



Research article**Qualitative behavior and impact of fractional parameter on soliton structures of the time fractional (2+1)-dimensional generalized Boussinesq equation via multi-reduction techniques****Nurzhan Serikbayev¹, Rajesh Kumar Gupta², Gaukhar Shaikhova^{1,*} and Sharmila²**

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Abstract: The fractional (2+1)-dimensional generalized Boussinesq equation, which incorporates the classical (1+1)- and (2+1)-dimensional Boussinesq and Benjamin-Ono equations as special cases, is investigated symbolically to find the infinitesimal generators and multi-reduction via invariance. The reduced equations are solved using the power series and analytical methods, and at the same time, we acquire a family of the soliton solutions, including singular, combined-singular, breather solitons, bright solitons, and two families of the analytic solutions via two different schemes. With respect to the wave field for the wave dynamics in atmospheric sciences and oceanography, our breather, kink, singular, and bright soliton solutions are closely linked to the nonlinear wave dynamics in the Boussinesq equation, including oscillating wave packets, step-like transitions, and stable localized pulses balanced by nonlinearity and dispersion. The analysis of the fractional parameter β 's influence on soliton dynamics through 3D and 2D plots for particular values of parameters shows how β affects the dispersion, amplitude, and intensity of waves. Furthermore, the qualitative behavior of the previously unstudied integer-order equation is investigated through chaotic analysis using a planar dynamical system approach, which has not been explored previously. Key novelties of this work are the derivation of new symmetries, the construction of diverse analytic solutions, and the analysis of the system's dynamical behavior.

Keywords: optimal system; (2+1)-dimensional generalized Boussinesq equation; chaotic analysis; generalized Sardar sub-equation technique; modified exp-function approach; power series solutions; soliton solutions

Mathematics Subject Classification: 39A33, 35C10, 35Q51

1. Introduction

Nonlinear partial differential equations (NLPDEs) have seen tremendous advancements in recent years. Numerous scientific and engineering fields have shown a great deal of interest in the theory of fractional calculus due to its many applications and significance in describing the complex dynamic behavior of real-world issues such as biological populations, diffusion systems, fluid flow, and trajectories [1–3]. The exact solutions of fractional partial differential equations (FPDEs) have been studied in a number of physical phenomena that occur in the real world. The concept of FPDEs originates from fractional-order derivatives. Fractional derivatives are defined in multiple ways, encompassing the Riemann–Liouville fractional derivative [4, 5], the Grünwald–Letnikov fractional derivative [6], etc. R. Khalil et al. [7] introduced the conformable fractional derivative, which offers a robust and consistent definition of fractional derivatives. Numerous attributes associated with this novel definition have been examined [8, 9].

Many researchers have focused their attention on examining different aspects of NLPDEs over the last few years. This involves investigating the general dynamic behavior, soliton solutions, and traveling wave solutions of these equations. Bifurcation analysis [10, 11] and chaotic analysis techniques [12] have also been employed to investigate the system's response to key parameters. These methods have played a vital role in revealing and investigating the chaotic behavior present in NLPDEs. Bifurcation analysis gives a better understanding of the creation and stability of spatial configuration in NLPDEs. Researchers can gain a better understanding of structures such as wave-fronts or spatially periodic configurations by looking at pattern-forming bifurcations. This analysis facilitates the investigation of various phenomena, including reaction-diffusion systems having localized structures. In NLPDEs, bifurcation phenomena can be examined using chaotic analysis. When a parameter results in a qualitatively different behavior from the system, bifurcations occur. The values of critical parameters at which bifurcations occur and chaotic behavior appears can be found with the help of chaotic analysis.

Exact solutions of time-fractional and space-time-fractional nonlinear partial differential equations, obtained via reductions using invariant analysis, were first investigated by Singla and Gupta [13, 14]. Also there are a variety of methods, such as Jacobi elliptic function expansion [15], similarity reduction method [16, 17], new extended G'/G expansion approach [18], generalized Arnous approach [19], new modified Sardar sub-equation method [20], generalized unified technique [21], Hirota bilinear method [22], and F-expansion technique [23].

Solitons have gained prominence in various scientific fields, including plasma physics, optical fiber research, and fluid dynamics. Due to a careful balance between nonlinearity and dispersion, soliton wave packets are self-reinforcing and retain their speed and shape as they propagate. In these fields, solitary waves have garnered considerable interest and attention. Many researchers [22, 24, 25] employed a variety of techniques to derive singular, lump, dark, bright, and other types of soliton solutions of NLPDEs.

Here, we examine the conformable time-fractional (2+1)-dimensional generalized Boussinesq equation [26]

$$\Gamma_{tt}^{2\beta} - a\Gamma_{xx} - b\Gamma_{yy} - 2\gamma\Gamma_x^2 - 2\gamma\Gamma\Gamma_{xx} - \delta\Gamma_{xxxx} = 0, \quad (1.1)$$

where β denotes the fractional derivative's order with respect to t and a , b , γ , and δ are arbitrary constants. It explains how gravity waves travel across water's surface, and specifically, the head-on

collision of oblique waves. In this study, the conformable time-fractional (2+1)-dimensional generalized Boussinesq equation is transformed into a nonlinear ordinary differential equation (ODE) via symmetry reduction. Several exact solutions, such as exponential solitary waves, one-singular solitary waves, bright solitons, and multi-singular solitary waves, are derived by employing the generalized Sardar sub-equation technique, modified exponential function expansion approach, sine-cosine method, and extended tanh method. In the existing studies, other researchers have investigated the governing equation and its special cases using different analytical methods.

Song and Shao [26] studied bifurcation behavior of (2+1)-dimensional generalized Boussinesq equation. Johnson [27] demonstrated the evolution of gravitational surface waves by examining the (2+1)-dimensional Boussinesq equation. Mitsotakis [28] investigated the higher-dimensional Boussinesq equation with a variable bottom and simulated wave propagation using surface water wave theory. Tian [29] obtained solitary wave solutions of the generalized Boussinesq water wave equation via symmetry reduction technique, which is a special case of the governing equation for $\beta = 1, b = 0, \gamma = b, \delta = -c$. Muhammad, et al. [30] derived traveling wave solutions of the Boussinesq equation by employing Sardar sub-equation approach, which is a special case of the considered equation for $\beta = 1, b = 0, a = \gamma = 1, \delta = \lambda$. The considered equation includes (2+1)- and (1+1)-dimensional Boussinesq and Benjamin–Ono equations as shown in table 1.

Table 1. Special cases of the time-fractional (2+1)-dimensional generalized Boussinesq equation.

Parameters	The corresponding equations
$\beta = 1, a = -\gamma, b = -\delta, \gamma = -\alpha, \delta = -\beta$	(2 + 1)-dimensional generalized Boussinesq equation [31]
$\beta = 1, a = \gamma, b = 1, \gamma = -\alpha, \delta = -\beta$	(2 + 1)-dimensional Boussinesq equation [32]
$\beta = 1, a = 1, b = 0, \gamma = -\alpha, \delta = -\beta$	(1 + 1)-dimensional Boussinesq equation [33]
$\beta = 1, \gamma = -\beta, \delta = -\gamma$	(1 + 1)-dimensional Benjamin-Ono equation [34]

1.1. Conformable fractional derivative (CFD)

The conformable fractional derivative has recently found diverse applications across multiple studies. Abdeljawad [9] extended its framework by formulating fractional versions of essential concepts such as integration by parts, the Laplace transform, the chain rule, Taylor series, exponential functions, Grönwall's inequality, and linear dynamical systems. Oqielat et al. [35] investigated the solutions of nonlinear time-fractional Schrödinger equations by comparing the Caputo and conformable derivatives. In addition to some novel definitions, Atangana et al. [36] determined some new theorems and properties of this definition of the CFD. For a function $f : (0, \infty) \rightarrow \mathbb{R}$, the CFD of order β is defined by [7]:

$$\mathcal{D}_t^\beta f(t) = \lim_{v \rightarrow 0} \frac{f(t + vt^{1-\beta}) - f(t)}{v}, \quad t > 0, 0 < \beta \leq 1.$$

If $f(t)$ is β -differentiable in some $(0, s)$, $s > 0$, and $\lim_{t \rightarrow 0^+} \mathcal{D}_t^\beta f(t)$ exists, then $\mathcal{D}_t^\beta f(0) = \lim_{t \rightarrow 0^+} \mathcal{D}_t^\beta f(t)$ exists. The following properties hold for the conformable fractional derivative of β -differentiable functions $f(t)$ and $g(t)$:

$$(1) \quad \mathcal{D}_t^\beta(pf + qg) = p\mathcal{D}_t^\beta(f) + q\mathcal{D}_t^\beta(g), \quad p, q \in \mathbb{R}.$$

$$(2) \quad \mathcal{D}_t^\beta(t^\Lambda) = \Lambda t^{\Lambda-\beta}, \quad \Lambda \in \mathbb{R}.$$

$$(3) \quad \mathcal{D}_t^\beta(fg) = f\mathcal{D}_t^\beta(g) + g\mathcal{D}_t^\beta(f).$$

$$(4) \quad \mathcal{D}_t^\beta\left(\frac{f}{g}\right) = \frac{g\mathcal{D}_t^\beta(f) - f\mathcal{D}_t^\beta(g)}{g^2}.$$

One of the main advantages of the conformable fractional derivative is that, unlike many other fractional derivatives, it retains the familiar rules of classical differentiation, including the quotient, product, and chain rules. It also facilitates the generalization of several well-known transforms, like the Laplace and Sumudu transforms, and enables the solution of fractional differential equations. The conformable framework has paved the way for developing various related fractional derivative definitions, such as those found in [37–39], thereby enriching the field of fractional calculus [40].

