



*Research article***Coefficient-matched random Lyapunov functionals and delay-robust small-gain criteria for stochastic delay systems****Zhao-sheng Wang^{1,*} and Guan-fa Li²**¹ Department of Mathematics, Ganzhou Teachers College, Ganzhou 341000, Jiangxi, China² School of Information Engineering, Gannan University of Science and Technology, Ganzhou 341000, Jiangxi, China*** Correspondence:** Email: WangZS3481@126.com.

Abstract: Delay and multiplicative noise complicate the analysis of long-time behavior in stochastic evolution equations. We study pathwise energy estimates when delayed states acquire random coefficients after an Ornstein–Uhlenbeck transformation. For positive delays and nonzero noise, the delay amplification factor is tempered but almost surely unbounded in time, so it has no uniform deterministic majorant. We construct a Lyapunov–Krasovskii functional whose history density uses the same random coefficient at a shifted time. Differentiation then reproduces the coefficient multiplying the delayed energy and closes the estimate. For a coupled fractional parabolic system with monotone polynomial damping and measure-valued memory, this gives $L_D < \Lambda_{\text{lin}}$, where L_D bounds the delay operator and Λ_{lin} is the smallest componentwise first-mode linear damping. The boundary is exact in the deterministic scalar first-mode linear subclass for each fixed delay, although the decay rate can deteriorate as the delay grows. Gaussian concentration yields an explicit sufficient noise threshold for positive mean dissipation. Under the stated assumptions, the resulting cocycle has a measurable tempered pullback absorbing family and a unique pullback random attractor, with upper semicontinuity as delay and noise vanish. A retarded finite-dimensional gradient system with diagonal linear drift and a separable convex potential illustrates the construction through time equicontinuity.

Keywords: stochastic delay system; fractional parabolic system; multiplicative Stratonovich noise; random Lyapunov–Krasovskii functional; delay-robust small-gain criterion; explicit noise threshold; pullback random attractor

Mathematics Subject Classification: 35R11, 35K55, 60H15, 37L55, 35B40, 34K50

1. Introduction

Fractional parabolic equations combine nonlocal spatial diffusion with the smoothing and dissipation of evolution equations. Random dynamical systems and pullback attractors provide the pathwise language for long-time stochastic behavior [1, 2]. The nonautonomous setting is developed in [3, 4]. The compactness arguments draw on the theory of dissipative partial differential equations [5–7]. Stochastic evolution, functional delay equations, and semigroup methods are used in the forms developed in [8–10], while the time-compactness step follows Simon’s criterion [11]. The spectral Dirichlet fractional operator is treated through its sub-Markov semigroup and the associated fractional Sobolev scale [12, 13].

Random attractors are already well developed for parabolic equations that combine fractional diffusion, delay, and multiplicative noise. Chen and Zhang established pullback attractors and upper semicontinuity for non-autonomous fractional stochastic parabolic equations with delay [14]. Zhang treated the delayed fractional stochastic p -Laplacian with multiplicative noise [15], and Wang et al. developed a foundational attractor theory for non-autonomous delay parabolic equations [16]. Subsequent work addresses unbounded domains [17], delayed reaction–diffusion equations [18, 19], dimensions and exponential attraction [20–22], and stability of attractors under delay or fractional perturbations [23–25]. Material memory and locally monotone stochastic partial differential equations (SPDEs) provide complementary settings [26, 27].

A second line of work studies stochastic functional differential equations through moment, asymptotic, and exponential stability. Shaikhet developed a systematic Lyapunov-functional approach for stochastic hereditary equations [28]; Lu et al. obtained comparison-principle criteria for stochastic delay equations [29]; and segmented Lyapunov–Krasovskii functionals reduce conservatism in linear time-delay tests [30]. These methods concern stability in probability or moments or deterministic matrix criteria. Here we instead seek pathwise energy estimates for a random cocycle.

Recent delay-system results use complementary methods. Ma et al. [31] and Yang et al. [32] establish practical exponential stability for impulsive stochastic functional systems through Lyapunov–Razumikhin arguments. Ma et al. [33] study inverse-optimal stabilization of recurrent neural networks with proportional delays. Wang et al. [34] use time translation for asynchronous consensus, while Tian and coauthors study nonlinear Halanay inequalities [35] and reachable-set estimates [36]. The present analysis instead concerns closure of a pathwise cocycle energy estimate with an unbounded transformed delay coefficient.

Related fractional-order control applications include guaranteed-cost fuzzy control of optical systems [37], fuzzy sliding-mode stabilization with input nonlinearities [38], sliding-mode synchronization of memristive neural networks [39], and adaptive model-free synchronization [40]. These studies provide Lyapunov-based control and synchronization context; they do not concern the spectral spatial operator and random delay coefficient considered here.

After the Ornstein–Uhlenbeck transformation, the delayed state carries a ratio of random exponential multipliers. For positive delays and nonzero noise, the amplification coefficient in the transformed delay estimate has finite moments and tempered growth but is almost surely unbounded along time, as shown in Proposition 2.1. The usual deterministic history energy

$$\int_{t-\tau}^t e^{-\delta(t-r)} \|Y(r)\|_H^2 dr$$

does not provide a deterministic uniform bound for that coefficient. Temperedness controls long-time

growth, not a pointwise bound uniform in time.

We consider the real two-component system

$$\begin{cases} du_i + \left[\nu_i (-\Delta_D)^{s_i} u_i + \gamma_i u_i + a_i |u_i|^{p_i-2} u_i - \mathcal{D}_{\tau,i}(U_t) - f_i(t) \right] dt = \varepsilon_i u_i \circ dW_i(t), \\ i = 1, 2, \end{cases} \quad (1.1)$$

where $U = (u_1, u_2)$, $U_t(\vartheta) = U(t + \vartheta)$ for $\vartheta \in [-\tau, 0]$, $0 < s_i < 1$, $\nu_i, \gamma_i > 0$, $a_i \geq 0$, and W_1, W_2 are independent two-sided Brownian motions. Homogeneous Dirichlet boundary conditions are encoded by the spectral fractional Laplacian $(-\Delta_D)^{s_i}$. The symbol θ_t is reserved for the driving metric flow, whereas ϑ denotes a delay coordinate. The delay is described by a Borel probability measure on $[-\tau, 0]$; distributed memory and the discrete delay $\delta_{-\tau}$ therefore enter the same framework.

Here “fractional” refers to the spatial spectral power $(-\Delta_D)^{s_i}$ with $0 < s_i < 1$, rather than to a fractional derivative in time. In the integer-order counterpart $s_i = 1$, this operator reduces to the classical Dirichlet Laplacian: on the m th eigenmode the diffusive contribution is $\nu_i \lambda_m$, whereas in the present model it is $\nu_i \lambda_m^{s_i}$. The energy factor is the spectral form domain $D((-\Delta_D)^{s_i/2})$ defined in Section 2.1; at $s_i = 1$ it is $H_0^1(D)$. Thus, the fractional model has spatially nonlocal diffusion and a different spectral and compactness scale. The coefficient-matching argument itself, however, does not rely on fractional diffusion; Section 5.5 applies the same pathwise closure mechanism to a finite-dimensional retarded gradient system.

To describe the construction, suppose that a nonnegative scalar energy X satisfies

$$X'(t) + a(\theta_t \omega) X(t) \leq b(\theta_t \omega) \int_{-\tau}^0 X(t + \vartheta) \mu(d\vartheta) + h(t). \quad (1.2)$$

Here a and b are scalar coefficients on the driving probability space, $b, h \geq 0$, and μ is a Borel probability measure on $[-\tau, 0]$. For $\delta > 0$, define

$$\mathcal{M}_b(t) = \int_{-\tau}^0 \int_{t+\vartheta}^t e^{-\delta(t-r)} b(\theta_{r-\vartheta} \omega) X(r) dr \mu(d\vartheta). \quad (1.3)$$

We call the functional (1.3) coefficient-matched with respect to b in (1.2). Its history density evaluates b at $r - \vartheta$; at the lower endpoint $r = t + \vartheta$, this argument is t . The lower-endpoint term in its derivative is therefore

$$-e^{\delta\vartheta} b(\theta_t \omega) X(t + \vartheta).$$

Thus, the coefficient multiplying the delayed energy in (1.2) is reproduced in the derivative. Theorem 2.2 gives the resulting differential inequality.

For the fractional system, retaining a fraction $\sigma \in (0, 1)$ of the spatial dissipation for compactness and optimizing the remaining spectral margin gives $L_D < \Lambda_{\text{lin}}$, where Λ_{lin} is the smallest componentwise first-mode linear damping. In the deterministic scalar first-mode linear subclass, this boundary is necessary and sufficient for exponential stability at each fixed delay; the decay rate need not remain uniform as the delay grows. Gaussian concentration and an estimate for the compact-window Ornstein–Uhlenbeck maximum give an explicit sufficient noise threshold for positive mean dissipation. The resulting energy estimates yield pullback random attractors and upper semicontinuity for the nonlinear fractional system. A retarded finite-dimensional gradient system with diagonal linear drift and a separable convex potential provides a second application.

The cited attractor results provide the dynamical framework for this analysis. In the finite-dimensional setting, Wu and Kloeden [41] study mean-square random attractors, whereas attraction here is pathwise. Lyapunov–Krasovskii history integrals are classical [9, 28]; the construction studied here concerns the shifted placement of the random coefficient in (1.3).

The manuscript has six main sections. Section 2 gives the model, assumptions, and the abstract coefficient-matching theorem. Section 3 proves global well-posedness and cocycle generation. Section 4 establishes the optimized small-gain condition, the exact scalar first-mode boundary, and the explicit noise threshold. Section 5 proves pullback attraction and upper semicontinuity and develops the fractional and finite-dimensional applications. Section 6 summarizes the conclusions and outlines extensions beyond the present two-application framework.

2. Model, assumptions, and the coefficient-matching principle

2.1. Fractional phase spaces and the delay operator

Let $D \subset \mathbb{R}^N$ be bounded with smooth boundary, and let $\{e_m\}_{m \geq 1}$ be an orthonormal eigenbasis of the Dirichlet Laplacian,

$$-\Delta e_m = \lambda_m e_m, \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_m \rightarrow \infty.$$

For $0 < s < 1$,

$$\begin{aligned} (-\Delta_D)^s w &= \sum_{m=1}^{\infty} \lambda_m^s (w, e_m)_{L^2} e_m, \\ D((-\Delta_D)^s) &= \left\{ w : \sum_{m=1}^{\infty} \lambda_m^{2s} |(w, e_m)_{L^2}|^2 < \infty \right\}. \end{aligned}$$

Set

$$H = L^2(D; \mathbb{R})^2, \quad V = V_1 \times V_2, \quad V_i = D((-\Delta_D)^{s_i/2}), \quad W = D((-\Delta_D)^{s_1}) \times D((-\Delta_D)^{s_2}). \quad (2.1)$$

Here each V_i is the quadratic-form domain of $(-\Delta_D)^{s_i}$ on real $L^2(D)$. This spectral definition incorporates the Dirichlet boundary convention, including the borderline order $s_i = 1/2$, without identifying the form domain with an unrestricted trace space. We use

$$\begin{aligned} \|U\|_H^2 &= \sum_{i=1}^2 \|u_i\|_{L^2}^2, \\ \|U\|_V^2 &= \sum_{i=1}^2 \nu_i \|(-\Delta_D)^{s_i/2} u_i\|_{L^2}^2, \\ \|U\|_W^2 &= \sum_{i=1}^2 \|(-\Delta_D)^{s_i} u_i\|_{L^2}^2. \end{aligned}$$

The compact chain

$$W \Subset V \Subset H \equiv H' \subset V' \quad (2.2)$$

holds on bounded D . For a fixed delay $\tau \geq 0$,

$$\mathcal{H}_\tau = C([- \tau, 0]; H), \quad \|\Phi\|_{\mathcal{H}_\tau} = \sup_{\vartheta \in [- \tau, 0]} \|\Phi(\vartheta)\|_H.$$

When $\tau = 0$, $\mathcal{H}_0$ is identified with H .

Let μ_τ be a Borel probability measure on $[- \tau, 0]$. The operator $\mathcal{D}_\tau : \mathcal{H}_\tau \rightarrow H$ satisfies, uniformly for the delay interval considered,

$$\|\mathcal{D}_\tau(\Phi) - \mathcal{D}_\tau(\Psi)\|_H^2 \leq L_D^2 \int_{-\tau}^0 \|\Phi(\vartheta) - \Psi(\vartheta)\|_H^2 \mu_\tau(d\vartheta), \quad (2.3)$$

$$\|\mathcal{D}_\tau(\Phi)\|_H^2 \leq L_D^2 \int_{-\tau}^0 \|\Phi(\vartheta)\|_H^2 \mu_\tau(d\vartheta) + c_D. \quad (2.4)$$

The constant $L_D > 0$ is a chosen common bound; a zero delay operator is included by choosing any positive bound. No density assumption is imposed on μ_τ ; in particular, $\mu_\tau = \delta_{-\tau}$ is allowed.

We impose the following standing hypotheses.

(H1) $0 < s_i < 1$, $\nu_i, \gamma_i > 0$, and $a_i \geq 0$. If $a_i = 0$, set $p_i = 2$. If $a_i > 0$, assume $p_i \geq 2$ and, when $N > 2s_i$,

$$2 \leq p_i < \frac{2N}{N - 2s_i}; \quad (2.5)$$

when $N \leq 2s_i$, every finite $p_i \geq 2$ is admissible.

(H2) The delay operators satisfy (2.3)-(2.4), and the map $\tau \mapsto \mathcal{D}_\tau(0)$ is measurable.

(H3) The deterministic forcing $f = (f_1, f_2) \in L_{\text{loc}}^2(\mathbb{R}; H)$ satisfies, for every $s \in \mathbb{R}$ and every $\alpha > 0$,

$$\int_{-\infty}^0 e^{\alpha r} \|f(s+r)\|_H^2 dr < \infty.$$

Condition (H3) is a standing assumption not a consequence of $f \in L_{\text{loc}}^2$. For example, if $\sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(r)\|_H^2 dr \leq C_f$, the integral in (H3) is at most $C_f \sum_{n \geq 0} e^{-\alpha n} < \infty$ for every $\alpha > 0$. The same argument applies when, for each s and $\eta > 0$, these backward unit-interval integrals are bounded by $C_{s,\eta} e^{\eta n}$. Requiring every positive weight keeps the forcing class independent of the positive dissipation rate chosen later.

