



Research article**Algebraic structures of picture fuzzy level operators with metric properties****Ümit Deniz***

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Abstract: This paper investigates the algebraic structure of picture fuzzy level operators. The concept of level subsets for picture fuzzy sets was introduced by Kankaew et al. [1] in the context of UP (BCC)-algebras, where they defined the upper and lower level subsets $\mathcal{U}_\mu(t)$, $\mathcal{U}_\eta(t)$, and $\mathcal{L}_\nu(t)$. Building upon this foundational work, we define picture fuzzy level operators $L_{\alpha,\beta,\gamma}$ and investigate their algebraic properties. Let $\mathcal{L} = \{L_{\alpha,\beta,\gamma} : \alpha, \beta, \gamma \in [0, 1], 0 \leq \alpha + \beta + \gamma \leq 1\}$ denote the admissible set of all picture fuzzy level operators, and let $\overline{\mathcal{L}} = \{L_{\alpha,\beta,\gamma} : \alpha, \beta, \gamma \in [0, 1]\}$ denote the extended operator space obtained by dropping the constraint $\alpha + \beta + \gamma \leq 1$. We define a partial order $\leq$ on $\overline{\mathcal{L}}$ by componentwise order with γ reversed. We prove that $(\overline{\mathcal{L}}, \leq)$ is a poset, with $\mathcal{L}$ as a sub-poset. We introduce the meet $\sqcap$ and join $\sqcup$ operations and show that $(\overline{\mathcal{L}}, \sqcap, \sqcup)$ forms a distributive lattice, and moreover a complete lattice. The admissible set $\mathcal{L}$ is characterized as closed under meet but not under join. We define composition $\circ$ of level operators and prove that $(\overline{\mathcal{L}}, \circ)$ forms a commutative idempotent monoid with identity $L_{0,0,1}$, with the interesting property that composition coincides with the join operation. We investigate inverse operators, proving that only the identity operator has a true inverse, and introduce dual operators satisfying natural duality properties. We establish metric structures, including Chebyshev, Manhattan, and Euclidean distances on $\overline{\mathcal{L}}$. Finally, we provide a geometric interpretation via the operator simplex—a tetrahedral representation in $\mathbb{R}^3$ that visualizes the admissible set of level operators. This work provides a systematic algebraic and geometric framework for understanding picture fuzzy level operators.

Keywords: picture fuzzy sets; level operators; level subsets; lattice theory; monoids; metric spaces; operator simplex; threshold operators

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1. Introduction

1.1. Background and motivation

Threshold operators have a long history in fuzzy set theory, dating back to Zadeh's foundational work on fuzzy sets [2] and their early applications in fuzzy group theory by Rosenfeld [3]. They take a fuzzy set and filter it, keep only what exceeds some cutoff, and everything else goes to zero. The result is a bridge from the graded world of fuzzy sets to the stark binary world of classical sets.

Zadeh [2] started it all with fuzzy sets. Rosenfeld [3] pushed the idea into algebra with fuzzy groups. Then Atanassov [4] added non-membership degrees, creating intuitionistic fuzzy sets (IFSs). Now two thresholds were needed: one for membership and one for non-membership [5, 6]. Others followed, extending these operators and exploring their algebraic consequences [7, 8].

Cuong's picture fuzzy sets (PFSs) [9] went further. Each element now carries three numbers: positive membership (μ), neutral membership (η), and negative membership (ν). The constraint? $\mu + \eta + \nu \leq 1$. What remains is the refusal degree π —a measure of how much an element neither belongs nor refuses. This tripartite structure fits decision problems where people hold mixed views simultaneously [10, 11].

Meanwhile, generalized fuzzy theories have multiplied. Neutrosophic sets [12–14] split membership into truth, indeterminacy, and falsity. Recent extensions, such as complex neutrosophic sets [15] have further enriched this framework. Plithogenic sets [16] unify aggregation. Hyperfuzzy sets [17] allow multiple membership values. Spherical fuzzy sets [18] and q -rung orthopair fuzzy sets [19] push generalization even further. We place our work within this landscape, studying level operators inside the picture fuzzy setting.

Beyond the classical and intuitionistic frameworks, several extensions of picture fuzzy sets have also been proposed. Bipolar picture fuzzy sets [20] capture both positive and negative aspects of membership in a unified framework, allowing a more nuanced representation of uncertainty in decision problems where both favorable and unfavorable evidence must be modeled simultaneously. Picture fuzzy soft sets [21] combine picture fuzzy sets with soft set theory, providing a parameterized tool for handling uncertainty. These extensions demonstrate the versatility of the picture fuzzy framework and motivate our study of its algebraic structure.

Level subsets for fuzzy sets—written $\mathcal{U}_\mu(t) = \{x \in X : \mu(x) \geq t\}$ —have been standard for decades. IFSs need two such subsets: one for membership and one for non-membership [5]. Kankaew et al. [1] extended the idea to picture fuzzy sets with three level subsets:

$$\mathcal{U}_\mu(t) = \{x \in X : \mu(x) \geq t\}, \quad \mathcal{U}_\eta(t) = \{x \in X : \eta(x) \geq t\}, \quad \mathcal{L}_\nu(t) = \{x \in X : \nu(x) \leq t\}.$$

They showed that picture fuzzy BCC-subalgebras, filters, and ideals correspond to classical structures through these level subsets. The line of work continues. Nakkhasen et al. [22] studied m -polar picture fuzzy bi-ideals in semigroups. Gunjan et al. [23] examined picture fuzzy subalgebras in Sheffer stroke UP-algebras.

But level subsets tell a static story, and level operators tell a dynamic one, and the operator $L_{\alpha,\beta,\gamma}$ thresholds all three membership degrees at once. And the collection

$$\mathcal{L} = \{L_{\alpha,\beta,\gamma} : \alpha, \beta, \gamma \in [0, 1], 0 \leq \alpha + \beta + \gamma \leq 1\}$$

carries structure—order-theoretic, algebraic, geometric. Moreover, to obtain a genuine lattice structure, we also consider the extended operator space

$$\overline{\mathcal{L}} = \{L_{\alpha,\beta,\gamma} : \alpha, \beta, \gamma \in [0, 1]\},$$

obtained by dropping the constraint. We take up the structure of both spaces here.

1.2. Motivation and significance

Why study the algebraic structure of level operators? Four reasons stand out.

Information fusion matters. In multi-criteria decision-making, criteria demand different thresholds. The lattice structure of level operators gives a principled way to combine them.

Optimization matters too. The metrics we develop on $\overline{\mathcal{L}}$ open the door to optimization over thresholding strategies.

Theoretical grounding is essential if these operators are to be used reliably.

And finally, the lattice and monoid structures reveal how picture fuzzy level operators inherit properties from both fuzzy logic and classical algebra—a bridge between worlds.

1.3. Related work and comparison

Level operators appear in several fuzzy settings. Classical fuzzy level operators L_α are well understood and applied to fuzzy groups [3]. Intuitionistic fuzzy level operators $L_{\alpha,\beta}$ were studied by Atanassov and others [5, 7]. We now compare these frameworks systematically, as summarized in Table 1.

Table 1. Comparison of level operators across different fuzzy frameworks.

Type	Parameters	Condition	Operator
Fuzzy sets	$\alpha \in [0, 1]$	—	L_α
Intuitionistic fuzzy sets	$\alpha, \gamma \in [0, 1]$	$\alpha + \gamma \leq 1$	$L_{\alpha,\gamma}$
Picture fuzzy sets	$\alpha, \beta, \gamma \in [0, 1]$	$\alpha + \beta + \gamma \leq 1$	$L_{\alpha,\beta,\gamma}$

To further clarify the position of the present work, Table 2 compares representative fuzzy, picture fuzzy, bipolar picture fuzzy, and neutrosophic models in terms of their membership structure and scope.

Table 2. Comparison of representative fuzzy models with the present framework.

Model	Membership degrees	Constraint	Reference
Fuzzy sets	μ	—	[2]
Intuitionistic fuzzy sets	μ, ν	$\mu + \nu \leq 1$	[4]
Picture fuzzy sets	μ, η, ν	$\mu + \eta + \nu \leq 1$	[9]
Bipolar picture fuzzy sets	μ^+, μ^-	—	[20]
Neutrosophic sets	T, I, F	—	[12]

Three differences separate our work from earlier contributions. Atanassov's operators act on IFSs with two degrees; ours act on picture fuzzy sets with three. Kankaew et al. studied level subsets but left the algebraic structure of the operators themselves untouched; we treat the operators as objects in

their own right. Further, we discover structures—lattice, monoid, metric space—that earlier work did not reach, including the striking coincidence $\circ = \sqcup$ between composition and join.

The lattice structure of fuzzy sets and IFSs has been studied extensively [24, 25]; However, the algebraic properties of level operators themselves have attracted little attention. Hasan [26] recently characterized picture fuzzy lattices, ideals, and their homomorphisms. Their work focuses on ideals as substructures, complementing our focus on the algebraic behavior of level operators.

1.4. Main results

Our results are as follows. First, $\overline{\mathcal{L}}$ forms a poset under a componentwise order with γ reversed, with $\mathcal{L}$ as a sub-poset (Section 3). Second, $\overline{\mathcal{L}}$ forms a distributive and complete lattice under meet and join, and the admissible set $\mathcal{L}$ is characterized as closed under meet but not under join (Section 4). Third, $\overline{\mathcal{L}}$ forms a commutative idempotent monoid under composition, and composition coincides with join— $\circ = \sqcup$ (Section 5). Fourth, only the identity has a true inverse; dual operators satisfy natural duality properties (Section 6). Fifth, three metrics (i.e., Chebyshev, Manhattan, and Euclidean) are defined on $\overline{\mathcal{L}}$, and lattice operations are continuous with respect to them (Section 7). Finally, the operator simplex offers a geometric interpretation (Section 8).