Although the conformable fractional derivative is local in nature and does not explicitly account for the long-memory effects described by nonlocal fractional derivatives, it offers a simple and useful framework for studying fractional-order nonlinear wave models while preserving several properties of the classical derivative. In the present study, it is useful for examining nonlinear wave propagation and dispersion, as well as the effect of the fractional order on the resulting wave patterns. Moreover, the fractional formulation can reveal additional or less-evident symmetry structures that may be used to obtain exact solutions not easily obtained from the corresponding integer-order model.

The manuscript is structured as follows:

- In Section 2, there are successive applications of symmetry reduction technique, a one-dimensional optimal system is constructed, and corresponding symmetry reduction of the governing equation into reduced PDEs and then into reduced ODEs.
- In Section 3, explicit power series solutions of reduced ODEs are obtained.
- In Section 4, various soliton solutions of the governing equation are obtained by applying generalized Sardar sub-equation, modified exp-function expansion techniques on reduced DEs and traveling wave solutions via analytical schemes.
- In Section 5, the chaotic nature of the governing equation is investigated through 2D, 3D-phase space, and time series trajectories.
- In Section 6, the influence of the conformable fractional parameter on some soliton solutions is investigated for varying values of β , through 2D and 3D plots with particular values of the parameters.
- In Section 7, final conclusions for the work are presented.

The overall framework, covering the conformable time-fractional generalized Boussinesq equation, symmetry reduction, analytical solution methods, wave solutions, chaotic analysis, and the effect of the fractional parameter, is given by Figure 1.

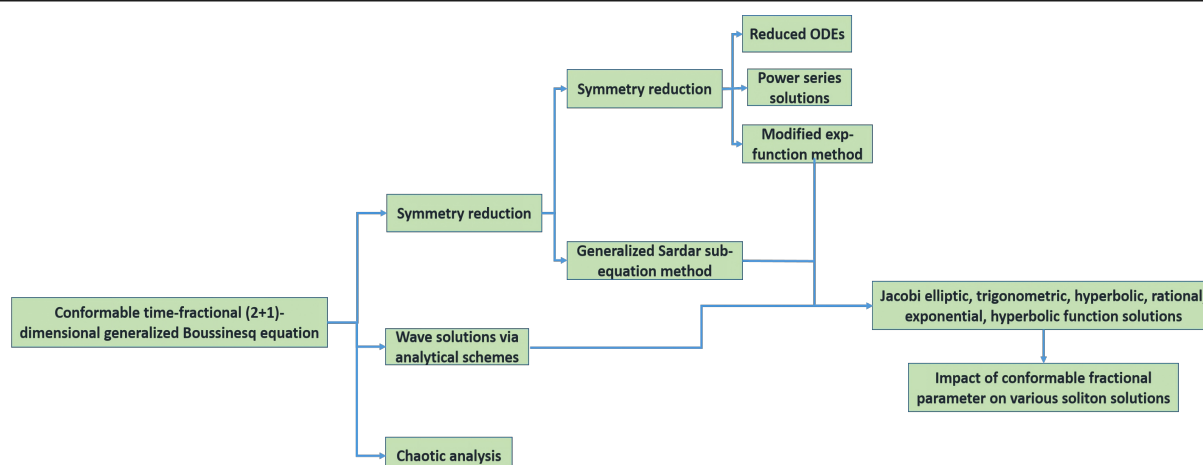


Figure 1. Schematic diagram of the paper's structure.

2. Conformable time-fractional (2+1) NLPDEs: optimal system and reduction via invariance

In this section, the invariance characteristics of the conformable time-fractional (2+1) NLPDE have been investigated using double symmetry-reduction technique.

The Lie group's infinitesimal generator [10, 16, 17] for the considered equation is provided by

$$\Xi = \zeta \frac{\partial}{\partial x} + \Phi \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \psi \frac{\partial}{\partial \Gamma}, \quad (2.1)$$

where the dependent variable Γ is expressed in terms of x , y , t , and the corresponding infinitesimals of x , y , t , and Γ are $\zeta(x, y, t, \Gamma)$, $\Phi(x, y, t, \Gamma)$, $\tau(x, y, t, \Gamma)$, and $\psi(x, y, t, \Gamma)$, respectively.

Equation (2.1)'s extended prolongation is as follows:

$$Pr^{(n,\beta)} = \zeta \frac{\partial}{\partial x} + \Phi \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \psi \frac{\partial}{\partial \Gamma} + \psi^{t\beta} \frac{\partial}{\partial \Gamma_t^\beta} + \psi^x \frac{\partial}{\partial \Gamma^x} + \psi^{xx} \frac{\partial}{\partial \Gamma^{xx}} + \psi^y \frac{\partial}{\partial \Gamma^y} + \psi^{yy} \frac{\partial}{\partial \Gamma^{yy}} + \psi^{t\beta t\beta} \frac{\partial}{\partial \Gamma_{tt}^{2\beta}} + \dots, \quad (2.2)$$

where the following expressions give the extended infinitesimals ψ^x , ψ^y , ψ^{xx} , and ψ^{yy} .

$$\begin{aligned} \psi^x &= D_x(\psi) - \Gamma_t D_x(\tau) - \Gamma_x D_x(\zeta) - \Gamma_y D_x(\Phi), \\ \psi^y &= D_y(\psi) - \Gamma_t D_y(\tau) - \Gamma_x D_y(\zeta) - \Gamma_y D_y(\Phi), \\ \psi^{xx} &= D_x(\psi^x) - \Gamma_{xt} D_x(\tau) - \Gamma_{xx} D_x(\zeta) - \Gamma_{xy} D_x(\Phi), \\ \psi^{yy} &= D_y(\psi^y) - \Gamma_{yt} D_y(\tau) - \Gamma_{yx} D_y(\zeta) - \Gamma_{yy} D_y(\Phi), \end{aligned} \quad (2.3)$$

where D_x and D_y denote the total derivative operators.

Additionally, $\psi^{t\beta t\beta}$ is the fractional extended infinitesimal that is given by

$$\psi^{t\beta t\beta} = t^{2-2\beta} \psi^{t,t} + (1-\beta) t^{1-2\beta} \psi^t + (2-2\beta) t^{1-2\beta} \tau \Gamma_{t,t} + (1-\beta)(1-2\beta) t^{-2\beta} \tau \Gamma_t, \quad (2.4)$$

where ψ^t and ψ^{tt} are provided by

$$\begin{aligned}\psi^t &= D_t(\psi) - \Gamma_t D_t(\tau) - \Gamma_x D_t(\zeta) - \Gamma_y D_t(\Phi), \\ \psi^{tt} &= D_t(\psi^t) - \Gamma_{tt} D_t(\tau) - \Gamma_{tx} D_t(\zeta) - \Gamma_{ty} D_t(\Phi).\end{aligned}$$

The following symmetry-determining equation was obtained by employing the 4th prolongation to Eq (1.1):

$$\psi^{t\beta t\beta} - a\psi^{xx} - b\psi^{yy} - 4\gamma\Gamma_x\psi^x - 2\gamma\Gamma\psi^{xx} - 2\gamma\psi\Gamma_{xx} - \delta\psi^{xxxx} = 0. \quad (2.5)$$

The following system of determining equations is obtained after substituting the values of extended infinitesimals:

$$\begin{aligned}\Phi_\Gamma &= \Phi_x = \tau_\Gamma = \tau_x = \zeta_\Gamma = \zeta_t = \zeta_y = \psi_{\Gamma\Gamma} = \Phi_{\Gamma\Gamma} = \tau_{\Gamma\Gamma} = 0, \\ 2b(\Phi_y - 2\zeta_x) &= 0, \\ -2\delta(2\psi_{x\Gamma} - 3\zeta_{xx}) &= 0, \\ -2\gamma(2\zeta_x + \psi_\Gamma) &= 0, \\ -2(\Phi_t)t^{2-2\beta} + 2(2\tau_y)b &= 0, \\ -2\tau(-1+\beta)t^{1-2\beta} + 4\left(\zeta_x - \frac{\tau_t}{2}\right)t^{2-2\beta} &= 0, \\ (\Phi_t)(-1+\beta)t^{1-2\beta} + (-2\psi_{y\Gamma} + \Phi_{yy})b - (\Phi_{tt})t^{2-2\beta} &= 0, \\ (2\psi_{\Gamma t} - \tau_{tt})t^{2-2\beta} + (\tau_{yy})b - 4\left(\zeta_x - \frac{\tau_t}{4}\right)(-1+\beta)t^{1-2\beta} + (2\beta-1)(-1+\beta)\tau t^{-2\beta} &= 0, \\ (-4\gamma\Gamma - 2a)\zeta_x - 2\gamma\psi + 4\delta\zeta_{xxx} - 6\delta\psi_{xx\Gamma} &= 0, \\ (-4\gamma\Gamma - 2a)\psi_{x\Gamma} + (2\gamma\Gamma + a)\zeta_{xx} - 4\psi_x\gamma - 4\delta\psi_{xxx\Gamma} + \delta\zeta_{xxxx} &= 0, \\ (-2\gamma\Gamma - a)\psi_{xx} - (\psi_t)(-1+\beta)t^{1-2\beta} - (\psi_{yy})b - \delta\psi_{xxxx} + t^{2-2\beta}\psi_{tt} &= 0.\end{aligned} \quad (2.6)$$

