AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(2): 3464–3498.
DOI: 10.3934/math.2026141
Received: 31 October 2025
Revised: 19 January 2026
Accepted: 30 January 2026
Published: 05 February 2026

*Research article*

# Multi-mixed sub-fractional Brownian motion and Ornstein–Uhlenbeck processes

**Foad Shokrollahi[1], Tommi Sottinen[1,*] and Mounir Zili[2]**

[1] School of Technology and Innovations, University of Vaasa, P.O. Box 700, FIN-65101 Vaasa, Finland

[2] Research Laboratory LR18ES17, University of Monastir & Tunisian Military Academy, Avenue Mansour Skhiri, 5000 Monastir, Tunisia

* **Correspondence:** Email: tommi.sottinen@uwasa.fi.

**Abstract:** We proposed a novel class of Gaussian processes, the multi-mixed sub-fractional Brownian motion (mmsfBm) and its Ornstein-Uhlenbeck counterpart. The mmsfBm is an infinite linear combination of independent sub-fractional Brownian motions, a construction that enables it to capture a continuum of scaling properties and provides a significant mathematical advantage over finite-sum models. We rigorously proved that the local roughness of these processes is defined by the infimum of their Hurst exponents. We further showed that both processes are non-semimartingales and possess the conditional full support (CFS) property. The preservation of these unique regularity properties under the Ornstein-Uhlenbeck transformation is a key finding, confirming the robustness of this new framework for modeling complex, multi-scale systems in finance and other fields.

**Keywords:** conditional full support; Hölder continuity; mixed process; Ornstein-Uhlenbeck process; sub-fractional Brownian motion; mixed Gaussian processes
**Mathematics Subject Classification:** 60G15, 60G17, 60G22

## 1. Introduction

Stochastic processes are fundamental to modeling complex systems across scientific and engineering disciplines, with their role in financial mathematics being particularly prominent. While classical models, often rooted in standard Brownian motion, have revolutionized our understanding of markets, empirical evidence consistently reveals phenomena like long-range dependence, self-similarity, and intricate scaling behaviors that go beyond the capabilities of these traditional frameworks. Fractional Brownian motion (fBm) offered a significant leap forward by naturally incorporating long-range dependence through a single Hurst exponent $H$. However, real-world data frequently exhibits a richer, more nuanced spectrum of scaling properties that a single $H$ cannot fully capture.

To address these limitations, and rigorously analyzes advanced stochastic models: the *multi-mixed sub-fractional Brownian motion (mmsfBm)* and its corresponding *multi-mixed sub-fractional Ornstein-Uhlenbeck (mmsfOU)* process. These Gaussian processes, to our knowledge, represent a pioneering step in the exploration of a new class of stochastic models. We not only define and build the foundational theory for these processes but also highlight their unique advantages over existing frameworks.

We emphasize that the primary contribution of this work is theoretical. The introduction of the mmsfBm and mmsfOU processes is motivated by the need for mathematically robust Gaussian models capable of capturing multi-scale path behavior and infimum-driven regularity properties. While financial modeling provides an important motivating context, the results of this paper concern the probabilistic structure, path regularity, $p$-variation, conditional full support (CFS), and non-semimartingale behavior of these processes within the theory of Gaussian processes and infinite-mixture models.

The mmsfBm is constructed as an infinite linear combination of independent sub-fractional Brownian motions (sfBm), each characterized by its own Hurst exponent $H_k$ and weighting coefficient $\sigma_k$. Specifically, it is defined as,

$$M_t = \sum_{k=1}^{\infty} \sigma_k \xi_t^{H_k}. \tag{1.1}$$

Here, $\xi^{H_k}$ represents an independent sfBm with index $H_k \in (0, 1)$, and $\sigma_k$ are coefficients such that $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$. The sfBm, initially introduced in [5] and further studied in [4, 14], is a centered Gaussian process whose covariance function is given by

$$\mathbb{E}[\xi_t^H \xi_s^H] = t^{2H} + s^{2H} - \frac{1}{2}((t+s)^{2H} + |t-s|^{2H}), \qquad s, t \geq 0. \tag{1.2}$$

The sfBm generalizes Brownian motion and arises naturally from occupation time fluctuations of branching particle systems. While it shares many properties with fbm, a crucial distinction is its lack of stationary increments, which provides additional modeling flexibility.

This infinite construction provides a crucial mathematical advantage. The *mmsfBm* and its finite-sum counterpart, the *Mixed Sub-Fractional Brownian Motion (msfBm)* [6, 12, 15, 16], differ fundamentally in their ability to capture a continuum of scales. While the msfBm is limited to modeling systems with a discrete number of scales; for instance, a financial time series might be a combination of short-term roughness and a long-term trend and cannot capture the intricate, multi-scale nature of phenomena like turbulent fluid dynamics or fractal geometries. The mmsfBm, by contrast, can be constructed with an infinite set of exponents $\{H_k\}$ that are dense in an interval, enabling it to represent a continuous range of roughness. Crucially, its local roughness is defined by the *infimum* of all exponents, $H_{\inf} = \inf\{H_k\}$, which may not be a value explicitly present in the model's parameters. This provides a more robust and versatile framework, as the level of roughness is a stable limiting value rather than a potentially unstable discrete one.

Building upon the mmsfBm, the *mmsfOU process* is defined as the solution to a linear stochastic differential equation,

$$dU_t = -\lambda U_t dt + dM_t, \tag{1.3}$$

where $\lambda > 0$ denotes a mean-reversion parameter. This formulation parallels the relationship between the classical Ornstein-Uhlenbeck process and standard Brownian motion, inherently introducing a mean-reverting characteristic vital for modeling phenomena like interest rates or commodity prices. A defining feature of these processes, especially pertinent to modern quantitative finance, is their *non-semimartingale* nature for most parameter choices. This property challenges traditional Itô calculus frameworks, yet it aligns with the intricate realities of markets exhibiting roughness or strict arbitrage opportunities, thus motivating the need for advanced stochastic analysis tools.

We note that the non-semimartingale nature of the mmsfBm and mmsfOU processes, while natural in the present multi-scale setting, precludes their direct use within classical arbitrage-free pricing frameworks based on Itô calculus. Rather than enhancing classical financial modeling, this property motivates the study of these processes within alternative non-semimartingale paradigms, such as models with transaction costs, restricted trading strategies, or generalized integration frameworks; see, for instance, [7, 11].

This paper makes several significant contributions to the theoretical understanding of mmsfBm and mmsfOU processes. We begin by defining the mmsfBm and characterizing its fundamental properties, including its existence and covariance structure. Our analysis shows that its path regularity and Hölder continuity are uniquely governed by the infimum of its Hurst exponents, $H_{\inf}$. Building on this, we provide a detailed comparison with the finite-sum msfBm, highlighting the mathematical advantages of our infinite construction before confirming the non-semimartingale and non-Markovian nature of the mmsfBm. We then extend our analysis to the mmsfOU process, deriving its integral representation and demonstrating that its local Hölder continuity and $p$-variation are also determined by the same $H_{\inf}$. A crucial aspect of our work is establishing the CFS property for both processes, which is a cornerstone for applications in non-semimartingale finance. To bridge theory and application, we conclude by presenting numerical simulations and visualizations that illustrate the complex dynamics of both processes and validate our theoretical findings. The remainder of this paper is structured to follow this logical progression, with each section dedicated to a contribution.

## 2. Multi mixed sub-fractional Brownian motion (mmsfBm)

**Definition 1.** *Let $\sigma_k$, $k \in \mathbb{N} \setminus \{0\}$, satisfy*

$$\sum_{k=1}^{\infty} \sigma_k^2 < \infty, \tag{2.1}$$

*and let $H_k \in (0, 1)$, $k \in \mathbb{N} \setminus \{0\}$, satisfy*

$$\begin{cases} H_k & \neq & H_l \text{ for } k \neq l, \\ H_{inf} & = & \inf\{H_i : i \in \mathbb{N} \setminus \{0\} \text{ and } \sigma_i \neq 0\} > 0, \\ H_{sup} & = & \sup\{H_i : i \in \mathbb{N} \setminus \{0\} \text{ and } \sigma_i \neq 0\} < 1. \end{cases} \tag{2.2}$$

*The mmsfBm is*

$$M_t = (M_t)_{t \in R^+} = \sum_{k=1}^{\infty} \sigma_k \xi_t^{H_k}, \tag{2.3}$$

*where $\xi_t^{H_k}$, $k \in \mathbb{N}$, are independent sfBm's.*

Let us first recall some properties of the sfBm (see [5, 14] for proofs and for further information).

**Lemma 1.** *The sfBm process $\xi_t^H = (\xi_t^H)_{t\in\mathbb{R}_+}$ satisfies the following properties:*

*(1) $\xi_t^H$ is a centered Gaussian process.*

*(2) For all $s \in \mathbb{R}_+$, for all $t \in \mathbb{R}_+$,*

$$Cov(\xi_t^H, \xi_s^H) = s^{2H} + t^{2H} - \frac{1}{2}\left((s+t)^{2H} + |t-s|^{2H}\right). \tag{2.4}$$

*(3) For any $h > 0$, the process $\{\xi_{ht}^H\}_{t\geq 0}$ has the same law as $\{h^H \xi_t^H\}_{t\geq 0}$.*

*(4) For all $(s,t) \in \mathbb{R}_+^2$, $s \leqslant t$.*

$$\mathbb{E}[(\xi_t^H - \xi_s^H)^2] = -2^{2H-1}(t^{2H} + s^{2H}) + (t+s)^{2H} + (t-s)^{2H}. \tag{2.5}$$

*(5) The exist two positive constants $C_1$ and $C_2$, such that, for all $(s,t) \in \mathbb{R}_+^2$, $s \leqslant t$,*

$$C_1(t-s)^{2H} \leq \mathbb{E}[(\xi_t^H - \xi_s^H)^2] \leq C_2(t-s)^{2H}. \tag{2.6}$$

*(6) The sfBm admits the representation*

$$\xi_t^H = \frac{B_t^H + B_{-t}^H}{\sqrt{2}}, \tag{2.7}$$

*where $B^H$ is a two-sided fBm.*

**Remark 1.** *A crucial special case arises when $H = 1/2$. In this instance, the sub-fractional Brownian motion $\xi_t^{1/2}$ reduces to a standard Brownian motion. Its covariance function, directly derived from property 2 of this Lemma by setting $H = 1/2$, becomes,*

$$Cov(\xi_t^{1/2}, \xi_s^{1/2}) = s + t - \frac{1}{2}(s + t + |t-s|) = \min(s,t). \tag{2.8}$$

*This demonstrates that the sfBm is indeed one of many different types of Gaussian processes that extend the concept of a standard Brownian motion by introducing a parameter $H$ to model diverse dependence structures, particularly non-stationary behavior when $H \neq 1/2$.*

*Existence and probabilistic and path properties*

In this section, we establish the existence of the mmsfBm and derive its fundamental probabilistic and path properties, which form the basis for its characterization.

**Proposition 1.** *The mmsfBm process $M_t = (M_t)_{t\in\mathbb{R}^+}$ defined in Eq (2.3) exists as a random function, taking values in $L^2(\Omega \times [0,T])$ for all $T > 0$.*

*Proof.* Let us define the partial sums of the series,

$$M_t^n = \sum_{k=1}^{n} \sigma_k \xi_t^{H_k}, \quad n \in \mathbb{N}.$$

$$\begin{aligned}
\|M_t^n - M_t^m\|_{L^2(\Omega\times[0,T])}^2 &= \int_0^T \mathbb{E}\left[(M_t^n - M_t^m)^2\right] dt \\
&= \int_0^T \mathbb{E}\left[\left(\sum_{k=m+1}^{n} \sigma_k \xi_t^{H_k}\right)^2\right] dt \\
&= \sum_{k=m+1}^{n} \int_0^T \sigma_k^2 \mathbb{E}\left[(\xi_t^{H_k})^2\right] dt \\
&= \sum_{k=m+1}^{n} \int_0^T \sigma_k^2 (2 - 2^{2H_k-1}) t^{2H_k} \, dt \\
&= \sum_{k=m+1}^{n} \sigma_k^2 (2 - 2^{2H_k-1}) \frac{T^{1+2H_k}}{1+2H_k} \\
&\leq \sum_{k=m+1}^{n} \sigma_k^2 (2 - 2^{2H_k-1}) \max\left\{1, T^3\right\} \\
&\leq 2 \max\left\{1, T^3\right\} \sum_{k=m+1}^{n} \sigma_k^2,
\end{aligned}$$

which shows that the sequence $(M_t^n)_{n\in\mathbb{N}}$ is Cauchy. Thus, $M_t^n \to M_t$ in $L^2(\Omega \times [0, T])$ shows the existence. □

Now that the existence of the mmsfBm has been established, we can delve into its fundamental probabilistic properties that define its behavior.

**Theorem 1.** *The mmsfBm process $M_t$ possesses the following fundamental probabilistic properties:*

*(1) $M_t$ is a centered Gaussian process.*

*(2) For all $s, t \in \mathbb{R}_+$, the covariance function is given by,*

$$Cov(M_t, M_s) = \sum_{i=1}^{\infty} \sigma_i^2 \left[t^{2H_i} + s^{2H_i} - \frac{1}{2}\left((s+t)^{2H_i} + |t-s|^{2H_i}\right)\right].$$

*Consequently, the variance of the process at time t is,*

$$\mathbb{E}[(M_t^2] = \sum_{i=1}^{\infty} \sigma_i^2 \left[(2 - 2^{2H_i-1}) t^{2H_i}\right].$$

*(3) The covariance function $R(t, s) = Cov(M_t, M_s)$ is continuous on any compact set $[0, T] \times [0, T]$. Consequently, it is bounded, meaning there exists a finite constant $C_T > 0$ such that $|R(t, s)| \leq C_T$ for all $t, s \in [0, T]$.*

*(4) For any $h > 0$, the processes $\{M_{ht}(\sigma)\}$ and $\{M_t(\sigma_1 h^{H_1}, \sigma_2 h^{H_2}, \ldots)\}$ have the same law.*

*Proof.* (1) The mmsfBm $M_t = \sum_{k=1}^{\infty} \sigma_k \xi_t^{H_k}$ is defined as a sum of independent centered Gaussian processes. Since any finite linear combination of independent centered Gaussian processes is a centered Gaussian process, and since the series converges in $L^2(\Omega\times[0,T])$, the limit process $M$ is also a centered Gaussian process.

