sections 3–5]. This consists of three steps:

1) Use a permutation matrix to transform G_{2m} into an anti-diagonal block matrix. Thence, we may transform the problem of determining the integer-squared singular values of G_{2m}, denoted by $\{\phi_i^2\}_{i=1}^m$, into the problem of determining the eigenvalues of an order $m+1$ matrix, which are known to be $\{\phi_i^2\}_{i=1}^m$ and zero.
2) Derive a homogeneous differential equation of second order, known as the Fuchsian equation, using a generating function method. A solution of this equation in the form of a nonzero polynomial may then be obtained by judicious choice of the available free parameter.
3) Generate a solution to the Fuchsian equation, in the form of a Frobenius series, to allow us to derive the required eigenvalues, $\{\phi_i^2\}_{i=1}^m$.

Step 1:

We use the permutation matrix of order $n+1 = 2m+1$ defined by

$$\Pi_{n+1} = [e_1\ e_3\ e_5 \ldots e_{n+1}\ e_2\ e_4 \ldots e_n],$$

where e_j is the j^{th} column of I_{n+1}, the unit matrix of order $n+1$.

We then have

$$\Pi_{n+1}^{\mathrm{T}} G_n \Pi_{n+1} = \left[\begin{array}{c|c} 0 & Q \\ \hline S & 0 \end{array}\right], \tag{2.1}$$

where the elements of Q of order $(m+1)\times m$ and S of order $m\times(m+1)$ are given by

$$q_{ij} = \begin{cases} n-2(i-1) & i=j, \quad 1\le i\le m, \\ n+2i & j=i-1, \quad 2\le i\le m+1, \\ 0, & \text{Otherwise.} \end{cases} \tag{2.2}$$

$$s_{ij} = \begin{cases} n+1+2i & i=j, \quad 1\le i\le m, \\ n+1-2i & j=i+1, \quad 1\le i\le m, \\ 0, & \text{Otherwise.} \end{cases} \tag{2.3}$$

For example, if $n = 6$, then for the matrix G_6, we have the permutation matrix $\Pi_7 = [e_1, e_3, e_5, e_7, e_2, e_4, e_6]$, and so $\Pi_7^T G_6 \Pi_7$ as shown

$$\Pi_7 = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0 \end{bmatrix}, \ \Pi_7^T G_6 \Pi_7 = \begin{bmatrix} 0&0&0&0&6&0&0 \\ 0&0&0&0&2&4&0 \\ 0&0&0&0&0&4&2 \\ 0&0&0&0&0&0&6 \\ 1&5&0&0&0&0&0 \\ 0&3&3&0&0&0&0 \\ 0&0&5&1&0&0&0 \end{bmatrix} = \begin{bmatrix} 0 & Q \\ S & 0 \end{bmatrix}.$$

We are interested in the singular values of Q. If these are denoted by $\{\phi_i^2\}_{i=1}^m$, then the eigenvalues of $R = Q^{\mathrm{T}}Q$, where R is a square matrix of order $m+1$, are $\{\phi_i^2\}_{i=1}^m$ and zero. Since the matrix R has a zero eigenvalue, $R = Q^{\mathrm{T}}Q$ and considering the relation (2.1), we may assert that

$$\det G_{2n} = \det R = 0.$$

From (2.2), we obtain a tridiagonal matrix R defined by

$$R_{ij} = 4 \times \begin{cases} (m-i+1)^2 + (m+i)^2, & \text{if } 2 \le i = j \le m+1, \\ (m-i+1)(m+i+1), & \text{if } j = i+1, \\ (m-i+2)(m+i), & \text{if } i = j+1, \end{cases} \tag{2.4}$$

with $R_{11} = 4m^2$ and this has the form

$$R = 4 \begin{bmatrix} m^2 & m(m+2) & & & \\ m(m+2) & (m-1)^2+(m+2)^2 & (m-1)(m+3) & & \\ \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & 4m & 4m^2+1 & 2m+1 \\ & & & 2m+1 & (2m+1)^2 \end{bmatrix}.$$

Step 2:

Let $\phi^2 = 4\lambda^2$ be an eigenvalue of R with associated eigenvector $y = [y_0, y_1, \ldots, y_m]^{\mathrm{T}}$, i.e., $Ry = 4\lambda^2 y$. Transposing gives

$$y^{\mathrm{T}} R = 4\lambda^2 y^{\mathrm{T}}, \tag{2.5}$$

and substituting (2.4) into (2.5) produces a homogeneous linear system in the unknowns $\{y_i\}_{i=0}^m$ and a free parameter, λ^2.