(H4) For the higher-regularity step, the delay term obeys the same measure estimate on V :

$$\begin{aligned} \|\mathcal{D}_\tau(\Phi) - \mathcal{D}_\tau(\Psi)\|_V^2 &\leq L_{D,V}^2 \int_{-\tau}^0 \|\Phi(\vartheta) - \Psi(\vartheta)\|_V^2 \mu_\tau(d\vartheta), \\ \|\mathcal{D}_\tau(\Phi)\|_V^2 &\leq L_{D,V}^2 \int_{-\tau}^0 \|\Phi(\vartheta)\|_V^2 \mu_\tau(d\vartheta) + c_{D,V}, \end{aligned}$$

whenever the histories take values in V . A directly verifiable case is $s_1 = s_2 = s$ and $\mathcal{D}_\tau(\Phi) = B \int_{-\tau}^0 \Phi(\vartheta) \mu_\tau(d\vartheta)$ for a constant matrix B and any probability measure. For $N_\nu = \text{diag}(\sqrt{\nu_1}, \sqrt{\nu_2})$, both estimates hold with $L_{D,V} = \|N_\nu B N_\nu^{-1}\|_{\mathcal{L}(\mathbb{R}^2)}$ and $c_{D,V} = 0$. Section 5.4 gives the proof, the H -constant, and a sufficient condition for unequal orders.

(H5) For the limit $\tau \rightarrow 0$, there is an instantaneous map $\mathcal{D}_0 : H \rightarrow H$ such that for every compact set $K \subset C([- \tau_*, 0]; H)$,

$$\limsup_{\tau \rightarrow 0} \sup_{\Phi \in K} \left\| \mathcal{D}_\tau(\Phi|_{[-\tau, 0]}) - \mathcal{D}_0(\Phi(0)) \right\|_H = 0. \quad (2.6)$$

The strict inequalities in (2.5) give the compact embeddings $V_i \Subset L^{p_i}(D)$. The identification of the polynomial weak limit in Section 3.1 uses almost-everywhere convergence and the L^{p_i} energy bound on each fixed sample path; it does not require strong convergence at a critical exponent.

2.2. Ornstein–Uhlenbeck conjugacy

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ be the canonical ergodic metric dynamical system generated by two independent two-sided Brownian motions. Let

$$dz_i + z_i dt = dW_i(t), \quad z_i(\theta_t \omega) = \int_{-\infty}^0 e^r dW_i(t+r)$$

be the stationary Ornstein–Uhlenbeck process. Define

$$\mathcal{Z}_\varepsilon(\omega) = \text{diag}(e^{\varepsilon_1 z_1(\omega)}, e^{\varepsilon_2 z_2(\omega)}).$$

For a solution beginning at time s , write

$$U(s+t) = \mathcal{Z}_\varepsilon(\theta_t \omega) Y(t), \quad t \geq 0. \quad (2.7)$$

The transformed equation is

$$\frac{dY}{dt} + \mathbb{A}_\varepsilon(\theta_t \omega) Y + \mathbb{F}_\varepsilon(\theta_t \omega, Y) - \mathbb{D}_{\tau, \varepsilon}(\theta_t \omega, Y_t) = \mathbb{G}_\varepsilon(s+t, \theta_t \omega), \quad (2.8)$$

where

$$\begin{aligned} (\mathbb{A}_\varepsilon(\omega) Y)_i &= \nu_i(-\Delta_D)^{s_i} y_i + (\gamma_i - \varepsilon_i z_i(\omega)) y_i, \\ (\mathbb{F}_\varepsilon(\omega, Y))_i &= a_i e^{(p_i-2)\varepsilon_i z_i(\omega)} |y_i|^{p_i-2} y_i, \\ \mathbb{G}_\varepsilon(t, \omega) &= \mathcal{Z}_\varepsilon(\omega)^{-1} f(t), \\ \mathbb{D}_{\tau, \varepsilon}(\omega, \Phi) &= \mathcal{Z}_\varepsilon(\omega)^{-1} \mathcal{D}_\tau(\vartheta \mapsto \mathcal{Z}_\varepsilon(\theta_\vartheta \omega) \Phi(\vartheta)). \end{aligned}$$

Define $\|\varepsilon\|_\infty = \max_i |\varepsilon_i|$ and the stationary amplification ratio

$$\kappa_{\tau, \varepsilon}(\omega) = \left\| \mathcal{Z}_\varepsilon(\omega)^{-1} \right\|_{\mathcal{L}(H)}^2 \sup_{\vartheta \in [-\tau, 0]} \left\| \mathcal{Z}_\varepsilon(\theta_\vartheta \omega) \right\|_{\mathcal{L}(H)}^2. \quad (2.9)$$

Then, (2.4) gives

$$\left\| \mathbb{D}_{\tau, \varepsilon}(\omega, \Phi) \right\|_H^2 \leq L_D^2 \kappa_{\tau, \varepsilon}(\omega) \int_{-\tau}^0 \left\| \Phi(\vartheta) \right\|_H^2 \mu_\tau(d\vartheta) + c_D \left\| \mathcal{Z}_\varepsilon(\omega)^{-1} \right\|_{\mathcal{L}(H)}^2. \quad (2.10)$$

The analogous difference estimate follows from (2.3). Since the Ornstein–Uhlenbeck process is continuous and Gaussian on compact intervals, for every $1 \leq m < \infty$, $\tau_* < \infty$, and bounded noise set,

$$\sup_{0 \leq \tau \leq \tau_*} \mathbb{E} \kappa_{\tau, \varepsilon}^m < \infty, \quad \sup_{0 \leq \tau \leq \tau_*} \left\| \kappa_{\tau, \varepsilon} - 1 \right\|_{L^m(\Omega)} \longrightarrow 0 \quad \text{as } \|\varepsilon\|_\infty \rightarrow 0. \quad (2.11)$$

These facts follow from Fernique’s theorem applied to the compact-time supremum in (2.9) [8].

2.3. Unboundedness of the transformed delay coefficient

The next proposition establishes unboundedness along almost every path.

Proposition 2.1. Fix $\tau > 0$. If $\varepsilon_i \neq 0$ for at least one component, then

$$\sup_{t \geq 0} \kappa_{\tau, \varepsilon}(\theta_t \omega) = \infty \quad \text{for } \mathbb{P}\text{-almost every } \omega. \quad (2.12)$$

Consequently, no deterministic constant can dominate the transformed delay coefficient for all time on a full-measure invariant set.

Proof. Choose an index i with $\varepsilon_i \neq 0$. By the definition of the operator norm,

$$\kappa_{\tau, \varepsilon}(\theta_t \omega) \geq \exp(2\varepsilon_i[z_i(\theta_{t-\tau} \omega) - z_i(\theta_t \omega)]).$$

The Gaussian process

$$\xi_i(t) = 2\varepsilon_i[z_i(\theta_{t-\tau} \omega) - z_i(\theta_t \omega)]$$

is stationary, ergodic, and nondegenerate because $\tau > 0$. Hence, for every $M > 1$,

$$\mathbb{P}\{e^{\xi_i(0)} > M\} > 0.$$

The continuous-time ergodic theorem applied to the indicator of this event shows that its exceedance times have positive asymptotic frequency almost surely. Thus, $e^{\xi_i(t)} > M$ for arbitrarily large t . Since M is arbitrary, (2.12) follows. $\square$

2.4. An abstract coefficient-matching theorem

The next theorem derives a closed scalar differential inequality. Here “coefficient-matched” refers specifically to the placement of $b(\theta_{r-\vartheta} \omega)$ in the functional (2.14), relative to the coefficient $b(\theta_t \omega)$ in (2.13). At the lower endpoint $r = t + \vartheta$, these coefficients agree. Let μ be a Borel probability measure on $[-\tau, 0]$.

Theorem 2.2 (Coefficient-matching principle). Let $X : [t_0 - \tau, \infty) \rightarrow [0, \infty)$ be continuous and locally absolutely continuous on $[t_0, \infty)$. Suppose that, for almost every $t \geq t_0$,

$$X'(t) + a(\theta_t \omega)X(t) \leq b(\theta_t \omega) \int_{-\tau}^0 X(t + \vartheta) \mu(d\vartheta) + h(t), \quad (2.13)$$

where $b, h \geq 0$ and the quantities below are locally integrable. For $\delta > 0$, define

$$\mathcal{M}_b(t) = \int_{-\tau}^0 \int_{t+\vartheta}^t e^{-\delta(t-r)} b(\theta_{r-\vartheta} \omega) X(r) dr \mu(d\vartheta) \quad (2.14)$$

and

$$B_b(\theta_t \omega) = \int_{-\tau}^0 b(\theta_{t-\vartheta} \omega) \mu(d\vartheta).$$

Then,

$$\frac{d}{dt} \mathcal{M}_b(t) = B_b(\theta_t \omega) X(t) - b(\theta_t \omega) \int_{-\tau}^0 e^{\delta\vartheta} X(t + \vartheta) \mu(d\vartheta) - \delta \mathcal{M}_b(t). \quad (2.15)$$

If $\chi e^{-\delta\tau} \geq 1$ and

$$\mathcal{E}_b(t) = X(t) + \chi \mathcal{M}_b(t),$$

then

$$\mathcal{E}'_b(t) + g_b(\theta_t\omega)\mathcal{E}_b(t) \leq h(t), \quad (2.16)$$

where

$$g_b(\omega) = \min\{a(\omega) - \chi B_b(\omega), \delta\}.$$

Equivalently, along the metric flow,

$$g_b(\theta_t\omega) = \min\{a(\theta_t\omega) - \chi B_b(\theta_t\omega), \delta\},$$

which is exactly the coefficient appearing in (2.16).

Proof. For fixed ϑ , the Leibniz rule gives

$$\begin{aligned} \frac{d}{dt} \int_{t+\vartheta}^t e^{-\delta(t-r)} b(\theta_{r-\vartheta}\omega) X(r) dr &= b(\theta_{t-\vartheta}\omega) X(t) \\ &\quad - e^{\delta\vartheta} b(\theta_t\omega) X(t+\vartheta) \\ &\quad - \delta \int_{t+\vartheta}^t e^{-\delta(t-r)} b(\theta_{r-\vartheta}\omega) X(r) dr. \end{aligned}$$

Integration with respect to μ yields (2.15). Adding χ times that identity to (2.13) produces

$$\begin{aligned} \mathcal{E}'_b(t) + [a(\theta_t\omega) - \chi B_b(\theta_t\omega)] X(t) + \chi \delta \mathcal{M}_b(t) \\ \leq b(\theta_t\omega) \int_{-\tau}^0 [1 - \chi e^{\delta\vartheta}] X(t+\vartheta) \mu(d\vartheta) + h(t). \end{aligned}$$

Since $e^{\delta\vartheta} \geq e^{-\delta\tau}$, the integral on the right is nonpositive. Finally, $g_b \leq a - \chi B_b$ and $g_b \leq \delta$, so the two nonnegative energy components imply (2.16). $\square$

Remark 2.3. If b were uniformly bounded, the standard deterministic Lyapunov–Krasovskii functional could be recovered by replacing the shifted coefficient in (2.14) with a constant majorant. Proposition 2.1 shows why that reduction is unavailable after a nontrivial Ornstein–Uhlenbeck conjugacy. Accordingly, the property used in (2.16) is the shifted placement of the coefficient in (2.14), rather than the presence of a history integral alone.

Dependence of cancellation on the memory measure. For the residual integral in the proof of Theorem 2.2, at a time when $b(\theta_t\omega) > 0$, nonpositivity for every nonnegative history is equivalent to

$$\chi e^{\delta\vartheta} \geq 1 \quad \mu\text{-almost everywhere,} \quad \text{or equivalently} \quad \chi \geq e^{\delta\tau_\mu}, \quad \tau_\mu = \operatorname{ess\,sup}_{\mu}(-\vartheta).$$

Indeed, if the coefficient $1 - \chi e^{\delta\vartheta}$ is positive on a set of positive measure, a nonnegative continuous history supported where it is positive makes the residual positive. Thus, $\chi e^{-\delta\tau} \geq 1$ is necessary when $\tau_\mu = \tau$, including $\mu = \delta_{-\tau}$, but only sufficient when the nominal interval exceeds the support. If $b(\theta_t\omega) = 0$, cancellation at that time is automatic. This is a condition for the displayed sign-definite cancellation, not a necessary condition for stability. Nonnegativity of $\mathcal{E}_b$ alone does not weaken this sign

condition. If $a, b \in L^1(\Omega)$, stationarity gives $\mathbb{E}B_b = \mathbb{E}b$, but this mean cannot replace $b(\theta, \omega)$ pointwise. In particular,

$$\mathbb{E}g_b = \mathbb{E} \min\{a - \chi B_b, \delta\}$$

is not determined by $\mathbb{E}b$ alone; positive mean of $a - \chi B_b$ need not give positive mean of its truncation at δ . A moment-based improvement must therefore also control negative fluctuations of the dissipation.