The identity $\circ = \sqcup$ stands out. It says that applying two filters in sequence gives exactly the same result as applying a single filter that combines the strictest conditions from both. This behavior is specific to picture fuzzy level operators. It does not appear in the classical or intuitionistic cases.

1.5. Contributions

We make the following contributions:

- (1) A partial order $\leq$ on $\overline{\mathcal{L}}$ under which $(\overline{\mathcal{L}}, \leq)$ is a poset, with the admissible set $\mathcal{L}$ as a sub-poset (Theorem 3.3).
- (2) Meet $\sqcap$ and join $\sqcup$ operations under which $(\overline{\mathcal{L}}, \sqcap, \sqcup)$ becomes a distributive, complete lattice, with the admissible set $\mathcal{L}$ characterized as meet-closed but not join-closed (Theorem 4.3, Corollary 4.5).
- (3) A composition $\circ$ turning $(\overline{\mathcal{L}}, \circ)$ into a commutative idempotent monoid (Theorem 5.3), with the identity $\circ = \sqcup$ (Corollary 5.5).
- (4) A proof that only $L_{0,0,1}$ has a true inverse (Theorem 6.2), plus dual operators with natural properties (Proposition 6.6).
- (5) Metric structures on $\overline{\mathcal{L}}$ —Chebyshev, Manhattan, and Euclidean (Theorem 7.2).
- (6) A geometric interpretation via the operator simplex, a tetrahedron in $\mathbb{R}^3$ (Section 8).

1.6. Organization

Section 2 recalls the definitions of picture fuzzy level operators and level subsets from Kankaew et al. [1]. Section 3 introduces the partial order. Section 4 develops the lattice structure on the extended space $\overline{\mathcal{L}}$ and characterizes the admissible set $\mathcal{L}$. Section 5 studies composition. Section 6 handles inverse and dual operators. Section 7 introduces metrics. Section 8 gives the geometric interpretation. Section 9 discusses applications. Section 10 concludes the paper.

2. Preliminaries

This section collects the foundational definitions and operations on picture fuzzy sets, level subsets, and level operators that are used throughout the paper. We begin with the basic notions of fuzzy, intuitionistic fuzzy, and picture fuzzy sets; then introduce level subsets; and finally define the picture fuzzy level operators that are the main object of study. Each definition is accompanied by a brief explanation of its role in the framework.

2.1. Picture fuzzy sets and basic operations

In this subsection, we recall the definitions of fuzzy, intuitionistic fuzzy, and picture fuzzy sets, together with their basic set-theoretic operations. These definitions form the foundation for the level subsets and level operators introduced later.

We begin with the classical notion of a fuzzy set [2].

Definition 2.1. [2] A *fuzzy set* A on a non-empty universe X is defined by a membership function $\mu_A : X \rightarrow [0, 1]$.

Intuitionistic fuzzy sets extend this by adding a non-membership degree, as the following definition recalls [4].

Definition 2.2. [4] An *intuitionistic fuzzy set* (IFS) A on a non-empty universe X is a pair (μ_A, ν_A) of functions $\mu_A, \nu_A : X \rightarrow [0, 1]$ satisfying $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

Building on these, Cuong introduced picture fuzzy sets with a third (neutral) membership degree [9].

Definition 2.3. [9] A *picture fuzzy set* (PFS) P on a non-empty universe X is a triple (μ_P, η_P, ν_P) of functions $\mu_P, \eta_P, \nu_P : X \rightarrow [0, 1]$ satisfying $\mu_P(x) + \eta_P(x) + \nu_P(x) \leq 1$ for all $x \in X$.

The following definitions establish the basic set-theoretic operations on picture fuzzy sets: complement, union, intersection, and the subset relation. These are needed for the lattice structure discussed in Section 4.

Definition 2.4. [9] [Complement of a PFS] For a picture fuzzy set $P = (\mu_P, \eta_P, \nu_P)$, the *complement* P^c is defined by:

$$\mu_{P^c}(x) = \nu_P(x), \quad \eta_{P^c}(x) = \eta_P(x), \quad \nu_{P^c}(x) = \mu_P(x).$$

Definition 2.5. [9] [Union of PFSs] For two picture fuzzy sets $P = (\mu_P, \eta_P, \nu_P)$ and $Q = (\mu_Q, \eta_Q, \nu_Q)$, their *union* $P \cup Q$ is defined componentwise as:

$$\begin{aligned} \mu_{P \cup Q}(x) &= \max\{\mu_P(x), \mu_Q(x)\}, \\ \eta_{P \cup Q}(x) &= \min\{\eta_P(x), \eta_Q(x)\}, \\ \nu_{P \cup Q}(x) &= \min\{\nu_P(x), \nu_Q(x)\}. \end{aligned}$$

Definition 2.6. [9] [Intersection of PFSs] For two picture fuzzy sets $P = (\mu_P, \eta_P, \nu_P)$ and $Q = (\mu_Q, \eta_Q, \nu_Q)$, their *intersection* $P \cap Q$ is defined componentwise as:

$$\begin{aligned} \mu_{P \cap Q}(x) &= \min\{\mu_P(x), \mu_Q(x)\}, \\ \eta_{P \cap Q}(x) &= \min\{\eta_P(x), \eta_Q(x)\}, \\ \nu_{P \cap Q}(x) &= \max\{\nu_P(x), \nu_Q(x)\}. \end{aligned}$$

The subset relation for picture fuzzy sets requires careful handling of the negative membership degree, as the following definition shows.

Definition 2.7. [9] [Subset and equality of PFSs] For two picture fuzzy sets P and Q , we say $P \subseteq Q$ if and only if:

$$\mu_P(x) \leq \mu_Q(x), \quad \eta_P(x) \leq \eta_Q(x), \quad \nu_P(x) \geq \nu_Q(x)$$

for all $x \in X$. Moreover, $P = Q$ if and only if $P \subseteq Q$ and $Q \subseteq P$.

Remark 2.8 (On the asymmetry of union and subset). The definitions of union and subset for picture fuzzy sets require careful interpretation. In the union $P \cup Q$, the neutral membership is combined via min, while in the subset relation $P \subseteq Q$, the condition is $\eta_P \leq \eta_Q$. At first glance, these two conventions might appear inconsistent: the union of two sets can have a smaller neutral membership than either operand, yet the subset relation treats smaller neutral membership as “more restrictive.”

The resolution lies in the interpretation of the neutral membership η as a degree of *indeterminacy*, rather than a degree of membership. Under the union, an element’s indeterminacy in $P \cup Q$ is the least indeterminacy it has in either set, reflecting the principle that evidence from either source can reduce uncertainty. Under the subset relation, $P \subseteq Q$ requires that P is at least as restrictive as Q , meaning P has no more indeterminacy than Q . These conventions follow Cuong’s original formulation [9] and are standard in the picture fuzzy literature.

Remark 2.9. The asymmetry for ν deserves a word. A smaller ν value means stronger membership—less negativity. So $P \subseteq Q$ says that P is more restrictive than Q : It demands at least as much positive and neutral membership, and at most as much negative membership.

We illustrate these operations with a simple example.

Example 2.10. Let P and Q be two picture fuzzy sets on $X = \{x_1, x_2\}$ defined by

	P			Q		
	μ_P	η_P	ν_P	μ_Q	η_Q	ν_Q
x_1	0.8	0.1	0.1	0.6	0.2	0.1
x_2	0.5	0.3	0.2	0.7	0.1	0.0

Then:

$$P \cap Q = \begin{array}{c|ccc} & \mu & \eta & \nu \\ \hline x_1 & 0.6 & 0.1 & 0.1 \\ x_2 & 0.5 & 0.1 & 0.2 \end{array} \quad \text{and} \quad P \cup Q = \begin{array}{c|ccc} & \mu & \eta & \nu \\ \hline x_1 & 0.8 & 0.1 & 0.1 \\ x_2 & 0.7 & 0.1 & 0.0 \end{array}.$$

Observe that $P \cap Q \subseteq P \subseteq P \cup Q$, matching intuition.

The following remark records the lattice structure of picture fuzzy sets, which will be useful for comparison with the lattice structure of level operators introduced in Section 4.

Remark 2.11 (Lattice structure of PFSs). The set of all picture fuzzy sets on X , together with $\cup$ and $\cap$, forms a distributive lattice [25], with partial order given by $\subseteq$. The top element is

$$P_{\top} = (\mu, \eta, \nu) = (1, 1, 0),$$

a “perfect” element with full positive and neutral membership and no negative membership. The bottom element is

$$P_{\perp} = (\mu, \eta, \nu) = (0, 0, 1),$$

an “empty” element. Note the asymmetry: $\nu = 1$ at the bottom, since ν measures negative membership.

2.2. Level subsets

Level subsets provide a way to decompose fuzzy sets into crisp sets at different thresholds. In this subsection, we recall the definitions of level subsets for fuzzy, intuitionistic fuzzy, and picture fuzzy sets, culminating in the three level subsets introduced by Kankaew et al. [1]. These definitions motivate the level operators studied in the remainder of the paper.

We begin with the classical fuzzy level subset.

Definition 2.12. [2] [Fuzzy level subset] For a fuzzy set A and $t \in [0, 1]$, the *level subset* is:

$$\mathcal{U}_\mu(t) = \{x \in X : \mu_A(x) \geq t\}.$$

For intuitionistic fuzzy sets, two level subsets are needed, one for each membership degree.