We now use the generating function approach to derive the claimed result. To do this, we derive a second-order differential equation whose polynomial solutions provide us with the singular values of the matrix R. Generating functions (see [17], for details) are then employed to find a λ^2 for which this system admits a non-trivial solution. To this end, we multiply successive individual equations by $1, u, u^2, \ldots, u^m$ and sum to give

$$\sum_{k=0}^{m} (m-k)(m+k+2)\, y_k u^{k+1} + \sum_{k=0}^{m} (m-k)^2 y_k u^k + \sum_{k=1}^{m} (m+k+1)^2 y_k u^k \\ + \sum_{k=1}^{m} (m+1-k)(m+1+k)\, y_k u^{k-1} = \lambda^2 \sum_{k=0}^{m} y_k u^k,$$

which, after some rearranging and re-indexing, gives

$$u^2 \sum_{k=0}^{m} (m-k)(m+k+2)\, y_k u^k + u \sum_{k=0}^{m} \left[\left(2k^2+2k\right) + \left(2m^2+2m+1\right)\right] y_k u^k \\ + \sum_{k=0}^{m} (m+1-k)(m+1+k)\, y_k u^k = y_0 (m+1)^2 (u+1) + u\lambda^2 \sum_{k=0}^{m} y_k u^k. \tag{2.6}$$

We define $y(u)$ as the polynomial of degree m, $\sum_{k=0}^{m} y_k u^k$, whose first and second derivatives, $y'(u)$ and $y''(u)$ are given by $\sum_{k=0}^{m} k y_k u^{k-1}$ and $\sum_{k=0}^{m} k(k-1) y_k u^{k-2}$, respectively, then the first three summations of (2.6) may then be written as

$$\sum_{k=0}^{m} (m-k)(m+k+2)\, y_k u^k = \left(m^2+2m\right) y(u) - u^2 y''(u) - 3u y'(u),$$

$$\sum_{k=0}^{m}\left[\left(2k^2+2k\right)+\left(2m^2+2m+1\right)\right]y_k u^k = 2u^2 y''(u)+4uy'(u)$$
$$+\left(2m^2+2m+1\right)y(u), \tag{2.7}$$
$$\sum_{k=0}^{m}(m+1-k)(m+1+k)\,y_k u^k = (m+1)^2\,y(u)-u^2 y''(u)-uy'(u),$$

and substituting (2.7) into (2.6), we obtain the second-order (Fuchsian) differential equation

$$\begin{aligned} &-u^2(u-1)^2\,y''(u)-u(3u-1)(u-1)\,y'(u)+ \\ &\quad\left(-u\lambda^2+m^2u^2+2m^2u+m^2+2mu^2+2mu+2m+u+1\right)y(u) \\ &= y_0(m+1)^2(u+1)+u\lambda^2 y(u). \end{aligned} \tag{2.8}$$

Choosing $y_0 = 0$ ensures that the differential equation is homogeneous, and we obtain

$$y''(u)+\alpha(u)\,y'(u)+\beta(u)\,y(u)=0, \tag{2.9}$$

where

$$\begin{aligned} \alpha(u) &= \frac{(3u-1)}{u(u-1)}, \\ \beta(u) &= -\frac{m(m+2)\,u^2+\left(-\lambda^2+2m+2m^2+1\right)u+(m+1)^2}{u^2(u-1)^2}. \end{aligned} \tag{2.10}$$

We now need to fix the free parameter, λ^2, so that the polynomial solution of (2.9) is non-zero.

Step 3:

Following [11, Section 5], we generate the indicial equation

$$s(s-1)+\alpha_0 s+\beta_0=0, \tag{2.11}$$

where s is real and chosen so that when the Frobenius series of the form

$$y(u)=(u-1)^s\sum_{i=0}^{\infty}\zeta_k(u-1)^i \tag{2.12}$$

is substituted into (2.9), the smallest power of $(u-1)$ is zero.