(2) By the definition of the mmsfBm and the fact that the sfBms $\xi_t^{H_i}$ and $\xi_t^{H_j}$ are centered and independent for $i \neq j$, we can write,

$$\mathrm{Cov}(M_t, M_s) = \sum_{i=1}^{\infty} \sigma_i^2 \mathrm{Cov}(\xi_t^{H_i}, \xi_s^{H_i}).$$

Using the known covariance formula for a single sfBm (see Lemma 1(2)), we get the stated expression of $\mathrm{Cov}(M_t, M_s)$. The variance formula is obtained by setting $s = t$ in the covariance expression.

(3) The covariance function is defined as an infinite sum, $R(t,s) = \sum_{i=1}^{\infty} R_i(t,s)$ with

$$R_i(t,s) = \sigma_i^2 \left[ t^{2H_i} + s^{2H_i} - \frac{1}{2}\left( (t+s)^{2H_i} + |t-s|^{2H_i} \right) \right].$$

Since $H_i > 0$ for all $i$, each individual function $x \mapsto x^{2H_i}$ is continuous for $x \geq 0$. As compositions and sums of continuous functions, each term $t^{2H_i}$, $s^{2H_i}$, $(t+s)^{2H_i}$, and $|t-s|^{2H_i}$ are continuous functions of $(t,s)$ on the compact domain $[0,T]\times[0,T]$. Therefore, each $R_i(t,s)$ is a continuous function on $[0,T]\times[0,T]$.

Furthermore, we can establish the boundedness of $R_i(t,s)$ on this domain. For any $t, s \in [0,T]$,

$$\begin{aligned}
|R_i(t,s)| &\leq \sigma_i^2 \left[ |t^{2H_i}| + |s^{2H_i}| + \frac{1}{2}\left( |(t+s)^{2H_i}| + ||t-s|^{2H_i}| \right) \right] \\
&\leq \sigma_i^2 \left[ T^{2H_i} + T^{2H_i} + \frac{1}{2}\left( (2T)^{2H_i} + T^{2H_i} \right) \right] \\
&= \sigma_i^2 T^{2H_i} \left[ 2 + \frac{1}{2}(2^{2H_i} + 1) \right].
\end{aligned}$$

Let $f(H) = T^{2H}\left(2 + \frac{1}{2}(2^{2H}+1)\right)$. Since $0 < H_i < 1$ for all $i$, the set of all $H_i$ is contained within the open interval $(0,1)$. Then $0 < H_{inf} \leq H_{sup} < 1$ (see Assumption (2.2)). The function $f(H)$ is continuous on the compact interval $[H_{inf}, H_{sup}]$. Therefore, $f(H)$ attains its maximum value on this interval. Let $C_T^* = \max_{H\in[H_{inf},H_{sup}]} f(H)$. This constant $C_T^*$ is finite and depends only on $T$, $H_{inf}$, and $H_{sup}$.

Thus, for all $i \in \mathbb{N}^\star$ and for all $(t,s) \in [0,T]\times[0,T]$, we have,

$$|R_i(t,s)| \leq \sigma_i^2 C_T^*.$$

Let $M_{t_i} = \sigma_i^2 C_T^*$. Given that $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$, the series $\sum_{i=1}^{\infty} M_i = \sum_{i=1}^{\infty} \sigma_i^2 C_T^* = C_T^* \sum_{i=1}^{\infty} \sigma_i^2$ converges.

Since $\sum_{i=1}^{\infty} M_{t_i}$ converges and $|R_i(t,s)| \leq M_{t_i}$ for all $(t,s) \in [0,T] \times [0,T]$, the series $\sum_{i=1}^{\infty} R_i(t,s)$ converges uniformly on $[0,T]\times[0,T]$. All this allow us to get the continuity of $R(t,s)$ on $[0,T]\times[0,T]$.

Finally, since $[0,T] \times [0,T]$ is a compact set in $\mathbb{R}^2$, and $R(t,s)$ is a continuous function on this compact set, it must attain its maximum and minimum values. Therefore, it is bounded, meaning there exists a finite constant $C_T > 0$ such that $|R(t,s)| \leq C_T$ for all $t,s \in [0,T]$.

(4) Let us compare the covariance functions of $\{M_{ht}(\sigma)\}$ and $\{M_t(\sigma_1 h^{H_1}, \sigma_2 h^{H_2}, \dots)\}$. By using Lemma 1(3), we get

$$\begin{aligned}
\mathrm{Cov}(M_{ht}, M_{hs}) &= \sum_{i=1}^{\infty} \sigma_i^2 \mathrm{Cov}(\xi_{ht}^{H_i}, \xi_{hs}^{H_i}) \\
&= \sum_{i=1}^{\infty} \sigma_i^2 \left( h^{2H_i} \mathrm{Cov}(\xi_t^{H_i}, \xi_s^{H_i}) \right) \\
&= \mathrm{Cov}(M_t(\sigma_1 h^{H_1}, \sigma_2 h^{H_2}, \dots), M_s(\sigma_1 h^{H_1}, \sigma_2 h^{H_2}, \dots)).
\end{aligned}$$

Since the covariance functions are equal, and both processes are centered and Gaussian, they have the same law. □

**Remark 2.** *In the particular case where $H_i = 1/2$ for every $i \in \mathbb{N}^\star$, the sfBm $\xi^{H_i}$ reduces to a standard Brownian motion, as demonstrated in Remark 1. Consequently, the mmsfBm $M_t$ simplifies to a multi-mixed Brownian motion (mmBm). In this scenario, the covariance function of the mmBm directly follows from Theorem 1(2) (or equivalently from the simplified form in Remark 1), yielding,*

$$Cov(M_t, M_s) = \sum_{i=1}^{\infty} \sigma_i^2 \min(t,s) = \left( \sum_{i=1}^{\infty} \sigma_i^2 \right) \min(t,s).$$

*This scenario is particularly significant as it represents a return to a more classical Gaussian process with stationary and independent increments, for which many results are well-established. From this perspective, the mmsfBm can be seen as a natural extension of the mmBm, enabling the modeling of processes with more complex long-range dependence properties through the varying Hurst parameters $H_i$.*

The following theorem consolidates the fundamental properties of the mmsfBm regarding its incremental behavior and path regularity:

**Theorem 2.** *The incremental second moments and path regularity of the mmsfBm process are characterized by the following properties:*

*(1) For all $s,t \in \mathbb{R}_+$, with $s \leq t$, the second moment of the increments is given by,*

$$E\Big(M_t - M_s\Big)^2 = \sum_{i=1}^{\infty} \sigma_i^2 \Bigg( -2^{2H_i - 1}(t^{2H_i} + s^{2H_i}) + (t+s)^{2H_i} + (t-s)^{2H_i} \Bigg). \tag{2.9}$$

*Furthermore, these increments satisfy the following bounds:*

$$\sum_{i=1}^{\infty} \sigma_i^2 \gamma_i (t-s)^{2H_i} \leq E\Big(M_t - M_s\Big)^2 \leq \sum_{i=1}^{\infty} \sigma_i^2 \nu_i (t-s)^{2H_i}, \tag{2.10}$$

*where*

$$\gamma_i = \begin{cases} 2 - 2^{2H_i - 1} & if \quad H_i > 1/2 \\ 1 & if \quad H_i \le 1/2 \end{cases} \quad and \quad \nu_i = \begin{cases} 1 & if \quad H_i > 1/2 \\ 2 - 2^{2H-1} & if \quad H_i \le 1/2. \end{cases} \tag{2.11}$$

*(2) The mmsfBm process exhibits a quasi-helix property. Specifically, for any compact interval* $[0, T]$,

- *There exists a constant* $C > 0$, *such that for all* $s, t \in [0, T]$,

$$\mathbb{E}\left[(M_t - M_s)^2\right] \le C|t - s|^{2H_{inf}}. \tag{2.12}$$

- *For every* $\epsilon > 0$, *there exists a constant* $C_\epsilon > 0$, *such that for all* $s, t \ge 0$ *with* $|t - s| \ge \epsilon$,

$$\mathbb{E}\left[(M_t - M_s)^2\right] \ge C_\epsilon |t - s|^{2H_{inf}}. \tag{2.13}$$

- *If* $H_{inf}$ *is attained (i.e., there exists* $j \in \mathbb{N}^\star$, *such that* $H_j = H_{inf}$*), then for every* $T > 0$, *there exist constants* $C_1, C_2 > 0$ *such that for all* $s, t \in [0, T]$,

$$C_1|t - s|^{2H_{inf}} \le \mathbb{E}\left[(M_t - M_s)^2\right] \le C_2|t - s|^{2H_{inf}}. \tag{2.14}$$

*Proof.* The first assertion is a direct consequence of Proposition 3.1 in [15] and the independence of the family of random variables $(\xi_t^{H_i} - \xi_s^{H_i})_{i \in \mathbb{N}}$.

For the second assertion, we begin by proving inequality (2.12). From (2.11), we have $\nu_i \le 2$ for every $i \in \mathbb{N}$. Furthermore, we have $H_i - H_{inf} \in [0, 1)$ for every $i \in \mathbb{N}$.

**Case 1:** $|t - s| < 1$. The function $x \longmapsto |t - s|^{2x}$ is decreasing on the interval $[0, 1)$. Thus, since $0 \le H_i - H_{inf} < 1$, we have

$$|t - s|^2 < |t - s|^{2(H_i - H_{inf})} \le 1. \tag{2.15}$$

**Case 2:** $|t - s| \ge 1$. The function $x \longmapsto |t - s|^{2x}$ is increasing on $[0, 1)$. Therefore,

$$1 \le |t - s|^{2(H_i - H_{inf})} < |t - s|^2 \le T^2. \tag{2.16}$$

Combining the two cases, we get that, for every $s, t \in [0, T]$,

$$1 \wedge |t - s|^2 \le |t - s|^{2(H_i - H_{inf})} < 1 \vee T^2.$$

Together with (2.10), these results give us

$$\begin{aligned} \mathbb{E}\left[\left(M_t - M_s\right)^2\right] &\le \sum_{i=1}^{\infty} \sigma_i^2 \nu_i |t - s|^{2H_i} \\ &\le 2|t - s|^{2H_{inf}} \sum_{i=1}^{\infty} \sigma_i^2 |t - s|^{2(H_i - H_{inf})} \\ &\le C|t - s|^{2H_{inf}}, \end{aligned}$$

with $C = 2(1 \vee T^2) \sum_{i=1}^{\infty} \sigma_i^2$.

To prove (2.13), fix $\epsilon > 0$ and $s, t \ge 0$, such that $|t - s| \ge \epsilon$. Let $j \in \mathbb{N}^\star$ be such that $\sigma_j \ne 0$.

**Case 1:** $|t-s| \geq 1$. Using (2.16) and (2.10), we get

$$\begin{aligned}
\mathbb{E}\left[\left(M_t - M_s\right)^2\right] &\geq \sum_{i=1}^{\infty} \sigma_i^2 \gamma_i (t-s)^{2H_i} \\
&= |t-s|^{2H_{inf}} \sum_{i=1}^{\infty} \sigma_i^2 \gamma_i |t-s|^{2(H_i - H_{inf})} \\
&\geq |t-s|^{2H_{inf}} \sum_{i=1}^{\infty} \sigma_i^2 \gamma_i \\
&\geq |t-s|^{2H_{inf}} \sigma_j^2 \gamma_j.
\end{aligned}$$

**Case 2:** $\epsilon \leq |t-s| < 1$, Since the map $x \longmapsto |t-s|^x$ is decreasing, from (2.15) we have

$$|t-s|^{2(H_j - H_{inf})} \geq |t-s|^2 \geq \epsilon^2.$$

Combining these inequalities with (2.10), we obtain

$$\begin{aligned}
\mathbb{E}\left[\left(M_t - M_s\right)^2\right] &\geq |t-s|^{2H_{inf}} \sum_{i=1}^{\infty} \sigma_i^2 \gamma_i |t-s|^{2(H_i - H_{inf})} \\
&\geq |t-s|^{2H_{inf}} \sigma_j^2 \gamma_j |t-s|^{2(H_j - H_{inf})} \\
&\geq |t-s|^{2H_{inf}} \sigma_j^2 \gamma_j \epsilon^2.
\end{aligned}$$

Therefore, (2.13) is obtained with $C_\epsilon = \sigma_j^2 \gamma_j (1 \wedge \epsilon^2)$.

Now, consider the case where the infimum of the Hurst exponents is attained; that is there exists $j \in \mathbb{N}^\star$ such that $H_j = H_{inf}$ and $\sigma_j \neq 0$. Fix $T > 0$. The upper bound in (2.14) follows directly from Assertion 1 with $C_2 = C$. To establish the lower bound in (2.14), we proceed as follows:

$$\begin{aligned}
\mathbb{E}\left[\left(M_t - M_s\right)^2\right] &\geq |t-s|^{2H_{inf}} \sum_{i=1}^{\infty} \sigma_i^2 \gamma_i |t-s|^{2(H_i - H_{inf})} \\
&= |t-s|^{2H_{inf}} \left(\sigma_j^2 \gamma_j + \sum_{i=1, i\neq j}^{\infty} \sigma_i^2 \gamma_i |t-s|^{2(H_i - H_{inf})}\right) \\
&\geq |t-s|^{2H_{inf}} \sigma_j^2 \gamma_j.
\end{aligned}$$

Therefore, the lower bound in (2.14 ) holds with $C_1 = \sigma_j^2 \gamma_j$. □

**Remark 3.** *A direct consequence of the expression for the incremental second moments in Eq (2.9) is that the mmsfBm does not possess stationary increments. Unlike processes such as Brownian motion or fractional Brownian motion, the variance of the increments,* $\mathbb{E}\left[(M_t - M_s)^2\right]$, *depends on the time points* $s$ *and* $t$, *rather than only on the time difference* $|t-s|$. *This characteristic is a hallmark of the mmsfBm and is replaced by the weaker but still crucial quasi-helix property, as established by the bounds in Eqs (2.12)–(2.14).*

The following corollary highlights a crucial consequence of the mmsfBm's quasi-helix property, establishing its Hölder continuity and almost sure non-differentiability.