Definition 2.13. [5] [Intuitionistic fuzzy level subset] For an IFS A and $t \in [0, 1]$, the level subsets are:

$$\mathcal{U}_\mu(t) = \{x \in X : \mu_A(x) \geq t\}, \quad \mathcal{L}_\nu(t) = \{x \in X : \nu_A(x) \leq t\}.$$

The following definition extends this to picture fuzzy sets, where three level subsets are needed.

Definition 2.14. [1] [Picture fuzzy level subsets] For a PFS P and $t \in [0, 1]$, define

$$\mathcal{U}_\mu(t) = \{x \in X : \mu_P(x) \geq t\}, \tag{1}$$

$$\mathcal{U}_\eta(t) = \{x \in X : \eta_P(x) \geq t\}, \tag{2}$$

$$\mathcal{L}_\nu(t) = \{x \in X : \nu_P(x) \leq t\}. \tag{3}$$

Remark 2.15. These three definitions go back to Kankaew et al. [1], who introduced them for UP (BCC)-algebras as natural generalizations of the fuzzy and intuitionistic level subsets. We adopt them for studying picture fuzzy level operators.

2.3. Picture fuzzy level operators

Having introduced level subsets, we now turn to level operators, which are the main object of study in this paper. In this subsection, we recall the fuzzy and intuitionistic fuzzy level operators and then define the picture fuzzy level operator $L_{\alpha,\beta,\gamma}$. We also establish that this operator is well-defined and discuss its basic properties.

The simplest case is the fuzzy level operator.

Definition 2.16 (Fuzzy level operator). For a fuzzy set A and $\alpha \in [0, 1]$, the *fuzzy level operator* L_α is defined by:

$$\mu_{L_\alpha(A)}(x) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) \geq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

For intuitionistic fuzzy sets, two thresholds are needed.

Definition 2.17. [5] [Intuitionistic fuzzy level operator] For an IFS A and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, the *intuitionistic fuzzy level operator* $L_{\alpha,\beta}$ is defined by

$$\mu_{L_{\alpha,\beta}(A)}(x) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) \geq \alpha, \\ 0 & \text{otherwise,} \end{cases} \quad \nu_{L_{\alpha,\beta}(A)}(x) = \begin{cases} \nu_A(x) & \text{if } \nu_A(x) \leq \beta, \\ 1 & \text{otherwise.} \end{cases}$$

We now introduce the central definition of this paper: the picture fuzzy level operator, which simultaneously thresholds all three membership degrees. A key feature of the definition is that the negative membership acts as a global filter: Whenever $\nu_P(x)$ exceeds the threshold γ , the element is fully rejected, and both $\mu_L(x)$ and $\eta_L(x)$ are set to zero. This asymmetric treatment is necessary to preserve the picture fuzzy constraint $\mu + \eta + \nu \leq 1$, as discussed in Remark 2.20 below.

Definition 2.18 (Picture fuzzy level operator). Let $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma \leq 1$. The *picture fuzzy level operator* $L_{\alpha, \beta, \gamma}$ transforms a picture fuzzy set $P = (\mu_P, \eta_P, \nu_P)$ into $L_{\alpha, \beta, \gamma}(P) = (\mu_L, \eta_L, \nu_L)$ defined by

$$\begin{aligned}\mu_L(x) &= \begin{cases} \mu_P(x) & \text{if } \mu_P(x) \geq \alpha \text{ and } \nu_P(x) \leq \gamma, \\ 0 & \text{otherwise,} \end{cases} \\ \eta_L(x) &= \begin{cases} \eta_P(x) & \text{if } \eta_P(x) \geq \beta \text{ and } \nu_P(x) \leq \gamma, \\ 0 & \text{otherwise,} \end{cases} \\ \nu_L(x) &= \begin{cases} \nu_P(x) & \text{if } \nu_P(x) \leq \gamma, \\ 1 & \text{otherwise.} \end{cases}\end{aligned}$$

Before proceeding, we verify that this operator produces valid picture fuzzy sets.

Proposition 2.19. *The picture fuzzy level operator $L_{\alpha, \beta, \gamma}$ is well-defined: For any picture fuzzy set P , the resulting $L_{\alpha, \beta, \gamma}(P)$ is again a picture fuzzy set.*

Proof. Let $x \in X$. We show that $\mu_L(x), \eta_L(x), \nu_L(x) \in [0, 1]$ and that

$$\mu_L(x) + \eta_L(x) + \nu_L(x) \leq 1.$$

Range. By construction, $\mu_L(x)$ is either $\mu_P(x) \in [0, 1]$ or 0, so $\mu_L(x) \in [0, 1]$. Similarly, $\eta_L(x) \in [0, 1]$. For $\nu_L(x)$: If $\nu_P(x) \leq \gamma$, then $\nu_L(x) = \nu_P(x) \in [0, 1]$; otherwise $\nu_L(x) = 1 \in [0, 1]$. Hence, all three components lie in $[0, 1]$.

Sum constraint. We consider two cases.

Case 1: $\nu_P(x) \leq \gamma$. Then, $\nu_L(x) = \nu_P(x)$, and Definition 2.18 gives $\mu_L(x) \leq \mu_P(x)$ and $\eta_L(x) \leq \eta_P(x)$. Therefore,

$$\mu_L(x) + \eta_L(x) + \nu_L(x) \leq \mu_P(x) + \eta_P(x) + \nu_P(x) \leq 1,$$

since P is a picture fuzzy set.

Case 2: $\nu_P(x) > \gamma$. Then, by Definition 2.18, $\mu_L(x) = 0$, $\eta_L(x) = 0$, and $\nu_L(x) = 1$. Hence,

$$\mu_L(x) + \eta_L(x) + \nu_L(x) = 0 + 0 + 1 = 1 \leq 1.$$

In both cases, the sum is at most 1. Therefore, $L_{\alpha, \beta, \gamma}(P)$ is a valid picture fuzzy set. $\square$

Remark 2.20. The asymmetric treatment of ν is essential for well-definedness. Suppose, by contrast, that we set $\mu_L(x) = \mu_P(x)$ whenever $\mu_P(x) \geq \alpha$ (independently of ν), $\eta_L(x) = \eta_P(x)$ whenever $\eta_P(x) \geq \beta$, and $\nu_L(x) = 1$ whenever $\nu_P(x) > \gamma$. Then, the resulting triple might violate the picture fuzzy constraint. For instance, let $P = (0.3, 0.3, 0.4)$ and $(\alpha, \beta, \gamma) = (0.2, 0.2, 0.3)$. Here $\mu_P(x) = 0.3 \geq \alpha$ and $\eta_P(x) = 0.3 \geq \beta$ would be preserved, while $\nu_P(x) = 0.4 > \gamma$ would give $\nu_L(x) = 1$. The resulting

sum is $0.3 + 0.3 + 1 = 1.6 > 1$, which is not a valid picture fuzzy set. Our definition avoids this by forcing $\mu_L(x) = \eta_L(x) = 0$ whenever $\nu_P(x) > \gamma$, so that the sum remains at most 1. This makes the negative membership degree a global filter on the element: high negativity overrides positive and neutral memberships.

The following remarks clarify the role of the consistency condition and the operational meaning of the operator.

Remark 2.21. The constraint $\alpha + \beta + \gamma \leq 1$ guarantees that the thresholds themselves form a valid picture fuzzy element. This consistency condition is inherited naturally from the definition of picture fuzzy sets.

Remark 2.22 (Operational interpretation). The operator $L_{\alpha,\beta,\gamma}$ acts on each $x \in X$ independently, in two stages:

- (1) **Global rejection test:** If $\nu_P(x) > \gamma$, the element is fully rejected: $\mu_L(x) = 0$, $\eta_L(x) = 0$, $\nu_L(x) = 1$.
- (2) **Conditional thresholding:** If $\nu_P(x) \leq \gamma$, then $\mu_P(x)$ is preserved when $\mu_P(x) \geq \alpha$ (otherwise 0), $\eta_P(x)$ is preserved when $\eta_P(x) \geq \beta$ (otherwise 0), and $\nu_P(x)$ is preserved.

In effect, the operator “filters” the picture fuzzy set according to (α, β, γ) , with the negative membership acting as a global rejection criterion.

Finally, we note the special cases that recover the fuzzy and intuitionistic operators.

Remark 2.23 (Special cases). Two classical cases emerge from this definition:

- If $\eta_P(x) = 0$ for all x and $\beta = 0$, the operator reduces to $L_{\alpha,\gamma}$.
- If additionally $\nu_P(x) = 0$ for all x and $\gamma = 0$, it reduces to L_α .

At the extremes:

- $L_{0,0,1}$: the identity operator (no filtering).
- $L_{1,1,0}$: the strictest operator in the extended space $\overline{\mathcal{L}}$. It preserves $\mu_P(x)$ only when $\mu_P(x) \geq 1$ (i.e., $\mu_P(x) = 1$) and $\nu_P(x) \leq 0$ (i.e., $\nu_P(x) = 0$); similarly for $\eta_P(x)$. Since $\mu_P(x) + \eta_P(x) + \nu_P(x) \leq 1$, no element can satisfy $\mu_P(x) = 1$ and $\eta_P(x) = 1$ simultaneously. Hence, $L_{1,1,0}(P)$ is the constant picture fuzzy set in which every element is fully rejected, that is, $L_{1,1,0}(P)(x) = (0, 0, 1)$ for all $x \in X$.

Let

$$\mathcal{L} = \{L_{\alpha,\beta,\gamma} : \alpha, \beta, \gamma \in [0, 1], 0 \leq \alpha + \beta + \gamma \leq 1\}$$

denote the *admissible set* of all picture fuzzy level operators.