The following infinitesimal values are obtained by solving the aforementioned system of determining equations:

$$\psi = -\frac{c_1(2\gamma\Gamma + a)}{2\gamma}, \quad \Phi = c_1y + c_2, \quad \tau = \frac{c_1t}{\beta} + c_3t^{1-\beta}, \quad \zeta = \frac{c_1x}{2} + c_4, \quad (2.7)$$

where the c_i s are arbitrary constants and $\gamma \neq 0$. The following infinitesimal vectors of Eq (1.1) are obtained from (2.7):

$$\begin{aligned}\Xi_1 &= -\frac{2\gamma\Gamma}{+}a2\gamma\frac{\partial}{\partial\Gamma} + y\frac{\partial}{\partial y} + \frac{t}{\beta}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x}, \\ \Xi_2 &= \frac{\partial}{\partial y}, \\ \Xi_3 &= t^{1-\beta}\frac{\partial}{\partial t}, \\ \Xi_4 &= \frac{\partial}{\partial x}.\end{aligned} \quad (2.8)$$

2.1. Optimal system

A one-dimensional optimal system of symmetry subalgebras is constructed, using the procedure described in [41]. The infinitesimals can be expressed as a linear combination of Ξ_j using the commutation relations between the four infinitesimals generators listed in Table 2.

$$\Xi = \alpha_1 \Xi_1 + \alpha_2 \Xi_2 + \alpha_3 \Xi_3 + \alpha_4 \Xi_4.$$

Furthermore, we obtain the adjoint relations as indicated in Table 3 by applying the Olver technique [42]. The commutator operation is given by

$$[\Xi_j, \Xi_k] = \Xi_j(\Xi_k) - \Xi_k(\Xi_j), \quad j, k = 1, 2, 3, 4.$$

The adjoint representation is shown by the following.

$$\text{Ad}(\exp(\epsilon \Xi_j)) \Xi_k = \Xi_k - \epsilon [\Xi_j, \Xi_k] + \frac{\epsilon^2}{2!} [\Xi_j, [\Xi_j, \Xi_k]] - \frac{\epsilon^3}{3!} [\Xi_j, [\Xi_j, [\Xi_j, \Xi_k]]] + \cdots, \quad \epsilon \in \mathbb{R}.$$

Table 2. Commutator table.

$[\Xi_j, \Xi_k]$	Ξ_1	Ξ_2	Ξ_3	Ξ_4
Ξ_1	0	$-\Xi_2$	$-\Xi_3$	$-\frac{\Xi_4}{2}$
Ξ_2	Ξ_2	0	0	0
Ξ_3	Ξ_3	0	0	0
Ξ_4	$\frac{\Xi_4}{2}$	0	0	0

Table 3. Adjoint table.

$\text{Ad}[\Xi_i, \Xi_j]$	Ξ_1	Ξ_2	Ξ_3	Ξ_4
Ξ_1	Ξ_1	$e^\epsilon \Xi_2$	$e^\epsilon \Xi_3$	$e^{\frac{\epsilon}{2}} \Xi_4$
Ξ_2	$\Xi_1 - \epsilon \Xi_2$	Ξ_2	Ξ_3	Ξ_4
Ξ_3	$\Xi_1 - \epsilon \Xi_3$	Ξ_2	Ξ_3	Ξ_4
Ξ_4	$\Xi_1 - \frac{\epsilon}{2} \Xi_4$	Ξ_2	Ξ_3	Ξ_4

Consequently, for Eq (1.1), the optimal system of Lie subalgebras is given by

$$\{\Xi_1, \Xi_2 + \alpha_3 \Xi_3 + \alpha_4 \Xi_4, \Xi_3 + \alpha_4 \Xi_4\}.$$

Next, we will perform the reduction of the considered PDE corresponding to this optimal system.

2.2. Dimensional reduction

Here, via double symmetry-reduction technique, we reduce the dimension of the governing equation corresponding to each vector field and subsequently transform reduced PDEs into ODEs.

$$(\mathbf{I}) < \Xi_1 >$$

The characteristic equations corresponding to this subalgebra are

$$\frac{dx}{\frac{x}{2}} = \frac{dy}{y} = \frac{dt}{\frac{t}{\beta}} = \frac{d\Gamma}{\frac{-2\gamma\Gamma+a}{2\gamma}}.$$

The solution corresponding to these equations is given by

$$\Gamma(x, y, t) = t^{-\beta} f(p_1, p_2) - \frac{a}{2\gamma},$$

where $f(p_1, p_2)$, $p_1 = x^2 t^{-\beta}$, and $p_2 = y t^{-\beta}$ are new dependent and similarity variables, respectively.

After substituting the above value of Γ in Eq (1.1), we have the following reduced PDE:

$$4\beta^2 p_1(f_{p_1}) + 4\beta^2 p_2(f_{p_2}) + 2\beta^2 f + \beta^2 p_1^2(f_{p_1, p_1}) + \beta^2 p_2^2(f_{p_2, p_2}) + 2\beta^2 p_1 p_2(f_{p_1, p_2}) - b(f_{p_2, p_2}) - 8\gamma p_1(f_{p_1}^2) - 4\gamma f(f_{p_1}) - 8\gamma p_1 f(f_{p_1, p_1}) - 48\delta p_1(f_{p_1, p_1, p_1}) - 12\delta(f_{p_1, p_1}) - 16\delta p_1^2(f_{p_1, p_1, p_1, p_1}) = 0. \quad (2.9)$$

In the same way that we performed the similarity analysis for the PDE (1.1), the new infinitesimals are obtained by applying the Lie transformation technique to Eq (2.9).

$$\zeta_{p_1}(p_1, p_2, f) = b_1 \beta^2 p_1 p_2, \quad \zeta_{p_2}(p_1, p_2, f) = b_1 (\beta^2 p_2^2 - b), \quad \psi_f(p_1, p_2, f) = -b_1 \beta^2 p_2 f. \quad (2.10)$$

Corresponding to Eq (2.10), the infinitesimal generator is

$$\Xi_5 = \beta^2 p_1 p_2 \frac{\partial}{\partial p_1} + (\beta^2 p_2^2 - b) \frac{\partial}{\partial p_2} - \beta^2 p_2 f \frac{\partial}{\partial f}. \quad (2.11)$$

The following ODE was obtained corresponding to $\langle \Xi_5 \rangle$ by reducing Eq (2.9).

$$32\gamma p_3^2 F F_{p_3, p_3} + (5616\delta p_3 + 4b) p_3 F_{p_3, p_3} + 32\gamma p_3^2 F^2 + (104\gamma F + 2904\delta) p_3 F_{p_3} + 4b F_{p_3} + 120\delta F + 20\gamma F^2 + 2432\delta p_3^3 F_{p_3, p_3, p_3} + 256\delta p_3^4 F_{p_3, p_3, p_3, p_3} = 0, \quad (2.12)$$

where $p_3 = \frac{\beta^2 p_2^2 - b}{\beta^2 p_1^2}$ is the similarity variable and $F(p_3)$ is the new dependent variable.

(II) $\langle \Xi_2 + \alpha_3 \Xi_3 + \alpha_4 \Xi_4 \rangle$

The corresponding characteristic equations for this subalgebra are

$$\frac{dx}{\alpha_4} = \frac{dy}{1} = \frac{dt}{\alpha_3 t^{(1-\beta)}} = \frac{d\Gamma}{0},$$

where $\alpha_3 \neq 0, \alpha_4 \neq 0$.

The solution of these equations is provided by

$$\Gamma(x, y, t) = f(p_1, p_2),$$

where the similarity variables are $p_1 = rx - s \frac{t^\beta}{\beta}$ and $p_2 = y - s \frac{t^\beta}{\beta}$, and $f(p_1, p_2)$ is the new dependent variable.

The following reduced PDE is obtained by replacing the above value of Γ in Eq (1.1).

$$l f_{p_1, p_1} + m f_{p_2, p_2} + 2s^2 f_{p_1, p_2} - 2\gamma r^2 f_{p_1}^2 - 2\gamma r^2 f f_{p_1, p_1} - \delta r^4 f_{p_1, p_1, p_1, p_1} = 0, \quad (2.13)$$

where $m = (s^2 - b)$, $l = (s^2 - ar^2)$, $s = \frac{1}{\alpha_3}$, and $r = \frac{1}{\alpha_4}$.

After applying the Lie transformation technique to Eq (2.13), the new infinitesimals are

$$\begin{aligned}\zeta_{p_1}(p_1, p_2, f) &= \frac{b_2 p_1}{2} + b_2 \tilde{p} p_2 + b_3, \quad \zeta_{p_2}(p_1, p_2, f) = b_2 p_2 + b_4, \\ \psi_f(p_1, p_2, f) &= -b_2(f + \tilde{l}),\end{aligned}\quad (2.14)$$

where $\tilde{p} = \frac{s^2}{2m}$ and $\tilde{l} = (\frac{s^4 - lm}{2\gamma r^2 m})$.