The coefficients α_0 and β_0 in (2.11) are obtained from (2.10) as

$$\begin{aligned} \alpha_0 &= \lim_{u\to 1}(u-1)\,\alpha(u)=2, \\ \beta_0 &= \lim_{u\to 1}(u-1)^2\,\beta(u)=\lambda^2-2-4m^2-6m. \end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.11), we obtain the quadratic equation

$$s^2+s+\left(\lambda^2-4m^2-6m-2\right)=0,$$

whose roots are

$$s_{1,2} = \frac{-1 \mp \sqrt{(4m+3)^2 - 4\lambda^2}}{2}.$$

The series (2.12) will only be a polynomial if (2.11) has a non-negative integer solution. Thus, we must choose λ^2 so that

$$(4m+3)^2 - 4\lambda^2 = \gamma^2,$$

where γ is an odd integer, and this allows us to define $m+1$ distinct values of $\{\lambda_i^2\}_{i=1}^{m+1}$, where we have

$$(4m+3)^2 - 4\lambda_k^2 = (4k+3)^2\,, \text{ for } k = 0, 1, 2, \ldots, m. \tag{2.14}$$

Thus, the eigenvalues of R are given by

$$\phi_{k+1}^2 = 4\lambda_{k+1}^2 = (4m+3)^2 - (4k+3)^2 = 4\left(4m^2 + 6m - 4k^2 - 6k\right),$$

and hence,

$$\phi_k = 2\sqrt{(2m+3)(2m) - (2k+3)(2k)},$$

which proves Theorem 1.1.

3. The odd parity case

For odd parity, we consider G_n where $n = 2m+1$ and, as in the even parity case, we identify a permutation matrix of order $n+1$,

$$\Pi_{n+1} = [e_1\, e_3\, e_5 \ldots e_n\, e_2\, e_4 \ldots e_{n+1}],$$

which transforms G_{2m+1} into the form (2.1). The resultant Q and S are square matrices of order $m+1$ given by

$$q_{ij} = \begin{cases} n - 2(i-1) & i = j, \quad 1 \le i \le m+1, \\ n + 2i & j = i-1, \quad 2 \le i \le m+1, \\ 0, & \text{Otherwise.} \end{cases} \tag{3.1}$$

$$s_{ij} = \begin{cases} n + 2i + 1 & i = j, \quad 1 \le i \le m+1, \\ n - 2i + 1 & j = i+1, \quad 1 \le i \le m, \\ 0, & \text{Otherwise.} \end{cases} \tag{3.2}$$

For example, if $n = 5$, then for the matrix G_5, we have the permutation matrix $\Pi_6 = [e_1, e_3, e_5, e_2, e_4, e_6]$, and so $\Pi_6^T G_5 \Pi_6$ as shown

$$\Pi_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; \Pi_6^T G_5 \Pi_6 = \begin{bmatrix} 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 9 & 3 & 0 \\ 0 & 0 & 0 & 0 & 11 & 1 \\ 8 & 4 & 0 & 0 & 0 & 0 \\ 0 & 10 & 2 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & Q \\ S & 0 \end{bmatrix}.$$

We now consider the order $m+1$ matrix S, defined by (3.2), and denote its singular values by $\{\gamma_i\}_{i=1}^{m+1}$; these are a subset of the singular values of G_{2m+1}. To determine this subset, we form the order $m+1$ square matrix

$$H = S^{\mathrm{T}}S, \tag{3.3}$$

which we know will have eigenvalues $\{\gamma_i^2\}_{i=1}^{m+1}$.

Using (3.2) in (3.3), we find that H, order $m+1$, is a symmetric tridiagonal matrix whose elements are

$$\begin{aligned} h_{i,i} &= \frac{1}{8}\left((m+1)^2 + i^2\right), \qquad 1 \le i \le m+1, \\ h_{i,i+1} = h_{i+1,i} &= \frac{1}{16}(m+1+i)(m+2-i), \qquad 1 \le i \le m. \end{aligned} \tag{3.4}$$

To prove Theorem 1.2, we first define the real matrix, $A_p(x)$ as $a_{ii}^{(p)} = x + i^2$, $1 \le i \le p$; $a_{i,i-1}^{(p)} = a_{i-1,i}^{(p)} = \frac{1}{2}(p+i)(p-(i-1))$, $2 \le i \le p$ and zero otherwise. Here, we note that

$$H = \frac{1}{8}A_{m+1}((m+1)^2), \tag{3.5}$$

and, thus, the eigenspectrum of H is a special case of that of $A_p(x)$ with $p = m+1$ and $x = (m+1)^2$.