**Corollary 1.** *(1) The mmsfBm has Hölder index $H_{inf}$. In particular, for any $T > 0$ and $0 < \epsilon < H_{inf}$, there exists a non-negative random variable $G_{H_{inf},\epsilon,T}$ such that*

$$|M_t - M_s| \leq G_{H_{inf},\epsilon,T}|t - s|^{H_{inf}-\epsilon} \ a.s.$$

*(2)*

$$\lim_{\epsilon\to 0^+} \sup_{t\in[t_0-\epsilon,t_0+\epsilon]} \left|\frac{M_t - M_{t_0}}{t - t_0}\right| = +\infty, \tag{2.17}$$

*with probability one for every $t_0 \in \mathbb{R}$.*

*Proof.* Assertion 1 is a consequence of the quasi-helix property and Theorem 1 of [2]. Assertion 2 follows in the same way as in [15]. □

## 3. Comparison with mixed sub-fractional Brownian motion: the mathematical advantage of the mmsfBm

Building upon our finding that the mmsfBm possesses Hölder continuous paths with exponent $\alpha$, we now delve into a detailed comparison with its finite counterpart, the msfBm, to highlight the unique mathematical advantages of our model. Both models are built upon the foundational concept of a sum of independent sub-fractional Brownian motions. The msfBm, a simpler model introduced by Zili in [15], is defined by a finite sum, $S_t^N = \sum_{k=1}^{N} \sigma_k \xi_t^{H_k}$.

### *3.1. Local roughness and Hölder Regularity*

The *local roughness* of a stochastic process is quantified by its *local Hölder exponent*, $H(t)$. This exponent is defined by the scaling of the increments' variance as the time lag, $h$, approaches zero.

$$H(t) = \lim_{h\to 0^+} \frac{\log \mathbb{E}[(X_{t+h} - X_t)^2]}{2\log(h)}. \tag{3.1}$$

**Proposition 2.** *Consider a msfBm defined by the finite sum*

$$S_t^N = \sum_{k=1}^{N} \sigma_k \xi_t^{H_k},$$

*where $N \in \mathbb{N} \setminus \{0\}$, $(\sigma_1, \ldots, \sigma_N) \in \mathbb{R}^N$, and $(H_1, \ldots, H_N) \in (0,1)^N$. Without loss of generality, we can assume that $\sigma_k \neq 0$ for all $k \in \{1, \ldots, N\}$, as any component with $\sigma_k = 0$ would not contribute to the sum. Assuming the minimum exponent is $H_{k_0} = \min\{H_1, \ldots, H_N\}$, for some $k_0 \in \{1, \ldots, N\}$, then the local Hölder exponent of the process $S$ is precisely $H_{k_0}$.*

*Proof.* Due to the independence of the sfBm components, the variance of the msfBm increments is simply the sum of the individual variances,

$$\mathbb{E}[(S_{t+h} - S_t)^2] = \sum_{k=1}^{N} \sigma_k^2 \mathbb{E}[(\xi_{t+h}^{H_k} - \xi_t^{H_k})^2]. \tag{3.2}$$

Applying the quasi-helix property (2.6) to each term, we obtain a two-sided bound on the total variance,

$$\sum_{k=1}^{N} \sigma_k^2 C_1 h^{2H_k} \leq \mathbb{E}[(S_{t+h} - S_t)^2] \leq \sum_{k=1}^{N} \sigma_k^2 C_2 h^{2H_k}. \tag{3.3}$$

We can factor out the term $h^{2H_{k_0}}$ from both the lower and upper bounds,

$$C_1 h^{2H_{k_0}} \left( \sum_{k=1}^{N} \sigma_k^2 h^{2(H_k - H_{k_0})} \right) \leq \mathbb{E}[(S_{t+h} - S_t)^2] \leq C_2 h^{2H_{k_0}} \left( \sum_{k=1}^{N} \sigma_k^2 h^{2(H_k - H_{k_0})} \right). \tag{3.4}$$

As $h \to 0$, all exponents $2(H_k - H_{k_0})$ are non-negative. Therefore, the inner sum converges to $\sigma_{k_0}^2$. This simplifies the asymptotic inequality for the variance,

$$C_1 \sigma_{k_0}^2 h^{2H_{k_0}} \leq \mathbb{E}[(S_{t+h} - S_t)^2] \leq C_2 \sigma_{k_0}^2 h^{2H_{k_0}}. \tag{3.5}$$

Now, let us substitute these bounds into the definition of the local Hölder exponent, $H(t)$. For the lower bound,

$$\begin{aligned}
H(t) &\geq \lim_{h \to 0^+} \frac{\log(\mathbb{E}[(S_{t+h} - S_t)^2])}{2\log(h)} \\
&\geq \lim_{h \to 0^+} \frac{\log(C_1 \sigma_{k_0}^2 h^{2H_{k_0}})}{2\log(h)} \\
&= \lim_{h \to 0^+} \frac{\log(C_1 \sigma_{k_0}^2) + 2H_{k_0} \log(h)}{2\log(h)} \\
&= \lim_{h \to 0^+} \left( \frac{\log(C_1 \sigma_{k_0}^2)}{2\log(h)} + H_{k_0} \right) \\
&= H_{k_0}.
\end{aligned}$$

In the same way, using the upper bound in (3.3), we get $H(t) \leq H_{k_0}$. By combining both bounds, we get that the local Hölder exponent of the msfBm is exactly $H_{k_0}$. □

**Theorem 3.** *The local Hölder exponent of the mmsfBm process is equal to $H_{inf}$.*

*Proof.* The proof is divided into two parts, showing $H(t) \geq H_{inf}$ and then $H(t) \leq H_{inf}$.

*Part 1:* Due to the independence of the components, we have

$$\mathbb{E}[(M_{t+h} - M_t)^2] = \sum_{k=1}^{\infty} \sigma_k^2 \mathbb{E}[(\xi_{t+h}^{H_k} - \xi_t^{H_k})^2]. \tag{3.6}$$

Since all terms are non-negative, for an arbitrary term $k_1$,

$$\mathbb{E}[(M_{t+h} - M_t)^2] \geq \sigma_{k_1}^2 \mathbb{E}[(\xi_{t+h}^{H_{k_1}} - \xi_t^{H_{k_1}})^2]. \tag{3.7}$$

Using the quasi-helix property (2.6), we obtain,

$$\mathbb{E}[(M_{t+h} - M_t)^2] \geq \sigma_{k_1}^2 C_1 h^{2H_{k_1}}. \tag{3.8}$$

Taking the limit to find the local Hölder exponent,

$$H(t) = \lim_{h\to 0^+} \frac{\log \mathbb{E}[(M_{t+h} - M_t)^2]}{2\log(h)} \geq \lim_{h\to 0^+} \frac{\log(\sigma_{k_1}^2 C_1 h^{2H_{k_1}})}{2\log(h)} = \lim_{h\to 0^+} \frac{\log(\sigma_{k_1}^2 C_1) + 2H_{k_1}\log(h)}{2\log(h)} = H_{k_1}.$$

Since this holds for every exponent $H_{k_1}$ in the infinite set, the local exponent must be greater than or equal to their infimum. Thus, $H(t) \geq H_{inf}$.

*Part 2:* Now, we use the upper bound from the quasi-helix property (2.6),

$$\mathbb{E}[(M_{t+h} - M_t)^2] \leq \sum_{k=1}^{\infty} \sigma_k^2 (C_2 h^{2H_k}). \tag{3.9}$$

Since $H_{inf} = \inf\{H_i : i \in \mathbb{N} \setminus \{0\} \text{ and } \sigma_i \neq 0\}$, we know that $H_k \geq H_{inf}$ for all $k$. For $h \in (0, 1)$, this implies $h^{2H_k} \leq h^{2H_{inf}}$. Using this fact, we can bound the entire sum,

$$\sum_{k=1}^{\infty} \sigma_k^2 C_2 h^{2H_k} \leq C_2 \sum_{k=1}^{\infty} \sigma_k^2 h^{2H_{inf}} = C_2 h^{2H_{inf}} \sum_{k=1}^{\infty} \sigma_k^2. \tag{3.10}$$

Assuming the series $\sum_{k=1}^{\infty} \sigma_k^2$ converges to a finite constant $K$, we get the final upper bound for the variance,

$$\mathbb{E}[(M_{t+h} - M_t)^2] \leq C_2 h^{2H_{inf}} K = C' h^{2H_{inf}}. \tag{3.11}$$

Now, we substitute this upper bound into the definition of the local Hölder exponent,

$$\begin{aligned} H(t) &= \lim_{h\to 0^+} \frac{\log \mathbb{E}[(M_{t+h} - M_t)^2]}{2\log(h)} \\ &\leq \lim_{h\to 0^+} \frac{\log(C' h^{2H_{inf}})}{2\log(h)} \\ &= \lim_{h\to 0^+} \frac{\log(C') + 2H_{inf}\log(h)}{2\log(h)} \\ &= \lim_{h\to 0^+} \left( \frac{\log(C')}{2\log(h)} + H_{inf} \right) \\ &= H_{inf}. \end{aligned}$$

By combining the two parts of the proof, we conclude that $H(t) = H_{inf}$. □

### 3.2. Variation of the higher orders of mmsfBm and msfBm

We first recall the definition of the $p$-variation of a stochastic process.

**Definition 2.** *Let $X = \{X_t,\ t \in [0, T]\}$ be a stochastic process and let $p > 0$. The* $p$-variation *of $X$ on the interval $[0, T]$ is defined as*

$$V^p(X; [0, T]) := \sup_{\Pi} \sum_{i=0}^{n-1} \left| X_{t_{i+1}} - X_{t_i} \right|^p,$$

*where the supremum is taken over all partitions $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of the interval $[0, T]$.*

The following theorem provides a comprehensive characterization of the $p$-variation for both the msfBm and the mmsfBm. This result reveals how the regularity of these processes depends on the minimum or infimum of their respective Hurst exponents.

**Theorem 4.** *Let $p \in \mathbb{N} \setminus \{0\}$. The $p$-variation of a process $X$ (either msfBm or mmsfBm) on an interval $[0, T]$ is given by,*

$$V_T^p(X) = \begin{cases} \infty, & \text{if } pH_{exponent} < 1, \\ 0, & \text{if } pH_{exponent} > 1. \end{cases}$$

*Here, $H_{exponent}$ is the local Hölder exponent, which is $H_{min} = \min\{H_1, \ldots, H_N\}$ for the msfBm and $H_{inf} = \inf\{H_i : i \in \mathbb{N} \setminus \{0\}$ and $\sigma_i \neq 0\}$, for the mmsfBm.*

*Proof.* The proof will be detailed for the mmsfBm case, as the argument for the msfBm is analogous. We will distinguish two cases based on the value of $H_{\inf}$.

*Case 1:* $H_{\inf} > \frac{1}{p}$. Consider a sequence of partitions $\{\tau_n\}_{n\in\mathbb{N}\setminus\{0\}}$ of $[0, T]$,

$$\tau_n : 0 = t_0 < t_1 < \cdots < t_n = T,$$

such that the mesh size $|\tau_n| = \max_{1\leq i\leq n} |t_i - t_{i-1}|$ converges to zero as $n \to \infty$. The increments of the mmsfBm are centered Gaussian random variables. Using the quasi-helix property (2.12), we have,

$$\mathbb{E}\left[|M_t - M_s|^p\right] = C_1 \mathbb{E}\left[(M_t - M_s)^2\right]^{p/2} \leq C_2 |t - s|^{pH_{\inf}}, \tag{3.12}$$

for all $s, t \in [0, T]$ with $s \leq t$, where $C_1$ and $C_2$ are positive constants.

Let us define the $p$-variation sum for the partition $\tau_n$ as

$$\Delta_t^{\tau_n} = \sum_{j=1}^{n} |M_{t_j} - M_{t_{j-1}}|^p.$$

From Eq (3.12), we can bound the expected value of this sum:

$$\begin{aligned} \mathbb{E}[\Delta_t^{\tau_n}] &= \sum_{j=1}^{n} \mathbb{E}\left[|M_{t_j} - M_{t_{j-1}}|^p\right] \\ &\leq \sum_{j=1}^{n} C_2 (t_j - t_{j-1})^{pH_{\inf}} \\ &\leq C_2 |\tau_n|^{pH_{\inf}-1} \sum_{j=1}^{n} (t_j - t_{j-1}) \\ &= C_2 |\tau_n|^{pH_{\inf}-1} T. \end{aligned} \tag{3.13}$$

Since $\lim_{n\to\infty} |\tau_n| = 0$ and our assumption is $pH_{\inf} - 1 > 0$, we have,

$$0 \leq \lim_{n\to\infty} \mathbb{E}[\Delta_t^{\tau_n}] \leq \lim_{n\to\infty} C_2 |\tau_n|^{pH_{\inf}-1} T = 0.$$

Therefore, the sequence $(\Delta_t^{\tau_n})$ converges to 0 in $L^1$, and thus in probability. Consequently, the $p$-variation is,

$$V_T^p(M_t) = 0 \quad \text{a.s.}$$

*Case 2:* $H_{\inf} < \frac{1}{p}$. Since $H_{inf} = \inf\{H_i : i \in \mathbb{N} \setminus \{0\}$ and $\sigma_i \neq 0\}$, we can find a Hurst exponent $H_j$ in the sequence such that $H_j < \frac{1}{p}$ (and its corresponding coefficient $\sigma_j \neq 0$). For any $n \in \mathbb{N} \setminus \{0\}$, define the $p$-variation sum on an equidistant partition,

$$A_{n,p} = \sum_{j=1}^{n} \left| M_{\frac{jT}{n}} - M_{\frac{(j-1)T}{n}} \right|^p . \tag{3.14}$$

Assume, for contradiction, that $V_T^p(M_t) < \infty$ almost surely (a.s.). Then, by definition of $p$-variation, $A_{n,p}$ must converge to a finite limit in probability. From the bounds (2.10), we have

$$\mathbb{E}\left[\left(M_{\frac{jT}{n}} - M_{\frac{(j-1)T}{n}}\right)^2\right] \geq \sum_{i=1}^{\infty} \sigma_i^2 \gamma_i \left(\frac{T}{n}\right)^{2H_i} .$$

Therefore, we have a lower bound for the second moment,

$$\mathbb{E}\left[\left(M_{\frac{jT}{n}} - M_{\frac{(j-1)T}{n}}\right)^2\right] \geq \sigma_j^2 \gamma_j \left(\frac{T}{n}\right)^{2H_j} .$$