The corresponding infinitesimal generators are expressed as

$$\begin{aligned}\Xi_6 &= -(f + \tilde{l}) \frac{\partial}{\partial f} + \left(\frac{p_1}{2} + \tilde{p} p_2 \right) \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2}, \\ \Xi_7 &= \frac{\partial}{\partial p_1}, \\ \Xi_8 &= \frac{\partial}{\partial p_2}.\end{aligned}\quad (2.15)$$

The combinations of these vector fields can also be studied; however, here we focus on the reduction corresponding to $\langle \Xi_6 \rangle$ to obtain the relevant solutions. So, using the vector field $\langle \Xi_6 \rangle$, Eq (2.13) is reduced into the following ODE:

$$(8\gamma r^2 F - m p_4^2) F_{p_4, p_4} - 8mF + 8\gamma r^2 F_{p_4}^2 - 7m p_4 F_{p_4} + 4\delta r^4 F_{p_4, p_4, p_4, p_4} = 0, \quad (2.16)$$

where $F(p_4)$ is the new dependent variable and $p_4 = -2\tilde{p} p_2^{\frac{1}{2}} + p_1 p_2^{\frac{-1}{2}}$ is the similarity variable.

(III) $\langle \Xi_3 + \alpha_4 \Xi_4 \rangle$

The characteristic equations for this subalgebra are expressed as

$$\frac{dx}{\alpha_4} = \frac{dy}{0} = \frac{dt}{t^{(1-\beta)}} = \frac{d\Gamma}{0}.$$

The following is the dependent variable for these equations:

$$\Gamma(x, y, t) = f(p_1, p_2),$$

where the similarity variables are $p_1 = rx - \frac{t^\beta}{\beta}$ and $p_2 = y$, and $f(p_1, p_2)$ is the new dependent variable.

By substituting the above value of Γ in Eq (1.1), the reduced PDE is as follows:

$$(1 - ar^2) f_{p_1, p_1} - b f_{p_2, p_2} - 2\gamma r^2 f_{p_1}^2 - 2\gamma r^2 f f_{p_1, p_1} - \delta r^4 f_{p_1, p_1, p_1, p_1} = 0. \quad (2.17)$$

After applying the Lie transformation approach to Eq (2.17), the new infinitesimals are as follows:

$$\begin{aligned}\zeta_{p_1}(p_1, p_2, f) &= \frac{b_5 p_1}{2} + b_6, \quad \zeta_{p_2}(p_1, p_2, f) = b_5 p_2 + b_7, \\ \psi_f(p_1, p_2, f) &= -b_5(f + L),\end{aligned}\quad (2.18)$$

where $L = \frac{ar-1}{2\gamma r^2}$.

For these infinitesimals, the corresponding infinitesimal generators are as follows:

$$\begin{aligned}\Xi_9 &= -(f + L)\frac{\partial}{\partial f} + \frac{p_1}{2}\frac{\partial}{\partial p_1} + p_2\frac{\partial}{\partial p_2}, \\ \Xi_{10} &= \frac{\partial}{\partial p_1}, \\ \Xi_{11} &= \frac{\partial}{\partial p_2}.\end{aligned}\tag{2.19}$$

Although combinations of these vector fields may also be considered, we restrict our analysis to the reduction associated with $\langle \Xi_9 \rangle$ to derive the relevant solutions. Using $\langle \Xi_9 \rangle$, Eq (2.17) reduces to the following ODE:

$$\begin{aligned}2bF + 4(bp_5 + \gamma r^2 F)F_{p_5} + 8\gamma r^2 p_5 F_{p_5}^2 + (bp_5^2 + 12r^4 \delta + 8\gamma r^2 p_5 F)F_{p_5, p_5} + 48\delta r^4 p_5 F_{p_5 p_5 p_5} \\ + 16\delta r^4 p_5^2 F_{p_5, p_5, p_5, p_5} = 0,\end{aligned}\tag{2.20}$$

where $F(p_5)$ and $p_5 = \frac{p_1^2}{p_2}$ are new dependent and similarity variables, respectively.

3. Power series solutions of reduced ODEs

Here, the explicit power series solutions corresponding to reduced ODEs (2.12), (2.16), and (2.20) obtained in the previous section are determined through the power series method [43].

3.1. Power series solution of ODE (2.12)

By assuming that $F(p_3)$ has the particular forms mentioned below, the reduced ODE can be explicitly solved in terms of power series.

$$F(p_3) = \sum_{\varpi=0}^{\infty} a_{\varpi} p_3^{\varpi},\tag{3.1}$$

where a_{ϖ} for $\varpi = 0, 1, 2, \dots$ are the coefficients to be obtained. From Eq (3.1), we have

$$\begin{aligned}F'(p_3) &= \sum_{\varpi=1}^{\infty} \varpi a_{\varpi} p_3^{\varpi-1}, \quad F''(p_3) = \sum_{\varpi=2}^{\infty} \varpi(\varpi-1) a_{\varpi} p_3^{\varpi-2}, \\ F'''(p_3) &= \sum_{\varpi=3}^{\infty} \varpi(\varpi-1)(\varpi-2) a_{\varpi} p_3^{\varpi-3}, \\ F''''(p_3) &= \sum_{\varpi=4}^{\infty} \varpi(\varpi-1)(\varpi-2)(\varpi-3) a_{\varpi} p_3^{\varpi-4}.\end{aligned}\tag{3.2}$$

After using expressions (3.2) in Eq (2.12), we obtain

$$\begin{aligned}
& 32\gamma \sum_{\varpi=2}^{\infty} \sum_{j=0}^{\varpi-2} \left[(\varpi-j-1)(\varpi-j)a_j a_{\varpi-j} + (j+1)(\varpi-j-1)a_{j+1} a_{\varpi-j-1} \right] p_3^{\varpi} + 5616\delta \sum_{\varpi=2}^{\infty} \varpi(\varpi-1) \\
& \times a_{\varpi} p_3^{\varpi} + 4b \sum_{\varpi=1}^{\infty} \varpi(\varpi+1)a_{\varpi+1} p_3^{\varpi} + 104\gamma \sum_{\varpi=1}^{\infty} \sum_{j=0}^{\varpi-1} (\varpi-j)a_j a_{\varpi-j} p_3^{\varpi} + 2904\delta \sum_{\varpi=1}^{\infty} \varpi a_{\varpi} p_3^{\varpi} + 4b \sum_{\varpi=0}^{\infty} (\varpi \\
& + 1)a_{\varpi+1} p_3^{\varpi} + 20\gamma \sum_{\varpi=0}^{\infty} \sum_{j=0}^{\varpi} a_j a_{\varpi-j} p_3^{\varpi} + 120\delta \sum_{\varpi=0}^{\infty} a_{\varpi} p_3^{\varpi} + 256\delta \sum_{\varpi=4}^{\infty} \varpi(\varpi-1)(\varpi-2)(\varpi-3)a_{\varpi} p_3^{\varpi} \\
& + 2432\delta \sum_{\varpi=3}^{\infty} \varpi(\varpi-1)(\varpi-2)a_{\varpi} p_3^{\varpi} = 0.
\end{aligned} \tag{3.3}$$

For $\varpi = 0$ in the above equation, we have

$$4ba_1 + 20\gamma a_0^2 + 120\delta a_0 = 0. \tag{3.4}$$

By simplifying Eq (3.4), we get

$$a_1 = \frac{-1}{b}(5\gamma a_0^2 + 30\delta a_0), \quad b \neq 0. \tag{3.5}$$

For $\varpi \geq 1$, we have

$$\begin{aligned}
a_{\varpi+1} = \frac{-1}{4b(\varpi+1)^2} & \left[(5616\delta\varpi(\varpi-1) + 2904\delta\varpi + 120\delta + 256\delta\varpi(\varpi-1)(\varpi-2)(\varpi-3) + 2432\delta\varpi \right. \\
& \times (\varpi-1)(\varpi-2))a_{\varpi} + 20\gamma \sum_{j=0}^{\varpi} a_j a_{\varpi-j} + 104\gamma \sum_{j=0}^{\varpi-1} (\varpi-j)a_j a_{\varpi-j} + 32\gamma \sum_{j=0}^{\varpi-2} ((\varpi-j-1)(\varpi-j) \\
& \times a_j a_{\varpi-j} + (j+1)(\varpi-j-1)a_{j+1} a_{\varpi-j-1}) \left. \right].
\end{aligned} \tag{3.6}$$

All coefficients a_{ϖ} , $\varpi \geq 1$, of the power series (3.1) can be determined from Eqs (3.5)–(3.6) as follows:

$$\begin{aligned}
a_2 &= \frac{-1}{b}(9\gamma a_0 a_1 + 189\delta a_1), \\
a_3 &= \frac{-1}{3b}(26\gamma a_0 a_2 + 1430\delta a_2 + 13\gamma a_1^2), \\
&\vdots
\end{aligned}$$

For any chosen constant number a_0 , the subsequent terms of the sequence $\{a_{\varpi}\}_{\varpi=0}^{\infty}$ can therefore be determined consecutively and uniquely through Eq (3.6).