Definition 2.24 (Extended operator space). Let

$$\overline{\mathcal{L}} = \{L_{\alpha,\beta,\gamma} : \alpha, \beta, \gamma \in [0, 1]\}$$

denote the *extended operator space*, obtained by dropping the picture fuzzy constraint $0 \leq \alpha + \beta + \gamma \leq 1$. The admissible set $\mathcal{L}$ is a proper subset of $\overline{\mathcal{L}}$.

Remark 2.25. The extended space $\overline{\mathcal{L}}$ is needed because the admissible set $\mathcal{L}$ is not closed under the join operation introduced in Section 4. The lattice structure is established on $\overline{\mathcal{L}}$, while $\mathcal{L}$ retains its picture fuzzy interpretation as the admissible subset.

3. Partial order

We now define a partial order on $\overline{\mathcal{L}}$ and show it forms a poset. The admissible set $\mathcal{L}$ inherits this order as a sub-poset. The order captures filtering strength: larger α and β , together with smaller γ , mean stronger filtering. This order will be used throughout the paper to describe the lattice, monoid, and metric structures.

We begin with the definition of the order.

Definition 3.1 (Partial order on $\overline{\mathcal{L}}$). For $L_{\alpha_1, \beta_1, \gamma_1}, L_{\alpha_2, \beta_2, \gamma_2} \in \overline{\mathcal{L}}$, define

$$L_{\alpha_1, \beta_1, \gamma_1} \leq L_{\alpha_2, \beta_2, \gamma_2} \quad \text{if and only if} \quad \alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \geq \gamma_2.$$

The following remark explains the intuition behind this definition.

Remark 3.2. The order is the product order on (α, β) combined with the reverse order on γ . This mirrors the asymmetry built into the definitions. Higher thresholds for μ and η shrink the surviving set. Higher thresholds for ν enlarge it—fewer elements get their negative membership set to 1.

We now verify that this order indeed makes $\overline{\mathcal{L}}$ a poset.

Theorem 3.3. $(\overline{\mathcal{L}}, \leq)$ is a partially ordered set (poset). The admissible set $\mathcal{L}$ is a sub-poset of $\overline{\mathcal{L}}$.

Proof. **Reflexivity:** For any $L_{\alpha, \beta, \gamma} \in \overline{\mathcal{L}}$, we have $\alpha \leq \alpha, \beta \leq \beta$, and $\gamma \geq \gamma$. Hence, $L_{\alpha, \beta, \gamma} \leq L_{\alpha, \beta, \gamma}$.

Antisymmetry: Suppose $L_1 = L_{\alpha_1, \beta_1, \gamma_1} \leq L_2 = L_{\alpha_2, \beta_2, \gamma_2}$ and $L_2 \leq L_1$. Then, $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$, so $\alpha_1 = \alpha_2$. Similarly $\beta_1 = \beta_2$. Also $\gamma_1 \geq \gamma_2$ and $\gamma_2 \geq \gamma_1$, so $\gamma_1 = \gamma_2$. Hence, $L_1 = L_2$.

Transitivity: Suppose $L_1 \leq L_2$ and $L_2 \leq L_3$. Then, $\alpha_1 \leq \alpha_2 \leq \alpha_3, \beta_1 \leq \beta_2 \leq \beta_3$, and $\gamma_1 \geq \gamma_2 \geq \gamma_3$. Hence, $L_1 \leq L_3$. $\square$

The following example illustrates the partial order.

Example 3.4. The chain

$$L_{0.6, 0.3, 0.1} \leq L_{0.4, 0.3, 0.2} \leq L_{0.3, 0.3, 0.3} \leq L_{0.1, 0.1, 0.5}$$

illustrates the partial order. As α and β decrease and γ increases, the operator becomes less restrictive. All four operators belong to the admissible set $\mathcal{L}$, since their component sums are 1, 0.9, 0.9, and 0.7, respectively.

4. Lattice structure

Next we introduce meet and join on $\overline{\mathcal{L}}$. These operations turn $\overline{\mathcal{L}}$ into a distributive lattice—in fact, a complete one. The lattice structure gives a systematic way to combine thresholding strategies. We begin with the definitions of meet and join, then prove that they yield a distributive lattice, and finally identify the top and bottom elements.

Before proceeding, we note that the admissible set $\mathcal{L}$ is not closed under the join operation defined below. To obtain a genuine lattice structure, we work in the extended operator space $\overline{\mathcal{L}}$ introduced in Definition 2.24.

4.1. Meet and join

We begin by defining the two basic lattice operations on $\overline{\mathcal{L}}$.

Definition 4.1 (Meet). For $L_{\alpha_1, \beta_1, \gamma_1}, L_{\alpha_2, \beta_2, \gamma_2} \in \overline{\mathcal{L}}$, the *meet* (greatest lower bound) is:

$$L_{\alpha_1, \beta_1, \gamma_1} \sqcap L_{\alpha_2, \beta_2, \gamma_2} = L_{\min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2), \max(\gamma_1, \gamma_2)}.$$

Definition 4.2 (Join). The *join* (least upper bound) is:

$$L_{\alpha_1, \beta_1, \gamma_1} \sqcup L_{\alpha_2, \beta_2, \gamma_2} = L_{\max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2), \min(\gamma_1, \gamma_2)}.$$

We now show that these operations make $\overline{\mathcal{L}}$ a distributive lattice.

Theorem 4.3. $(\overline{\mathcal{L}}, \sqcap, \sqcup)$ forms a distributive lattice. Moreover, the admissible set $\mathcal{L}$ is closed under meet, but not under join.

Proof. The operations act componentwise: min/max on α and β , reversed on γ . Since $[0, 1]$ with min and max is a distributive lattice [27, 28], so is $\overline{\mathcal{L}} = [0, 1]^3$.

For an explicit check, let $L_1 = L_{\alpha_1, \beta_1, \gamma_1}$, $L_2 = L_{\alpha_2, \beta_2, \gamma_2}$, and $L_3 = L_{\alpha_3, \beta_3, \gamma_3}$. Then,

$$L_1 \sqcap (L_2 \sqcup L_3) = L_{\min(\alpha_1, \max(\alpha_2, \alpha_3)), \min(\beta_1, \max(\beta_2, \beta_3)), \max(\gamma_1, \min(\gamma_2, \gamma_3))}.$$

By distributivity of min over max and max over min on $[0, 1]$, this equals:

$$L_{\max(\min(\alpha_1, \alpha_2), \min(\alpha_1, \alpha_3)), \max(\min(\beta_1, \beta_2), \min(\beta_1, \beta_3)), \min(\max(\gamma_1, \gamma_2), \max(\gamma_1, \gamma_3))},$$

which is precisely $(L_1 \sqcap L_2) \sqcup (L_1 \sqcap L_3)$.

It remains to verify the closure properties of $\mathcal{L}$. If $L_1, L_2 \in \mathcal{L}$, then $\alpha_i + \beta_i + \gamma_i \leq 1$ for $i = 1, 2$. Let $j \in \{1, 2\}$ be such that $\max(\gamma_1, \gamma_2) = \gamma_j$. Since $\min(\alpha_1, \alpha_2) \leq \alpha_j$ and $\min(\beta_1, \beta_2) \leq \beta_j$, the meet satisfies

$$\min(\alpha_1, \alpha_2) + \min(\beta_1, \beta_2) + \max(\gamma_1, \gamma_2) \leq \alpha_j + \beta_j + \gamma_j \leq 1,$$

so $L_1 \sqcap L_2 \in \mathcal{L}$. □

Example 4.4 (Non-closure of $\mathcal{L}$ under join). Let $L_1 = L_{0.6, 0.3, 0.1}$ and $L_2 = L_{0.5, 0.4, 0.1}$. Both belong to $\mathcal{L}$ since $0.6 + 0.3 + 0.1 = 1$ and $0.5 + 0.4 + 0.1 = 1$. However,

$$L_1 \sqcup L_2 = L_{0.6, 0.4, 0.1},$$

and $0.6 + 0.4 + 0.1 = 1.1 > 1$, so $L_1 \sqcup L_2 \notin \mathcal{L}$. Thus, $\mathcal{L}$ is not closed under join.

The completeness of the lattice follows from the completeness of $\overline{\mathcal{L}}$.

Corollary 4.5. $(\overline{\mathcal{L}}, \sqcap, \sqcup)$ is a complete lattice with

$$\bigwedge_{i \in I} L_{\alpha_i, \beta_i, \gamma_i} = L_{\inf_i \alpha_i, \inf_i \beta_i, \sup_i \gamma_i}, \quad \bigvee_{i \in I} L_{\alpha_i, \beta_i, \gamma_i} = L_{\sup_i \alpha_i, \sup_i \beta_i, \inf_i \gamma_i}.$$

Proof. The completeness of $[0, 1]$ under infimum and supremum guarantees that arbitrary meets and joins exist in $\overline{\mathcal{L}}$ [29]. □

The following proposition connects the order structure on $\overline{\mathcal{L}}$ with the behavior of the operators on picture fuzzy sets: A stronger filter (larger in the order $\leq$) produces a more restrictive output.

Proposition 4.6. *Let $L_1 = L_{\alpha_1, \beta_1, \gamma_1}, L_2 = L_{\alpha_2, \beta_2, \gamma_2} \in \overline{\mathcal{L}}$ with $L_1 \leq L_2$. Then, for every picture fuzzy set P on X ,*

$$L_2(P) \subseteq L_1(P),$$

where $\subseteq$ is the picture fuzzy subset relation of Definition 2.7.