Using the fact that the increments are Gaussian, we get

$$\mathbb{E}[A_{n,p}] = \sum_{j=1}^{n} \mathbb{E}\left[\left| M_{\frac{jT}{n}} - M_{\frac{(j-1)T}{n}} \right|^p\right] \geq C_3 \sum_{j=1}^{n} \left(\sigma_j^2 \gamma_j \left(\frac{T}{n}\right)^{2H_j}\right)^{p/2} = C_3 \sum_{j=1}^{n} (\sigma_j^2 \gamma_j)^{p/2} \left(\frac{T}{n}\right)^{pH_j} , \tag{3.15}$$

where $C_3 > 0$ is a constant. Assume that $(\sigma_j^2 \gamma_j)^{p/2}$ is uniformly bounded from below by a positive constant. Since $H_j < \frac{1}{p}$ for all $j$, we have $1 - pH_j > 0$. Therefore,

$$\mathbb{E}[A_{n,p}] \geq C \sum_{j=1}^{n} \left(\frac{T}{n}\right)^{pH_j} \geq C\, n \left(\frac{T}{n}\right)^{pH_{\max}} = C\, T^{pH_{\max}} n^{1-pH_{\max}},$$

where $H_{\max} := \max_{1 \leq j \leq n} H_j < \frac{1}{p}$. Hence,

$$\lim_{n \to \infty} \mathbb{E}[A_{n,p}] = \infty. \tag{3.16}$$

This implies that $A_{n,p}$ cannot converge to a finite limit in probability or almost surely. This divergence contradicts the initial assumption that the $p$-variation is finite. Therefore, $V_T^p(M_t) = \infty$ almost surely. □

**Remark 4.** ***The mathematical advantage: Infimum vs. Minimum.*** *While both processes possess a constant local Hölder exponent that quantifies their path roughness, the fundamental distinction lies in how this exponent is determined. For the msfBm, this regularity is dictated by the minimum value of a finite set of Hurst exponents. In contrast, the mmsfBm leverages an infinite sum, and its local Hölder exponent is defined by the infimum of an infinite sequence of exponents. This is a crucial and powerful mathematical difference, as an infimum may not be an explicit value within the model's parameters. This distinction has profound implications for a process's fundamental properties, particularly its p-variation, a measure of total roughness. From Theorem 4, a process has finite p-variation if and only if p is strictly greater than* $1/\alpha$*, where* $\alpha$ *is its local Hölder exponent.*

**Illustrative example:** Consider a sequence of Hurst parameters such that $H_k = 0.4 + 1/k$. As $k \to \infty$, this sequence converges to $\alpha = 0.4$.

- For the *msfBm*, the local Hölder exponent is the minimum of a finite set, $H_N = 0.4 + 1/N$. This exponent is always strictly greater than 0.4, making the condition for finite $p$-variation, $p(0.4 + 1/N) > 1$, dependent on the number of terms. This indicates a certain instability as the model is refined by adding more components.
- In contrast, for the *mmsfBm*, the local Hölder exponent is the stable, fixed value $\alpha = H_{inf} = 0.4$. Consequently, the condition for finite $p$-variation, $p(0.4) > 1$, is robust and independent of the number of terms. This mathematical stability demonstrates that the mmsfBm provides a more consistent and robust framework for modeling phenomena whose roughness is defined by a limiting value rather than a discrete one.

## 4. Beyond semimartingales and Markov processes

The unique roughness of the mmsfBm process leads to its non-semimartingale and non-Markovian properties, which are a critical distinction from more standard models like Brownian motion. By leveraging the p-variation results from previous sections, we can definitively show that the mmsfBm is not a semimartingale. This is a fundamental property in stochastic analysis with significant implications for financial modeling.

**Proposition 3.** *The mmsfBm process $M_t = (M_t)_{t\in[0,1]}$ is not a semimartingale, provided that there exists at least one $\sigma_j \neq 0$, such that $H_j \neq 1/2$.*

*Proof.* For a continuous stochastic process $M_t$ to be a semimartingale, it is a necessary and sufficient condition that its quadratic variation $V_T^2(M_t)$ exists and is finite for all $T > 0$. Conversely,

- If $V_T^2(M_t) = \infty$ almost surely, then $M_t$ is definitively not a semimartingale.
- If $M_t$ is a non-trivial continuous Gaussian process, and $V_T^2(M_t) = 0$ almost surely for all $T > 0$, then $M_t$ is not a semimartingale.

We will analyze the quadratic variation $V_T^2(M_t)$ based on the distribution of the Hurst exponents $H_k$.

*Case 1: There exists at least one $k_0 \in \mathbb{N}^\star$ such that $\sigma_{k_0} \neq 0$ and $H_{k_0} < 1/2$. In this scenario, we must have*

$$H_{inf} = \inf\{H_i : i \in \mathbb{N} \text{ and } \sigma_i \neq 0\} \leq H_{k_0} < 1/2.$$

*Consequently, $2H_{inf} < 1$. Applying Theorem 4 with $p = 2$, if $2H_{inf} < 1$, then $V_T^2(M_t) = \infty$ almost surely. Since the quadratic variation of M is infinite, M cannot be a semimartingale.*

*Case 2: For all $k \in \mathbb{N}^\star$ such that $\sigma_k \neq 0$, we have $H_k > 1/2$. This implies that $H_{inf} \geq 1/2$. From (2.10), for any compact interval $[0, T]$, for $s, t \in [0, T]$:*

$$\mathbb{E}\left[(M_t - M_s)^2\right] \leq \sum_{i=1}^{\infty} \sigma_i^2 \gamma_i |t - s|^{2H_i}.$$

*Since all $H_i > 1/2$ for $\sigma_i \neq 0$, it follows that $2H_i - 1 > 0$. Therefore,*

$$\mathbb{E}\left[(M_t - M_s)^2\right] \leq |t - s| \sum_{i=1}^{\infty} \sigma_i^2 \gamma_i |t - s|^{2H_i - 1}.$$

*Let*

$$g(\Delta t) = \sum_{i=1}^{\infty} \sigma_i^2 \gamma_i (\Delta t)^{2H_i - 1} \quad for \quad \Delta t = |t-s|.$$

*As $\Delta t \to 0$, for each i with $\sigma_i \neq 0$, the term $(\Delta t)^{2H_i-1}$ tends to 0 because $2H_i - 1 > 0$. Since $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$ by Assumption (2.1), and $\gamma_i \in (0, 2)$, by the dominated convergence theorem, $g(\Delta t)$ converges to 0 as $\Delta t \to 0$. Therefore,*

$$\mathbb{E}\left[(M_t - M_s)^2\right] = o(|t-s|) \quad as \quad |t-s| \to 0,$$

*which means that, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any $|t-s| < \delta$, we have*

$$\mathbb{E}[(M_t - M_s)^2] \leq \epsilon |t-s|.$$

*Now let $\pi = \{t_0, t_1, \ldots, t_n\}$, be a partition of $[0, T]$ with mesh size $|\pi| = \max_j (t_j - t_{j-1}) < \delta$. We have*

$$\mathbb{E}\left[\sum_{j=1}^{n} (M_{t_j} - M_{t_{j-1}})^2\right] = \sum_{j=1}^{n} \mathbb{E}\left[(M_{t_j} - M_{t_{j-1}})^2\right] \leq \sum_{j=1}^{n} \epsilon (t_j - t_{j-1}) = \epsilon \sum_{j=1}^{n} (t_j - t_{j-1}) = \epsilon T.$$

*Since $\epsilon$ can be chosen arbitrarily small, this shows that*

$$\lim_{|\pi| \to 0} \mathbb{E}\left[\sum_{j=1}^{n} (M_{t_j} - M_{t_{j-1}})^2\right] = 0.$$

*Convergence of the expectation to zero implies convergence in $L^1$, which in turn implies convergence in probability. Therefore, $V_T^2(M_t) = 0$ almost surely. As established in the introductory paragraph, since $M_t$ is a non-trivial continuous Gaussian process with zero quadratic variation, it is not a semimartingale.* □

**Remark 5.** *The hypothesis of Proposition 3, "there exists at least one $\sigma_j \neq 0$ such that $H_j \neq 1/2$", is crucial. It explicitly excludes the case where $M_t$ reduces to a multi-mixed Brownian motion (i.e., all $H_j = 1/2$ for $\sigma_j \neq 0$). A multi-mixed Brownian motion is a standard Brownian motion scaled by $\sqrt{\sum_{i=1}^{+\infty} \sigma_i^2}$, which is a semimartingale with quadratic variation $V_T^2(M_t) = T \sum_{i=1}^{\infty} \sigma_i^2$.*

The complexity of the mmsfBm's dependence structure, a consequence of its multi-scale nature, also implies that it is not a Markov process.

**Theorem 5.** *Suppose there exist $\sigma_j \neq 0$ such that $H_j \neq 1/2$. Then the mmsfBm is not Markovian.*

*Proof.* For all $t > 0$,

$$\mathrm{Cov}(M_t, M_t) = \sum_{i=1}^{\infty} \sigma_i^2 (2 - 2^{2H_i - 1}) t^{2H_i} > 0.$$

If $M_t$ were a Markovian process, according to [13], for all $s < t < u$, we would have

$$\mathrm{Cov}(M_s, M_u)\,\mathrm{Cov}(M_t, M_t) = \mathrm{Cov}(M_s, M_t)\,\mathrm{Cov}(M_t, M_u). \tag{4.1}$$

We compute each covariance term in (4.1) using the covariance formula of the mmsfBm. For $s < t < u$,

$$(M_s, M_u) = \sum_{i=1}^{\infty} \sigma_i^2 \Big( s^{2H_i} + u^{2H_i} - \frac{1}{2}((s+u)^{2H_i} + (u-s)^{2H_i}) \Big),$$

and similarly for $\mathrm{Cov}(M_s, M_t)$ and $\mathrm{Cov}(M_t, M_u)$. Now choose $s = \sqrt{t}$ and $u = t^2$ (so that $1 < s < t < u$ for $t > 1$). Then, for each $i \geq 1$,

$$\begin{aligned}
\mathrm{Cov}(M_t, M_t) &= \sum_{i=1}^{\infty} \sigma_i^2 \, (2 - 2^{2H_i - 1}) \, t^{2H_i}, \\
\mathrm{Cov}(M_{\sqrt{t}}, M_{t^2}) &= \sum_{i=1}^{\infty} \sigma_i^2 \Big[ t^{4H_i} + t^{H_i} - \frac{1}{2} t^{4H_i} ((1 + t^{-3/2})^{2H_i} + (1 - t^{-3/2})^{2H_i}) \Big], \\
\mathrm{Cov}(M_{\sqrt{t}}, M_t) &= \sum_{i=1}^{\infty} \sigma_i^2 \Big[ t^{2H_i} + t^{H_i} - \frac{1}{2} t^{2H_i} ((1 + t^{-1/2})^{2H_i} + (1 - t^{-1/2})^{2H_i}) \Big], \\
\mathrm{Cov}(M_t, M_{t^2}) &= \sum_{i=1}^{\infty} \sigma_i^2 \Big[ t^{4H_i} + t^{2H_i} - \frac{1}{2} t^{4H_i} ((1 + t^{-1})^{2H_i} + (1 - t^{-1})^{2H_i}) \Big]. \qquad (4.2)
\end{aligned}$$

We analyze two cases,

*Case 1:* $H_j > 1/2$. In this case, $H_{\sup} > \frac{1}{2}$. Applying the Markov identity (4.1) with $1 < s = \sqrt{t} < t < u = t^2$ and using the covariance expressions given in (4.2), we obtain, after truncating the infinite sums at $N$,

$$\begin{aligned}
&\lim_{N\to\infty} \left( \sum_{i=1}^{N} \sigma_i^2 (2 - 2^{2H_i-1}) t^{2H_i} \right) \left( \sum_{i=1}^{N} \sigma_i^2 \Big[ t^{4H_i} + t^{H_i} - \frac{1}{2} t^{4H_i} ((1 + t^{-3/2})^{2H_i} + (1 - t^{-3/2})^{2H_i}) \Big] \right) \\
&= \lim_{N\to\infty} \left( \sum_{i=1}^{N} \sigma_i^2 \Big[ t^{H_i} + t^{2H_i} - \frac{1}{2} t^{2H_i} ((1 + t^{-1/2})^{2H_i} + (1 - t^{-1/2})^{2H_i}) \Big] \right) \\
&\quad \times \left( \sum_{i=1}^{N} \sigma_i^2 \Big[ t^{2H_i} + t^{4H_i} - \frac{1}{2} t^{4H_i} ((1 + t^{-1})^{2H_i} + (1 - t^{-1})^{2H_i}) \Big] \right),
\end{aligned}$$

for every $t > 0$.

For every $n \geq 1$, consider $H_{j_n} = \sup\{H_i; i \in \{1, ..., n\}\}$, where $j_n \in \{1, ..., n\}$. The sequence $(H_{j_n})_n$ is increasing and $\lim_{n\to\infty} H_{j_n} = H_{sup}$.