Thus, the expression for the power series solution of Eq (1.1) is as follows:

$$\Gamma(x, y, t) = \frac{1}{x^2} \left[a_0 + a_1 + \sum_{\varpi=1}^{\infty} a_{\varpi+1} p_3^{\varpi+1} \right] - \frac{a}{2\gamma}.$$

After inserting Eq (3.6), we have

$$\begin{aligned}\Gamma(x, y, t) = & \frac{1}{x^2} \left[a_0 + a_1 p_3 - \sum_{\varpi=1}^{\infty} \frac{1}{4b(\varpi+1)^2} \left((5616\delta\varpi(\varpi-1) + 2904\delta\varpi + 120\delta + 256\delta\varpi(\varpi-1)) \right. \right. \\ & \times (\varpi-2)(\varpi-3) + 2432\delta\varpi(\varpi-1)(\varpi-2)a_{\varpi} + 20\gamma \sum_{j=0}^{\varpi} a_j a_{\varpi-j} + 104\gamma \sum_{j=0}^{\varpi-1} (\varpi-j)a_j a_{\varpi-j} \\ & \left. \left. + 32\gamma \sum_{j=0}^{\varpi-2} ((\varpi-j-1)(\varpi-j)a_j a_{\varpi-j} + (j+1)(\varpi-j-1)a_{j+1} a_{\varpi-j-1}) \right) p_3^{\varpi+1} \right] - \frac{a}{2\gamma},\end{aligned}$$

where $p_3 = \frac{\beta^2 y^2 t^{-2\beta} - b}{\beta^2 x^4 t^{-2\beta}}$.

3.2. Power series solution of ODEs (2.16) and (2.20)

In a similar way, we obtain power series solutions of ODEs (2.16) and (2.20).

After substituting expressions (3.2) into Eq (2.16), we obtain

$$\begin{aligned}8\gamma r^2 \sum_{\varpi=0}^{\infty} \sum_{j=0}^{\varpi} ((\varpi-j+1)(\varpi-j+2)a_j a_{\varpi-j+2} + (\varpi-j+1)(j+1)a_{j+1} a_{\varpi-j+1}) p_4^{\varpi} - 8m \sum_{\varpi=0}^{\infty} a_{\varpi} p_4^{\varpi} \\ + 4\delta r^4 \sum_{\varpi=0}^{\infty} (\varpi+1)(\varpi+2)(\varpi+3)(\varpi+4)a_{\varpi+4} p_4^{\varpi} - m \sum_{\varpi=0}^{\infty} \varpi(\varpi-1)a_{\varpi-1} p_4^{\varpi} - 7m \sum_{\varpi=0}^{\infty} \varpi a_{\varpi} p_4^{\varpi} = 0.\end{aligned}\quad (3.7)$$

Setting $\varpi = 0$ in the above equation and simplifying, we obtain

$$a_4 = \frac{1}{12\delta r^4} (ma_0 - 2\gamma r^2 a_0 a_2 - \gamma r^2 a_1^2), \quad \delta \neq 0, \quad r \neq 0. \quad (3.8)$$

Therefore, the power series solution of Eq (1.1) is given by

$$\begin{aligned}\Gamma(x, y, t) = & -\tilde{t} + \frac{\beta}{(\beta y - s t^{-\beta})} \left[a_0 + a_1 p_4 + a_2 p_4^2 + a_3 p_4^3 + a_4 p_4^4 + \sum_{\varpi=1}^{\infty} \frac{1}{4\delta r^4 (\varpi+1)(\varpi+2)(\varpi+3)(\varpi+4)} \right. \\ & \times \left(m(\varpi+2)(\varpi+4)a_{\varpi} - 8\gamma r^2 \sum_{j=0}^{\varpi} ((\varpi-j+2)(\varpi-j+1)a_j a_{\varpi-j+2} + (\varpi-j+1)(j+1) \right. \\ & \left. \left. \times a_{j+1} a_{\varpi-j+1}) \right) p_4^{\varpi+4} \right],\end{aligned}$$

where $p_4 = -2\tilde{p} \left(y - s \frac{t^{\beta}}{\beta} \right)^{\frac{1}{2}} + \left(rx - s \frac{t^{\beta}}{\beta} \right) \left(y - s \frac{t^{\beta}}{\beta} \right)^{-\frac{1}{2}}$.

Upon substituting expressions (3.2) into Eq (2.20), we obtain

$$\begin{aligned}
& 8\gamma r^2 \sum_{\varpi=1}^{\infty} \sum_{j=0}^{\varpi-1} ((\varpi-j)(\varpi-j+1)a_j a_{\varpi-j+1} + (\varpi-j)(j+1)a_{j+1} a_{\varpi-j}) p_5^{\varpi} + 16\delta r^4 \sum_{\varpi=2}^{\infty} (\varpi+2)(\varpi+1)\varpi \\
& \times (\varpi-1)a_{\varpi+2} p_5^{\varpi} + 48\delta r^4 \sum_{\varpi=1}^{\infty} (\varpi+1)(\varpi+2)\varpi a_{\varpi+2} p_5^{\varpi} + 12\delta r^4 \sum_{\varpi=0}^{\infty} (\varpi+2)(\varpi+1)a_{\varpi+2} p_5^{\varpi} \\
& + 4\gamma r^2 \sum_{\varpi=0}^{\infty} \sum_{j=0}^{\varpi} (\varpi-j+1)a_j a_{\varpi-j+1} p_5^{\varpi} + b \sum_{\varpi=0}^{\infty} \varpi(\varpi-1)a_{\varpi} p_5^{\varpi} + 4b \sum_{\varpi=0}^{\infty} \varpi a_{\varpi} p_5^{\varpi} + 2b \sum_{\varpi=0}^{\infty} a_{\varpi} p_5^{\varpi} = 0.
\end{aligned} \tag{3.9}$$

Setting $\varpi = 0$ in the above equation and simplifying, we obtain

$$a_2 = \frac{-1}{12\delta r^4} (2\gamma r^2 a_0 a_1 + b a_0). \tag{3.10}$$

Therefore, the power series solution of Eq (1.1) is given by

$$\begin{aligned}
\Gamma(x, y, t) = & -L + \frac{1}{y} \left[a_0 + a_1 p_5 + a_2 p_5^2 - \sum_{\varpi=1}^{\infty} \frac{1}{4\delta r^4 (\varpi+1)(\varpi+2)(4\varpi^2 + 8\varpi + 3)} \left(b(\varpi+1)(\varpi+2)a_{\varpi} \right. \right. \\
& + 4\gamma r^2 \sum_{j=0}^{\varpi} (\varpi-j+1)a_j a_{\varpi-j+1} + 8\gamma r^2 \sum_{j=0}^{\varpi-1} ((\varpi-j)(\varpi-j+1)a_j a_{\varpi-j+1} + (\varpi-j)(j+1) \\
& \left. \left. \times a_{j+1} a_{\varpi-j}) \right) p_5^{\varpi+2} \right],
\end{aligned}$$

where $p_5 = \frac{\left(r x - \frac{\beta}{\beta} \right)^2}{y}$.

4. Soliton solutions of reduced DEs using analytical techniques

In this section, we obtain various soliton solutions of reduced PDEs (2.13) and (2.17) that are reduced through Lie transformation using the Maple by utilizing the generalized Sardar sub-equation approach [43] and the modified exp-function approach [44]. Several soliton solutions in terms of trigonometric, hyperbolic, and rational functions are derived. Bright, exponential solitary wave, single soliton, and multi-singular solutions to the reduced DEs are acquired using these analytical techniques.

4.1. Generalized Sardar sub-equation method

Here, we use the generalized Sardar sub-equation method (GSSEM), which is a generalization of the Sardar sub-equation method first formulated by Rezazadeh et al. [45]. The GSSEM is used to derive various soliton solutions of the governing equation from Eq (2.13) in positive as well as negative powers.

Let us utilize the following transformation:

$$f(p_1, p_2) = U(\chi), \quad \chi = k p_1 - c p_2. \tag{4.1}$$

Now, Eq (2.13) is transformed into the consequent ODE

$$(k^2 l + c^2 m - 2ck s^2)U''' - 2\gamma r^2 k^2 U'^2 - 2\gamma r^2 k^2 U U'' - \delta r^4 k^4 U'''' = 0. \quad (4.2)$$

We assume that the solution of Eq (4.2) is as follows:

$$U(\chi) = \sum_{i=0}^q \tilde{a}_i Z^i(\chi) + \sum_{i=1}^q \tilde{b}_i Z^{-i}(\chi), \quad (4.3)$$

where $\tilde{a}_i, \tilde{b}_i$ ($i = 0, 1, \dots, q$) are arbitrary constants that will be computed later and $Z(\chi)$ satisfies the following ODE:

$$Z'(\chi) = \sqrt{\Omega + \Upsilon (Z(\chi))^2 + \Delta (Z(\chi))^4}, \quad (4.4)$$

where Ω, Υ , and Δ are constants. Using the balancing principle, we have $q = 2$. So the assumed solution becomes

$$U(\chi) = \frac{\tilde{b}_1}{Z(\chi)} + \frac{\tilde{b}_2}{(Z(\chi))^2} + \tilde{a}_0 + \tilde{a}_1 Z(\chi) + \tilde{a}_2 (Z(\chi))^2. \quad (4.5)$$

After inserting Eq (4.5) into Eq (4.2) and equating the coefficients for different powers of $Z(\chi)$, we derived several algebraic equations. These are solved using Maple, giving us the values of $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2$:

$$\tilde{a}_0 = -\frac{4\delta \Upsilon k^4 r^4 + 2ck s^2 - c^2 m - k^2 l}{2\gamma k^2 r^2}, \tilde{a}_1 = 0, \tilde{a}_2 = -\frac{6\delta \Delta r^2 k^2}{\gamma}, \tilde{b}_1 = 0, \tilde{b}_2 = -\frac{6\delta \Omega r^2 k^2}{\gamma}. \quad (4.6)$$

By replacing these constant values and returning the solution to its original variables, we can derive the following soliton solutions of Eq (1.1).