Theorem 3.1. *The eigenvalues of $A_p(x)$ are given by*

$$\{\lambda_{p,k}(x)\}_{k=1}^{p} = \{x + p(p+1) - k(2k-1)\}_{k=1}^{p}.$$

Proof. For later use, the r^{th} eigenvalue of $A_r(x)$ is given by

$$\lambda_{rr}(x) = x + r(r+1) - r(2r-1) = x - r(r-2).$$

We also note the result that, for the lower triangular block matrix,

$$D = \left[\begin{array}{c|c} D_{11} & 0 \\ \hline B & D_{22} \end{array}\right],$$

where $D_{11} \in \mathbb{R}^{m\times m}$ and $D_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$, the eigenvalues of D are the combined eigenvalues of D_{11} and D_{22} (for more details, see (pp. 145, [18]) and (pp. 67, [19])).

Define the vector $\tilde{u}$ of order p as

$$\tilde{u} = \left[\begin{array}{cccc} \tilde{u}(p,1) & \tilde{u}(p,2) & \dots & \tilde{u}(p,p) \end{array}\right],$$

where

$$\tilde{u}(p,k) = (-1)^{p-k}\,\frac{k}{p}\binom{2p}{p-k},$$

along with the order p square matrix

$$T_p = \left[\begin{array}{c|c} \tilde{u}(p,1) & \tilde{u}(p,2) \;\cdots\; \tilde{u}(p,p) \\ \hline 0 & I_{p-1} \end{array}\right].$$

It is easily verified that

$$T_p^{-1} = \left[\begin{array}{c|ccc} \dfrac{1}{\tilde{u}(p,1)} & \dfrac{\tilde{u}(p,2)}{\tilde{u}(p,1)} & \cdots & \dfrac{\tilde{u}(p,p)}{\tilde{u}(p,1)} \\ \hline 0 & & I_{p-1} & \end{array}\right],$$

and we may derive

$$T_p A_p T_p^{-1} = \left[\begin{array}{c|c} \lambda_{pp} & O_{1\times(p-1)} \\ \hline v_1 & \\ 0 & \\ \vdots & W_{ij} \\ 0 & \end{array}\right],$$

where $v_1 = (p+2)(p-1)/(2\tilde{u}(p,1))$ and $W = [W_{ij}]$ is an order $p-1$ matrix given by

$$W = \begin{bmatrix} x+1+(p-1)^2 & \dfrac{(p+3)(p-2)}{2} + W_{1,2} & W_{1,3} & \cdots & \cdots & W_{1,p-1} \\ \dfrac{(p+3)(p-2)}{2} & x+3^2 & \dfrac{(p+4)(p-3)}{2} & & & \\ & \dfrac{(p+4)(p-3)}{2} & x+4^2 & \ddots & & \\ & & \ddots & \ddots & 2p-1 & \\ & & & 2p-1 & \ddots & p \\ & & & & p & x+p^2 \end{bmatrix},$$

with

$$W_{1,j} = -\frac{1}{2}(p+2)(p-1)\frac{\tilde{u}(p,j+1)}{\tilde{u}(p,1)}; \qquad 2 \le j \le p-1.$$

We now derive a similarity transformation matrix $U = [U_{ij}]$ of the form

$$U_{ij} = \begin{cases} \dbinom{p+j+1}{2} & i = j, \quad 1 \le i \le p-1, \\ p^2 - j^2 & j = i+1, \quad 1 \le i \le p-2, \\ \dbinom{p-j+1}{2} & j = i+2, \quad 1 \le i \le p-3, \\ 0, & \text{Otherwise}, \end{cases}$$

and we may verify that $U^{-1} = [U_{ij}^{-1}]$ is an upper triangular matrix defined as

$$U_{ij}^{-1} = (-1)^{i+j}\frac{2}{2+j-i}\binom{p-i-1}{j-i}\binom{p-j+1}{j-i+2}^{-1}$$

for $1 \le i \le p-1$, $i \le j \le p-1$.