Proof. Let $x \in X$. We must show

$$\mu_{L_2(P)}(x) \leq \mu_{L_1(P)}(x), \quad \eta_{L_2(P)}(x) \leq \eta_{L_1(P)}(x), \quad \nu_{L_2(P)}(x) \geq \nu_{L_1(P)}(x).$$

Since $L_1 \leq L_2$, we have $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2$, and $\gamma_1 \geq \gamma_2$. We distinguish three cases.

Case 1: $\nu_P(x) > \gamma_1$. Then, $\nu_P(x) > \gamma_1 \geq \gamma_2$, so the global rejection test triggers for both operators:

$$L_1(P)(x) = (0, 0, 1) \quad \text{and} \quad L_2(P)(x) = (0, 0, 1).$$

Hence, $L_1(P)(x) = L_2(P)(x)$, and the subset relation holds with equality.

Case 2: $\gamma_2 < \nu_P(x) \leq \gamma_1$. Then, L_1 passes the global test (since $\nu_P(x) \leq \gamma_1$), while L_2 triggers it (since $\nu_P(x) > \gamma_2$). Therefore,

$$L_1(P)(x) = (\mu_1, \eta_1, \nu_P(x)), \quad \text{where } \mu_1 = \mu_P(x) \text{ if } \mu_P(x) \geq \alpha_1 \text{ else } 0,$$

and similarly for η_1 , while $L_2(P)(x) = (0, 0, 1)$. Since $0 \leq \mu_1, 0 \leq \eta_1$, and $1 \geq \nu_P(x)$, the subset relation holds.

Case 3: $\nu_P(x) \leq \gamma_2$. Then, $\nu_P(x) \leq \gamma_2 \leq \gamma_1$, so both operators pass the global test. For the positive membership: if $\mu_P(x) \geq \alpha_2$, then $\mu_P(x) \geq \alpha_1$ (since $\alpha_1 \leq \alpha_2$), so both operators preserve $\mu_P(x)$ and $\mu_{L_1(P)}(x) = \mu_{L_2(P)}(x)$. If $\mu_P(x) < \alpha_2$, then $\mu_{L_2(P)}(x) = 0 \leq \mu_{L_1(P)}(x)$ regardless of whether $\mu_{L_1(P)}(x)$ equals $\mu_P(x)$ or 0. In all subcases, $\mu_{L_2(P)}(x) \leq \mu_{L_1(P)}(x)$. The same argument applies to η . For the negative membership, both operators preserve $\nu_P(x)$, so $\nu_{L_1(P)}(x) = \nu_{L_2(P)}(x) = \nu_P(x)$.

In all three cases, the three componentwise inequalities hold, so $L_2(P) \subseteq L_1(P)$. $\square$

Remark 4.7. Proposition 4.6 confirms that the order $\leq$ on $\overline{\mathcal{L}}$ is consistent with the filtering behavior of the operators: Moving up in the order (larger α, β and smaller γ) yields a more restrictive output. This justifies the interpretation of $(\overline{\mathcal{L}}, \leq)$ as a lattice of filtering strategies and connects the algebraic structure with the underlying picture fuzzy semantics.

4.2. Top and bottom elements

We now identify the extreme elements of the lattice.

Proposition 4.8. *The extended space $\overline{\mathcal{L}}$ has:*

- **Bottom element:** $\perp = L_{0,0,1}$.
- **Top element:** $\top = L_{1,1,0}$.

The admissible set $\mathcal{L}$ has the same bottom element $L_{0,0,1}$ (the identity operator), but no top element.

Proof. For any $L_{\alpha,\beta,\gamma} \in \overline{\mathcal{L}}$, we have $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, and $0 \leq \gamma \leq 1$. Since the order on γ is reversed, the smallest value of γ in the order corresponds to $\gamma = 1$, and the largest to $\gamma = 0$. Hence, $L_{0,0,1} \leq L_{\alpha,\beta,\gamma} \leq L_{1,1,0}$.

For the admissible set $\mathcal{L}$: The identity operator $L_{0,0,1}$ is the smallest element under $\leq$ since $0 \leq \alpha$, $0 \leq \beta$, and $1 \geq \gamma$ for all $L_{\alpha,\beta,\gamma} \in \mathcal{L}$. However, $L_{1,1,0}$ does not belong to $\mathcal{L}$ because $1 + 1 + 0 = 2 > 1$. In fact, $\mathcal{L}$ has no top element: For any $L_{\alpha,\beta,\gamma} \in \mathcal{L}$ with $\alpha + \beta + \gamma < 1$, we can increase α or β (or decrease γ) to obtain a larger element still in $\mathcal{L}$, and the boundary $\alpha + \beta + \gamma = 1$ is approached but never attained by a unique maximum. $\square$

Remark 4.9. The bottom element $L_{0,0,1}$ is the identity operator—the same in both $\overline{\mathcal{L}}$ and $\mathcal{L}$. The top element $L_{1,1,0}$ of $\overline{\mathcal{L}}$ filters out everything except elements with $\mu_P(x) \geq 1$, $\eta_P(x) \geq 1$, and $\nu_P(x) \leq 0$. Within the admissible set $\mathcal{L}$, no top element exists because the constraint $\alpha + \beta + \gamma \leq 1$ prevents the extreme configuration $L_{1,1,0}$.

4.3. Examples

We conclude this section with an explicit computation.

Example 4.10. Let $L_1 = L_{0.5,0.3,0.2}$ and $L_2 = L_{0.4,0.3,0.1}$. Both belong to $\mathcal{L}$ since $0.5 + 0.3 + 0.2 = 1.0 \leq 1$ and $0.4 + 0.3 + 0.1 = 0.8 \leq 1$. Then,

$$L_1 \sqcap L_2 = L_{\min(0.5,0.4),\min(0.3,0.3),\max(0.2,0.1)} = L_{0.4,0.3,0.2},$$

$$L_1 \sqcup L_2 = L_{\max(0.5,0.4),\max(0.3,0.3),\min(0.2,0.1)} = L_{0.5,0.3,0.1}.$$

Both results belong to $\mathcal{L}$, since $0.4 + 0.3 + 0.2 = 0.9 \leq 1$ and $0.5 + 0.3 + 0.1 = 0.9 \leq 1$.

5. Composition

Now to composition. We show that $(\overline{\mathcal{L}}, \circ)$ is a commutative idempotent monoid and—perhaps surprisingly—that composition coincides with join. The admissible set $\mathcal{L}$ is not closed under composition; the monoid structure is established on $\overline{\mathcal{L}}$. This section introduces the composition operation, proves the composition formula, establishes the monoid structure, and identifies the striking identity $\circ = \sqcup$.

5.1. Definition and formula

We begin with the definition of composition.

Definition 5.1 (Composition). For $L_1, L_2 \in \overline{\mathcal{L}}$, their *composition* is defined by:

$$(L_1 \circ L_2)(P) = L_1(L_2(P))$$

for any picture fuzzy set P .

The following theorem gives the explicit formula for the composition of two level operators. The proof relies on the two-stage structure of Definition 2.18: Composition of two filters is equivalent to a single filter whose parameters are the componentwise extrema.

Theorem 5.2 (Composition formula). *For any $L_{\alpha_1, \beta_1, \gamma_1}, L_{\alpha_2, \beta_2, \gamma_2} \in \overline{\mathcal{L}}$,*

$$L_{\alpha_1, \beta_1, \gamma_1} \circ L_{\alpha_2, \beta_2, \gamma_2} = L_{\max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2), \min(\gamma_1, \gamma_2)}.$$

Proof. Let P be any picture fuzzy set and let $Q = L_{\alpha_2, \beta_2, \gamma_2}(P)$. We compute $R = L_{\alpha_1, \beta_1, \gamma_1}(Q)$ and compare it with $S = L_{\alpha^*, \beta^*, \gamma^*}(P)$, where $\alpha^* = \max(\alpha_1, \alpha_2)$, $\beta^* = \max(\beta_1, \beta_2)$, and $\gamma^* = \min(\gamma_1, \gamma_2)$.

Fix $x \in X$. We distinguish three cases.

Case 1: $v_P(x) > \gamma_2$. Then, by Definition 2.18, Q has $\mu_Q(x) = 0$, $\eta_Q(x) = 0$, and $v_Q(x) = 1$. Applying $L_{\alpha_1, \beta_1, \gamma_1}$ to Q : since $v_Q(x) = 1 \geq \gamma_1$, the global rejection test either triggers (if $\gamma_1 < 1$) or passes with $v_Q(x) = 1 \leq \gamma_1$ (if $\gamma_1 = 1$). In both subcases, the resulting R has $\mu_R(x) = 0$, $\eta_R(x) = 0$, $v_R(x) = 1$. On the other hand, $v_P(x) > \gamma_2 \geq \min(\gamma_1, \gamma_2) = \gamma^*$, so the global rejection test for S also triggers, giving $\mu_S(x) = 0$, $\eta_S(x) = 0$, $v_S(x) = 1$. Hence, $R = S$.

Case 2: $v_P(x) \leq \gamma_2$ and $v_P(x) > \gamma_1$. Then, $\gamma_1 < \gamma_2$, so $\gamma^* = \min(\gamma_1, \gamma_2) = \gamma_1$. Since $v_P(x) \leq \gamma_2$, the operator $L_{\alpha_2, \beta_2, \gamma_2}$ preserves $v_Q(x) = v_P(x)$, and thresholds $\mu_Q(x)$ and $\eta_Q(x)$ conditionally: $\mu_Q(x) = \mu_P(x)$ if $\mu_P(x) \geq \alpha_2$, else 0; similarly for $\eta_Q(x)$. Now $v_Q(x) = v_P(x) > \gamma_1$, so the global rejection test for $L_{\alpha_1, \beta_1, \gamma_1}$ triggers, giving $\mu_R(x) = 0$, $\eta_R(x) = 0$, $v_R(x) = 1$. Similarly, $v_P(x) > \gamma_1 = \gamma^*$, so the global rejection test for S also triggers, giving $\mu_S(x) = 0$, $\eta_S(x) = 0$, $v_S(x) = 1$. Hence, $R = S$.