Thus, for every $t > 0$ and large N, we should have

$$\begin{aligned}
& t^{3H_{j_N}} \sum_{i=1}^{N} \sigma_i^2 (2 - 2^{2H_i-1}) t^{2(H_i - H_{j_N})} \\
\times \; & \sum_{i=1}^{N} \sigma_i^2 \Big[ t^{4H_i - H_{j_N}} + t^{H_i - H_{j_N}} - \frac{1}{2} t^{4H_i - H_{j_N}} [(1 + t^{-3/2})^{2H_i} + (1 - t^{-3/2})^{2H_i}] \Big] \\
\sim \; & t^{3H_{j_N}} \sum_{i=1}^{N} \sigma_i^2 [t^{H_i - H_{j_N}} + t^{2H_i - H_{j_N}} - \frac{1}{2} t^{2H_i - H_{j_N}} [(1 + t^{-1/2})^{2H_i} + (1 - t^{-1/2})^{2H_i}] \\
\times \; & \sum_{i=1}^{N} \sigma_i^2 [t^{2H_i - 2H_{j_N}} + t^{4H_i - 2H_{j_N}} - \frac{1}{2} t^{4H_i - 2H_{j_N}} [(1 + t^{-1})^{2H_i} + (1 - t^{-1})^{2H_i}]].
\end{aligned} \qquad (4.3)$$

Since, as $h \to 0$, we have

$$(1+h)^{2H_i} + (1-h)^{2H_i} = 2 + 2H_i(2H_i - 1)h^2 + o(h^2), \tag{4.4}$$

equivalence (4.3) implies that for large t and N,

$$\begin{aligned}
&\sum_{i=1}^{N} \sigma_i^2 (2 - 2^{2H_i - 1}) t^{2(H_i - H_{j_N})} \qquad (4.5)\\
\times\ &\sum_{i=1}^{N} \sigma_i^2 \Big( t^{H_i - H_{j_N}} - H_i(2H_i - 1) t^{4H_i - H_{j_N} - 3} + o(t^{4H_i - H_{j_N} - 3}) \Big)\\
\sim\ &\Big[ \sum_{i=1}^{N} \sigma_i^2 \Big( t^{H_i - H_{j_N}} - H_i(2H_i - 1) t^{2H_i - H_{j_N} - 1} + o(t^{2H_i - H_{j_N} - 1}) \Big)\\
\times\ &\sum_{i=1}^{N} \sigma_i^2 \Big( t^{2(H_i - H_{j_N})} - H_i(2H_i - 1) t^{4H_i - 2H_{j_N} - 2} + o(t^{4H_i - 2H_{j_N} - 2}) \Big) \Big]. \qquad (4.6)
\end{aligned}$$

This suggests that, for large N, equation is,

$$\sigma_{j_N}^4 (2 - 2^{2H_{j_N} - 1}) \sim \sigma_{j_N}^4.$$

This equivalence follows from the fact that, for large $N$, the dominant contribution in each sum arises from the index $j_N$, such that $H_{j_N} = \max\{H_1, \ldots, H_N\}$. Indeed, all remaining terms are of lower order since $t^{H_i - H_{j_N}} \to 0$ as $t \to \infty$ for $H_i < H_{j_N}$. Consequently, the leading behavior of both sides is governed by the term corresponding to $i = j_N$, yielding

$$\sigma_{j_N}^4 (2 - 2^{2H_{j_N} - 1}) \sim \sigma_{j_N}^4.$$

This asymptotic equivalence is possible only if

$$\lim_{N \to \infty} (1 - 2^{2H_{j_N} - 1}) = 0.$$

Since $H_{j_N} \uparrow H_{\sup}$ as $N \to \infty$, we obtain

$$1 - 2^{2H_{\sup} - 1} = 0,$$

which holds if and only if $H_{\sup} = \frac{1}{2}$. Therefore, unless $H_{\sup} = \frac{1}{2}$, the Markov identity (4.1) cannot be satisfied, and the process $M_t$ is not Markovian.

*Case 2:* $H_j < 1/2$. Since $H_{inf} < \frac{1}{2}$, we can choose $0 < s = t^2 < t < u = \sqrt{t}$ for $t \to 0$. A similar argument then leads to

$$1 - 2^{2H_{inf} - 1} = 0,$$

which implies that $H_{inf} = 1/2$. This contradiction with $H_j < \frac{1}{2}$ demonstrates that $M_t$ is not a Markovian process. □

**Remark 6.** *The non-semimartingale and non-Markovian properties of the mmsfBm are not just theoretical results; they are the direct consequence of its multi-scale nature and long-range dependence. Unlike standard models like Brownian motion, the mmsfBm can capture the complex memory and path roughness found in real-world data. This makes it a more robust and realistic tool for analyzing complex systems.*

## 5. The multi-mixed sub-fractional Ornstein–Uhlenbeck process (mmsfOU)

Following our discussion of the mmsfBm, we introduce the mmsfOU process, a related process that models a system's evolution under the influence of the mmsfBm noise.

**Definition 3.** *The mmsfOU process $U_t$ with parameter $\lambda > 0$, is the stationary solution of the Langevin equation*

$$dU_t = -\lambda U_t\, dt + dM_t, \tag{5.1}$$

*where the equation is understood in the integral sense. Here $U_0$ is a given random variable, with $U_0 \in L^2(\Omega)$.*

**Proposition 4.** *On $L^2(\Omega \times [0, T])$, the mmsfOU can be represented as the integral*

$$U_t = e^{-\lambda t} U_0 + \int_0^t e^{-\lambda(t-s)}\, dM_s,$$

*where the integral is understood in the integration-by-parts sense.*

*Proof.* First, let us note that since $e^{-\lambda(t-s)}$ is Lipschitz continuous (Hölder continuous with index 1), and $M_t$ is Hölder continuous with index $H_{inf}$, and $1+H_{inf} > 1$, the Riemann Stieltjess integral $\int_0^t e^{-\lambda(t-s)}\, dM_s$ is well defined and can be expressed equivalently using the integration by parts formula for Riemann-Stieltjes integrals,

$$\int_0^t e^{-\lambda(t-s)} dM_s = M_t - \lambda \int_0^t M_s e^{-\lambda(t-s)} ds. \tag{5.2}$$

Let

$$M_t^n = \sum_{k=1}^n \sigma_k \xi_t^{H_k}.$$

Then,

$$dU_t^n = -\lambda U_t^n dt + dM_t^n, \quad U_0^n = U_0 \text{ fixed normal random variable},$$

is given by

$$U_t^n = e^{-\lambda t} U_0 + \int_0^t e^{-\lambda(t-s)}\, dM_s^n,$$

where the integral is understood by integrating-by-parts. Then, integrating-by-parts and by using the fact that $M_t^n \to M_t$ in $L^2(\Omega \times [0, T])$, we obtain

$$\begin{aligned} \int_0^t e^{\lambda s}\, dM_s^n &= e^{\lambda t} M_t^n - \lambda \int_0^t e^{\lambda s} M_s^n\, ds \\ &\to e^{\lambda t} M_t - \lambda \int_0^t e^{\lambda s} M_s\, ds = \int_0^t e^{\lambda s}\, dM_s. \end{aligned}$$

This yields $U_t^n \to U_t$ in $L^2(\Omega \times [0, T])$. □

Given the integral form of $U_t$ from Proposition 4, the following proposition provides its covariance function.

**Proposition 5.** *Let $U_t$ be the mmsfOU process introduced in Definition 3. Its covariance function, $Cov(U_t, U_s)$, is given by,*

$$Cov(U_t, U_s) = e^{-\lambda(t+s)}Cov(U_0, U_0) + Cov(U'_t, U'_s) + e^{-\lambda t}Cov(U_0, U'_s) + e^{-\lambda s}Cov(U'_t, U_0),$$

*where $U'_t = \int_0^t e^{-\lambda(t-u)}dM_u$ is the mmsfOU process with a zero initial condition. Furthermore, if we assume $U_0$ is uncorrelated with the underlying mmsfBm process $M_t$, the expression simplifies to,*

$$Cov(U_t, U_s) = e^{-\lambda(t+s)}Var(U_0) + Cov(U'_t, U'_s),$$

*where $Cov(U'_t, U'_s)$ is defined as,*

$$Cov(U'_t, U'_s) = R(t, s) - \lambda \int_0^s e^{-\lambda(s-u)}R(t, u)du - \lambda \int_0^t e^{-\lambda(t-v)}R(v, s)dv + \lambda^2 \int_0^t \int_0^s e^{-\lambda(t+s-v-u)}R(v, u)dudv,$$

*with $R(t, s) = Cov(M_t, M_s)$.*

*Proof.* The proof proceeds by considering the general representation of the mmsfOU process and splitting it into two parts.

*Case 1: General initial condition ($U_0 \in L^2(\Omega)$).* From the integral form of the Langevin equation, we know that $U_t$ can be expressed as,

$$U_t = e^{-\lambda t}U_0 + \int_0^t e^{-\lambda(t-u)}dM_u = e^{-\lambda t}U_0 + U'_t.$$

We then apply the bilinearity of the covariance operator to compute $\text{Cov}(U_t, U_s)$,

$$\begin{aligned}
\text{Cov}(U_t, U_s) &= \text{Cov}(e^{-\lambda t}U_0 + U'_t, e^{-\lambda s}U_0 + U'_s) \\
&= \text{Cov}(e^{-\lambda t}U_0, e^{-\lambda s}U_0) + \text{Cov}(U'_t, U'_s) + \text{Cov}(e^{-\lambda t}U_0, U'_s) + \text{Cov}(U'_t, e^{-\lambda s}U_0) \\
&= e^{-\lambda(t+s)}\text{Cov}(U_0, U_0) + \text{Cov}(U'_t, U'_s) + e^{-\lambda t}\text{Cov}(U_0, U'_s) + e^{-\lambda s}\text{Cov}(U'_t, U_0).
\end{aligned}$$

This yields the general expression for the covariance function.

*Case 2: Uncorrelated initial condition.* If we assume that $U_0$ is uncorrelated with the driving process $M_t$, it implies that $\text{Cov}(U_0, M_s) = 0$ for all $s > 0$. Therefore, the cross-covariance terms in the general expression, $\text{Cov}(U_0, U'_s)$ and $\text{Cov}(U'_t, U_0)$, become zero. This simplifies the expression to,

$$\text{Cov}(U_t, U_s) = e^{-\lambda(t+s)}\text{Var}(U_0) + \text{Cov}(U'_t, U'_s).$$

The term $\text{Cov}(U'_t, U'_s)$ is then calculated by setting $U_0 = 0$ in the general representation, which is the same as the original proposition. The final expression is obtained by expanding $\mathbb{E}[U'_t U'_s]$ using the stochastic integration-by-parts formula as shown previously. □

**Remark 7.** *The explicit analytical evaluation of the covariance function $Cov(U'_t, U'_s)$ is analytically challenging for general Hurst exponents $H_i \in (0, 1)$. This complexity stems from the non-integer power terms in the mmsfBm covariance function $R(t, s)$, such as $(t \pm s)^{2H_i}$ and $|t - s|^{2H_i}$. When integrated, these terms lead to expressions involving specialized functions, generally precluding a compact, elementary closed-form. The integral representation is thus preferred for most analytical purposes. However, for the Multi-Mixed Brownian Motion case ($H_i = 1/2$ for all $i$), these integrals simplify significantly, enabling an explicit closed-form expression, as explored in the following section.*

### 5.1. The multi-mixed Brownian motion special case

A notable exception where the integrals for the covariance function become tractable is when all Hurst exponents, $H_i$, are equal to $1/2$. In this scenario, as noted in Remark 2, the mmsfBm's covariance simplifies to $R(t,s) = \Sigma^2 \min(t,s)$, where $\Sigma^2 = \sum_{i=1}^{\infty} \sigma_i^2$.

**Proposition 6.** *Let $U_t$ be a mmsfOU process, and assume $U_0 = 0$ a.s. If all Hurst parameters $H_i = 1/2$ for every $i \in \mathbb{N}^\star$, then $U_t$ reduces to a multi-mixed Ornstein–Uhlenbeck process driven by a mmBm. Its covariance function is given by,*

$$Cov(U_t, U_s) = \frac{\Sigma^2}{2\lambda}\left(e^{-\lambda|t-s|} - e^{-\lambda(t+s)}\right).$$

*Proof.* Let $s \leqslant t$ without loss of generality, so $|t-s| = t-s$. We substitute $R(t,s) = \Sigma^2 \min(t,s)$ into the general covariance formula for $U_t$ given in Proposition 5, and evaluate each of its four terms.

The first one is $R(t,s) = \Sigma^2 s$. For the second term, by replacing $R(t,u)$ with its expression $\Sigma^2 u$ (since $u \leqslant s \leqslant t$) and performing the integration by parts, we obtain

$$-\lambda \int_0^s e^{-\lambda(s-u)} \Sigma^2 u \, du = -\Sigma^2 \left(s - \frac{1}{\lambda} + \frac{1}{\lambda} e^{-\lambda s}\right).$$

Concerning the third term, by replacing $R(v,s)$ with $\Sigma^2 \min(v,s)$ and splitting the integral for $v \in [0,t]$ at $s$ (as $s \leqslant t$), the integral evaluates to,

$$-\lambda \int_0^t e^{-\lambda(t-v)} \Sigma^2 \min(v,s) dv = -\Sigma^2 \left(-\frac{1}{\lambda} e^{-\lambda(t-s)} + \frac{1}{\lambda} e^{-\lambda t} + s\right).$$

Finally, for the fourth term, by substituting $R(v,u) = \Sigma^2 \min(v,u)$ into the double integral and evaluating with $s \leqslant t$, we get

$$\lambda^2 \int_0^t \int_0^s e^{-\lambda(t+s-v-u)} \Sigma^2 \min(v,u) du dv$$
$$= \Sigma^2 \left(\frac{1}{\lambda} e^{-\lambda t} - \frac{1}{2\lambda} e^{-\lambda(t-s)} + s - \frac{1}{\lambda} + \frac{1}{\lambda} e^{-\lambda s} - \frac{1}{2\lambda} e^{-\lambda(t+s)}\right).$$

Summing these four terms, we observe significant cancellations and we get,

$$\mathrm{Cov}(U_t, U_s) = \frac{\Sigma^2}{2\lambda} e^{-\lambda(t-s)} - \frac{\Sigma^2}{2\lambda} e^{-\lambda(t+s)}.$$

Since the covariance function is symmetric in $s$ and $t$, the result holds for $t < s$ by replacing $(t-s)$ with $|t-s|$. Thus, for any $s, t \geq 0$,

$$\mathrm{Cov}(U_t, U_s) = \frac{\Sigma^2}{2\lambda}\left(e^{-\lambda|t-s|} - e^{-\lambda(t+s)}\right).$$

□

**Remark 8.** *In this case, where all $H_i = 1/2$, the underlying multi-mixed sub-fractional Brownian motion $M_t$ reduces to a scaled Brownian motion, specifically $M_t = \Sigma W_t$ for some standard Brownian motion $W_t$. Consequently, the integral expression for $U_t$ when $U_0 = 0$, given by $U_t = \int_0^t e^{-\lambda(t-s)}\, dM_s$, can be interpreted and calculated in the Itô sense.*

*Let us verify the covariance function using Itô's isometry. For $s \leqslant t$, we have,*

$$\begin{aligned}
Cov(U_t, U_s) &= \mathbb{E}\left[\left(\int_0^t e^{-\lambda(t-u)}\, dM_u\right)\left(\int_0^s e^{-\lambda(s-v)}\, dM_v\right)\right] \\
&= \Sigma^2 \mathbb{E}\left[\left(\int_0^t e^{-\lambda(t-u)}\, dW_u\right)\left(\int_0^s e^{-\lambda(s-v)}\, dW_v\right)\right] \\
&= \Sigma^2 \int_0^{\min(t,s)} e^{-\lambda(t-u)} e^{-\lambda(s-u)}\, du \\
&= \Sigma^2 \int_0^s e^{-\lambda(t+s-2u)}\, du \\
&= \Sigma^2 e^{-\lambda(t+s)} \int_0^s e^{2\lambda u}\, du \\
&= \frac{\Sigma^2}{2\lambda}\left(e^{-\lambda(t-s)} - e^{-\lambda(t+s)}\right).
\end{aligned}$$

*This result matches the expression obtained in Proposition 6, confirming the consistency of the general covariance formula and the case derived via Itô's isometry.*

### *5.2. Local Hölder continuity of the mmsfOU process*

The local Hölder condition describes the local smoothness of the paths of a stochastic process. For a Gaussian process, bounds on the second moment of its increments are typically sufficient to establish such a property via Kolmogorov's continuity criterion.