A kernel-dependent alternative is to use

$$\begin{aligned}\widehat{\mathcal{M}}_b(t) &= \int_{-\tau}^0 \int_{t+\vartheta}^t e^{-\delta(t-r+\vartheta)} b(\theta_{t-\vartheta}\omega) X(r) dr \mu(d\vartheta), \\ \widehat{B}_b(\omega) &= \int_{-\tau}^0 e^{-\delta\vartheta} b(\theta_{-\vartheta}\omega) \mu(d\vartheta).\end{aligned}$$

The Leibniz rule gives $\widehat{\mathcal{M}}'_b = \widehat{B}_b(\theta_t\omega)X - b(\theta_t\omega) \int X(t+\vartheta) \mu(d\vartheta) - \delta \widehat{\mathcal{M}}_b$. Hence, $\widehat{\mathcal{E}}_b = X + \widehat{\mathcal{M}}_b$ satisfies

$$\widehat{\mathcal{E}}'_b + \min\{a(\theta_t\omega) - \widehat{B}_b(\theta_t\omega), \delta\} \widehat{\mathcal{E}}_b \leq h.$$

For a prescribed family of probability measures, let

$$\bar{k}_\delta = \sup_{0 \leq \tau \leq \tau_*} \int_{-\tau}^0 e^{-\delta\vartheta} \mu_\tau(d\vartheta) \leq e^{\delta\tau_*}.$$

For a fixed μ , stationarity gives $\mathbb{E}\widehat{B}_b = k_\delta(\mu)\mathbb{E}b$, where $k_\delta(\mu) = \int e^{-\delta\vartheta} \mu(d\vartheta)$. In the PDE application, the same Gaussian bound used in Theorem 4.5 bounds $\widehat{B}_b$ by $b_0 \bar{k}_\delta e^{4eZ_{q,\tau_*}}$. Consequently, whenever $a_0 - \bar{k}_\delta b_0 > 0$, formula (4.14) remains a sufficient certificate with χ_* replaced by $\bar{k}_\delta$ and g_0 replaced by $\min\{a_0 - \bar{k}_\delta b_0, \delta\}$. This can improve the certificate when the measure is known; the original theorem retains the measure-independent choice.

3. Global well-posedness and cocycle generation

3.1. Galerkin construction and monotonicity

Lemma (Delay Gronwall estimate). *Let $T > 0$, let μ be a Borel probability measure on $[-\tau, 0]$, and let $X : [-\tau, T] \rightarrow [0, \infty)$ be continuous and locally absolutely continuous on $[0, T]$. Suppose that, for almost every $t \in [0, T]$,*

$$X'(t) \leq A(t)X(t) + B(t) \int_{-\tau}^0 X(t+\vartheta) \mu(d\vartheta) + H(t),$$

where $A \in L^1(0, T)$, $B, H \in L^1(0, T)$, and $B, H \geq 0$ almost everywhere. Then, for every $t \in [0, T]$,

$$\sup_{r \in [-\tau, t]} X(r) \leq \left(\sup_{r \in [-\tau, 0]} X(r) + \int_0^t H(r) dr \right) \exp \left(\int_0^t [A^+(r) + B(r)] dr \right), \quad A^+(r) = \max\{A(r), 0\}.$$

Proof. Set $M(t) = \sup_{r \in [-\tau, t]} X(r)$. Since μ is a probability measure, $\int_{-\tau}^0 X(t+\vartheta) \mu(d\vartheta) \leq M(t)$. For $0 \leq u \leq t$, integration of the differential inequality up to u , followed by $X(r) \leq M(r)$, gives

$$X(u) \leq M(0) + \int_0^u [A^+(r) + B(r)] M(r) dr + \int_0^u H(r) dr.$$

Taking the supremum over $u \in [0, t]$ and using $H \geq 0$ yields

$$M(t) \leq M(0) + \int_0^t [A^+(r) + B(r)]M(r) dr + \int_0^t H(r) dr.$$

The integral Gronwall inequality gives the stated estimate. $\square$

Let P_m denote the orthogonal projection onto the first m Dirichlet eigenfunctions in each component. For original initial history Φ , prescribe $Y_m(\vartheta) = P_m[\mathcal{Z}_\varepsilon(\theta_\vartheta\omega)^{-1}\Phi(\vartheta)]$ on $[-\tau, 0]$. These projected histories converge uniformly in H to the transformed initial history. The Galerkin approximation of (2.8) is

$$\frac{dY_m}{dt} + P_m\mathbb{A}_\varepsilon(\theta_t\omega)Y_m + P_m\mathbb{F}_\varepsilon(\theta_t\omega, Y_m) - P_m\mathbb{D}_{\tau,\varepsilon}(\theta_t\omega, (Y_m)_t) = P_m\mathbb{G}_\varepsilon(s+t, \theta_t\omega). \quad (3.1)$$

The polynomial damping is monotone in the real state variable. For $p \geq 2$,

$$(|r|^{p-2}r - |q|^{p-2}q)(r - q) \geq c_p(|r| + |q|)^{p-2}|r - q|^2, \quad r, q \in \mathbb{R}, \quad (3.2)$$

and the random factor $a_i e^{(p_i-2)\varepsilon_i z_i}$ is nonnegative and is positive whenever the polynomial damping is present. Accordingly, the damping coefficients are assumed real and nonnegative.

Fix $\sigma \in (0, 1)$. In the energy decomposition below, the term $2\sigma \|Y\|_V^2$ is kept as spatial dissipation for the later compactness argument, while the complementary fraction $1 - \sigma$ contributes through Poincaré's inequality to the H -level spectral damping. With this convention, put

$$\Lambda_{\text{lin}} = \min_{i=1,2} (\gamma_i + \nu_i \lambda_1^{s_i}), \quad \Lambda_\sigma = \min_{i=1,2} (\gamma_i + (1 - \sigma)\nu_i \lambda_1^{s_i}). \quad (3.3)$$

Taking the H -inner product of (3.1) with Y_m and using Poincaré's inequality componentwise gives

$$\begin{aligned} \frac{d}{dt} \|Y_m\|_H^2 + 2\sigma \|Y_m\|_V^2 + 2\Lambda_\sigma \|Y_m\|_H^2 + 2 \sum_{i=1}^2 a_i e^{(p_i-2)\varepsilon_i z_i(\theta_t\omega)} \|y_{im}\|_{L^{p_i}}^{p_i} \\ \leq \alpha_\varepsilon(\theta_t\omega) \|Y_m\|_H^2 + 2(\mathbb{D}_{\tau,\varepsilon}, Y_m)_H + 2(\mathbb{G}_\varepsilon, Y_m)_H, \end{aligned}$$

where

$$\alpha_\varepsilon(\omega) = 2 \max_{i=1,2} |\varepsilon_i z_i(\omega)|.$$

For arbitrary $\eta_d, \eta_f > 0$, (2.10) and Young's inequality yield

$$\begin{aligned} 2(\mathbb{D}_{\tau,\varepsilon}, Y_m)_H &\leq \eta_d \|Y_m\|_H^2 + \frac{L_D^2}{\eta_d} \kappa_{\tau,\varepsilon}(\theta_t\omega) \int_{-\tau}^0 \|Y_m(t+\vartheta)\|_H^2 \mu_\tau(d\vartheta) \\ &\quad + \frac{c_D}{\eta_d} \|\mathcal{Z}_\varepsilon(\theta_t\omega)^{-1}\|_{\mathcal{L}(H)}^2, \\ 2(\mathbb{G}_\varepsilon, Y_m)_H &\leq \eta_f \|Y_m\|_H^2 + \frac{1}{\eta_f} \|\mathbb{G}_\varepsilon(s+t, \theta_t\omega)\|_H^2. \end{aligned}$$

Therefore, with $X_m(t) = \|Y_m(t)\|_H^2$,

$$X'_m(t) + 2\sigma \|Y_m(t)\|_V^2 + \alpha_\varepsilon(\theta_t\omega)X_m(t) \leq b_{\tau,\varepsilon}(\theta_t\omega) \int_{-\tau}^0 X_m(t+\vartheta) \mu_\tau(d\vartheta) + h_\varepsilon(s+t, \theta_t\omega), \quad (3.4)$$

where

$$\begin{aligned} a_\varepsilon(\omega) &= 2\Lambda_\sigma - \eta_d - \eta_f - \mathfrak{q}_\varepsilon(\omega), \\ b_{\tau,\varepsilon}(\omega) &= \frac{L_D^2}{\eta_d} \kappa_{\tau,\varepsilon}(\omega), \\ h_\varepsilon(t, \omega) &= \frac{c_D}{\eta_d} \|\mathcal{Z}_\varepsilon(\omega)^{-1}\|_{\mathcal{L}(H)}^2 + \frac{1}{\eta_f} \|\mathcal{Z}_\varepsilon(\omega)^{-1} f(t)\|_H^2. \end{aligned}$$

On each prescribed finite interval $[0, T]$, the Ornstein–Uhlenbeck-dependent coefficients are bounded and $h_\varepsilon(s + \cdot, \theta_s \omega) \in L^1(0, T)$. Applying the delay Gronwall lemma above to (3.4) with $A(t) = -a_\varepsilon(\theta_t \omega)$, $B(t) = b_{\tau,\varepsilon}(\theta_t \omega)$, and $H(t) = h_\varepsilon(s + t, \theta_t \omega)$ first controls $\sup_{r \in [-\tau, T]} X_m(r)$. Integrating the preceding full energy inequality then controls the retained V -dissipation and the monotone polynomial term. Consequently,

$$\sup_{r \in [-\tau, T]} \|Y_m(r)\|_H^2 + \int_0^T \|Y_m(r)\|_V^2 dr + \sum_i \int_0^T \|y_{im}(r)\|_{L^{p_i}}^{p_i} dr \leq C_{T,\omega,s,\Phi}. \quad (3.5)$$

The strict subcriticality in (2.5) implies that the Nemytskii maps $|y_i|^{p_i-2} y_i$ send bounded V -sets into bounded subsets of V' . Equations (3.1), (3.5), and the delay estimate also give, for some $\varrho > 1$ independent of m ,

$$\|\partial_t Y_m\|_{L^p(0,T;V')} \leq C_{T,\omega,s,\Phi}.$$

Aubin–Lions compactness and extraction therefore yield

$$\begin{aligned} Y_m &\rightharpoonup Y \quad \text{in } L^2(0, T; V), \\ Y_m &\rightarrow Y \quad \text{in } L^2(0, T; H), \\ |y_{im}|^{p_i-2} y_{im} &\rightharpoonup \Xi_i \quad \text{in } L^{p'_i}(0, T; L^{p'_i}(D)). \end{aligned} \quad (3.6)$$

Fix ω in the invariant set of continuous Ornstein–Uhlenbeck paths and a finite $T > 0$. For $a_i > 0$, put

$$c_i(t, \omega) = a_i e^{(p_i-2)\varepsilon_i z_i(\theta_t \omega)}, \quad 0 < c_{i,T,\omega} \leq c_i(t, \omega) \leq C_{i,T,\omega} < \infty.$$

There is no monotonicity assumption in ω : (3.2) concerns the real state variable. The positive factor $c_i(t, \omega)$ preserves this monotonicity for each fixed (t, ω) . From the strong $L^2(0, T; H)$ convergence in (3.6), a subsequence converges almost everywhere on $Q_T = (0, T) \times D$. Moreover, (3.5) gives

$$\sup_m \| |y_{im}|^{p_i-2} y_{im} \|_{L^{p'_i}(Q_T)} < \infty, \quad p'_i = \frac{p_i}{p_i - 1} > 1.$$

The polynomial terms therefore converge almost everywhere to $|y_i|^{p_i-2} y_i$ and are uniformly integrable in $L^1(Q_T)$. Vitali's theorem gives L^1 convergence. Testing first against bounded functions and then using their density in $L^{p_i}(Q_T)$, together with the uniform $L^{p'_i}$ bound, identifies the weak limit as $\Xi_i = |y_i|^{p_i-2} y_i$. Multiplication by the bounded function $c_i(\cdot, \omega)$ preserves this weak convergence. For $a_i = 0$ the nonlinear term is absent. The argument is pathwise and uses no uniform-in- ω strong convergence; uniqueness makes the solution obtained from any convergent Galerkin subsequence the same.

For the delayed term, (2.3) and Fubini's theorem give

$$\int_0^T \|\mathbb{D}_{\tau,\varepsilon}(\theta_t \omega, (Y_m)_t) - \mathbb{D}_{\tau,\varepsilon}(\theta_t \omega, Y_t)\|_H^2 dt \leq C_{T,\omega} \int_{-\tau}^T \|Y_m(r) - Y(r)\|_H^2 dr \rightarrow 0. \quad (3.7)$$

Thus, the Galerkin limit solves (2.8).

3.2. Uniqueness and continuous dependence

Fix an arbitrary finite horizon $T > 0$. Let $Y^{(1)}, Y^{(2)}$ be two solutions and $R = Y^{(1)} - Y^{(2)}$. The monotonicity (3.2) eliminates the polynomial contribution. The delay difference estimate yields, for every $\eta > 0$,

$$\frac{d}{dt} \|R(t)\|_H^2 \leq C_{T,\omega,\eta} \|R(t)\|_H^2 + \frac{L_D^2}{\eta} \kappa_{\tau,\varepsilon}(\theta_t \omega) \int_{-\tau}^0 \|R(t+\vartheta)\|_H^2 \mu_\tau(d\vartheta). \quad (3.8)$$

Taking the supremum over the recent history and applying Gronwall's inequality gives

$$\|R_t\|_{\mathcal{H}_\tau}^2 \leq C_{T,\omega} \|R_0\|_{\mathcal{H}_\tau}^2, \quad 0 \leq t \leq T. \quad (3.9)$$

Here $T > 0$ is arbitrary. For each prescribed finite T , after fixing the auxiliary parameter $\eta > 0$ in (3.8), the Gronwall constant is finite and independent of $t \in [0, T]$; its dependence on T, ω, η , and the fixed model parameters is absorbed into the notation $C_{T,\omega}$. Hence, (3.9) holds on every bounded time interval.

Theorem 3.1 (Global well-posedness). *Assume (H1)–(H3). For every $s \in \mathbb{R}$, every initial history $\Phi \in \mathcal{H}_\tau$, and every ω in a full-measure invariant set, system (1.1) has a unique global pathwise Stratonovich solution. Its conjugate Y satisfies*

$$Y \in C([-\tau, T]; H) \cap L^2(0, T; V), \quad y_i \in L^{p_i}(0, T; L^{p_i}(D))$$

for every $T > 0$, and the solution depends continuously on the initial history in $\mathcal{H}_\tau$.