Case 3: $v_P(x) \leq \gamma_1$ and $v_P(x) \leq \gamma_2$. Then, $v_P(x) \leq \min(\gamma_1, \gamma_2) = \gamma^*$. Both operators preserve the negative membership: $v_Q(x) = v_P(x) \leq \gamma_1$, so $L_{\alpha_1, \beta_1, \gamma_1}$ also preserves it, giving $v_R(x) = v_P(x)$; and $v_S(x) = v_P(x)$. For the positive membership: $\mu_R(x) = \mu_P(x)$ if and only if $\mu_P(x) \geq \alpha_2$ (for L_2 to preserve) and $\mu_P(x) \geq \alpha_1$ (for L_1 to preserve), which is equivalent to $\mu_P(x) \geq \max(\alpha_1, \alpha_2) = \alpha^*$. Similarly $\mu_S(x) = \mu_P(x)$ if and only if $\mu_P(x) \geq \alpha^*$. Hence, $\mu_R(x) = \mu_S(x)$. The same argument gives $\eta_R(x) = \eta_S(x)$. Therefore, $R = S$.

In all cases, $L_{\alpha_1, \beta_1, \gamma_1}(L_{\alpha_2, \beta_2, \gamma_2}(P))(x) = L_{\alpha^*, \beta^*, \gamma^*}(P)(x)$ for every $x \in X$, establishing the claim. $\square$

5.2. Monoid structure

We now show that composition gives $\overline{\mathcal{L}}$ the structure of a commutative idempotent monoid.

Theorem 5.3. *$(\overline{\mathcal{L}}, \circ)$ is a commutative idempotent monoid. The admissible set $\mathcal{L}$ is not closed under composition.*

Proof. • **Associativity:** For any $L_1, L_2, L_3 \in \overline{\mathcal{L}}$,

$$(L_1 \circ L_2) \circ L_3 = L_1 \circ (L_2 \circ L_3)$$

follows from the associativity of max and min [30].

• **Commutativity:**

$$L_1 \circ L_2 = L_2 \circ L_1$$

follows from the commutativity of max and min.

• **Identity:** $L_{0,0,1}$ is the identity since

$$L_{0,0,1} \circ L_{\alpha, \beta, \gamma} = L_{\max(0, \alpha), \max(0, \beta), \min(1, \gamma)} = L_{\alpha, \beta, \gamma}.$$

• **Idempotence:**

$$L_{\alpha, \beta, \gamma} \circ L_{\alpha, \beta, \gamma} = L_{\max(\alpha, \alpha), \max(\beta, \beta), \min(\gamma, \gamma)} = L_{\alpha, \beta, \gamma}.$$

To see that $\mathcal{L}$ is not closed under composition, it suffices to note that composition coincides with join (Corollary 5.5 below), and $\mathcal{L}$ is not closed under join (Example 4.4). $\square$

Remark 5.4. This monoid structure connects to the broader theory of fuzzy semigroups and monoids [31]. Classical fuzzy monoids live on fuzzy sets with a single membership degree. Our structure extends the idea to picture fuzzy level operators with three parameters, preserving associativity, commutativity, and idempotence. The three-parameter setting thus generalizes the classical fuzzy algebraic picture. Notably, the monoid structure holds under the asymmetric Definition 2.18, which is a non-trivial consequence of the two-stage filtering process.

5.3. Relationship with join

The following result is perhaps the most striking in the paper: composition coincides with the lattice join.

Corollary 5.5. For any $L_1, L_2 \in \overline{\mathcal{L}}$,

$$L_1 \circ L_2 = L_1 \sqcup L_2.$$

Proof. Compare the two formulas:

$$L_1 \circ L_2 = L_{\max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2), \min(\gamma_1, \gamma_2)}$$

and

$$L_1 \sqcup L_2 = L_{\max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2), \min(\gamma_1, \gamma_2)}.$$

They are identical. Note that this identity is established at the level of parameters; by Definition 2.18, operators with identical parameters produce identical outputs on every picture fuzzy set. $\square$

Remark 5.6 (Intuitive interpretation). The identity $\circ = \sqcup$ has a clean intuitive reading. Applying two filters in sequence is the same as applying a single filter that combines the strictest conditions from both. Concretely,

- For μ , an element must satisfy both $\mu \geq \alpha_1$ and $\mu \geq \alpha_2$, so $\mu \geq \max(\alpha_1, \alpha_2)$.
- For η , likewise $\eta \geq \max(\beta_1, \beta_2)$.
- For ν , the condition is $\nu \leq \gamma_1$ and $\nu \leq \gamma_2$, so $\nu \leq \min(\gamma_1, \gamma_2)$.

Thus composition is just the join in the lattice structure.

5.4. Distributivity properties

We close this section with the distributivity of composition over meet and join.

Proposition 5.7. For any $L_1, L_2, L_3 \in \overline{\mathcal{L}}$,

- (1) $L_1 \circ (L_2 \sqcap L_3) = (L_1 \circ L_2) \sqcap (L_1 \circ L_3)$,
- (2) $L_1 \circ (L_2 \sqcup L_3) = (L_1 \circ L_2) \sqcup (L_1 \circ L_3)$.

Proof. Since $\circ = \sqcup$ and the lattice is distributive (Theorem 4.3), both properties follow immediately. $\square$

6. Inverse and dual operators

In this section, we study two further operator-theoretic notions: inverses and duals. We show that only the identity operator has a true inverse, and we introduce a dual operator that is defined for all elements of $\overline{\mathcal{L}}$. We also discuss a structural feature concerning the range of the dual operator.

6.1. Inverse operators

We begin with the definition of an inverse.

Definition 6.1 (Inverse). The *inverse* of $L_{\alpha,\beta,\gamma} \in \overline{\mathcal{L}}$ is $L_{\alpha,\beta,\gamma}^{-1} \in \overline{\mathcal{L}}$ such that

$$L_{\alpha,\beta,\gamma} \circ L_{\alpha,\beta,\gamma}^{-1} = L_{0,0,1} \quad \text{and} \quad L_{\alpha,\beta,\gamma}^{-1} \circ L_{\alpha,\beta,\gamma} = L_{0,0,1}.$$

The following theorem characterizes which operators admit an inverse.

Theorem 6.2. $L_{\alpha,\beta,\gamma} \in \overline{\mathcal{L}}$ has an inverse if and only if $\alpha = 0$, $\beta = 0$, and $\gamma = 1$. In that case, $L_{0,0,1}^{-1} = L_{0,0,1}$.

Proof. From the composition formula (Theorem 5.2),

$$L_{\alpha,\beta,\gamma} \circ L_{\alpha',\beta',\gamma'} = L_{\max(\alpha,\alpha'),\max(\beta,\beta'),\min(\gamma,\gamma')}.$$

Setting this equal to $L_{0,0,1}$ forces $\max(\alpha,\alpha') = 0$, $\max(\beta,\beta') = 0$, and $\min(\gamma,\gamma') = 1$. The first two give $\alpha = \alpha' = 0$ and $\beta = \beta' = 0$. The third gives $\gamma = \gamma' = 1$. Hence, $L_{\alpha,\beta,\gamma} = L_{0,0,1}$. $\square$

Remark 6.3. Only the identity being invertible reflects the unavoidable information loss built into thresholding. Once elements are set to 0 or 1, the original values cannot be recovered. This is intrinsic to level operators and cannot be circumvented. The asymmetric Definition 2.18 does not change this conclusion: the composition formula remains the same, so the inverse classification is unaffected.

6.2. Dual operators

Although a true inverse exists only for the identity, a useful dual operator can be defined on all of $\overline{\mathcal{L}}$. We introduce it now.

Definition 6.4 (Dual operator). For $L_{\alpha,\beta,\gamma} \in \overline{\mathcal{L}}$, the *dual* operator is defined as:

$$L_{\alpha,\beta,\gamma}^{\diamond} = L_{1-\alpha, 1-\beta, 1-\gamma}.$$

This is always well-defined since $1 - \alpha, 1 - \beta, 1 - \gamma \in [0, 1]$.

Remark 6.5. Unlike the inverse, the dual is defined for every element of $\overline{\mathcal{L}}$. Note that the dual does not preserve the admissible set $\mathcal{L}$: even if $L_{\alpha,\beta,\gamma} \in \mathcal{L}$, the dual $L_{1-\alpha,1-\beta,1-\gamma}$ may fail the constraint $\alpha + \beta + \gamma \leq 1$. Indeed, the sum

$$(1 - \alpha) + (1 - \beta) + (1 - \gamma) \geq 2$$

always holds under the constraint $0 \leq \alpha + \beta + \gamma \leq 1$ (indeed, $3 - (\alpha + \beta + \gamma) \geq 2$), so the dual of an admissible operator is generally not admissible.

One way forward is to extend the operator space to the full cube $[0, 1]^3$. This is precisely the extended space $\overline{\mathcal{L}}$ introduced in Definition 2.24, which lets the dual act as a self-map, at the cost of dropping some constraints that characterize picture fuzzy level operators. This trade-off between algebraic completeness and structural specificity deserves future investigation, in the same spirit as extensions to Neutrosophic [12], Plithogenic [16], and other generalized fuzzy frameworks [23, 35].