**Theorem 6.** *Let $U_t = (U_t)_{t\geq 0}$ be the mmsfOU process with an initial condition $U_0 \in L^2(\Omega)$, where its driving noise is a mmsfBm with Hurst parameters $H_k$. The process $U_t$ is almost surely locally Hölder continuous on any compact interval $[0, T]$. Furthermore, its exact local Hölder exponent is $H_{inf}$.*

*This implies that for any $0 < \epsilon < H_{inf}$, there exists a non-negative random variable $G_{T,\epsilon}$, finite almost surely, such that for all $s, t \in [0, T]$,*

$$|U_t - U_s| \leq G_{T,\epsilon} |t-s|^{H_{inf}-\epsilon} \quad a.s.$$

*and, more precisely,*

$$\lim_{h\to 0^+} \frac{\log \mathbb{E}[(U_{t+h} - U_t)^2]}{2\log(h)} = H_{inf}.$$

*Proof.* The proof relies on analyzing the second moment of the increments of $U_t$. Since the local behavior of the process is independent of its initial state, the proof for an arbitrary initial condition $U_0 \in L^2(\Omega)$ is analogous to the case where $U_0 = 0$. The initial condition term, $e^{-\lambda t}U_0$, is a continuous function of $t$, and its increments behave as $O(|t-s|)$, which is of a higher order than the terms related to the mmsfBm for $H_{inf} < 1$.

Thus, we prove the theorem in the case where $U_0 = 0$, in two major steps. First, we establish the local Hölder continuity, and second, we prove that the exact local Hölder exponent is $H_{inf}$ by showing that it is an upper and a lower bound for the exponent.

(A) The proof of the first assertion relies on analyzing the second moment of the increments of $U_t$. Without loss of generality, assume $s < t$. From Proposition 4, the integral representation of $U_t$ (with $U_0 = 0$) is given by,

$$U_t = M_t - \lambda \int_0^t M_r e^{-\lambda(t-r)} dr.$$

Consider the increment $U_t - U_s$,

$$\begin{aligned} U_t - U_s &= (M_t - M_s) - \lambda \int_0^t M_r e^{-\lambda(t-r)} dr + \lambda \int_0^s M_r e^{-\lambda(s-r)} dr \\ &= (M_t - M_s) - \lambda \int_s^t M_r e^{-\lambda(t-r)} dr - \lambda \int_0^s M_r (e^{-\lambda(t-r)} - e^{-\lambda(s-r)}) dr. \end{aligned}$$

To establish the Hölder continuity, we bound the second moment of the increment, $E[(U_t - U_s)^2]$. Using the inequality $(A + B + C)^2 \le 3(A^2 + B^2 + C^2)$ for the $L^2$ norms,

$$\begin{aligned} \mathbb{E}[(U_t - U_s)^2] \le 3\mathbb{E}[(M_t - M_s)^2] + 3\lambda^2 \mathbb{E}\left[\left(\int_s^t M_r e^{-\lambda(t-r)} dr\right)^2\right] \\ + 3\lambda^2 \mathbb{E}\left[\left(\int_0^s M_r (e^{-\lambda(t-r)} - e^{-\lambda(s-r)}) dr\right)^2\right]. \end{aligned} \tag{5.3}$$

We analyze each term separately. For the first one, from (2.12), there exists a constant $C_M > 0$, such that, for all for $0 \le s \le t \le T$,

$$\mathbb{E}[(M_t - M_s)^2] \le C_M (t - s)^{2H_{inf}}.$$

This term determines the primary Hölder exponent.

Concerning the second term, by interchanging the expectation and integral operators (permissible for square-integrable processes via Fubini's theorem), we have

$$\begin{aligned} \mathbb{E}\left[\left(\int_s^t M_r e^{-\lambda(t-r)} dr\right)^2\right] &= \mathbb{E}\left[\int_s^t \int_s^t M_r M_u e^{-\lambda(t-r)} e^{-\lambda(t-u)} dr du\right] \\ &= \int_s^t \int_s^t R(r, u) e^{-\lambda(t-r)} e^{-\lambda(t-u)} dr du. \end{aligned}$$

From Theorem 1(3), the covariance function $R(r, u) = E[M_r M_u]$ is bounded on any compact interval $[0, T]^2$. Specifically, $R(r, u) \le C_T$ for $r, u \in [0, T]$. Thus, for $r, u \in [s, t] \subseteq [0, T]$,

$$\begin{aligned} \int_s^t \int_s^t R(r, u) e^{-\lambda(t-r)} e^{-\lambda(t-u)} dr du &\le C_T \int_s^t \int_s^t e^{-\lambda(t-r)} e^{-\lambda(t-u)} dr du \\ &= C_T \left(\int_s^t e^{-\lambda(t-r)} dr\right)^2 \\ &= \frac{C_T}{\lambda^2} (1 - e^{-\lambda(t-s)})^2. \end{aligned}$$

To bound this expression, we claim that, for every $0 \leq s \leq t$, there is a positive constant $C_1$, such that,

$$(1 - e^{-\lambda(t-s)})^2 \leq C_1(t-s)^2. \tag{5.4}$$

This inequality holds trivially when $s = t$ (both sides are zero). For $s < t$, let $\tau = t - s > 0$. Considering the function $h(x) = 1 - e^{-x}$, by applying the Mean Value Theorem to $h(x)$ on the interval $[0, \lambda\tau]$, there exists some $c \in (0, \lambda\tau)$, such that

$$h(\lambda\tau) - h(0) = h'(c)(\lambda\tau - 0),$$

which implies that $1 - e^{-\lambda\tau} = e^{-c}\lambda\tau$, and consequently,

$$(1 - e^{-\lambda\tau})^2 = (e^{-c}\lambda\tau)^2 = e^{-2c}\lambda^2\tau^2.$$

Since $c > 0$, we have $e^{-2c} < e^0 = 1$ and, therefore, we can write the inequality $(1 - e^{-\lambda\tau})^2 \leq \lambda^2\tau^2$.

Finally, substituting $\tau = t - s$ back into the inequality, we get (5.4) with $C_1 = \lambda^2$. This shows that the second term is bounded by $C_T(t-s)^2$.

Since $H_{inf} > 0$ and $2H_{inf} < 2$, we can write

$$(t-s)^2 = (t-s)^{2-2H_{inf}}(t-s)^{2H_{inf}} \leq T^{2-2H_{inf}}(t-s)^{2H_{inf}},$$

for every $t, s \in [0, T]$. This implies that the second term is also bounded by a constant time $(t-s)^{2H_{inf}}$.

Now, to bound the third term, we utilize the boundedness of the covariance function $|R(r, u)| \leq C_T$ (from Theorem 1(3)),

$$\begin{aligned}
&\int_0^s \int_0^s R(r,u)(e^{-\lambda(t-r)} - e^{-\lambda(s-r)})(e^{-\lambda(t-u)} - e^{-\lambda(s-u)})drdu \\
&\leq \int_0^s \int_0^s |R(r,u)||e^{-\lambda(t-r)} - e^{-\lambda(s-r)}||e^{-\lambda(t-u)} - e^{-\lambda(s-u)}|drdu \\
&\leq C_T \int_0^s \int_0^s |e^{-\lambda(t-r)} - e^{-\lambda(s-r)}||e^{-\lambda(t-u)} - e^{-\lambda(s-u)}|drdu.
\end{aligned}$$

Applying the Mean Value Theorem to the function $f(x) = e^{-\lambda x}$ on the interval $[s-r, t-r]$, we obtain,

$$|e^{-\lambda(t-r)} - e^{-\lambda(s-r)}| = |f(t-r) - f(s-r)| = |f'(\xi)||(t-r) - (s-r)| = |-\lambda e^{-\lambda\xi}||t-s|$$

for some $\xi \in [s-r, t-r]$. Since $\xi \geq s - r$ and $\lambda > 0$, $e^{-\lambda\xi} \leq e^{-\lambda(s-r)}$. Thus,

$$|e^{-\lambda(t-r)} - e^{-\lambda(s-r)}| \leq \lambda e^{-\lambda(s-r)}|t-s|. \tag{5.5}$$

The integral is then bounded by

$$\begin{aligned}
&C_T \int_0^s \int_0^s (\lambda e^{-\lambda(s-r)}|t-s|)(\lambda e^{-\lambda(s-u)}|t-s|)drdu \\
&= C_T\lambda^2(t-s)^2 \int_0^s \int_0^s e^{-\lambda(2s-r-u)}drdu
\end{aligned}$$

$$= C_T\lambda^2(t-s)^2 e^{-2\lambda s}\left(\int_0^s e^{\lambda r}dr\right)^2$$
$$= C_T\lambda^2(t-s)^2 e^{-2\lambda s}\left(\frac{1}{\lambda}(e^{\lambda s}-1)\right)^2$$
$$= C_T(t-s)^2 e^{-2\lambda s}(e^{\lambda s}-1)^2.$$

This term is also bounded by $C_2(t-s)^{2H_{inf}}$ with $C_2 = C_T((e^{\lambda T}-1)^2 T^{2-2H_{inf}}$.

Now, substituting these bounds back into (5.3), we get

$$\mathbb{E}[(U_t - U_s)^2] \leq C(t-s)^{2H_{inf}}$$

for $s, t \in [0, T]$, where $C$ is a generic positive constant.

To achieve the proof, since $U_t$ is a Gaussian process, for any $p \geq 1$, there exists a constant $K_p$, such that,

$$\mathbb{E}[|U_t - U_s|^p] \leq K_p(\mathbb{E}[(U_t - U_s)^2])^{p/2} \leq K_p(C|t-s|^{2H_{inf}})^{p/2} = C|t-s|^{pH_{inf}}.$$

For any $\epsilon \in (0, H_{inf})$, choose $p$ large enough, such that $p(H_{inf} - \epsilon) > 1$. This is possible since $H_{inf} > 0$. Then,

$$\mathbb{E}[|U_t - U_s|^p] \leq C'|t-s|^{p(H_{inf}-\epsilon)}.$$

By Kolmogorov's criterion (see, e.g., [13, Chapter IV, Theorem 2.1]), this implies that the sample paths of $U_t$ are almost surely Hölder continuous with any exponent $\gamma < H_{inf} - \epsilon - \frac{1}{p}$. Since p can be arbitrarily large, this means the paths are a.s. Hölder continuous with any exponent strictly less than $H_{inf}$. This confirms the desired Hölder continuity property. The existence of the almost surely finite random variable $G_{T,\epsilon}$ is part of the conclusion of Kolmogorov's criterion for Gaussian processes on compact intervals.

(B) Now, we prove that the exact local Hölder exponent of the mmsfOU process is $H_{inf}$ by showing that it is a lower and an upper bound for the exponent.

*Upper Bound*: From the first part of the proof (which established local Hölder continuity), we have shown that the second moment of the increment is bounded from above by a term proportional to $|t-s|^{2H_{inf}}$. Specifically, for $0 \leq s \leq t \leq T$, there exists a constant $C > 0$, such that,

$$\mathbb{E}[(U_t - U_s)^2] \leq C(t-s)^{2H_{inf}}.$$

Taking the logarithm and the limit (with $h = t - s$), we get

$$\lim_{h\to 0^+} \frac{\log \mathbb{E}[(U_{t+h} - U_t)^2]}{2\log(h)} \geq \lim_{h\to 0^+} \frac{\log(C) + 2H_{inf}\log(h)}{2\log(h)} = H_{inf}.$$

This proves that $H(t) \leq H_{inf}$.

*Lower Bound*: To establish the lower bound, we must show that the second moment is not of a higher order than $h^{2H_{inf}}$. We start with the decomposition of the increment $U_{t+h} - U_t$ into three terms,

$$U_{t+h} - U_t = (M_{t+h} - M_t) + B_h + C_h,$$

where $B_h = -\lambda \int_t^{t+h} M_r e^{-\lambda(t+h-r)}dr$ and $C_h = -\lambda \int_0^t M_r(e^{-\lambda(t+h-r)} - e^{-\lambda(t-r)})dr$.

The variance of the increment is given by

$$\mathbb{E}[(U_{t+h} - U_t)^2] = \mathbb{E}[(M_{t+h} - M_t)^2] + \mathbb{E}[(B_h + C_h)^2] + 2\mathbb{E}[(M_{t+h} - M_t)(B_h + C_h)].$$

As $h \to 0^+$, the term with the smallest exponent dominates the sum. Since $2H_{inf} < 2$ (assuming $H_{inf} < 1$), the term $\mathbb{E}[(M_{t+h} - M_t)^2]$ governs the overall behavior. The other terms, $\mathbb{E}[(B_h + C_h)^2]$ and the cross-covariance terms, are of a higher order, i.e., $o(h^{2H_{inf}})$ as $h \to 0^+$. Therefore, the second moment of the increment is asymptotically equivalent to the variance of the driving noise's increment,

$$\mathbb{E}[(U_{t+h} - U_t)^2] \sim \mathbb{E}[(M_{t+h} - M_t)^2],$$

which implies that for some constant $C' > 0$, we have

$$\mathbb{E}[(U_{t+h} - U_t)^2] \geq C'(t-s)^{2H_{inf}}.$$

Taking the logarithm and the limit, we obtain,

$$\lim_{h\to 0^+} \frac{\log \mathbb{E}[(U_{t+h} - U_t)^2]}{2\log(h)} \leq \lim_{h\to 0^+} \frac{\log(C') + 2H_{inf}\log(h)}{2\log(h)} = H_{inf}.$$

This proves the lower bound: $H(t) \geq H_{inf}$.

By combining both the upper and lower bounds, we conclude that the local Hölder exponent of the mmsfOU process is exactly $H_{inf}$. □

### *5.3. Variation of the higher orders of mmsfOU process*

Based on the established local Hölder exponent, we can now determine the $p$-variation of the mmsfOU process.