Proof. The finite-dimensional solutions exist locally. Estimate (3.5) prevents finite-time blow-up and supplies the compactness used in (3.6). The delay convergence is (3.7); the polynomial limit follows from the pathwise identification after (3.6). Hence, a global weak solution exists. Uniqueness and continuous dependence follow from (3.8)–(3.9). The inverse multiplication by $\mathcal{Z}_\varepsilon$ recovers the unique Stratonovich solution of (1.1). $\square$

For $t \geq 0$, define the history solution map

$$\varphi_{\tau,\varepsilon}(t, s, \omega)\Phi = U_{s+t}(\cdot; s, \omega, \Phi) \in \mathcal{H}_\tau. \quad (3.10)$$

Uniqueness and the Brownian shift identity imply

$$\varphi_{\tau,\varepsilon}(t+r, s, \omega) = \varphi_{\tau,\varepsilon}(t, s+r, \theta_r \omega) \circ \varphi_{\tau,\varepsilon}(r, s, \omega).$$

Measurability follows from the Galerkin construction and passage to the limit, while (3.9) gives continuity in Φ .

Corollary 3.2. *Under (H1)–(H3), (3.10) is a continuous non-autonomous random cocycle on $\mathcal{H}_\tau$ over the base $(\mathbb{R} \times \Omega, (s, \omega) \mapsto (s+t, \theta_t \omega))$.*

4. Random dissipativity, delay-robust small-gain criteria, and explicit noise margins

4.1. The random energy inequality

Apply Theorem 2.2 to (3.4). Define

$$\begin{aligned}\mathcal{M}_{\tau,\varepsilon}(t) &= \int_{-\tau}^0 \int_{t+\vartheta}^t e^{-\delta(t-r)} b_{\tau,\varepsilon}(\theta_{r-\vartheta}\omega) X(r) dr \mu_{\tau}(d\vartheta), \\ B_{\tau,\varepsilon}(\omega) &= \int_{-\tau}^0 b_{\tau,\varepsilon}(\theta_{-\vartheta}\omega) \mu_{\tau}(d\vartheta), \\ \mathcal{G}_{\tau,\varepsilon}(t) &= X(t) + \chi \mathcal{M}_{\tau,\varepsilon}(t).\end{aligned}$$

For $\chi e^{-\delta\tau} \geq 1$,

$$\frac{d}{dt} \mathcal{G}_{\tau,\varepsilon}(t) + g_{\tau,\varepsilon}(\theta_t\omega) \mathcal{G}_{\tau,\varepsilon}(t) + 2\sigma \|Y(t)\|_V^2 \leq h_{\varepsilon}(s+t, \theta_t\omega), \quad (4.1)$$

where

$$g_{\tau,\varepsilon}(\omega) = \min \left\{ 2\Lambda_{\sigma} - \eta_d - \eta_f - q_{\varepsilon}(\omega) - \chi B_{\tau,\varepsilon}(\omega), \delta \right\}. \quad (4.2)$$

This stationary process is integrable by (2.11).

4.2. A tempered ergodic lower envelope

The next lemma gives the precise pathwise form of the positive-mean condition.

Lemma 4.1 (Tempered ergodic envelope). *Let $g \in L^1(\Omega)$ be stationary and ergodic with $\bar{g} = \mathbb{E}g > 0$. For every $\alpha \in (0, \bar{g})$, define*

$$C_{\alpha}(\omega) = \exp \left(\sup_{T \geq 0} \left\{ \alpha T - \int_{-T}^0 g(\theta_r\omega) dr \right\} \right). \quad (4.3)$$

Then, $1 \leq C_{\alpha}(\omega) < \infty$ almost surely,

$$\int_{s-t}^s g(\theta_r\omega) dr \geq \alpha t - \log C_{\alpha}(\theta_s\omega), \quad t \geq 0, \quad (4.4)$$

and C_{α} is tempered

$$\lim_{|t| \rightarrow \infty} \frac{\log^+ C_{\alpha}(\theta_t\omega)}{|t|} = 0. \quad (4.5)$$

Proof. Write $F(t, \omega) = \int_0^t [g(\theta_r\omega) - \bar{g}] dr$ for positive and negative t . Birkhoff's theorem gives $F(t, \omega)/|t| \rightarrow 0$ as $t \rightarrow \pm\infty$. Since $\bar{g} - \alpha > 0$, the expression in braces in (4.3) tends to $-\infty$ as $T \rightarrow \infty$, proving finiteness. Formula (4.4) is the definition of C_{α} applied to the shifted sample $\theta_s\omega$.

For temperedness, put $\beta = \bar{g} - \alpha > 0$. A change of variables gives

$$\log C_{\alpha}(\theta_t\omega) = \sup_{u \geq 0} \{-\beta u - F(t, \omega) + F(t-u, \omega)\}.$$

For each $\epsilon \in (0, \beta)$, continuity and sublinear growth give a finite $K_{\epsilon}(\omega)$ such that $|F(r, \omega)| \leq \epsilon|r| + K_{\epsilon}(\omega)$ for every real r . Consequently,

$$0 \leq \log C_{\alpha}(\theta_t\omega) \leq \sup_{u \geq 0} \{2\epsilon|t| + 2K_{\epsilon} - (\beta - \epsilon)u\} = 2\epsilon|t| + 2K_{\epsilon}.$$

Dividing by $|t|$ and then letting $\epsilon \downarrow 0$ proves (4.5). In the preceding supremum representation, the expression tends to $-\infty$ as $u \rightarrow \infty$, uniformly for t in a compact interval. Its maximum is therefore attained on a locally uniform compact u -interval. Continuity of F then implies continuity of $t \mapsto C_\alpha(\theta_t\omega)$, which also justifies the finite-window supremum used in Theorem 4.7. $\square$

4.3. Intrinsic small gain and arbitrary bounded delay intervals

At zero noise, $q_0 = 0$, $\kappa_{\tau,0} = 1$, and

$$b_0 = \frac{L_D^2}{\eta_d}, \quad B_{\tau,0} = b_0.$$

Put

$$a_0 = 2\Lambda_\sigma - \eta_d - \eta_f.$$

The underlying deterministic balance is $a_0 > b_0$ after a suitable choice of Young parameters; a fixed finite delay bound arises only after fixing an auxiliary rate δ .

Theorem 4.2 (Delay-robust small gain). *Assume (H1)–(H3). Suppose that there exist $\eta_d, \eta_f > 0$ such that*

$$S := \frac{(2\Lambda_\sigma - \eta_d - \eta_f)\eta_d}{L_D^2} > 1. \quad (4.6)$$

Let $\tau_* > 0$ be arbitrary. Choose $\rho > 0$ and $\delta > 0$ so that

$$1 + \rho < S, \quad 0 < \delta < \frac{1}{\tau_*} \log \frac{S}{1 + \rho}, \quad (4.7)$$

and set

$$\chi_* = (1 + \rho)e^{\delta\tau_*}. \quad (4.8)$$

Then, uniformly for $0 \leq \tau \leq \tau_*$,

$$m_0 := a_0 - \chi_* \frac{L_D^2}{\eta_d} > 0, \quad g_{\tau,0} \equiv \min\{m_0, \delta\} > 0. \quad (4.9)$$

Moreover, for some $\varepsilon_0 > 0$ depending on τ_*, ρ, δ and the fixed deterministic parameters, there is a single stationary ergodic process $\underline{g}_{\varepsilon_0} \in L^1(\Omega)$ such that

$$g_{\tau,\varepsilon}(\omega) \geq \underline{g}_{\varepsilon_0}(\omega) \quad \text{for all } 0 \leq \tau \leq \tau_*, \max_i |\varepsilon_i| \leq \varepsilon_0, \quad \mathbb{E} \underline{g}_{\varepsilon_0} > 0. \quad (4.10)$$

Hence, $\mathbb{E}_{\tau,\varepsilon}$ has a common positive-mean stationary lower dissipation rate on every prescribed bounded delay interval.

Proof. For $\tau \leq \tau_*$, (4.8) gives $\chi_* e^{-\delta\tau} \geq 1 + \rho > 1$, so Theorem 2.2 applies. The first inequality in (4.7) and the second one imply $\chi_* < S$, which is equivalent to (4.9).

For $\varepsilon > 0$, define the compact-window random variables

$$Q_e(\omega) = 2e \max_{i=1,2} |z_i(\omega)|,$$

$$K_{e,\tau_*}(\omega) = \exp \left(4e \max_{i=1,2} \sup_{|r| \leq \tau_*} |z_i(\theta_r \omega)| \right).$$

If $\max_i |\varepsilon_i| \leq e$ and $0 \leq \tau \leq \tau_*$, then

$$q_\varepsilon(\omega) \leq Q_e(\omega), \quad B_{\tau,\varepsilon}(\omega) \leq \frac{L_D^2}{\eta_d} K_{e,\tau_*}(\omega).$$

The second bound follows because the time indices entering $\kappa_{\tau,\varepsilon}(\theta_{-\vartheta}\omega)$ lie in $[-\tau_*, \tau_*]$. Set

$$\underline{g}_e(\omega) = \min \left\{ a_0 - Q_e(\omega) - \chi_* \frac{L_D^2}{\eta_d} K_{e,\tau_*}(\omega), \delta \right\}.$$

This process is stationary and ergodic, and the preceding bounds give $g_{\tau,\varepsilon} \geq \underline{g}_e$. Fernique's theorem implies $Q_e \rightarrow 0$ and $K_{e,\tau_*} \rightarrow 1$ in $L^1(\Omega)$ as $e \rightarrow 0$. Since the minimum map is Lipschitz,

$$\underline{g}_e \longrightarrow \min\{m_0, \delta\} \quad \text{in } L^1(\Omega).$$

The limit is strictly positive by (4.9). Choosing $e = \varepsilon_0$ sufficiently small proves (4.10). $\square$

Corollary 4.3 (Spectrally optimized intrinsic criterion). *For fixed $\sigma \in (0, 1)$ and $\eta_f \in (0, 2\Lambda_\sigma)$, a value $\eta_d \in (0, 2\Lambda_\sigma - \eta_f)$ satisfying (4.6) exists if and only if*

$$L_D < \Lambda_\sigma - \frac{\eta_f}{2}.$$

Consequently, admissible parameters σ, η_f, η_d exist if and only if

$$L_D < \Lambda_{\text{lin}}. \quad (4.11)$$

Proof. For fixed σ and η_f , the concave quadratic

$$\eta_d(2\Lambda_\sigma - \eta_f - \eta_d)$$

attains its maximum $(2\Lambda_\sigma - \eta_f)^2/4$ at $\eta_d = (2\Lambda_\sigma - \eta_f)/2$. This proves the first assertion. Since $\Lambda_\sigma \uparrow \Lambda_{\text{lin}}$ as $\sigma \downarrow 0$, sufficiently small positive σ and η_f exist precisely under (4.11). $\square$

Meaning of delay robustness. The condition $L_D < \Lambda_{\text{lin}}$ does not depend on a prescribed finite τ_* . For each such window, Theorem 4.2 selects δ, χ_* , and a corresponding $\varepsilon_0 > 0$ so that a common positive-mean lower dissipation rate holds for $0 \leq \tau \leq \tau_*$ and $\|\varepsilon\|_\infty \leq \varepsilon_0$. This does not assert a single positive decay rate or noise radius as $\tau_* \rightarrow \infty$. In (4.1)-(4.2), the general parameter χ is then fixed at χ_* . Table 1 summarizes the constants and their dependence on the prescribed delay window.

Table 1. Constants and dependence on the prescribed delay window.

Quantity	Definition or choice	Dependence and role
σ	Fraction in $(0, 1)$; (3.3)	Can be fixed from the spectral data before choosing τ_* .
η_d, η_f	Positive Young parameters with $S > 1$	Can be fixed before τ_* in Theorem 4.2; reoptimization for a fixed window can introduce window dependence.
S, ρ	$S = (2\Lambda_\sigma - \eta_d - \eta_f)\eta_d/L_D^2$; $0 < \rho < S - 1$	Independent of τ_* under the preceding fixed-parameter construction.
δ, χ_*	(4.7)-(4.8)	Chosen for τ_* ; $\chi_* = (1 + \rho)e^{\delta\tau_*}$.
q_ε	$2 \max_i \varepsilon_i z_i $	Depends on ε, ω , not directly on τ_* .
a_0, b_0	$2\Lambda_\sigma - \eta_d - \eta_f, L_D^2/\eta_d$	Fixed when the spectral and Young parameters are fixed.
g_0	$\min\{a_0 - \chi_* b_0, \delta\}$	Inherits dependence through δ, χ_* and any reoptimized parameters.
$\varepsilon_0, \varepsilon_{\text{crit}}^{(q)}$	Theorems 4.2 and 4.5	Depend on the window and deterministic parameters; the explicit certificate also uses q, β .

Theorem 4.4 (Exact boundary on the first fractional mode). *Consider the deterministic one-component linear equation*

$$\partial_t u + \nu(-\Delta_D)^s u + \gamma u = Lu(t - \tau), \quad u|_{\partial D} = 0, \quad (4.12)$$

with $\nu, \gamma > 0$, $L \geq 0$, and $\tau \geq 0$. On the invariant first-eigenfunction subspace $u(t) = x(t)e_1$, put $d = \gamma + \nu\lambda_1^s$. Then,

$$L < d \iff x = 0 \text{ is exponentially stable for each fixed } \tau \geq 0. \quad (4.13)$$

If $L = d$, zero is a characteristic root; if $L > d$, a positive real characteristic root exists for every $\tau \geq 0$. Hence the optimized boundary $L_D < \Lambda_{\text{lin}}$ cannot be enlarged in the scalar first-mode class without additional structure. This is an exact fixed-delay stability boundary, not a uniform-in- τ decay-rate theorem.