Case 1: If $\Upsilon > 0, \Omega = 0$,

$$\Gamma_1(x, y, t) = \frac{K_1}{(\operatorname{sech}(\mathcal{G}_1))^2} + \tilde{a}_0 + K_2 (\operatorname{sech}(\mathcal{G}_1))^2, \quad (4.7)$$

$$\Gamma_2(x, y, t) = -\frac{K_1}{(\operatorname{csch}(\mathcal{G}_1))^2} + \tilde{a}_0 - K_2 (\operatorname{csch}(\mathcal{G}_1))^2. \quad (4.8)$$

Case 2: If $\Upsilon < 0, \Delta > 0, \Omega = 0$,

$$\Gamma_3(x, y, t) = \frac{K_1}{(\sec(\mathcal{G}_2))^2} + \tilde{a}_0 + K_2 (\sec(\mathcal{G}_2))^2, \quad (4.9)$$

$$\Gamma_4(x, y, t) = \frac{K_1}{(\csc(\mathcal{G}_2))^2} + \tilde{a}_0 + K_2 (\csc(\mathcal{G}_2))^2. \quad (4.10)$$

Case 3: If $\Upsilon < 0, \Delta > 0, \Omega = \frac{\Upsilon^2}{4\Delta}$,

$$\begin{aligned}
\Gamma_5(x, y, t) &= \frac{K_1}{2 \left(\tanh \left(\frac{\mathcal{G}_2}{\sqrt{2}} \right) \right)^2} + \tilde{a}_0 + 2K_2 \left(\tanh \left(\frac{\mathcal{G}_2}{\sqrt{2}} \right) \right)^2, \\
\Gamma_6(x, y, t) &= \frac{K_1}{2 \left(\coth \left(\frac{\mathcal{G}_2}{\sqrt{2}} \right) \right)^2} + \tilde{a}_0 + 2K_2 \left(\coth \left(\frac{\mathcal{G}_2}{\sqrt{2}} \right) \right)^2, \\
\Gamma_7(x, y, t) &= -\frac{K_1}{2 \left(\tanh \left(\sqrt{2}\mathcal{G}_2 \right) \pm \iota \operatorname{sech} \left(\sqrt{2}\mathcal{G}_2 \right) \right)^2} + \tilde{a}_0 + 2K_2 \left(\tanh \left(\sqrt{2}\mathcal{G}_2 \right) \pm \iota \operatorname{sech} \left(\sqrt{2}\mathcal{G}_2 \right) \right)^2, \\
\Gamma_8(x, y, t) &= \frac{K_1}{2 \left(\coth \left(\sqrt{2}\mathcal{G}_2 \right) \pm \operatorname{csch} \left(\sqrt{2}\mathcal{G}_2 \right) \right)^2} + \tilde{a}_0 + 2K_2 \left(\coth \left(\sqrt{2}\mathcal{G}_2 \right) \pm \operatorname{csch} \left(\sqrt{2}\mathcal{G}_2 \right) \right)^2, \\
\Gamma_9(x, y, t) &= \frac{K_1}{8 \left(\tanh \left(\frac{\mathcal{G}_2}{2\sqrt{2}} \right) \pm \coth \left(\frac{\mathcal{G}_2}{2\sqrt{2}} \right) \right)^2} + \tilde{a}_0 + 8K_2 \left(\tanh \left(\frac{\mathcal{G}_2}{2\sqrt{2}} \right) \pm \coth \left(\frac{\mathcal{G}_2}{2\sqrt{2}} \right) \right)^2.
\end{aligned} \tag{4.11}$$

Case 4: If $\Upsilon > 0$, $\Delta > 0$, $\Omega = \frac{\Upsilon^2}{4\Delta}$,

$$\begin{aligned}
\Gamma_{10}(x, y, t) &= -\frac{K_1}{2 \left(\tan \left(\frac{\mathcal{G}_1}{\sqrt{2}} \right) \right)^2} + \tilde{a}_0 - 2K_2 \left(\tan \left(\frac{\mathcal{G}_1}{\sqrt{2}} \right) \right)^2, \\
\Gamma_{11}(x, y, t) &= -\frac{K_1}{2 \left(\cot \left(\frac{\mathcal{G}_1}{\sqrt{2}} \right) \right)^2} + \tilde{a}_0 - 2K_2 \left(\cot \left(\frac{\mathcal{G}_1}{\sqrt{2}} \right) \right)^2, \\
\Gamma_{12}(x, y, t) &= -\frac{K_1}{2 \left(\tan \left(\frac{\mathcal{G}_1}{\sqrt{2}} \right) \pm \sec \left(\frac{\mathcal{G}_1}{\sqrt{2}} \right) \right)^2} + \tilde{a}_0 - 2K_2 \left(\tan \left(\frac{\mathcal{G}_1}{\sqrt{2}} \right) \pm \sec \left(\frac{\mathcal{G}_1}{\sqrt{2}} \right) \right)^2, \\
\Gamma_{13}(x, y, t) &= -\frac{K_1}{2 \left(\cot \left(\frac{\mathcal{G}_1}{\sqrt{2}} \right) \pm \csc \left(\frac{\mathcal{G}_1}{\sqrt{2}} \right) \right)^2} + \tilde{a}_0 - 2K_2 \left(\cot \left(\frac{\mathcal{G}_1}{\sqrt{2}} \right) \pm \csc \left(\frac{\mathcal{G}_1}{\sqrt{2}} \right) \right)^2,
\end{aligned} \tag{4.12}$$

$$\Gamma_{14}(x, y, t) = -\frac{K_1}{8 \left(\tan \left(\frac{\mathcal{G}_1}{2\sqrt{2}} \right) \pm \cot \left(\frac{\mathcal{G}_1}{2\sqrt{2}} \right) \right)^2} + \tilde{a}_0 - 8K_2 \left(\tan \left(\frac{\mathcal{G}_1}{2\sqrt{2}} \right) \pm \cot \left(\frac{\mathcal{G}_1}{2\sqrt{2}} \right) \right)^2, \tag{4.13}$$

where $K_1 = \frac{6\delta k^2 r^2 \Upsilon}{\gamma}$, $K_2 = \frac{6\delta k^2 r^2 \Omega \Delta}{\gamma \Upsilon}$, $\mathcal{G}_1 = \sqrt{\Upsilon}\chi$, $\mathcal{G}_2 = \sqrt{-\Upsilon}\chi$, and $\chi = -c \left(y - \frac{st^\beta}{\beta} \right) + k \left(rx - \frac{st^\beta}{\beta} \right)$.

4.2. Modified exp-function method

In this subsection, various exact soliton solutions of the considered equation are constructed by utilizing the modified exp-function technique on reduced PDE (2.17), which was first proposed by Cai et al. [46].

Let us utilize the following transformation:

$$f(p_1, p_2) = U(\chi), \quad \chi = Kp_1 - Cp_2. \tag{4.14}$$

Now, Eq (2.17) is transformed to the consequent ODE

$$((1 - ar^2)K^2 - bc^2)U'' - 2\gamma r^2 K^2 U'^2 - 2\gamma r^2 K^2 U U'' - \delta r^4 K^4 U'''' = 0. \quad (4.15)$$

The solution of this reduced ODE has the following structure:

$$U(\chi) = \sum_{i=0}^q \sigma_i [\exp(-\vartheta(\chi))]^i = \sigma_0 + \sigma_1 \exp(-\vartheta(\chi)) + \dots + \sigma_q \exp(q(-\vartheta(\chi))), \quad (4.16)$$

where ϑ is the solution of the following ODE:

$$\dot{\vartheta} = \exp(-\vartheta(\chi)) + \Lambda \exp(\vartheta(\chi)) + \Theta. \quad (4.17)$$

Using the balancing principle, we obtain $q = 2$, and therefore, it is possible to rewrite Eq (4.23) as

$$U(\chi) = \sigma_0 + \sigma_1 \exp(-\vartheta(\chi)) + \sigma_2 (\exp(-\vartheta(\chi)))^2. \quad (4.18)$$

After substituting Eq (4.25) along with Eq (4.24) into Eq (4.22), and comparing the different powers of $\exp(\vartheta(\chi))$, we are left with nonlinear algebraic equations. The following constants can be computed by solving these nonlinear equations.

$$\sigma_0 = -\frac{K^4 \Theta^2 \delta r^4 + 8 K^4 \Lambda \delta r^4 + K^2 a r^2 + C^2 b - K^2}{2 K^2 \gamma r^2}, \quad \sigma_1 = -\frac{6 K^2 \Theta \delta r^2}{\gamma}, \quad \sigma_2 = -\frac{6 r^2 \delta K^2}{\gamma}, \quad (4.19)$$

where Λ , Θ , K , and C are arbitrary.