The following proposition lists the main properties of the dual operator.

Proposition 6.6. *The dual operator satisfies*

- (1) **Involution:** $(L_{\alpha,\beta,\gamma}^\diamond)^\diamond = L_{\alpha,\beta,\gamma}$.
- (2) **Order-reversing:** If $L_1 \leq L_2$ in $\overline{\mathcal{L}}$, then $L_1^\diamond \geq L_2^\diamond$ (in the componentwise order on $[0, 1]^3$).
- (3) **Duality with meet/join:**

$$(L_1 \sqcap L_2)^\diamond = L_1^\diamond \sqcup L_2^\diamond, \quad (L_1 \sqcup L_2)^\diamond = L_1^\diamond \sqcap L_2^\diamond.$$

Proof. All three follow directly from the definitions and the properties of min, max, and complement on $[0, 1]$. $\square$

7. Metric structures

Now we equip $\overline{\mathcal{L}}$ with metrics and show that the lattice operations are continuous with respect to them—essential for optimization applications. Since $\mathcal{L} \subset \overline{\mathcal{L}}$, the metrics restrict to the admissible set. This section defines three metrics on $\overline{\mathcal{L}}$, proves that they are indeed metrics, and establishes the continuity of the lattice operations.

7.1. Basic metrics

We begin by defining three standard metrics on $\overline{\mathcal{L}}$.

Definition 7.1 (Distance between level operators). For $L_1 = L_{\alpha_1,\beta_1,\gamma_1}$, $L_2 = L_{\alpha_2,\beta_2,\gamma_2} \in \overline{\mathcal{L}}$, define:

- (1) **Chebyshev (supremum) distance:**

$$d_\infty(L_1, L_2) = \max\{|\alpha_1 - \alpha_2|, |\beta_1 - \beta_2|, |\gamma_1 - \gamma_2|\}.$$

- (2) **Manhattan (taxicab) distance:**

$$d_1(L_1, L_2) = |\alpha_1 - \alpha_2| + |\beta_1 - \beta_2| + |\gamma_1 - \gamma_2|.$$

- (3) **Euclidean distance:**

$$d_2(L_1, L_2) = \sqrt{(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2 + (\gamma_1 - \gamma_2)^2}.$$

The following theorem confirms that these are indeed metrics.

Theorem 7.2. d_∞ , d_1 , and d_2 are metrics on $\overline{\mathcal{L}}$.

Proof. For each metric, the properties follow directly from the corresponding properties of absolute value and the Euclidean norm on $\mathbb{R}^3$ [32, 33]:

- **Non-negativity:** $d(L_1, L_2) \geq 0$, and $d(L_1, L_2) = 0$ iff $L_1 = L_2$.
- **Symmetry:** $d(L_1, L_2) = d(L_2, L_1)$.
- **Triangle inequality:** $d(L_1, L_3) \leq d(L_1, L_2) + d(L_2, L_3)$.

$\square$

Remark 7.3. These metrics resemble those used for orthogonal fuzzy metric spaces [34]. The novelty lies in the target: We apply them to the parameter space of level operators, opening the door to optimization over threshold values.

We now show that the lattice operations are continuous with respect to these metrics.

Proposition 7.4. *The lattice operations $\sqcap$ and $\sqcup$ are continuous with respect to d_∞ , d_1 , and d_2 .*

Proof. Both $\sqcap$ and $\sqcup$ are defined componentwise via min and max, and both min and max are continuous on $[0, 1]$. The result follows at once. For d_∞ ,

$$d_\infty(L_1 \sqcap L_2, L'_1 \sqcap L'_2) \leq \max\{d_\infty(L_1, L'_1), d_\infty(L_2, L'_2)\}.$$

The same holds for d_1 and d_2 . □

We illustrate the metrics with an explicit example.

Example 7.5. For $L_1 = L_{0.5,0.3,0.2}$ and $L_2 = L_{0.4,0.3,0.1}$,

$$d_\infty(L_1, L_2) = \max\{0.1, 0.0, 0.1\} = 0.1,$$

$$d_1(L_1, L_2) = 0.1 + 0.0 + 0.1 = 0.2,$$

$$d_2(L_1, L_2) = \sqrt{0.1^2 + 0.0^2 + 0.1^2} = \sqrt{0.02} \approx 0.141.$$

7.2. Distance with respect to a picture fuzzy set

We close this section with a distance that depends on a fixed picture fuzzy set.

Definition 7.6. For a fixed picture fuzzy set P on a universe X and $L_1, L_2 \in \overline{\mathcal{L}}$, define:

$$d_P(L_1, L_2) = \sup_{x \in X} (|\mu_{L_1(P)}(x) - \mu_{L_2(P)}(x)| + |\eta_{L_1(P)}(x) - \eta_{L_2(P)}(x)| + |\nu_{L_1(P)}(x) - \nu_{L_2(P)}(x)|).$$

Proposition 7.7. *For any picture fuzzy set P , d_P is a pseudometric on $\overline{\mathcal{L}}$. It is a metric if and only if the mapping $L \mapsto L(P)$ is injective.*

Proof. d_P satisfies every metric property except possibly the identity condition: $d_P(L_1, L_2) = 0$ implies $L_1(P) = L_2(P)$, but not necessarily $L_1 = L_2$. If $L \mapsto L(P)$ is injective, then d_P is a metric. □

8. Geometric interpretation

We close the theoretical development with a geometric picture. The operator simplex is a tetrahedron in $\mathbb{R}^3$ that visualizes the entire family of level operators and their algebraic operations. We begin by defining the simplex, then interpret the operations geometrically, and finally summarize the results in a table.

8.1. The operator simplex

We begin with the definition of the operator simplex.

Definition 8.1 (Operator simplex). Each level operator $L_{\alpha,\beta,\gamma}$ corresponds to a point (α, β, γ) in the operator simplex:

$$C = \{(\alpha, \beta, \gamma) \in [0, 1]^3 : 0 \leq \alpha + \beta + \gamma \leq 1\}.$$

The extended space $\overline{\mathcal{L}}$ corresponds to the full cube $[0, 1]^3$, while the admissible set $\mathcal{L}$ corresponds to the simplex C .

Remark 8.2. C is a 3-dimensional simplex (tetrahedron) with vertices:

$$(0, 0, 0), \quad (1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1).$$

Several features deserve note:

- C is convex—it is defined by linear inequalities.
- It is a simplex because it is the convex hull of four affinely independent points.
- The simplex structure reflects the constraint $\alpha + \beta + \gamma \leq 1$ on the parameters.
- Because the simplex is affine, convex combinations of operator parameters correspond to convex combinations of the corresponding points in $\mathbb{R}^3$.

Remark 8.3 (Geometric interpretation of vertices).

- $(0, 0, 1)$: identity operator $L_{0,0,1}$ (no filtering).
- $(1, 0, 0)$: keeps only elements with $\mu_P(x) \geq 1$ (perfect positive membership).
- $(0, 1, 0)$: keeps only elements with $\eta_P(x) \geq 1$ (perfect neutral membership).
- $(0, 0, 0)$: keeps only elements with $\nu_P(x) \leq 0$ (zero negative membership).
- Face $\alpha + \beta + \gamma = 1$: refusal degree $\pi_P(x)$ becomes zero for surviving elements.

Remark 8.4. The operator simplex is analogous to the use of barycentric coordinates in convex geometry. It gives a visual handle on the parameter space and on convex combinations of thresholds.

8.2. Geometric interpretation of operations

Under the simplex representation, the algebraic operations acquire a direct geometric meaning.

Proposition 8.5. Under the representation $L_{\alpha,\beta,\gamma} \leftrightarrow (\alpha, \beta, \gamma)$:

(1) **Meet** $\sqcap$:

$$(\alpha_1, \beta_1, \gamma_1) \sqcap (\alpha_2, \beta_2, \gamma_2) = (\min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2), \max(\gamma_1, \gamma_2)).$$

(2) **Join** $\sqcup$:

$$(\alpha_1, \beta_1, \gamma_1) \sqcup (\alpha_2, \beta_2, \gamma_2) = (\max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2), \min(\gamma_1, \gamma_2)).$$

(3) **Composition** $\circ$: Same as join.

(4) **Order** $\leq$:

$$(\alpha_1, \beta_1, \gamma_1) \leq (\alpha_2, \beta_2, \gamma_2) \iff \alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \geq \gamma_2.$$

(5) **Dual** $\cdot^\circ$:

$$(\alpha, \beta, \gamma)^\circ = (1 - \alpha, 1 - \beta, 1 - \gamma).$$

Remark 8.6. The identity $\circ = \sqcup$ is the most striking feature of this table. It turns the monoid operation on $(\bar{\mathcal{L}}, \circ)$ into a commutative, idempotent operation identical to the lattice join, considerably simplifying the algebraic structure.

8.3. Summary of geometric and algebraic properties

Table 3 summarizes the operations, formulas, and their geometric readings.

Table 3. Summary of operations, formulas, and geometric interpretations.

Operation	Algebraic formula	Geometric interpretation
Identity	$L_{0,0,1}$	Vertex $(0, 0, 1)$
Meet $\sqcap$	$(\min \alpha, \min \beta, \max \gamma)$	Componentwise min/max
Join $\sqcup$	$(\max \alpha, \max \beta, \min \gamma)$	Componentwise max/min
Composition $\circ$	$(\max \alpha, \max \beta, \min \gamma)$	Same as join
Order $\leq$	$\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \geq \gamma_2$	Componentwise with γ reversed
Dual $\cdot^\diamond$	$(1 - \alpha, 1 - \beta, 1 - \gamma)$	Reflection through $(0.5, 0.5, 0.5)$
Distance d_∞	$\max(\Delta\alpha , \Delta\beta , \Delta\gamma)$	Chebyshev distance
Distance d_1	$ \Delta\alpha + \Delta\beta + \Delta\gamma $	Manhattan distance
Distance d_2	$\sqrt{(\Delta\alpha)^2 + (\Delta\beta)^2 + (\Delta\gamma)^2}$	Euclidean distance

9. Potential applications

The framework we have built lends itself to several application areas. In this section, we discuss these applications and provide a detailed illustrative example.