Proof. The first mode satisfies $x'(t) + dx(t) = Lx(t - \tau)$, with characteristic equation

$$\lambda + d - Le^{-\lambda\tau} = 0.$$

If $L < d$ and $\text{Re } \lambda \geq 0$, then

$$|\lambda + d| \geq d > L \geq |Le^{-\lambda\tau}|,$$

a contradiction. For each fixed τ , choose $\eta \in (0, d)$ with $Le^{\eta\tau} < d - \eta$. The same modulus argument excludes every root with $\text{Re } \lambda \geq -\eta$, giving a positive spectral gap and hence exponential stability [9]. If $L = d$, $\lambda = 0$ is a root. If $L > d$, the real function $F(\lambda) = \lambda + d - Le^{-\lambda\tau}$ satisfies $F(0) < 0$ and $F(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$, so it has a positive zero. Taking the component that attains the minimum in (3.3) identifies $d = \Lambda_{\text{lin}}$. $\square$

Theorem 4.5 (Explicit noise threshold). *Fix the parameters of Theorem 4.2, let $q \in \{1, 2, \dots\}$ be the number of independent Ornstein–Uhlenbeck components, set $b_0 = L_D^2/\eta_d$, and put*

$$g_0 = \min\{a_0 - \chi_* b_0, \delta\} > 0, \quad Z_{q, \tau_*}(\omega) = \max_{1 \leq j \leq q} \sup_{|\tau| \leq \tau_*} |z_j(\theta_r \omega)|.$$

Define

$$N_{q,\tau_*} = q \max\{1, \lceil 2\tau_* \rceil\}, \quad \mathcal{N}_{q,\tau_*} = \max\{e, N_{q,\tau_*}\}, \quad c_{\text{OU}} = \frac{1}{\sqrt{\pi}} + 2\sqrt{\frac{e^2 - 1}{\pi}},$$

and

$$\bar{m}_{q,\tau_*} = c_{\text{OU}} + \sqrt{\log \mathcal{N}_{q,\tau_*}} + \frac{1}{2\sqrt{\log \mathcal{N}_{q,\tau_*}}}.$$

Then, $\mathbb{E}Z_{q,\tau_*} \leq \bar{m}_{q,\tau_*}$. For any $\beta > 0$, define

$$C_{\text{OU}}^{(q)}(\beta, \tau_*) = 2\bar{m}_{q,\tau_*} + \frac{4\chi_* b_0}{\beta} \left[\exp\left(\beta \bar{m}_{q,\tau_*} + \frac{\beta^2}{4}\right) - 1 \right].$$

The explicit sufficient threshold

$$\varepsilon_{\text{crit}}^{(q)}(\beta, \tau_*) = \min \left\{ \frac{\beta}{4}, \frac{g_0}{2C_{\text{OU}}^{(q)}(\beta, \tau_*)} \right\} \quad (4.14)$$

ensures that, whenever $e := \max_{1 \leq j \leq q} |\varepsilon_j| \leq \varepsilon_{\text{crit}}^{(q)}$, one has $\mathbb{E}g_e \geq g_0/2 > 0$. Every quantity in (4.14) is an elementary function of the prescribed delay window, the number of noise components, and deterministic Lyapunov parameters; no unknown stochastic moment remains. For the fractional system $q = 2$, whereas the retarded gradient application uses $q = m$.

Proof. Partition $[-\tau_*, \tau_*]$ into at most $\max\{1, \lceil 2\tau_* \rceil\}$ intervals of length at most one and include all q independent components. On each block, stationarity and

$$z(t_0 + t) = e^{-t}z(t_0) + e^{-t}\widetilde{W}_{(e^{2t}-1)/2}, \quad 0 \leq t \leq 1,$$

give

$$\mathbb{E} \sup_{0 \leq t \leq 1} |z(t_0 + t)| \leq \mathbb{E}|z(t_0)| + \mathbb{E} \sup_{0 \leq u \leq (e^2-1)/2} |\widetilde{W}_u| \leq c_{\text{OU}}.$$

Here $\text{Var } z(t) = 1/2$. Treating the absolute value by the doubled Gaussian index set and applying Borell–TIS therefore yields, for every block maximum M_j [42],

$$\mathbb{P}\{M_j > c_{\text{OU}} + u\} \leq e^{-u^2}, \quad u \geq 0.$$

A union bound over N_{q,τ_*} blocks and integration of the tail, split at $a = \sqrt{\log \mathcal{N}_{q,\tau_*}}$, give

$$\mathbb{E}Z_{q,\tau_*} \leq c_{\text{OU}} + \int_0^\infty \min\{1, N_{q,\tau_*} e^{-u^2}\} du \leq \bar{m}_{q,\tau_*}.$$

Borell–TIS on the full index set $(j, r, \pm)$ also gives the centered sub-Gaussian moment bound

$$\mathbb{E} \exp\{\beta(Z_{q,\tau_*} - \mathbb{E}Z_{q,\tau_*})\} \leq \exp(\beta^2/4),$$

and hence

$$\mathbb{E} e^{\beta Z_{q,\tau_*}} \leq \exp\left(\beta \bar{m}_{q,\tau_*} + \frac{\beta^2}{4}\right).$$

If $4e \leq \beta$, convexity yields

$$e^{4eZ_{q,\tau_*}} - 1 \leq \frac{4e}{\beta}(e^{\beta Z_{q,\tau_*}} - 1).$$

The lower process in Theorem 4.2 satisfies pointwise

$$\underline{g}_e \geq g_0 - 2eZ_{q,\tau_*} - \chi_* b_0(e^{4eZ_{q,\tau_*}} - 1).$$

Taking expectations gives $\mathbb{E}\underline{g}_e \geq g_0 - eC_{OU}^{(q)}(\beta, \tau_*)$, and (4.14) proves the half-margin estimate. The component count enters the certificate only through N_{q,τ_*} . $\square$

Optimization of the Young parameters in (4.14). Fix $\sigma, \tau_*, \rho, \delta, \beta$ and an admissible $\eta_f > 0$, with $L_D > 0$. Write

$$D_* = 2\Lambda_\sigma - \eta_f, \quad k = \chi_* L_D^2, \quad x = \eta_d.$$

The uncapped margin is $m(x) = D_* - x - k/x$. It is positive precisely on (x_-, x_+) , where

$$x_{\pm} = \frac{1}{2}(D_* \pm \sqrt{D_*^2 - 4k}), \quad D_* > 2\sqrt{k}.$$

Its maximum occurs at $x = \sqrt{k} = \sqrt{\chi_*} L_D$, giving

$$\max_x g_0(x) = \min\{D_* - 2\sqrt{k}, \delta\}.$$

This differs from maximizing S in (4.6) and need not maximize the noise certificate because its denominator also depends on x . For the latter, set

$$u = 2\overline{m}_{q,\tau_*}, \quad v = \frac{4k}{\beta} \left[\exp\left(\beta\overline{m}_{q,\tau_*} + \frac{\beta^2}{4}\right) - 1 \right] > 0.$$

Then, the exact one-variable objective is

$$\mathcal{E}_{\text{crit}}^{(q)}(x) = \min \left\{ \frac{\beta}{4}, \frac{\min\{D_*x - x^2 - k, \delta x\}}{2(ux + v)} \right\}.$$

The derivative of $(D_*x - x^2 - k)/(ux + v)$ has the sign of $D_*v - ux^2 - 2vx + uk$, whose unique positive zero is

$$x_c = \frac{\sqrt{v^2 + uvD_* + u^2k} - v}{u}.$$

The derivative is positive at x_- and negative at x_+ , so $x_c \in (x_-, x_+)$. Since $x_c > \sqrt{k}$ and $\delta x/(ux + v)$ is increasing, a maximizing choice is $x_{\text{opt}} = x_c$ when $D_* - 2\sqrt{k} < \delta$. Otherwise it is

$$x_{\text{opt}} = \max\{x_c, x_\delta^+\}, \quad x_\delta^+ = \frac{1}{2}(D_* - \delta + \sqrt{(D_* - \delta)^2 - 4k}).$$

The outer cap $\beta/4$ may make the maximizer nonunique. Reducing η_f cannot decrease the certificate, so optimization over the open constraint $\eta_f > 0$ has its supremum as $\eta_f \downarrow 0$, although a saturated cap may already attain it for positive η_f . With nonzero forcing one must retain $\eta_f > 0$, because the absorbing bound contains $\|f\|_H^2/\eta_f$. Thus, there is a closed-form optimum in η_d for fixed remaining parameters, and a closed-form supremum of g_0 over the Young parameters, namely $\min\{2\Lambda_\sigma - 2\sqrt{\chi_*}L_D, \delta\}$; no universal attained optimum over all auxiliary parameters is asserted.

Remark 4.6. For a larger prescribed τ_* , one generally chooses a smaller δ , so the lower rate $\min\{m_0, \delta\}$ may deteriorate. The criterion therefore gives stability on every prescribed bounded delay interval, not a delay-independent exponential rate.

Interpretation of the explicit threshold. The quantity $\varepsilon_{\text{crit}}^{(q)}$ in (4.14) is a rigorous sufficient certificate and is not expected to be sharp in general. Conservatism enters through the blockwise union bound for Z_{q,τ_*} , the replacement of $\mathbb{E}Z_{q,\tau_*}$ by the explicit upper envelope $\bar{m}_{q,\tau_*}$, the convexity estimate in the exponential moment step, and the final half-margin requirement $\mathbb{E}g_e \geq g_0/2$. In particular, when $\tau_* \leq 1/2$, the block count already equals q , so the present union-bound certificate should not be interpreted as asymptotically exact as $\tau_* \downarrow 0$. With the remaining parameters fixed, as $L_D \downarrow 0$ one has $b_0 = L_D^2/\eta_d \downarrow 0$, so the delay-amplification contribution becomes less restrictive, but the direct random-damping term q_ε and the half-margin step remain. Thus, (4.14) is intended primarily as an explicit, fully computable robustness certificate rather than an optimal noise boundary. Optimizing the Young parameters within this certificate does not remove the preceding probabilistic overestimates.

Numerical illustration of the exact first-mode boundary. To illustrate Theorem 4.4 and the above meaning of delay robustness, we integrate

$$x'(t) + 3x(t) = Lx(t - \tau), \quad x(t) \equiv 1 \text{ for } t \in [-\tau, 0],$$

on $0 \leq t \leq 30$ by the method of steps using classical RK4 stages, step size $h = 2 \times 10^{-3}$, and linear interpolation of delayed values. Figure 1 shows that for $L = 2.4 < 3$ the solution decays for $\tau = 0.5, 2$, and 6, while the decay becomes slower as the delay increases. For $L = 3$ and $\tau = 2$, the chosen history generates the exact neutral solution $x \equiv 1$; for $L = 3.6 > 3$ and $\tau = 2$, the solution grows. The simulation concerns the deterministic linear subclass only; it is not a stochastic simulation or a numerical test of the sharpness of $\varepsilon_{\text{crit}}^{(q)}$, and it is not used in any proof. The use of linear interpolation means that fourth-order global accuracy is not asserted for the full delay scheme. The reproducible calculation includes step-halving checks.

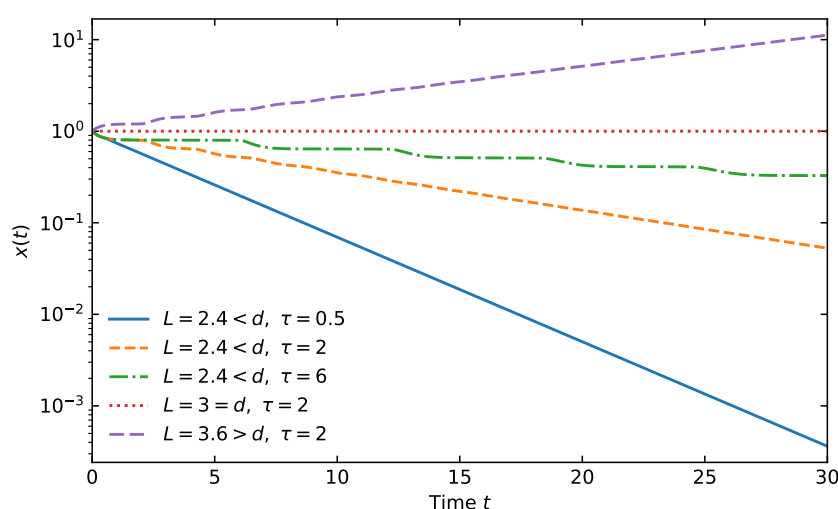


Figure 1. Numerical illustration of Theorem 4.4. The stable case $L < d = 3$ persists for the three displayed fixed delays, with slower decay for larger τ ; $L = d$ is neutral for the constant history, whereas $L > d$ produces growth.

4.4. Pullback absorption

Let $\mathfrak{D}_\tau$ denote the universe of tempered families $B(s, \omega) \subset \mathcal{H}_\tau$:

$$\lim_{t \rightarrow \infty} e^{-ct} \sup_{\Phi \in B(s-t, \theta_{-t}\omega)} \|\Phi\|_{\mathcal{H}_\tau}^2 = 0 \quad \text{for every } c > 0. \quad (4.15)$$

Let Φ be the history of the original variable and set $\Psi(\vartheta) = \mathcal{Z}_\varepsilon(\theta_\vartheta\omega)^{-1}\Phi(\vartheta)$ for the transformed equation. Since $r - \vartheta \in [0, \tau_*]$ for $\vartheta \in [-\tau, 0]$ and $r \in [0, \tau_*]$, and $e^{\delta r} \leq 1$, one has

$$\mathcal{M}_{\tau, \varepsilon}(0) \leq \tau_* \sup_{0 \leq r \leq \tau_*} b_{\tau, \varepsilon}(\theta_r\omega) \|\Psi\|_{\mathcal{H}_\tau}^2, \quad 0 \leq \tau \leq \tau_*. \quad (4.16)$$

Define $Z_*(\omega) = \max_{i=1,2} \sup_{|r| \leq \tau_*} |z_i(\theta_r\omega)|$ and $b_0 = L_D^2/\eta_d$. Every time entering $\kappa_{\tau, \varepsilon}(\theta_r\omega)$, with $0 \leq r \leq \tau_*$, lies in $[-\tau_*, \tau_*]$. Thus, uniformly for $0 \leq \tau \leq \tau_*$ and $\|\varepsilon\|_\infty \leq \varepsilon_0$,

$$\begin{aligned} \sup_{0 \leq r \leq \tau_*} \kappa_{\tau, \varepsilon}(\theta_r\omega) &\leq e^{4\varepsilon_0 Z_*(\omega)}, & \|\Psi\|_{\mathcal{H}_\tau}^2 &\leq e^{2\varepsilon_0 Z_*(\omega)} \|\Phi\|_{\mathcal{H}_\tau}^2, \\ \mathfrak{C}_{\tau, \varepsilon}(0) &\leq \Gamma_*(\omega) \|\Phi\|_{\mathcal{H}_\tau}^2, & \Gamma_* &= (1 + \chi_* \tau_* b_0) e^{6\varepsilon_0 Z_*}. \end{aligned}$$

Here $\chi = \chi_*$ is the choice in Theorem 4.2. The required fact is the sublinear path growth $Z_*(\theta_t\omega)/|t| \rightarrow 0$, not merely logarithmic temperedness of z_i . To verify it, apply the Gaussian compact-window tail bound to $\sup_{r \in [n-\tau_*, -1, n+\tau_*+1]} |z_i(\theta_r\omega)|$. For each fixed positive threshold times $|n|$, the resulting probabilities are summable in $n \in \mathbb{Z}$. Borel–Cantelli and continuity, followed by a countable choice of thresholds, give the stated sublinear growth for all real t . Therefore, Γ_* is measurable and tempered, uniformly on the full parameter set, including $\tau = \tau_*$.