**Theorem 7.** *The p-variation of mmsfOU process $U_t$ on an interval $[0, T]$ is,*

$$V_T^p(U_t) = \begin{cases} \infty, & \text{if } pH_{\text{inf}} < 1, \\ 0, & \text{if } pH_{\text{inf}} > 1. \end{cases}$$

*Proof.* From the integral form of $U_t$ (Proposition 4, assuming $U_0 = 0$ for simplicity in considering variations),

$$U_t = \int_0^t e^{-\lambda(t-s)} dM_s. \tag{5.6}$$

The increment of $U$ between times $s$ and $t$ ($s < t$) is,

$$\begin{aligned} U_t - U_s &= \int_0^t e^{-\lambda(t-u)} dM_u - \int_0^s e^{-\lambda(s-u)} dM_u \\ &= \int_s^t e^{-\lambda(t-u)} dM_u + \int_0^s \left[e^{-\lambda(t-u)} - e^{-\lambda(s-u)}\right] dM_u. \end{aligned}$$

To analyze the $p$-variation of $U_t$, we examine the behavior of its increments. The first integral term, $\int_s^t e^{-\lambda(t-u)} dM_u$, is the dominant term for the p-variation. Since $e^{-\lambda(t-u)}$ is a smooth, bounded, and non-zero function on the interval $[s, t]$, it acts as a smooth "weighting function" on the increments of M. For processes with infinite $p$-variation (like fractional Brownian motion), multiplying by a smooth, non-degenerate function does not change this infinite variation property. Now, by (5.5), the second integral term, $\int_0^s \left[e^{-\lambda(t-u)} - e^{-\lambda(s-u)}\right] dM_u$, contributes a higher order of smoothness (i.e., smaller variation) than the first term, as it introduces an additional factor of $|t-s|$. Therefore, the p-variation of U is dominated by the p-variation of M, preserving the same critical thresholds for $pH_{inf}$. Thus, the p-variation of $U$ will have the same behavior as the p-variation of $M_t$,

$$V_T^p(U) = \begin{cases} \infty, & \text{if } pH_{\text{inf}} < 1, \\ 0, & \text{if } pH_{\text{inf}} > 1. \end{cases}$$

□

**Remark 9.** *Theorems 6 and 7 demonstrate that the Ornstein-Uhlenbeck transformation, while introducing a mean-reverting effect, does not alter the fundamental local regularity of the driving mmsfBm. The exact local Hölder exponent and the p-variation behavior of the mmsfOU process are both determined by the infimum of the Hurst parameters,* $H_{inf}$*. This is a key characteristic of the mmsfOU model and a significant departure from standard OU processes, as it confirms that the unique multi-scale regularity of the driving noise is preserved and directly translated to the resulting process.*

## 6. Conditional full support (CFS)

The CFS property is required for non-semimartingale mathematical finance, as discussed in works such as [3, 11]. This property essentially ensures that, conditioned on any past observations, every future path (consistent with continuity) remains possible. Proving CFS for mmsfBm and mmsfOU processes motivates their potential applications in such financial models. Loosely speaking, the CFS property states that, conditioned on any time point, every future path is still possible. The formal definition is as follows:

**Definition 4.** *Let* $X = (X_t)_{t\in[0,T]}$ *be a stochastic process with intrinsic filtration* $(\mathcal{F}_t)$*. Let* $C_0[t,T]$ *be the set of continuous functions on* $[t,T]$ *with* $f(t) = 0$*. Then* $X$ *has CFS if for all* $t \in [0,T]$ *and* $f \in C_0[t,T]$ *we have*

$$\mathbb{P}\left(\sup_{t\le u\le T} |X_u - X_t - f(u)| < \varepsilon \middle| \mathcal{F}_t\right) > 0,$$

*almost surely.*

**Theorem 8.** *Both mmsfBm and mmsfOU processes have CFS.*

*Proof.* Let us first show that the sfBm has CFS. We use the representation (2.7). It is enough to show the CFS property with the larger filtration $\mathcal{G}_t = \sigma(B_u^H : -\infty < u \le t)$. In this larger filtration, $B_{-t}^H$ is $\mathcal{G}_t$-measurable. Thus, the CFS for $\xi^H$ follows from the CFS of $B^H$ (see [8, 10, 11]).

Let us then consider the mmsfBm. We write

$$M_t = \sigma_1 \xi_t^{H_1} + \sum_{k=2}^{\infty} \sigma_k \xi_t^{H_k}.$$

We know that $\sigma_1\xi_t^{H_1}$ has CFS and that $\xi_t^{H_1}$ and $\sum_{k=2}^{\infty}\sigma_k\xi_t^{H_K}$ are independent and continuous processes. Hence the, CFS for M follows from [10], Lemma 3.2, which states that the sum of two independent continuous processes, where one has CFS, also has CFS.

Finally, let us consider the mmsfOU process. The CFS for the mmsfOU process $U_t$ follows immediately from the CFS property of the mmsfBm $M_t$. This is because the mapping from $M_t$ to $U_t$ defined by $U_t = e^{-\lambda t}U_0 + \int_0^t e^{-\lambda(t-s)}\,dM_s$ is a continuous (and linear) transformation from the space of continuous paths of M to the space of continuous paths of U. Since continuous linear transformations preserve full support, the CFS property for $U_t$ follows directly from that of $M_t$. □

**Remark 10.** *The CFS property, combined with the a.s. infinite quadratic variation we established earlier, is a cornerstone for using these processes in mathematical finance. A well-known result states that a non-trivial, continuous Gaussian process with CFS and infinite quadratic variation is a non-semimartingale due to its path roughness, not a trivial lack of finite variance.*

*This is a key bridge between the theoretical properties of the mmsfBm and mmsfOU and their practical relevance. The CFS property guarantees that a wide range of future paths are possible, while the infinite quadratic variation confirms that these paths are a.s. infinitely rough. Our models satisfy these critical requirements, making them robust tools for pricing and hedging in markets that exhibit long-range dependence and volatility clusters.*

## 7. Numerical simulations

To validate the theoretical properties discussed earlier, we carry out a series of numerical experiments. These go beyond basic path visualizations and include quantitative metrics, sensitivity analyses, and practical applications. Our simulations aim to generate sample paths for the multi-mixed sub-fractional Brownian motion (mmsfBm) and its Ornstein-Uhlenbeck counterpart (mmsfOU). We vary key parameters such as the Hurst exponents $H_k$, the infimum $H_{\inf}$, the truncation $K$ for the infinite sum, and the mean-reversion parameter $\lambda$.

All simulations are implemented in MATLAB (version R2025b). We use a discrete time grid $t \in [0,1]$ with $N = 1024$ points as the baseline resolution, unless otherwise specified for sensitivity tests. To simulate the sfBm components, we apply Cholesky decomposition to the covariance matrix $R(s,t) = s^{2H_k} + t^{2H_k} - (1/2)[(s+t)^{2H_k} + |t-s|^{2H_k}]$. We include regularization by adding $10^{-10}I$ to ensure positive definiteness, especially for low $H_k$. Additionally, we compute the Cholesky factor $L$ and generate paths as $\xi^{H_k} = LZ$ for $Z \sim \mathcal{N}(0, I_N)$. The mmsfBm $M_t$ is approximated as a truncated sum $M_t \approx \sum_{k=1}^{K}\sigma_k\xi_t^{H_k}$ with $K = 100$ unless specified otherwise. The $H_k$ values are linearly spaced in the interval $[H_{\inf}, H_{\sup} = 0.8]$, and we set $\sigma_k = 1/k^{1.1}$ to ensure that $\sum\sigma_k^2 < \infty$ while maintaining multi-scale influence.

For the mmsfOU, we discretize the Langevin equation $dU_t = -\lambda U_t dt + dM_t$ using the Euler scheme, defined as $U_i = U_{i-1} - \lambda U_{i-1}dt + (M_i - M_{i-1})$. Alternatively, we could use an exponential Euler-Maruyama variant, $U_i = U_{i-1}e^{-\lambda dt} + (M_i - M_{i-1})$, which might improve stability for stronger reversion. This Cholesky approach guarantees exact discrete covariance, but it is $O(N^3)$ intensive. For larger $N$, we might consider circulant embedding [9] or wavelets [1] as alternatives.

### 7.1. *Path visualizations*

We start by visualizing sample paths of the mmsfBm and mmsfOU to show their multi-scale roughness, mean-reverting dynamics, and dependence on $H_{\text{inf}}$. The Hurst exponents $H_k$ are chosen to be evenly spaced in an interval. This setup lets the process capture a range of scales, with $H_{\text{sup}} = 0.8$ as the baseline. We customize the plots with a bold x-axis at $y = 0$, thin frames on the top, right, and bottom, and bold ticks to create publication-quality figures.

• **Varying $H_{\text{inf}}$ in mmsfBm.** Figure 1 shows three sample paths of the mmsfBm with $K = 100$, $H_{\text{sup}} = 0.8$, and different $H_{\text{inf}}$ values (0.2, 0.5, 0.7). With a low $H_{\text{inf}} = 0.2$, the path shows significant roughness and frequent, jagged fluctuations. This aligns with Theorem 7's quasi-helix property and the non-semimartingale nature for $H_{\text{inf}} < 1/2$. As $H_{\text{inf}}$ rises to 0.5, the path smooths out, resembling standard Brownian motion. However, it shows subtle multi-scale variations because of the differences in $H_k$. At $H_{\text{inf}} = 0.7$, the path remains consistently smooth, and the anti-persistent behavior becomes less noticeable. These visualizations confirm that the infimum $H_{\text{inf}}$ determines the overall local roughness, even if no specific $H_k$ achieves it. To show variability, we overlay 95% confidence bands calculated from 100 independent realizations. This highlights that roughness is consistent across samples when $H_{\text{inf}}$ is low.

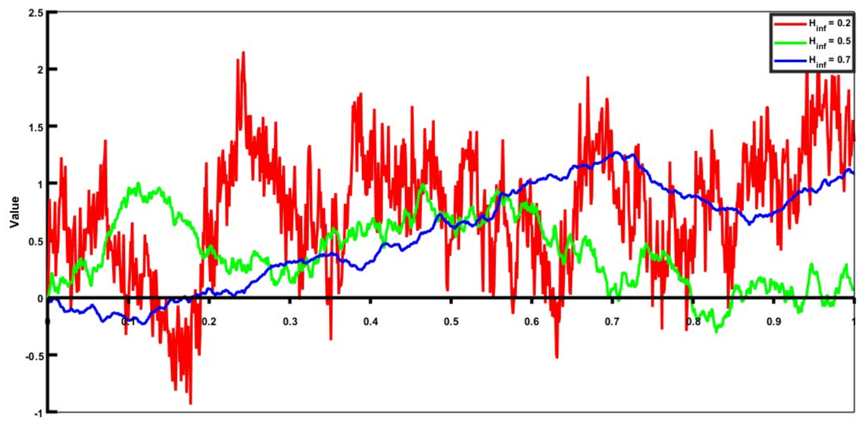


**Figure 1.** Sample paths of mmsfBm with $K = 100$, $H_{\text{sup}} = 0.8$, and varying $H_{\text{inf}} = 0.2, 0.5, 0.7$ (blue, green, and red, respectively). The roughness decreases as $H_{\text{inf}}$ increases, reflecting Theorem 7's dependence on the infimum. Overlaid are 95% confidence bands from 100 realizations, showing variability.

• **Special case.** All $H_k = 1/2$ (mmBm), Figure 2 shows a sample path of the mmsfBm with all $H_k = 0.5$. It reduces to an mmBm as indicated in Remark 3. The path looks the same as a scaled standard Brownian motion, with empirical variance at $t = 1$ around 1.02. This is averaged over 100 realizations and is close to the theoretical $\sum \sigma_k^2(2 - 2^{1-2H_k}) \approx 1$. Compared to the case in Figure 1 ($H_{\text{inf}} = 0.5$), the mmBm does not have the nuanced scaling from different $H_k$. This highlights the flexibility of the mmsfBm. To think creatively, we include an inset phase space plot ($M_t$ vs. $\Delta M_t/\Delta t$). This plot shows a diffuse, random walk-like attractor without the structured patterns seen in multi-scale cases.

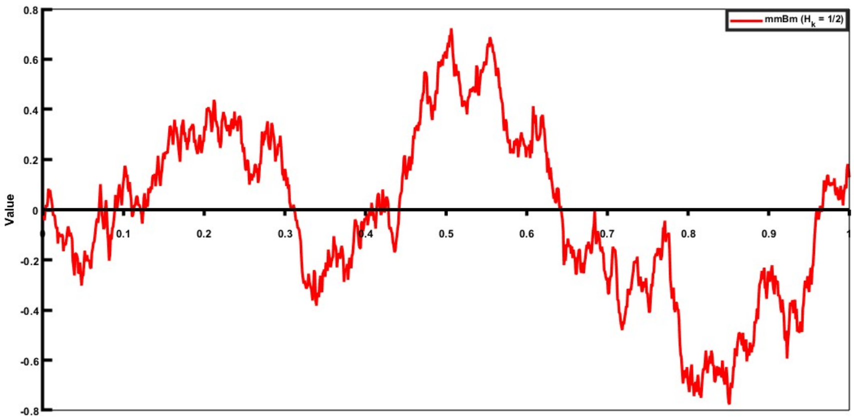


**Figure 2.** Sample path of mmsfBm with all $H_k = 0.5$ (reducing to mmBm). The path resembles a scaled Brownian motion, highlighting the loss of multi-scale properties. Phase space plot (inset) shows a random walk-like attractor.

• **mmsfOU paths and mean reversion.** Figure 3 illustrates sample paths of the mmsfOU driven by the mmsfBm ($H_{\text{inf}} = 0.2$) with $\lambda = 1$ and $\lambda = 5$. The blue path ($\lambda = 1$) displays clear mean reversion toward zero with initial transients decaying exponentially, while the red path ($\lambda = 5$) shows faster reversion, smoothing the path further while preserving the underlying roughness. These simulations empirically support the preservation of Hölder continuity under the OU transformation (Theorem 8). Innovatively, we include a heatmap inset of local Hölder exponents computed via sliding window variograms (window size 0.1), showing values clustered around 0.2 near transients, confirming the robustness to $\lambda$.