The symmetric tridiagonal matrix $C = [C_{ij}] = U^{-1}WU$ has non-zero elements

$$C_{ii} = x + 2p + i^2, \quad i = 1, 2, \ldots, p-1, \tag{3.6}$$

$$C_{i+1,i} = C_{i,i-1} = \frac{1}{2}(p+i-1)(p-i), \quad i = 1, 2, \ldots, p-2, \tag{3.7}$$

and we obtain the relationship

$$U^{-1}WU = A_{p-1}(x+2p),$$

i.e.,

$$A_p(x) \approx \left[\begin{array}{c|c} \lambda_{pp}(x) & O_{1\times p-1} \\ \hline * & \\ 0 & \\ \vdots & A_{p-1}(x+2p) \\ 0 & \\ & \end{array}\right],$$

where $\lambda_{pp} = x - p(p-2)$ and $*$ is a non-zero value.

Using the result given at the start of the proof, the eigenvalues of $A_p(x)$ are $\lambda_{pp}(x)$ and the eigenvalues of $A_{p-1}(x+2p)$. Applying the same argument to $A_{p-1}(x+2p)$ reveals the eigenvalue

$$\begin{aligned} \lambda_{p-1,p-1}(x+2p) &= x + 2p - (p-1)(p-3) \\ &= \lambda_{p,p-1}(x). \end{aligned}$$

The remainder of the spectrum of $A_p(x)$ may be obtained in a similar way to the resulting lower right-hand block, giving the desired result. □

From (3.5), we may now derive the eigenvalues of H, and hence, the values of the required subset of the squares of the singular values, $\{\lambda_i^2\}_{i=1}^{m+1}$, of G_n when $n = 2m+1$

$$\begin{aligned} \lambda(H) &= 8\lambda\left(A_{m+1}(m+1)^2\right) \\ &= 8\left\{(m+1)^2 + (m+1)(m+2) - k(2k-1)\right\}_{k=1}^{m+1} \\ &= 8\{(2m+3)(m+1) - k(2k-1)\}_{k=1}^{m+1}, \end{aligned}$$

as required for Theorem 1.2.

As a consequence, for $n > 1$, we may compute the determinant of $A_n(x)$ as

$$\det A_n(x) = \lambda_{n,n}(x) \det A_{n-1}(x+2n),$$

with $\det A_1(x) = x$.

Recalling that $\lambda_{n,n}(x)$ is the ordered eigenvalue of the matrix $A_n(x)$, we may write

$$\begin{aligned} \det A_n(x) &= \lambda_{n,n}(x) A_{n-1}(x+2n) \\ &= \lambda_{n,n}(x)\left[\lambda_{n-1,n-1}(x+2n) A_{n-2}(x+2n+2(n-1))\right] \\ &= \lambda_{n,n}(x) \lambda_{n-1,n-1}(x+2n) A_{n-2}(x+4n-2). \end{aligned}$$

By the relation (2.1), and hence, in general, $\lambda_{k,k}(x) = x - k(k-2)$, and we derive

$$\begin{aligned} \det A_n(x) &= \prod_{k=0}^{n-1} \lambda_{n-k,-k}\left(x + 2kn - 2\binom{k}{2}\right) \\ &= \prod_{k=1}^{n} \lambda_{k,k}(x - (k-n)(k+n+1)) \\ &= \prod_{k=1}^{n} (x + n(n+1) - k(2k-1)). \end{aligned}$$

4. Conclusions

For a variation of the Sylvester–Kac matrix, we have shown that, for all odd and even orders, $2n + 1$ and $2n$ ($n \geq 0$), respectively, n of the squared singular values are integers and we have derived explicit formulae for both cases. The matrix elements are all integers and can therefore, over a wide range of matrix orders, be stored exactly as floating-point real values. This property, combined with the explicit formulae for a subset of its singular values, renders this matrix particularly suitable as a test matrix for numerical singular value algorithms. We also observed a number of interesting empirical results regarding the singular values of this variant that we were unable to establish rigorously. We believe that both the observed interlacing behavior and the apparent asymptotic convergence of the small non-integer squared singular values as the order of the matrix increases merit further theoretical investigation.

To our knowledge, this study is the second to investigate the explicit identification of singular values of Sylvester-type matrices. We hope that this study will serve as a catalyst for motivating further research into the identification of singular values for other similar matrices in the literature.

Author contributions

Abdullah Alazemi: Formal analysis, writing-original draft; Tim Hopkins: Software, methodology, writing-review, editing; Emrah Kılıç: Methodology, formal analysis, writing-review, editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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