Theorem 4.7 (Tempered pullback absorption). *Assume the hypotheses and parameter choices of Theorem 4.2. Put*

$$\bar{g} = \mathbb{E} g_{-\varepsilon_0} > 0$$

and fix $\alpha \in (0, \bar{g})$. Define

$$\begin{aligned} M_{\varepsilon_0}(\omega) &= \exp\left(2\varepsilon_0 \max_{i=1,2} |z_i(\omega)|\right), \\ h_*(t, \omega) &= M_{\varepsilon_0}(\omega) \left(\frac{c_D}{\eta_d} + \frac{1}{\eta_f} \|f(t)\|_H^2 \right). \end{aligned}$$

Then, $h_\varepsilon \leq h_*$ throughout the parameter set. For every $B \in \mathfrak{D}_\tau$ and $s \in \mathbb{R}$ there is a pullback time $T_B(s, \omega)$ such that

$$\|Y_s(\cdot; s-t, \theta_{-t}\omega, \Phi)\|_{\mathcal{H}_\tau}^2 \leq R_0(s, \omega) \quad (4.17)$$

for $t \geq T_B(s, \omega)$ and $\Phi \in B(s-t, \theta_{-t}\omega)$. Here $Y_s(\vartheta) = \mathcal{Z}_\varepsilon(\theta_\vartheta\omega)^{-1}U(s+\vartheta)$ denotes the transformed terminal history of that pullback solution, for $\vartheta \in [-\tau, 0]$. Put $C_{\alpha, \tau_*}(\omega) = \sup_{-\tau_* \leq v \leq 0} C_\alpha(\theta_v\omega)$. A parameter-independent squared radius is

$$R_0(s, \omega) = 2e^{\alpha\tau_*} C_{\alpha, \tau_*}(\omega) \left[1 + \int_{-\infty}^0 e^{\alpha r} h_*(s+r, \theta_r\omega) dr \right]. \quad (4.18)$$

Here C_α is the envelope from Lemma 4.1 for $\underline{g}_{-\varepsilon_0}$. Its proof, using the continuous sublinear primitive of that process, also gives local boundedness along the flow; hence, C_{α, τ_*} is finite, measurable, and tempered. The balls with squared radius R_0 are uniformly tempered for the transformed histories. For the original cocycle the squared radius is $R_U(s, \omega) = e^{2\varepsilon_0 Z_*(\omega)} R_0(s, \omega)$, which has the same properties.

Proof. By (4.10), (4.1) implies

$$\mathcal{E}'(t) + g_{\underline{\varepsilon}_0}(\theta_t \omega) \mathcal{E}(t) \leq h_*(s+t, \theta_t \omega).$$

Integrating from $-t$ to 0 in cocycle time gives

$$\begin{aligned} \mathcal{E}(0) &\leq e^{-\int_{-t}^0 g_{\underline{\varepsilon}_0}(\theta_r \omega) dr} \mathcal{E}(-t) \\ &\quad + \int_{-t}^0 e^{-\int_r^0 g_{\underline{\varepsilon}_0}(\theta_\ell \omega) d\ell} h_*(s+r, \theta_r \omega) dr. \end{aligned}$$

For a terminal cocycle time $v \in [-\tau, 0]$ and $t > \tau_*$, the same integration and Lemma 4.1 give

$$\mathcal{E}(v) \leq C_{\alpha, \tau_*}(\omega) e^{\alpha \tau_*} \left[e^{-\alpha t} \mathcal{E}(-t) + \int_{-\infty}^0 e^{\alpha r} h_*(s+r, \theta_r \omega) dr \right].$$

The initial term tends to zero uniformly in v because $\Gamma_*(\theta_{-t} \omega)$ and B are tempered. This proves (4.17) with (4.18). For completeness, write

$$I(s, \omega) = \int_{-\infty}^0 e^{\alpha r} M_{\varepsilon_0}(\theta_r \omega) F_s(r) dr, \quad F_s(r) = c_D/\eta_d + \|f(s+r)\|_H^2/\eta_f.$$

For any small $a, b > 0$ with $a+b < \alpha$, there is a finite $K_a(\omega)$ such that $M_{\varepsilon_0}(\theta_u \omega) \leq K_a(\omega) e^{-au}$ for $u \leq 0$, and (H3) gives $\int_{-\infty}^0 e^{bu} F_s(u) du < \infty$. With $u = r - t$,

$$\begin{aligned} I(s-t, \theta_{-t} \omega) &= e^{\alpha t} \int_{-\infty}^{-t} e^{\alpha u} M_{\varepsilon_0}(\theta_u \omega) F_s(u) du \\ &\leq K_a(\omega) e^{(a+b)t} \int_{-\infty}^0 e^{bu} F_s(u) du. \end{aligned}$$

Since $a+b$ can be arbitrarily small, this proves the required backward temperedness. The envelope and conjugacy factors are tempered as well. Multiplying by $\mathcal{Z}_\varepsilon$ on the terminal window gives the stated original-history radius. $\square$

5. Pullback random attractors, upper semicontinuity, and two applications

5.1. Uniform higher regularity and compact histories

The pure-power damping is nonnegative in the V -energy estimate. In the terminal-time estimates below, $Y(r) = \mathcal{Z}_\varepsilon(\theta_{r-s} \omega)^{-1} U(r)$ is written in physical time; replacing r by $s+v$ gives the terminal cocycle coordinate v . We first record the required operator inequality.

Lemma 5.1 (Spectral Dirichlet Stroock–Varopoulos inequality). *Let $L = -\Delta_D$ be the Dirichlet Laplacian on a bounded smooth domain, let $0 < s < 1$, and let $p \geq 2$. For every real w for which the two sides are finite,*

$$\int_D |w|^{p-2} w L^s w dx \geq \frac{4(p-1)}{p^2} \|L^{s/2}(|w|^{p/2})\|_{L^2}^2 \geq 0. \quad (5.1)$$

Proof. The sub-Markov property belongs to the semigroup generated by $-L^s$, rather than to L^s viewed as a pointwise map. Let $T(r) = e^{-rL}$. For $t > 0$, the stable subordinator law $\eta_{s,t}$ is a probability measure with Laplace transform $\int_0^\infty e^{-\lambda r} \eta_{s,t}(dr) = e^{-t\lambda^s}$. Spectral calculus therefore gives

$$e^{-tL^s} w = \int_0^\infty T(r)w \eta_{s,t}(dr).$$

Since $0 \leq T(r)w \leq 1$ whenever $0 \leq w \leq 1$, the same implication holds for e^{-tL^s} . Symmetry and L^2 contractivity also pass to this integral. The Dirichlet realization of L supplies all boundary information; no exterior-value condition for a different fractional operator is introduced [12].

To see the inequality including the loss of mass at the boundary, write $c_p = 4(p-1)/p^2 \leq 1$. For all real a, b ,

$$(|a|^{p-2}a - |b|^{p-2}b)(a - b) \geq c_p(|a|^{p/2} - |b|^{p/2})^2.$$

The positive symmetric Dirichlet heat kernel and its mass deficit $1 - T(r)1 \geq 0$ give

$$\langle |w|^{p-2}w, (I - T(r))w \rangle \geq c_p \langle |w|^{p/2}, (I - T(r))|w|^{p/2} \rangle.$$

Indeed, the kernel part follows from the scalar inequality, and the mass-deficit part follows from $|w|^p \geq c_p|w|^p$. Integrating against $s r^{-1-s} dr / \Gamma(1-s)$ gives (5.1) by the spectral representation of L^s . Truncation and closure of the quadratic form extend the computation from bounded admissible functions to the stated domain. This proof applies to the spectral Dirichlet operator, including its boundary killing term; the whole-space fractional-power estimate in [43] alone would not specify that term. $\square$

This is the sign needed in the V -energy estimate.

Test the Galerkin equation with AY_m , where $A = \text{diag}(\nu_1(-\Delta_D)^{s_1}, \nu_2(-\Delta_D)^{s_2})$. This choice agrees with the ν_i -weighted V -norm in Section 2.1. By (5.1), the nonlinear term is nonnegative. The random linear term is bounded by $Cq_\varepsilon \|Y_m\|_V^2$, and Young's inequality gives

$$\frac{d}{dt} \|Y_m\|_V^2 + c_W \|Y_m\|_W^2 \leq C[1 + q_\varepsilon(\theta_t\omega)] \|Y_m\|_V^2 + C \left\| \mathbb{D}_{\tau,\varepsilon}(\theta_t\omega, (Y_m)_t) \right\|_H^2 + C \|\mathbb{G}_\varepsilon(s+t, \theta_t\omega)\|_H^2. \quad (5.2)$$

The transformed delay is controlled by (2.10). Integrating (4.1) on terminal unit intervals first yields

$$\int_{s-1}^s \|Y(r)\|_V^2 dr \leq R_1(s, \omega)$$

for solutions whose initial time is sufficiently far in the past. Uniform Gronwall applied to (5.2) then gives

$$\sup_{r \in [s-\tau, s]} \|Y(r)\|_V^2 + \int_{s-\tau}^s \|Y(r)\|_W^2 dr \leq R_2(s, \omega), \quad (5.3)$$

where R_2 is tempered and uniform on the compact parameter set from Theorem 4.2. Assumption (H4) is convenient for propagating V -regularity through the complete history, while the H -growth estimate is sufficient for the W -integral in (5.3).

The equation and strict subcriticality imply, for some $\varrho > 1$,

$$\int_{s-\tau}^s \|\partial_t Y(r)\|_V^\varrho dr \leq R_3(s, \omega). \quad (5.4)$$

Indeed, $AY \in L^2(H)$, the delay and force belong to $L^2(H)$, and the polynomial maps are bounded in $L^{p'_i}(V')$ on the V -bounded set. By (2.2), (5.3), (5.4), and Simon's compactness theorem [11], the transformed terminal histories are relatively compact in $C([-\tau, 0]; H)$. The transfer to the original cocycle requires the continuous conjugacy, as made explicit below; no time-Sobolev regularity of the original stochastic histories is asserted.

Lemma 5.2 (Pullback asymptotic compactness). *Assume (H1)–(H4) and Theorem 4.2. For every $B \in \mathfrak{D}_\tau$, every $s \in \mathbb{R}$, every sequence $t_n \rightarrow \infty$, and every $\Phi_n \in B(s - t_n, \theta_{-t_n}\omega)$, the sequence*

$$\varphi_{\tau,\varepsilon}(t_n, s - t_n, \theta_{-t_n}\omega)\Phi_n$$

has a convergent subsequence in $\mathcal{H}_\tau$.

Proof. The absorbing estimate bounds the transformed solutions in a common H -ball. In terminal cocycle coordinates $r \in [-\tau, 0]$, estimates (5.3)–(5.4) bound their histories in

$$L^\infty(-\tau, 0; V) \cap W^{1,\varrho}(-\tau, 0; V').$$

Since $V \Subset H$ and $\varrho > 1$, Simon's theorem gives a convergent subsequence of transformed histories in $C([-\tau, 0]; H)$. The map $Y(r) \mapsto \mathcal{Z}_\varepsilon(\theta_r\omega)Y(r)$ is continuous on this space for the fixed path and terminal window. The corresponding original histories therefore converge as well. For $\tau = 0$, the terminal V -bound and $V \Subset H$ give the same conclusion directly. $\square$

Theorem 5.3 (Pullback random attractor). *Under (H1)–(H4) and the positive-mean condition of Theorem 4.2, the cocycle $\varphi_{\tau,\varepsilon}$ possesses a unique $\mathfrak{D}_\tau$ -pullback random attractor $\mathcal{A}_{\tau,\varepsilon}(s, \omega)$. It is compact in $\mathcal{H}_\tau$, invariant,*

$$\varphi_{\tau,\varepsilon}(t, s, \omega)\mathcal{A}_{\tau,\varepsilon}(s, \omega) = \mathcal{A}_{\tau,\varepsilon}(s + t, \theta_t\omega),$$

and attracts every $B \in \mathfrak{D}_\tau$ in the pullback sense.

Proof. Theorem 4.7 gives a closed tempered absorbing family, and Lemma 5.2 gives pullback asymptotic compactness. Measurability of the cocycle follows from the measurable Ornstein–Uhlenbeck coefficients, Galerkin measurability, uniqueness, and passage to the limit; the absorbing radius is measurable because it is an integral of measurable random variables. The standard omega-limit construction therefore produces a measurable compact invariant pullback attractor [1, 2, 4]. $\square$

5.2. A common compact history space

Fix $\tau_* > 0$ and define

$$\mathcal{H}_* = C([-\tau_*, 0]; H).$$

For $0 \leq \tau \leq \tau_*$, define $\mathcal{E}_\tau : \mathcal{H}_\tau \rightarrow \mathcal{H}_*$ by

$$(\mathcal{E}_\tau\Phi)(\vartheta) = \Phi(\max\{\vartheta, -\tau\}), \quad -\tau_* \leq \vartheta \leq 0.$$

Thus $\mathcal{E}_\tau\Phi$ agrees with Φ on $[-\tau, 0]$ and equals $\Phi(-\tau)$ on $[-\tau_*, -\tau]$. At $\tau = 0$, it maps a current value to the corresponding constant history. This extension map is distinct from the scalar energy $\mathcal{E}_{\tau,\varepsilon}$. The parameter-uniform versions of (5.3)–(5.4) imply the following.