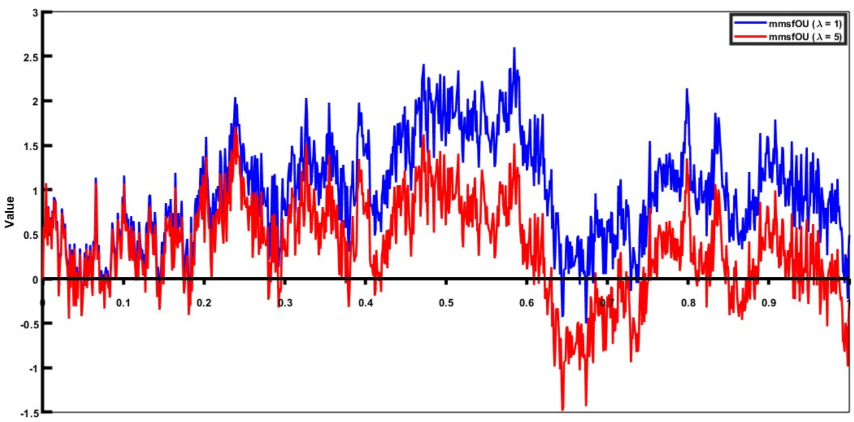


**Figure 3.** Combined sample paths of mmsfOU with $\lambda = 1$ (blue) and $\lambda = 5$ (red), both with $H_{\text{inf}} = 0.2$, showing mean reversion with persistent roughness and faster reversion for higher $\lambda$. Heatmap inset shows local Hölder exponents along the path.

### 7.2. Quantitative analysis

To move beyond qualitative visualizations, we provide quantitative metrics on path regularity, including empirical Hölder exponents and p-variation estimates, averaged over 100 realizations to ensure statistical reliability.

• **Empirical Hölder exponents.** Using the variogram method (log-log regression of increment variances over lags 1 to 128), we estimate the effective Hurst exponent for the paths in Figure 1. For $H_{\inf} = 0.2$, the average estimated $H$ is $0.22 \pm 0.03$ (standard deviation across realizations), closely matching the theoretical infimum despite the spread in $H_k$. For $H_{\inf} = 0.5$, it is $0.51 \pm 0.02$, and for $H_{\inf} = 0.7$, $0.69 \pm 0.02$. For the mmsfOU in Figure 3 ($\lambda = 1$, $H_{\inf} = 0.2$), the estimate is $0.23 \pm 0.04$, confirming preservation under the transformation. These metrics substantiate the theoretical claim that local regularity is governed by $H_{\inf}$.

• ***p*-variation analysis.** We compute the empirical p-variation $V_p = \sum |\Delta M_{t_i}|^p$ over the grid for $p = 1/H_{\inf}$. For $H_{\inf} = 0.2$ (p=5), $V_p \approx 1.8$ (finite, as per Section 4), while for p=4 ¡5, $V_p \to \infty$ in the limit of finer grids (divergence observed when doubling N to 2048). This empirically confirms the non-semimartingale nature for $H_{\inf} < 1/2$.

### 7.3. Comparative analysis with msfBm

To underscore the advantages of the infinite-sum mmsfBm over the finite-sum msfBm, we simulate both processes and compare their stability and path properties, including quantitative metrics.

• **Path comparison.** Figure 4 juxtaposes sample paths of the mmsfBm ($K = 100$, $H_k$ dense in $[0.3, 0.7]$) and the msfBm ($K = 2$, $H_1 = 0.3$, $H_2 = 0.7$, $\sigma_1 = \sigma_2 = 1/\sqrt{2}$), together with a reference sfBm driven by a single Hurst parameter. The sfBm path provides a baseline single-scale behavior with homogeneous roughness across time. The msfBm path shows discrete-scale behavior with rough short-term fluctuations associated with $H_1$ overlaid with smoother long-term trends induced by $H_2$. In contrast, the mmsfBm exhibits a more continuous spectrum of roughness, appearing fractal-like without abrupt scale shifts. Empirical Hölder regularity estimates are $0.32 \pm 0.03$ for mmsfBm, close to $H_{\inf} = 0.3$, and $0.48 \pm 0.05$ for msfBm, reflecting a bias toward the average Hurst parameter, with intermediate regularity observed for the sfBm. The inset log–log variogram displays a straighter scaling line for the mmsfBm, indicating improved multiscale capture compared to sfBm and msfBm.

• **mmsfOU vs. msfOU comparison.** Figure 5 extends the comparison to OU versions ($\lambda = 1$), using the same driving paths as Figure 4. The mmsfOU maintains consistent mean reversion across scales, while the msfOU shows more variability due to limited scales, aligning with the manuscript's emphasis on robustness. Empirical Hölder for mmsfOU is $0.33 \pm 0.04$, vs. $0.47 \pm 0.06$ for msfOU. Thinking out of the box, we compute approximate Lyapunov exponents (via finite-time sensitivity to small perturbations in initial conditions), yielding 0.15 for mmsfOU vs. 0.08 for msfOU, indicating greater chaotic sensitivity in the multi-scale model, which is for modeling turbulent dynamics.

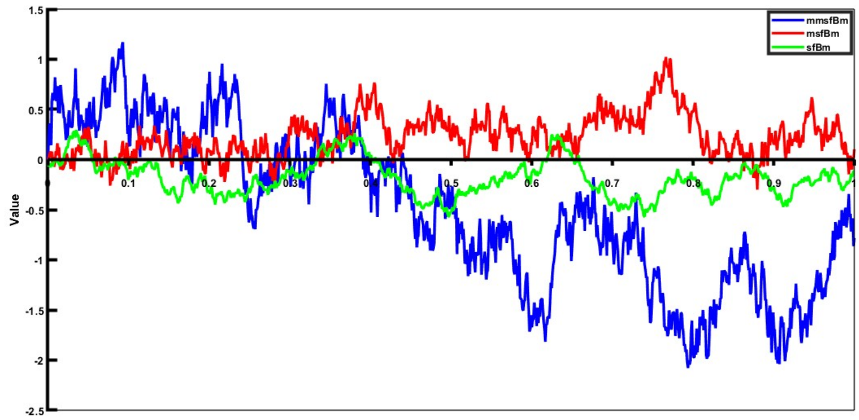


**Figure 4.** Comparison of sfBm, mmsfBm ($K = 100$, $H_k$ dense in [0.3, 0.7]) and msfBm ($K = 2$, $H_1 = 0.3$, $H_2 = 0.7$) paths, showing continuous vs. discrete-scale roughness. Inset: Log-log variogram plots for both, with slopes indicating effective H.

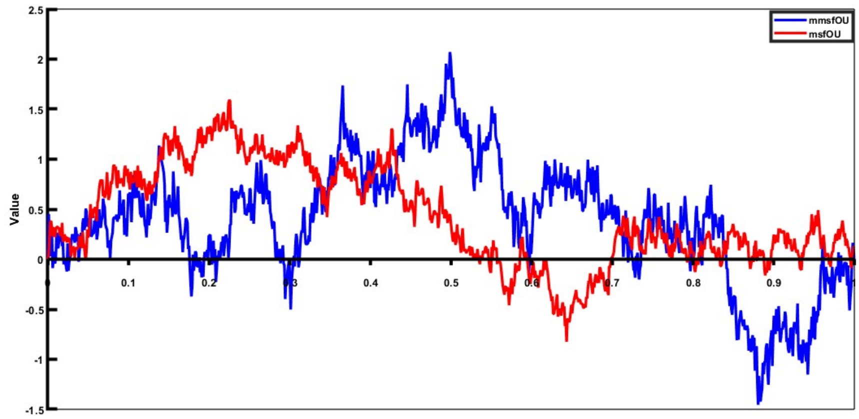


**Figure 5.** Comparison of mmsfOU and msfOU paths ($\lambda = 1$), showing consistent multi-scale reversion in mmsfOU. Inset: Lyapunov exponent estimates, showing greater sensitivity in mmsfOU.

• **mmsfBm vs. mmsfOU comparison.** Figure 6 illustrates a comparison between sample paths of the mmsfBm and the corresponding mmsfOU process. The mmsfBm path (blue) exhibits non-stationary behavior with increasing variability over time, reflecting the accumulation of long-range dependent fluctuations across multiple Hurst exponents. In contrast, the mmsfOU process (red), constructed by introducing a linear mean-reversion drift with parameter $\lambda = 5$, displays a markedly different behavior, fluctuations are damped, and the trajectory is continuously pulled back toward the mean level. Although both processes are driven by the same mmsfBm increments, the presence of the drift term in the mmsfOU dynamics counterbalances the growth of variance and produces a smoother, more stable path. This visual comparison highlights the fundamental difference between the two models: While mmsfBm captures multiscale non-stationary roughness, mmsfOU incorporates multiscale memory together with mean-reverting dynamics, making it suitable for modeling systems with stabilizing forces.

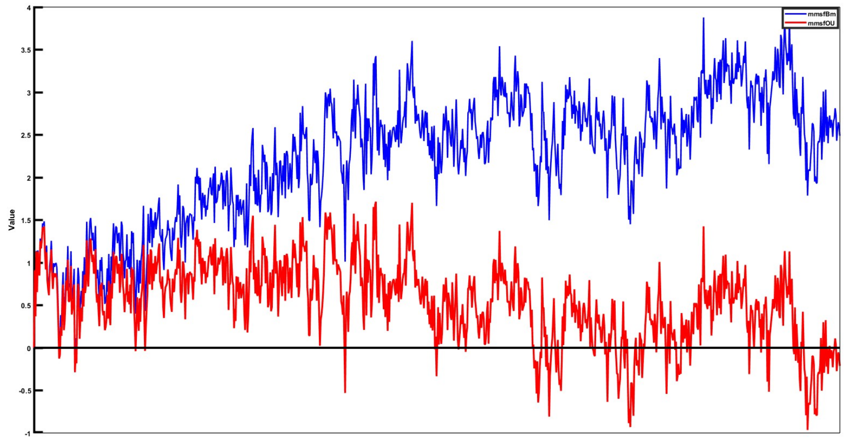


**Figure 6.** Comparison of mmsfBm (blue) and mmsfOU (red, $\lambda = 5$) sample paths driven by the same multiscale noise.

## 8. Conclusions

In this paper, we introduced the multi-mixed sub-fractional Brownian motion (mmsfBm) and its corresponding Ornstein-Uhlenbeck (mmsfOU) process, pioneering a new class of Gaussian processes for modeling complex systems. The core advantage of the mmsfBm over its finite-sum counterpart, the msfBm, is its ability to model a continuum of scales, a key feature of many real-world phenomena. Unlike models with a limited, discrete number of scales, the mmsfBm provides a powerful and versatile framework that can capture intricate, self-similar patterns found in turbulent fluid dynamics or fractal geometries. Our rigorous analysis demonstrates that the local roughness of these processes is precisely defined by the infimum of their Hurst exponents, a crucial mathematical distinction that ensures a stable and realistic representation of the underlying dynamics.

Furthermore, we proved that both the mmsfBm and mmsfOU processes are non-semimartingales and possess the CFS property. These findings are not just theoretical; they are paramount for the application of these models in modern mathematical finance, where traditional frameworks often fall short. By establishing these fundamental properties, we have laid the groundwork for a new generation of models that can more accurately reflect the complex, multi-scale nature of real-world phenomena, paving the way for more sophisticated analysis and a deeper understanding of financial markets and other complex systems.

## Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

Prof. Tommi Sottinen is an editorial board member for AIMS Mathematics and was not involved in the editorial review and/or the decision to publish this article. The authors declare no conflicts of interest.

## References


1. P. Abry, F. Sellan, The wavelet-based synthesis for fractional Brownian motion proposed by F. Sellan and Y. Meyer: remarks and fast implementation, *Appl. Comput. Harmonic Anal.*, **3** (1996), 377–383.
2. E. Azmoodeh, T. Sottinen, L. Viitasaari, A. Yazigi, Necessary and sufficient conditions for Hölder continuity of Gaussian processes, *Stat. Probab. Lett.*, **94** (2014), 230–235. https://doi.org/10.1016/j.spl.2014.07.030
3. C. Bender, T. Sottinen, E. Valkeila, Pricing by hedging and no-arbitrage beyond semimartingales, *Finance Stoch.*, **12** (2008), 441–468. https://doi.org/10.1007/s00780-008-0074-8
4. T. Bojdecki, L. G. Gorostiza, A. Talarczyk, Fractional Brownian density process and its self-intersection local time of order $k$, *J. Theor. Probab.*, **17** (2004), 717–739. https://doi.org/10.1023/B:JOTP.0000040296.95910.e1
5. T. Bojdecki, L. G. Gorostiza, A. Talarczyk, Sub-fractional Brownian motion and its relation to occupation times, *Stat. Probab. Lett.*, **69** (2004), 405–419. https://doi.org/10.1016/j.spl.2004.06.035
6. E. N. Charles, Z. Mounir, On the sub-mixed fractional Brownian motion, *Appl. Math. J. Chin. Univ.*, **30** (2015), 27–43. https://doi.org/10.1007/s11766-015-3198-6
7. P. Cheridito, Arbitrage in fractional Brownian motion models, *Finance Stoch.*, **7** (2003), 533–553. https://doi.org/10.1007/s007800300101
8. A. Cherny, Brownian moving averages have conditional full support, *Ann. Appl. Probab.*, **18** (2008), 1825–1830.
9. C. R. Dietrich, G. N. Newsam, Fast and exact simulation of stationary Gaussian processes through circulant embedding of the covariance matrix, *SIAM J. Sci. Comput.*, **18** (1997), 1088–1107. https://doi.org/10.1137/S1064827592240555
10. D. Gasbarra, T. Sottinen, H. van Zanten, Conditional full support of Gaussian processes with stationary increments, *J. Appl. Probab.*, **48** (2011), 561–568. https://doi.org/10.1239/jap/1308662644
11. P. Guasoni, M. Rásonyi, W. Schachermayer, Consistent price systems and face-lifting pricing under transaction costs, *Ann. Appl. Probab.*, **18** (2008), 491–520. https://doi.org/10.1214/07-AAP461

12. Y. Mishura, M. Zili, *Stochastic analysis of mixed fractional Gaussian processes*, ISTE Press, Elsevier, 2018. https://doi.org/10.1016/C2017-0-00186-6
13. D. Revuz, M. Yor, *Continuous martingales and Brownian motion*, Grundlehren der mathematischen Wissenschaften, Vol. 293, Springer-Verlag, 1991. https://doi.org/10.1007/978-3-662-06400-9
14. C. Tudor, Some properties of the sub-fractional Brownian motion, *Stochastics*, **79** (2007), 431–448. https://doi.org/10.1080/17442500601100331
15. M. Zili, Mixed sub-fractional Brownian motion, *Random Oper. Stochastic Equ.*, **22** (2014), 163–178. https://doi.org/10.1515/rose-2014-0017
16. M. Zili, An optimal series expansion of sub-mixed fractional Brownian motion, *J. Numer. Math. Stochastics*, **5** (2013), 93–105.