With the constant values given above and using the solutions of Eq (4.24), we are able to get the following solutions to the equation under consideration.

- When $\Lambda \neq 0, \Theta^2 - 4\Lambda > 0$,

$$\Gamma_{15}(x, y, t) = \sigma_0 + \frac{12 K^2 \Theta \delta r^2 \Lambda}{\gamma (\sqrt{\Theta^2 - 4\Lambda} \tanh(\mathcal{G}_3) + \Theta)} - \frac{24 K^2 \delta r^2 \Lambda^2}{\gamma (\sqrt{\Theta^2 - 4\Lambda} \tanh(\mathcal{G}_3) + \Theta)^2}. \quad (4.20)$$

- When $\Lambda \neq 0, \Theta^2 - 4\Lambda < 0$,

$$\Gamma_{16}(x, y, t) = \sigma_0 - \frac{12 K^2 \Theta \delta r^2 \Lambda}{\gamma (\sqrt{-\Theta^2 + 4\Lambda} \tan(\mathcal{G}_4) - \Theta)} - \frac{24 K^2 \delta r^2 \Lambda^2}{\gamma (\sqrt{-\Theta^2 + 4\Lambda} \tan(\mathcal{G}_4) - \Theta)^2}. \quad (4.21)$$

- When $\Lambda = 0, \Theta \neq 0, \Theta^2 - 4\Lambda > 0$,

$$\Gamma_{17}(x, y, t) = \sigma_0 - \frac{6 K^2 \Theta^2 \delta r^2}{\gamma (e^{\mathcal{G}_5} - 1)} - \frac{6 K^2 \Theta^2 \delta r^2}{\gamma (e^{\mathcal{G}_5} - 1)^2}. \quad (4.22)$$

- When $\Lambda \neq 0, \Theta \neq 0, \Theta^2 - 4\Lambda = 0$,

$$\Gamma_{18}(x, y, t) = \sigma_0 + \frac{6 K^2 \Theta^2 \delta r^2 \mathcal{G}_5}{\gamma (4 + 2\mathcal{G}_5 \Theta)} - \frac{3 K^2 \delta r^2 \Theta^2 \mathcal{G}_5^2}{2\gamma (2 + \mathcal{G}_5)^2}. \quad (4.23)$$

- When $\Lambda = 0, \Theta = 0$,

$$\Gamma_{19}(x, y, t) = \sigma_0 - \frac{6 K^2 \delta r^2}{\gamma \mathcal{G}_5^2}, \quad (4.24)$$

where $\mathcal{G}_3 = \frac{\sqrt{\Theta^2 - 4\Lambda}}{2} \left(K(rx - \frac{t^\beta}{\beta}) - Cy + E \right)$, $\mathcal{G}_4 = \frac{\sqrt{-\Theta^2 + 4\Lambda}}{2} \left(K(rx - \frac{t^\beta}{\beta}) - Cy + E \right)$, and $\mathcal{G}_5 = \left(K(rx - \frac{t^\beta}{\beta}) - Cy + E \right)$.

4.3. Wave solutions obtained via analytical schemes

In this section, we obtained exact solutions of the governing equation using the Maple by utilizing the sine-cosine method and extended tanh method. The various solutions obtained using these techniques are represented in Table 4.

The periodic solutions in the form of combined singular soliton, breather soliton and solitary wave for $\Gamma_{1,1}(x, y, t)$, $\Gamma_{1,2}(x, y, t)$, $\Gamma_{1,3}(x, y, t)$ and $\Gamma_{1,4}(x, y, t)$, given in Table 4, are illustrated in Figures 2 and 3 by varying space variable.

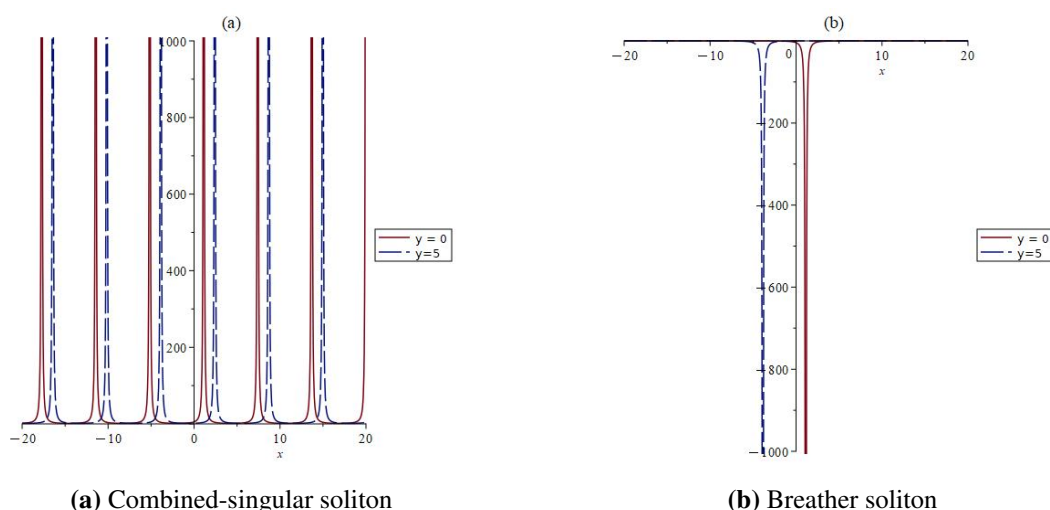


Figure 2. **a)** Periodic (breather) solution of $\Gamma_{1,1}(x, y, t)$ with $a = 1, b = 1, \lambda = 1, \gamma = 1, \delta = 1, \beta = 0.5, \mu = \frac{1}{2}, \alpha = \frac{3}{2}, t = 1$; **b)** solitary waves for $y = 0$ (solid line), $y = 5$ (dotted line) of $\Gamma_{1,2}(x, y, t)$ with $a = 1, b = 1, \lambda = 1, \gamma = 1, \delta = -1, \beta = 0.5, \alpha = \frac{3}{2}, t = 1$.

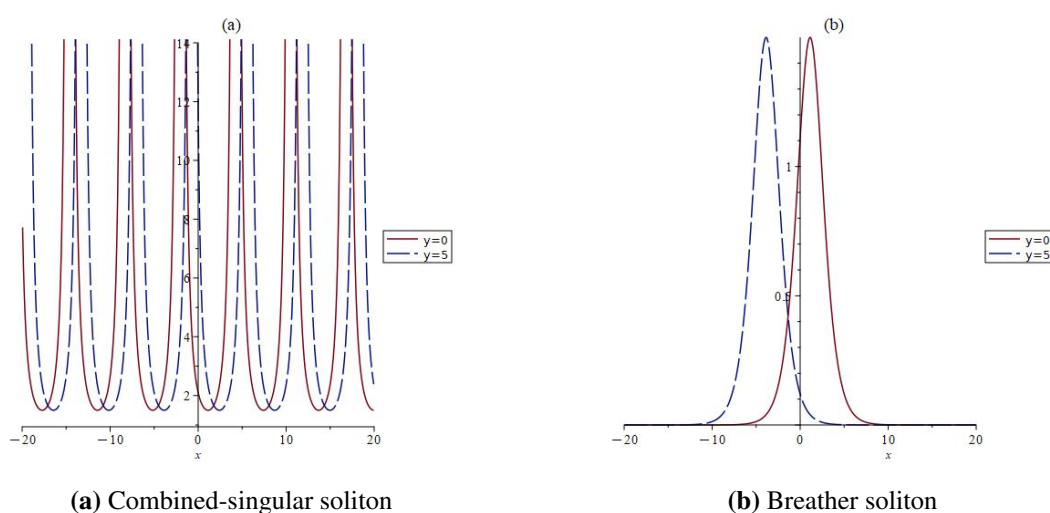


Figure 3. **a)** Periodic (breather) solution of $\Gamma_{1,3}(x, y, t)$ with $a = 1, b = 1, \lambda = 1, \gamma = 1, \delta = 1, \beta = 0.5, \mu = \frac{1}{2}, \alpha = \frac{3}{2}, t = 1$; **b)** solitary waves for $y = 0$ (solid line), $y = 5$ (dotted line) of $\Gamma_{1,4}(x, y, t)$ with $a = 1, b = 1, \lambda = 1, \gamma = 1, \delta = -1, \beta = 0.5, \alpha = \frac{3}{2}, t = 1$.

Table 4. Wave solutions.