Proposition 5.4 (Uniform compact-history set). *Under the hypotheses of Theorem 4.2 and (H4), for every fixed s there is a compact set $K_*(s, \omega) \subset \mathcal{H}_*$ such that*

$$\bigcup_{\substack{0 \leq \tau \leq \tau_* \\ \max_i |\varepsilon_i| \leq \varepsilon_0}} \mathcal{E}_\tau \mathcal{A}_{\tau, \varepsilon}(s, \omega) \subset K_*(s, \omega).$$

The same compact set contains the terminal histories obtained by evolving the common absorbing family for one fixed positive time. It can be chosen as a measurable compact family whose radius in $\mathcal{H}_$ is tempered; consequently, its current-value projection is a tempered family in H .*

Proof. Use terminal cocycle coordinates $r \in [-\tau, 0]$. The common bounds $R_2(s, \omega)$ and $R_3(s, \omega)$ in (5.3)-(5.4) concern Y . Hölder's inequality in V' and

$$\|w\|_H^2 \leq C \|w\|_V \|w\|_{V'}$$

give these transformed histories a common H -modulus $m_Y(\ell) \rightarrow 0$ as $\ell \downarrow 0$. Constant-left extension preserves that modulus. On the fixed larger window define

$$\begin{aligned} M_Z(\omega) &= \sup_{\|\varepsilon\|_\infty \leq \varepsilon_0} \sup_{-\tau_* \leq r \leq 0} \|\mathcal{Z}_\varepsilon(\theta_r \omega)\|_{\mathcal{L}(H)}, \\ m_Z(\ell, \omega) &= \sup_{\|\varepsilon\|_\infty \leq \varepsilon_0} \sup_{\substack{r, v \in [-\tau_*, 0] \\ |r-v| \leq \ell}} \|\mathcal{Z}_\varepsilon(\theta_r \omega) - \mathcal{Z}_\varepsilon(\theta_v \omega)\|_{\mathcal{L}(H)}. \end{aligned}$$

Continuity of the Ornstein–Uhlenbeck paths and compactness of the noise-parameter box imply $M_Z < \infty$ and $m_Z(\ell, \omega) \rightarrow 0$. On the active history interval, the original histories thus have modulus

$$m_U(\ell) = M_Z m_Y(\ell) + \sqrt{R_0(s, \omega)} m_Z(\ell, \omega).$$

Their constant-left extensions have the same modulus because $\vartheta \mapsto \max\{\vartheta, -\tau\}$ is 1-Lipschitz. Diagonal multiplication preserves V with norm at most M_Z , so pointwise relative compactness follows from the common V -bound and $V \Subset H$ (also for $\tau = 0$). Arzelà–Ascoli gives compactness in $\mathcal{H}_*$.

To include images of the common absorbing family, fix $T_0 > \tau_* + 1$ and apply the finite-window estimates over that interval. Use the larger of the resulting bounds and the attractor bounds in the preceding modulus calculation, including its H -bound. The terminal-history estimate in Theorem 4.7 bounds their squared H -radius by

$$\begin{aligned} \widehat{R}_U(s, \omega) &= R_U(s, \omega) + e^{2\varepsilon_0 Z_*(\omega)} C_{\alpha, \tau_*}(\omega) e^{\alpha \tau_* - \alpha T_0} \\ &\quad \times \Gamma_*(\theta_{-T_0} \omega) R_U(s - T_0, \theta_{-T_0} \omega). \end{aligned}$$

All factors are tempered by Theorem 4.7. Take $K_*(s, \omega)$ to be the closed set of histories satisfying the enlarged common V -bounds and continuity modulus, and $\|\Phi\|_{\mathcal{H}_*}^2 \leq \widehat{R}_U(s, \omega)$. Invariance also places the attractor histories in this set. The defining bounds and moduli are measurable and may be tested on countable dense time sets, giving a measurable compact family on the completed Wiener base. Its radius is at most $\sqrt{\widehat{R}_U}$, so its current-value projection is tempered as well. $\square$

The set K_* therefore supplies the compact history family required in (2.6); a bounded V -ball alone is not enough.

5.3. Finite-time convergence and upper semicontinuity

For nonempty subsets A, B of a normed space X , we use the one-sided Hausdorff semidistance $\text{dist}_X(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_X$.

Let $\varphi_{0,0}$ be the deterministic non-autonomous cocycle generated by the instantaneous limit

$$\frac{dU}{dt} + \text{diag}(v_i(-\Delta_D)^{s_i} + \gamma_i)U + (a_i|u_i|^{p_i-2}u_i)_{i=1}^2 - \mathcal{D}_0(U) = f(t).$$

Under the same dissipativity condition, it has a pullback attractor $\mathcal{A}_{0,0}(s) \subset H$, identified with constant histories in $\mathcal{H}_*$.

Lemma 5.5 (Uniform finite-time convergence). *Assume (H1)–(H5). Let $K \subset \mathcal{H}_*$ be compact and $T > 0$. If the initial history for the τ -problem is the restriction of $\Phi \in K$ and the initial value for the limit problem is $\Phi(0)$, then for almost every ω ,*

$$\lim_{\tau + \|\varepsilon\|_\infty \rightarrow 0} \sup_{\Phi \in K} \sup_{0 \leq t \leq T} \|\mathcal{E}_\tau \varphi_{\tau,\varepsilon}(t, s, \omega)(\Phi|_{[-\tau, 0]}) - \mathcal{E}_0 \varphi_{0,0}(t, s)\Phi(0)\|_{\mathcal{H}_*} = 0. \quad (5.5)$$

Proof. Let U_Φ^0 be the deterministic limit solution with initial value $\Phi(0)$, extended by this value to negative times. Continuous dependence on the compact set $\{\Phi(0) : \Phi \in K\}$ implies that the translated histories of these reference solutions, for $0 \leq t \leq T$, form a compact subset of $\mathcal{H}_*$. Apply (H5) on that reference set, not on a presumed V -bounded set of arbitrary initial histories. The initial restrictions converge to $\Phi(0)$ uniformly over K by equicontinuity. Also, $\mathcal{Z}_\varepsilon \rightarrow I$ uniformly on $[-\tau_*, T]$ for the fixed path.

Put $R = Y_{\tau,\varepsilon} - U_\Phi^0$, with the transformed initial history for negative times. Split the delay difference into a Lipschitz difference from the reference history, the multiplier error, and the (H5) error. For $a_i > 0$, write $N_i(u) = |u|^{p_i-2}u$. The polynomial difference is

$$c_i^\varepsilon(N_i(Y_i) - N_i(U_i^0)) + (c_i^\varepsilon - a_i)N_i(U_i^0), \quad c_i^\varepsilon = a_i e^{(p_i-2)\varepsilon_i z_i}.$$

The first term controls a positive multiple of $\|R_i\|_{L^{p_i}}^{p_i}$. Young's inequality absorbs the second with remainder $C|c_i^\varepsilon - a_i|^{p'_i}\|U_i^0\|_{L^{p_i}}^{p_i}$, whose integral tends to zero uniformly over K . This term is absent if $a_i = 0$. The linear and forcing errors tend to zero on the finite interval as well. Consequently,

$$\frac{d}{dt}\|R(t)\|_H^2 \leq C_{T,K,\omega} \sup_{-\tau \leq r \leq t} \|R(r)\|_H^2 + r_{\tau,\varepsilon}(t), \quad \int_0^T r_{\tau,\varepsilon}(t) dt \rightarrow 0.$$

A history Gronwall estimate gives uniform convergence in $C([0, T]; H)$. The compact reference trajectories have a common time modulus, so comparison with their current values also yields convergence after constant-left extension. This proves (5.5). $\square$

Theorem 5.6 (Upper semicontinuity). *Under (H1)–(H5) and the uniform small-gain condition of Theorem 4.2, for every fixed $s \in \mathbb{R}$ and almost every ω ,*

$$\lim_{\tau + \|\varepsilon\|_\infty \rightarrow 0} \text{dist}_{\mathcal{H}_*}(\mathcal{E}_\tau \mathcal{A}_{\tau,\varepsilon}(s, \omega), \mathcal{E}_0 \mathcal{A}_{0,0}(s)) = 0. \quad (5.6)$$

The convergence is qualitative under (H5); no parameter modulus is prescribed by that hypothesis.

Proof. We use a sequential argument, which avoids applying pullback attraction to a compact set that moves with the pullback time. Suppose that (5.6) fails. Then, there are $(\tau_n, \varepsilon_n) \rightarrow (0, 0)$ and $\Phi_n \in \mathcal{A}_{\tau_n, \varepsilon_n}(s, \omega)$ whose extended distance from $\mathcal{E}_0 \mathcal{A}_{0,0}(s)$ is bounded below by some $\eta > 0$. Proposition 5.4 yields, after extraction,

$$\mathcal{E}_{\tau_n} \Phi_n \longrightarrow \Phi \quad \text{in } \mathcal{H}_*.$$

For each n , choose a complete orbit ξ_n on the perturbed attractor with $\xi_n(s) = \Phi_n$. For every integer $k \geq 1$, the sequence $\mathcal{E}_{\tau_n} \xi_n(s - k)$ lies in the fixed compact family $K_*(s - k, \theta_{-k}\omega)$. A diagonal extraction gives limits $\Phi_{-k} \in \mathcal{H}_*$. Lemma 5.5, applied on these fixed compact sets, passes the orbit identities to the limit and gives, for $k > \ell \geq 0$,

$$\Phi_{-\ell} = \mathcal{E}_0 \varphi_{0,0}(k - \ell, s - k) \Phi_{-k}(0), \quad \Phi_0 = \Phi.$$

The tempered radius in Proposition 5.4 shows that $\{\Phi_{-k}(0)\}_{k \geq 1}$ has tempered pullback growth. Interpolating between integer times by the deterministic cocycle produces a complete tempered orbit of the limit system through $\Phi(0)$. By the complete-orbit characterization of pullback attractors, $\Phi(0) \in \mathcal{A}_{0,0}(s)$, and therefore $\Phi \in \mathcal{E}_0 \mathcal{A}_{0,0}(s)$. This contradicts the lower distance bound. Hence, (5.6) holds. The quantitative information needed for a rate is discussed below; the sequential argument itself supplies no prescribed parameter modulus. $\square$

What is needed for a convergence rate. In the deterministic instantaneous limit, $\mathcal{D}_0$ is L_D -Lipschitz by (H2) and (H5). For two solutions with the same forcing, monotonicity and Poincaré's inequality give

$$\|U^{(1)}(s + T) - U^{(2)}(s + T)\|_H \leq e^{-(\Lambda_{\text{lin}} - L_D)T} \|U^{(1)}(s) - U^{(2)}(s)\|_H.$$

This contraction does not quantify the parameter error in (H5). For example, in a scalar first-mode problem $x' + dx = r(\tau)$, take a bounded function r with $r(0) = 0$ and $r(\tau) \rightarrow 0$, and the constant memory map $\mathcal{D}_\tau(\Phi) = r(\tau)$. These maps have Lipschitz constant zero (and hence satisfy any chosen positive Lipschitz bound); their growth constants can be chosen uniformly. Attraction is exponential with rate d , but the equilibria differ by $|r(\tau)|/d$. The function r may tend to zero arbitrarily slowly while the compact-uniform memory condition holds.

More generally, let $h = \tau + \|\varepsilon\|_\infty$. If on the common pullback families one additionally has a limit-attraction estimate Ce^{-cT} and a finite-time perturbation estimate $Ce^{\ell T}r(h)$ in the extended history norm, with $c > 0$, $\ell \geq 0$, $r(h) \rightarrow 0$, and constants independent of T and h , invariance and the triangle inequality yield

$$\text{dist}_{\mathcal{H}_*}(\mathcal{E}_\tau \mathcal{A}_{\tau, \varepsilon}, \mathcal{E}_0 \mathcal{A}_{0,0}) \leq C(e^{-cT} + e^{\ell T}r(h)) \leq 2C r(h)^{c/(c+\ell)}$$

by taking $T = \log(1/r(h))/(c + \ell)$ when $0 < r(h) < 1$. Time and sample arguments are fixed in this display. These additional uniform estimates are not consequences of (H5) alone. The possible loss of decay as $\tau \rightarrow \infty$ does not preclude a local modulus near $\tau = 0$; the missing input is quantitative parameter control, not an unbounded-delay stability theorem.

5.4. A nonlinear system with measure-valued memory

Let the two fractional orders be equal,

$$s_1 = s_2 = s \in (0, 1),$$

so that $V = D((-\Delta_D)^{s/2})^2$. Let $B \in \mathbb{R}^{2 \times 2}$ be a constant matrix. An affine offset $d_0 \in V$ is absorbed into the forcing by writing $\tilde{f} = f + d_0$; the memory operator used in the small-gain estimate is therefore the homogeneous map

$$\tilde{\mathcal{D}}_\tau(\Phi) = B \int_{-\tau}^0 \Phi(\vartheta) \mu_\tau(d\vartheta).$$

Moving the offset preserves the bound $L_D = \|B\|$ in (2.4); estimating the affine map directly generally requires a larger growth coefficient. For the V -estimate define

$$N_\nu = \text{diag}(\sqrt{\nu_1}, \sqrt{\nu_2}), \quad b_V = \|N_\nu B N_\nu^{-1}\|_{\mathcal{L}(\mathbb{R}^2)}.$$

The weights are necessary because the V -norm contains ν_i . Jensen's inequality and commutation of a constant matrix with the common scalar operator give

$$\begin{aligned} \|\tilde{\mathcal{D}}_\tau(\Phi) - \tilde{\mathcal{D}}_\tau(\Psi)\|_H^2 &\leq \|B\|_{\mathcal{L}(\mathbb{R}^2)}^2 \int_{-\tau}^0 \|\Phi(\vartheta) - \Psi(\vartheta)\|_H^2 \mu_\tau(d\vartheta), \\ \|\tilde{\mathcal{D}}_\tau(\Phi) - \tilde{\mathcal{D}}_\tau(\Psi)\|_V^2 &\leq b_V^2 \int_{-\tau}^0 \|\Phi(\vartheta) - \Psi(\vartheta)\|_V^2 \mu_\tau(d\vartheta). \end{aligned}$$

Thus, (H2) and (H4) hold with $L_{D,V} = b_V$ and $c_D = c_{D,V} = 0$. For $B \neq 0$, take $L_D = \|B\|_{\mathcal{L}(\mathbb{R}^2)}$; for $B = 0$, choose any positive L_D as specified in Section 2.1. In particular $b_V = \|B\|$ if $\nu_1 = \nu_2$, as in the numerical example below.