- **Multi-criteria decision-making:** Each criterion can be represented as a picture fuzzy set (positive, neutral, negative satisfaction). Level operators filter alternatives based on simultaneous thresholds, while the lattice structure helps aggregate different decision-makers' thresholds [11, 36].
- **Image processing:** Pixels can be represented as picture fuzzy sets based on color channels or intensity levels. Level operators then threshold pixel memberships across three channels simultaneously, enabling sophisticated filtering.
- **Pattern recognition and classification:** The metrics d_∞, d_1, d_2 on $\bar{\mathcal{L}}$ measure similarity between thresholding strategies—useful for finding optimal threshold parameters for classification.
- **Fuzzy control systems:** because $\circ = \sqcup$, combining multiple thresholding rules is efficient: applying two operators in sequence is the same as applying a single operator with componentwise extremal thresholds.
- **Optimization over threshold spaces:** The metrics allow us to formulate optimization problems over thresholding strategies, with direct applications in risk-aware decision-making and multi-objective optimization [37].

9.1. Illustrative example: multi-criteria decision-making

Consider a decision problem with three alternatives A_1 – A_3 evaluated by three criteria C_1 – C_3 . Each criterion is a picture fuzzy set, with μ for satisfaction, η for neutrality, and ν for dissatisfaction [38]:

	C_1	C_2	C_3
A_1	(0.8, 0.1, 0.1)	(0.6, 0.3, 0.1)	(0.4, 0.2, 0.4)
A_2	(0.7, 0.2, 0.1)	(0.9, 0.1, 0.0)	(0.5, 0.3, 0.2)
A_3	(0.5, 0.3, 0.2)	(0.4, 0.4, 0.2)	(0.8, 0.1, 0.1)

Suppose the decision-maker applies $L_{0.6,0.2,0.2}$, requiring satisfaction ≥ 0.6 , neutrality ≥ 0.2 , and dissatisfaction ≤ 0.2 . Under Definition 2.18, the global rejection test applies first: if $\nu > 0.2$, the element is fully rejected. The filtered sets become:

	C_1	C_2	C_3
A_1	(0.8, 0, 0.1)	(0.6, 0.3, 0.1)	(0, 0, 1)
A_2	(0.7, 0.2, 0.1)	(0.9, 0, 0)	(0, 0.3, 0.2)
A_3	(0, 0.3, 0.2)	(0, 0.4, 0.2)	(0.8, 0, 0.1)

Let us verify a few entries. For A_1 under C_1 : $\nu = 0.1 \leq 0.2$, so the global test passes; $\mu = 0.8 \geq 0.6$ is preserved; $\eta = 0.1 < 0.2$ is set to 0. For A_1 under C_3 : $\nu = 0.4 > 0.2$, so the element is fully rejected: (0, 0, 1). For A_2 under C_1 : $\nu = 0.1 \leq 0.2$ passes; $\mu = 0.7 \geq 0.6$ preserved; $\eta = 0.2 \geq 0.2$ preserved. For A_2 under C_3 : $\nu = 0.2 \leq 0.2$ passes; $\mu = 0.5 < 0.6$ is set to 0; $\eta = 0.3 \geq 0.2$ preserved; $\nu_L = 0.2$. Result: (0, 0.3, 0.2). For A_3 under C_1 : $\nu = 0.2 \leq 0.2$ passes; $\mu = 0.5 < 0.6$ is set to 0; $\eta = 0.3 \geq 0.2$ preserved; $\nu_L = 0.2$. Result: (0, 0.3, 0.2). For A_3 under C_2 : $\nu = 0.2 \leq 0.2$ passes; $\mu = 0.4 < 0.6$ is set to 0; $\eta = 0.4 \geq 0.2$ preserved; $\nu_L = 0.2$. Result: (0, 0.4, 0.2).

Now the table is fully consistent with Definition 2.18. Alternative A_2 satisfies the threshold for C_1 fully and for C_2 partially (positive membership preserved, neutrality dropped to 0). A_3 satisfies C_3 fully. Yet no alternative satisfies all three criteria simultaneously under these strict thresholds—a good illustration of how level operators identify which alternatives meet minimum requirements across multiple criteria. Note also the global rejection effect: A_1 under C_3 is fully rejected because its negative membership exceeds 0.2.

The decision-maker can then:

- (1) Use the lattice structure to combine thresholds from different decision-makers (meet/join operations).
- (2) Use the metrics d_∞, d_1, d_2 to measure distances between threshold strategies.
- (3) Adjust thresholds systematically to find an optimal filtering strategy.

This example shows how the algebraic structures developed here provide a rigorous foundation for practical decision-making [39].

10. Conclusions

We have systematically investigated the algebraic and geometric structure of picture fuzzy level operators, building on the level subset theory of Kankaew et al. [1] and extending the classical fuzzy and intuitionistic level operator theories. In this final section, we summarize our main contributions, discuss the significance of the results, and outline directions for future research.

Our main contributions are as follows:

- (1) A partial order $\leq$ on $\overline{\mathcal{L}}$ with componentwise order on (α, β) and reversed order on γ , and the identification of $\mathcal{L}$ as a sub-poset (Theorem 3.3).
- (2) A distributive, complete lattice structure $(\overline{\mathcal{L}}, \sqcap, \sqcup)$, with the admissible set $\mathcal{L}$ characterized as closed under meet but not under join (Theorem 4.3, Corollary 4.5).
- (3) A commutative idempotent monoid $(\overline{\mathcal{L}}, \circ)$ with $\circ = \sqcup$, and the observation that $\mathcal{L}$ is not closed under composition (Theorem 5.3, Corollary 5.5).
- (4) The classification that only the identity $L_{0,0,1}$ has a true inverse (Theorem 6.2).
- (5) Dual operators with natural duality properties (Proposition 6.6).
- (6) Metric structures (Chebyshev, Manhattan, and Euclidean) on $\overline{\mathcal{L}}$ with continuity of lattice operations (Theorem 7.2, Proposition 7.4).
- (7) A geometric interpretation via the operator simplex, a tetrahedron in $\mathbb{R}^3$ (Section 8).

This work provides a coherent framework for understanding picture fuzzy level operators as mathematical objects with rich algebraic and geometric structure, generalizing both the classical fuzzy and intuitionistic fuzzy cases. Recent developments in picture fuzzy algebraic structures [22, 23] reinforce the relevance of this direction.

10.1. Discussion of results

Several of our results—particularly the lattice and monoid structures—arise from the componentwise application of operations on $[0, 1]$. That might suggest these results are merely inherited from existing structures. We emphasize three points in response:

- (1) The identity $\circ = \sqcup$ is a novel, non-trivial observation specific to picture fuzzy level operators.
- (2) The distinction between the extended space $\overline{\mathcal{L}}$ and the admissible set $\mathcal{L}$ clarifies the closure properties of the algebraic operations and highlights the role of the picture fuzzy constraint.
- (3) The operator simplex offers a new perspective on the parameter space of level operators.

10.2. Future research directions

Several directions suggest themselves:

- **Categorical properties:** Investigating the categorical structure of level operators, including functorial relationships between fuzzy, intuitionistic, and picture fuzzy level operators.
- **Extensions to other frameworks:** Generalizing to Neutrosophic sets [12–14], complex neutrosophic sets [15], Plithogenic sets [16], Hyperfuzzy sets [17], spherical fuzzy sets [18], and q -rung orthopair fuzzy sets [19]. The three-parameter picture fuzzy setting is a special case of these broader frameworks, so the algebraic properties established here may serve as a foundation elsewhere. The dual operator issue discussed in Section 6.2, in particular, suggests that extending the operator space (as in Neutrosophic or Plithogenic frameworks) may yield more complete algebraic properties.
- **HyperAlgebra and SuperHyperAlgebra:** The algebraic framework developed here could be further extended to HyperAlgebra and SuperHyperAlgebra [40], as well as to SuperHyperfuzzy Sets, where level operators may admit richer structural properties. These generalized frameworks offer a broader setting in which the admissible set and the extended space could be naturally embedded.

- **Applications:** Developing decision-making algorithms based on the metric and lattice structures. The metric spaces $(\overline{\mathcal{L}}, d_\infty)$, $(\overline{\mathcal{L}}, d_1)$, and $(\overline{\mathcal{L}}, d_2)$ allow optimization over threshold parameters, which is valuable for automated decision systems [37].
- **Probabilistic interpretations:** Exploring probabilistic models of level operators and their relationship to random thresholding.
- **Adjoint operators:** Investigating a full adjunction in the categorical sense for the dual operator, perhaps by extending the operator space to $\overline{\mathcal{L}} = [0, 1]^3$ as discussed in Section 6.2.
- **Operator learning:** Using the metrics to develop learning algorithms that automatically select optimal thresholds from data.
- **Generalized picture fuzzy structures:** Extending the results to m -polar picture fuzzy structures [22] and Type-3 order picture fuzzy ideals [35].

Author contributions

Ümit Deniz: Conceptualization, Formal analysis, Investigation, Methodology, Writing–original draft, Writing–review and editing.

The author has read and approved the final version of the manuscript.

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Conflict of interest

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