For unequal orders, every constant matrix still acts boundedly on $H = L^2(D)^2$, so it also maps $V \subset H$ into H . The possible failure is $V \rightarrow V$, not $V \rightarrow H$. A sufficient condition for (H4) is $B_{ij} = 0$ whenever $s_j < s_i$. Indeed, if $s_j \geq s_i$, spectral calculus gives

$$\|(-\Delta_D)^{s_i/2} w\|_{L^2} \leq \lambda_1^{(s_i-s_j)/2} \|(-\Delta_D)^{s_j/2} w\|_{L^2}.$$

Define the nonnegative matrix C_B by $(C_B)_{ij} = |B_{ij}| \sqrt{\nu_i/\nu_j} \lambda_1^{(s_i-s_j)/2}$ for the allowed entries and zero otherwise. The triangle inequality gives $\|BU\|_V \leq \|C_B\|_{\mathcal{L}(\mathbb{R}^2)} \|U\|_V$, and Jensen's inequality yields (H4) with this constant for any probability measure. Conversely, an entry $B_{ij} \neq 0$ with $s_j < s_i$ gives output-to-input ratios proportional to $\lambda_m^{(s_i-s_j)/2} \rightarrow \infty$ on histories supported in the j th component and the m th eigenmode. Such a matrix cannot be covered by (H4) without additional regularity or smoothing.

The nonlinear terms may be genuinely superlinear. Choose any $p_i > 2$ satisfying (2.5). Their H -level monotonicity follows from (3.2), and their V -level sign follows from (5.1). Hence, the model includes, for example, cubic damping when $p_i = 4$ satisfies (2.5).

For $\tau > 0$, two representative choices of memory are covered:

$$\mu_\tau(d\vartheta) = \frac{1}{\tau} \mathbf{1}_{[-\tau,0]}(\vartheta) d\vartheta \quad (\text{uniform distributed delay}), \quad (5.7)$$

$$\mu_\tau = \delta_{-\tau} \quad (\text{discrete delay}). \quad (5.8)$$

For every compact $K \subset C([-\tau_*, 0]; H)$,

$$\sup_{\Phi \in K} \|\tilde{\mathcal{D}}_\tau(\Phi|_{[-\tau,0]}) - B\Phi(0)\|_H \leq \|B\| \sup_{\Phi \in K} \sup_{\vartheta \in [-\tau,0]} \|\Phi(\vartheta) - \Phi(0)\|_H \rightarrow 0,$$

so (H5) holds with $\mathcal{D}_0(x) = Bx$ for both (5.7) and (5.8). At $\tau = 0$, both families are defined by $\mu_0 = \delta_{\{0\}}$. For a concrete choice of admissible parameters, take

$$D = (0, \pi), \quad s = 0.75, \quad \nu_1 = \nu_2 = 2, \quad \gamma_1 = \gamma_2 = 1, \quad p_1 = p_2 = 4,$$

and $\|B\| = L_D = 0.4$. Since $\lambda_1 = 1$, the full spectral margin is $\Lambda_{\text{lin}} = 3$. Choose $\sigma = 0.2$, so $\Lambda_\sigma = 2.6$, and choose $\eta_f = 0.2$ and $\eta_d = (2\Lambda_\sigma - \eta_f)/2 = 2.5$, which maximizes S in (4.6), not necessarily the noise threshold. Then

$$S = \frac{(5.2 - 2.5 - 0.2)(2.5)}{0.4^2} = 39.0625 > 1.$$

For the prescribed interval $0 \leq \tau \leq 10$, take $\rho = 0.1$ and $\delta = 0.30$. Since

$$(1 + \rho)e^{\delta\tau_*} = 1.1e^3 \approx 22.09 < S,$$

Theorem 4.2 applies uniformly on $[0, 10]$. The deterministic margin is $g_0 = 0.30$. Theorem 4.5 with $q = 2$ gives $N_{2,10} = 40$ and $\bar{m}_{2,10} < 5.598$; direct one-dimensional maximization of (4.14) gives a maximum near $\beta = 0.014$, yielding the rigorous conservative certificate $\varepsilon_{\text{crit}}^{(2)} > 3.39 \times 10^{-3}$. The strict inequality $L_D < \Lambda_{\text{lin}}$ also certifies every other prescribed finite delay interval after reselecting the memory rate.

Table 2 records sensitivity to β with the other displayed parameters fixed; its entries are not jointly optimized over the Young parameters.

Table 2. Sensitivity of the explicit two-component noise certificate at $\tau_* = 10$.

β	0.010	0.012	0.0136	0.014	0.018
$\varepsilon_{\text{crit}}^{(2)}$	2.50×10^{-3}	3.00×10^{-3}	3.40×10^{-3}	3.398×10^{-3}	3.368×10^{-3}

5.5. A finite-dimensional retarded gradient application

The coefficient-matching mechanism is not tied to fractional compactness. Let $x(t) \in \mathbb{R}^m$ solve

$$dx + \left[Ax + \nabla \Psi(x) - B \int_{-\tau}^0 x(t + \vartheta) \mu_\tau(d\vartheta) - f(t) \right] dt = \text{diag}(\varepsilon_1 x_1, \dots, \varepsilon_m x_m) \circ dW(t), \quad (5.9)$$

where $A = \text{diag}(\alpha_1, \dots, \alpha_m)$ with $\alpha_j \geq \alpha_* > 0$, $\Psi(x) = \sum_{j=1}^m \psi_j(x_j)$ with each $\psi_j \in C^1(\mathbb{R})$ convex, ψ'_j locally Lipschitz, and $r\psi'_j(r) \geq 0$, $B \in \mathbb{R}^{m \times m}$, $W = (W_1, \dots, W_m)$ has independent two-sided components, and f satisfies the finite-dimensional analogue of hypothesis (H3). The separable potential preserves monotonicity under the diagonal Ornstein–Uhlenbeck conjugacy, and the measure μ_τ may again contain atoms.

Theorem 5.7 (Retarded gradient systems). *Assume the diagonal A and separable potential specified above, and $\|B\| < \alpha_*$. For every prescribed $\tau_* > 0$, one can choose the Young and memory parameters so that (5.9) generates continuous measurable cocycles with unique tempered pullback random attractors simultaneously for $0 \leq \tau \leq \tau_*$ and*

$$\max_{1 \leq j \leq m} |\varepsilon_j| \leq \varepsilon_{\text{crit}}^{(m)}(\beta, \tau_*).$$

Here (4.14) is used with $q = m$ and Λ_σ replaced by α_* . For $B \neq 0$, take $L_D = \|B\|$; for $B = 0$, choose any $L_D \in (0, \alpha_*)$. This positive bound is used in the Young and memory parameter choices. If $\mu_\tau \rightarrow \delta_{\{0\}}$,

where $\delta_{\{0\}}$ is the Dirac mass at zero, so that the memory maps converge to Bx uniformly on compact histories, these attractors are upper semicontinuous at $(\tau, \varepsilon) = (0, 0)$. In the deterministic isotropic linear subclass ($\varepsilon = 0$) $A = \alpha_* I$, $\Psi = 0$, and $B = LI$ with $L \geq 0$, the boundary $L < \alpha_*$ is exact for each fixed delay; the decay rate may deteriorate as $\tau \rightarrow \infty$.

Proof. Because A is diagonal, $\mathcal{Z}_\varepsilon^{-1} A \mathcal{Z}_\varepsilon = A$, so its coercive lower bound is unchanged. After the same conjugacy, the j th nonlinear term is

$$e^{-\varepsilon_j z_j} \psi'_j(e^{\varepsilon_j z_j} y_j).$$

For positive multipliers, scalar convexity gives both

$$e^{-\varepsilon_j z_j} [\psi'_j(e^{\varepsilon_j z_j} y_j) - \psi'_j(e^{\varepsilon_j z_j} v_j)](y_j - v_j) \geq 0$$

and $e^{-\varepsilon_j z_j} \psi'_j(e^{\varepsilon_j z_j} y_j) y_j \geq 0$. Thus, local solutions are unique and depend continuously on histories. The energy $X(t) = |y(t)|^2$ satisfies (2.13) with deterministic linear margin $2\alpha_*$, delay strength $\|B\|$, and the same amplification ratio as in (2.9). The inequality furnished by Theorem 2.2 prevents finite-time blow-up and extends the local solution globally, thereby generating a measurable cocycle.

Theorems 4.2 and 4.5, with $q = m$, and the original-history radius in Theorem 4.7, with componentwise maxima over $1 \leq j \leq m$, give a common measurable tempered absorbing ball. On a fixed terminal interval, the transformed variable y satisfies a parameter-uniform $W^{1,2}$ bound: its drift is bounded on the absorbing ball apart from the square-integrable forcing. The local Lipschitz property of the potential derivatives suffices on that ball. The compact embedding into $C([-\tau, 0]; \mathbb{R}^m)$, followed by multiplication by the continuous matrices $\mathcal{Z}_\varepsilon(\theta, \omega)$, gives compact original histories. No $W^{1,2}$ regularity is asserted for the stochastic variable x itself. This proves pullback asymptotic compactness and existence of the attractor.

For the parameter limit, use the extension in Section 5.2 with H replaced by $\mathbb{R}^m$ to extend the original histories to $[-\tau_*, 0]$. The bounds for y and the common continuity modulus of $\mathcal{Z}_\varepsilon$ give a compact family with tempered radius in $C([-\tau_*, 0]; \mathbb{R}^m)$. Uniform convergence of the diagonal random multipliers and compact-uniform convergence of the memory maps give finite-time convergence by Gronwall's inequality. The sequential complete-orbit argument used in Theorem 5.6 then proves upper semicontinuity. Finally, for $\varepsilon = 0$ the isotropic linear system decomposes into scalar equations with characteristic equation $\lambda + \alpha_* - L \int_{-\tau}^0 e^{\lambda \theta} \mu_\tau(d\theta) = 0$. If $L < \alpha_*$, choose $\eta \in (0, \alpha_*)$ with $Le^{\eta\tau} < \alpha_* - \eta$; the integral has modulus at most $e^{\eta\tau}$ when $\operatorname{Re} \lambda \geq -\eta$, which excludes roots in that half-plane. At $L = \alpha_*$, zero is a root; for $L > \alpha_*$, the real characteristic function is negative at zero and tends to infinity, hence has a positive root. The fixed-delay argument of Theorem 4.4 therefore applies to any of the stated probability measures. $\square$

Limits of monotonicity under diagonal conjugacy. For a non-separable convex potential, even a globally Lipschitz gradient need not retain Euclidean monotonicity under $F_Z(y) = Z^{-1} \nabla \Psi(Zy)$. For example, let

$$Q = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \quad \Psi(x) = \frac{1}{2} x^\top Q x, \quad Z = \operatorname{diag}(4, 1), \quad v = (1, -1)^\top.$$

The matrix Q is positive definite and $\nabla \Psi$ is globally Lipschitz, but

$$v^\top Z^{-1} Q Z v = -\frac{1}{8} < 0.$$

Thus separability cannot simply be replaced by convexity and a Lipschitz gradient in Theorem 5.7. The same computation shows why a general symmetric positive definite A does not retain its Euclidean coercivity under independent diagonal multipliers; the theorem therefore assumes diagonal A . In contrast, for a common scalar multiplier $Z = cI$ with $c > 0$, monotonicity follows from $\langle F_Z(y) - F_Z(v), y - v \rangle = c^{-2} \langle \nabla \Psi(cy) - \nabla \Psi(cv), cy - cv \rangle \geq 0$. That is a different noise structure and is not claimed as part of the present theorem.

6. Conclusions

For positive delays and nonzero multiplicative noise, the Ornstein–Uhlenbeck conjugacy produces a stationary, tempered amplification factor in the delay estimate that is almost surely unbounded in time. The coefficient-matched history functional uses the shift $r - \vartheta$ so that the lower endpoint $r = t + \vartheta$ reproduces the current-time coefficient. The resulting identity closes the random energy inequality without a deterministic uniform majorant.

Optimizing the first-mode damping gives the intrinsic criterion $L_D < \Lambda_{\text{lin}}$. In the deterministic scalar first-mode linear class with nonnegative feedback, this boundary is exact for each fixed delay: equality yields a neutral root and violation yields a positive root, while the stable rate can approach zero as the delay grows. Gaussian concentration further converts the positive deterministic margin into the explicit sufficient threshold (4.14); the certificate contains only deterministic parameters, the delay window, and the number of noise components.

For the fractional parabolic system, these estimates lead to global well-posedness, a continuous random cocycle, tempered pullback absorption, pullback asymptotic compactness, a unique pullback random attractor, and qualitative upper semicontinuity as delay and noise vanish. Measure-valued memory covers both distributed and discrete delays. For the stated diagonal linear drift and separable potential, the finite-dimensional system reaches the same conclusions through $W^{1,2}$ bounds on the transformed histories and continuity of the conjugacy, rather than fractional spatial smoothing.

The exact boundary is proved for the scalar first-mode class. Matrix feedback may permit sharper criteria based on positivity, spectral radius, or directional damping. Natural extensions include operator-valued coefficient matching, uniform attraction rates and exponential attractors, state-dependent delays, non-diagonal multiplicative noise, and unbounded spatial domains.

Author contributions

Zhao-sheng Wang: Conceptualization, Methodology, Writing—original draft. Guan-fa Li: Validation, Writing—review & editing. Both authors have read and approved the final version of the manuscript for publication.

Data availability

No empirical datasets were used. The time series for Figure 1 were generated by the stated delay-equation solver. The numerical script, parameter settings, trajectory data, and step-halving checks are supplied as supplementary computational material.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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