Methods	Wave transformation	ODE	Values of constants	Corresponding solutions
Sine method	$\Gamma(x, y, t) = u(\xi) = \alpha \sin^p(\mu\xi)$, $\xi = x + y - \lambda \frac{t^p}{(p!)}$	$u'' + k_1 u^2 + k_2 u = 0$, where $k_1 = \frac{\gamma}{\delta}$, $k_2 = \frac{a+b-\lambda^2}{\delta}$	$\alpha = -\frac{3(a+b-\lambda^2)}{2\gamma}$, $\mu = \frac{1}{2} \sqrt{\frac{a+b-\lambda^2}{\delta}}$	$\Gamma_{1,1}(x, y, t) = -\frac{3(a+b-\lambda^2)}{2\gamma} \operatorname{cosec}^2 \left(\sqrt{\frac{a+b-\lambda^2}{4\delta}} (x + y - \lambda \frac{t^p}{(p!)}) \right)$, with $\frac{a+b-\lambda^2}{\delta} > 0$, $\Gamma_{1,2}(x, y, t) = \frac{3(a+b-\lambda^2)}{2\gamma} \operatorname{csch}^2 \left(\sqrt{\frac{a+b-\lambda^2}{4\delta}} (x + y - \lambda \frac{t^p}{(p!)}) \right)$, with $\frac{a+b-\lambda^2}{\delta} > 0$, $\Gamma_{1,3}(x, y, t) = -\frac{3(a+b-\lambda^2)}{2\gamma} \sec^2 \left(\sqrt{\frac{a+b-\lambda^2}{4\delta}} (x + y - \lambda \frac{t^p}{(p!)}) \right)$, with $\frac{a+b-\lambda^2}{\delta} > 0$, $\Gamma_{1,4}(x, y, t) = -\frac{3(a+b-\lambda^2)}{2\gamma} \operatorname{sech}^2 \left(\sqrt{\frac{a+b-\lambda^2}{4\delta}} (x + y - \lambda \frac{t^p}{(p!)}) \right)$, with $\frac{a+b-\lambda^2}{\delta} > 0$, $\Gamma_{2,1}(x, y, t) = -\frac{3(a+b-\lambda^2)}{4\gamma} + \frac{3(a+b-\lambda^2)}{8\gamma} \tanh^2 \left(\sqrt{\frac{a+b-\lambda^2}{16\delta}} \xi \right) +$ $\frac{3(a+b-\lambda^2)}{8\gamma \tanh^2 \left(\sqrt{\frac{a+b-\lambda^2}{16\delta}} \xi \right)}$, $\Gamma_{2,2}(x, y, t) = -\frac{(a+b-\lambda^2)}{4\gamma} - \frac{3(a+b-\lambda^2)}{8\gamma} \tanh^2 \left(\sqrt{\frac{a+b-\lambda^2}{16\delta}} \xi \right) -$ $\frac{3(a+b-\lambda^2)}{8\gamma \tanh^2 \left(\sqrt{\frac{a+b-\lambda^2}{16\delta}} \xi \right)}$, $\Gamma_{2,3}(x, y, t) = -\frac{3(a+b-\lambda^2)}{2\gamma} + \frac{3(a+b-\lambda^2)}{2\gamma \tanh^2 \left(\sqrt{\frac{a+b-\lambda^2}{4\delta}} \xi \right)}$, $\Gamma_{2,4}(x, y, t) = \frac{(a+b-\lambda^2)}{2\gamma} - \frac{3(a+b-\lambda^2)}{2\gamma \tanh^2 \left(\sqrt{\frac{a+b-\lambda^2}{4\delta}} \xi \right)}$, $\Gamma_{2,5}(x, y, t) = -\frac{3(a+b-\lambda^2)}{2\gamma} + \frac{3(a+b-\lambda^2)}{2\gamma} \tanh^2 \left(\sqrt{\frac{a+b-\lambda^2}{4\delta}} \xi \right)$, $\Gamma_{2,6}(x, y, t) = \frac{(a+b-\lambda^2)}{2\gamma} - \frac{3(a+b-\lambda^2)}{2\gamma} \tanh^2 \left(\sqrt{\frac{a+b-\lambda^2}{4\delta}} \xi \right)$, where $\xi = x + y - \lambda \frac{t^p}{(p!)}$.
Cosine method	$\Gamma(x, y, t) = u(\xi) = \alpha \cos^p(\mu\xi)$, $\xi = x + y - \lambda \frac{t^p}{(p!)}$		$\alpha = -\frac{3(a+b-\lambda^2)}{2\gamma}$, $\mu = \frac{1}{2} \sqrt{\frac{a+b-\lambda^2}{\delta}}$	
Tanh method	$\Gamma(x, y, t) = u(\xi) = \sum_{i=0}^2 \mathfrak{B}_i Y^i$ $+ \sum_{i=1}^2 \mathfrak{C}_i Y^{-i}$, $\xi = x + y - \lambda \frac{t^p}{(p!)}$		Set 1: $\mathfrak{B}_0 = -\frac{3(a+b-\lambda^2)}{4\gamma}$, $\mathfrak{B}_1 = 0$, $\mathfrak{B}_2 = \frac{3(a+b-\lambda^2)}{8\gamma}$, $\mathfrak{C}_1 = 0$, $\mathfrak{C}_2 = \frac{3(a+b-\lambda^2)}{8\gamma}$, $\mu = \pm \sqrt{\frac{a+b-\lambda^2}{16\delta}}$, Set 2: $\mathfrak{B}_0 = -\frac{a+b-\lambda^2}{4\gamma}$, $\mathfrak{B}_1 = 0$, $\mathfrak{B}_2 = -\frac{3(a+b-\lambda^2)}{8\gamma}$, $\mathfrak{C}_1 = 0$, $\mathfrak{C}_2 = -\frac{3(a+b-\lambda^2)}{8\gamma}$, $\mu = \pm \sqrt{\frac{a+b-\lambda^2}{16\delta}}$, Set 3: $\mathfrak{B}_0 = -\frac{3(a+b-\lambda^2)}{2\gamma}$, $\mathfrak{B}_1 = 0$, $\mathfrak{B}_2 = 0$, $\mathfrak{C}_1 = 0$, $\mathfrak{C}_2 = \frac{3(a+b-\lambda^2)}{2\gamma}$, $\mu = \pm \sqrt{\frac{a+b-\lambda^2}{4\delta}}$, Set 4: $\mathfrak{B}_0 = \frac{a+b-\lambda^2}{2\gamma}$, $\mathfrak{B}_1 = 0$, $\mathfrak{B}_2 = 0$, $\mathfrak{C}_1 = 0$, $\mathfrak{C}_2 = -\frac{3(a+b-\lambda^2)}{2\gamma}$, $\mu = \pm \sqrt{\frac{a+b-\lambda^2}{4\delta}}$, Set 5: $\mathfrak{B}_0 = -\frac{3(a+b-\lambda^2)}{2\gamma}$, $\mathfrak{B}_1 = 0$, $\mathfrak{B}_2 = \frac{3(a+b-\lambda^2)}{2\gamma}$, $\mathfrak{C}_1 = 0$, $\mathfrak{C}_2 = 0$, $\mu = \pm \sqrt{\frac{a+b-\lambda^2}{4\delta}}$, Set 6: $\mathfrak{B}_0 = \frac{a+b-\lambda^2}{2\gamma}$, $\mathfrak{B}_1 = 0$, $\mathfrak{B}_2 = -\frac{3(a+b-\lambda^2)}{2\gamma}$, $\mathfrak{C}_1 = 0$, $\mathfrak{C}_2 = 0$, $\mu = \pm \sqrt{\frac{a+b-\lambda^2}{4\delta}}$.	

5. Chaotic analysis

Bifurcation and chaos analysis provide valuable insight into the complex dynamical behavior and transitions that can occur in nonlinear systems. Bifurcation analysis provides valuable insights into changes in the dynamical behavior, while chaotic behavior reflects the complex and often irregular dynamics that may emerge from such transitions.

Consider the conformable time-fractional (2+1)-dimensional generalized Boussinesq equation

$$\Gamma_{tt}^{2\beta} - a\Gamma_{xx} - b\Gamma_{yy} - 2\gamma\Gamma_x^2 - 2\gamma\Gamma\Gamma_{xx} - \delta\Gamma_{xxx} = 0. \quad (5.1)$$

Consider the following wave transformation:

$$\Gamma(x, y, t) = u(\xi), \quad \xi = x + y - \lambda \frac{t^\beta}{(\beta)!}, \quad (5.2)$$

and by substituting Eq (5.2) into Eq (5.1), followed by two successive integrations with respect to ξ and taking the integration constants to be zero, reduces Eq (5.1) to the following ODE:

$$u'' + k_1 u^2 + k_2 u = 0, \quad (5.3)$$

where $k_1 = \frac{\gamma}{\delta}$, $k_2 = \frac{a+b-\lambda^2}{\delta}$.

The bifurcation behavior of the corresponding reduced ODE (5.3) has already been investigated by Song and Shao [26]. In particular, for $g = 0$, the reduced ODE (2.3) and the corresponding planar dynamical system (3.1) as reported in [26] are identical in our study. Therefore, the associated bifurcation diagrams and qualitative dynamical behavior would be the same as those already presented in the existing literature. Hence, to complete the existing literature, we will discuss the chaotic behavior only in this study. Chaotic dynamics, which exhibit strong sensitivity to nonlinear interactions and initial conditions, have important applications across diverse fields. Despite its unpredictable nature, chaos theory offers valuable tools for understanding and addressing the complexities of modern systems.

Many researchers have recently become interested in studying these dynamical systems. A number of well-known PDEs have been specifically studied for these topics, such as the fractional complex Ginzburg–Landau model [47]. Chaotic analysis is commonly used to analyze dynamical systems. In this section, to investigate the occurrence of chaotic dynamics, a perturbation term is introduced into the resulting dynamical system, and the corresponding two- and three-dimensional phase portraits are analyzed.

Equation (5.3) is rewritten as a 2D planar system using the transformation given below:

$$\begin{aligned} u' &= z, \\ z' &= -k_1 u^2 - k_2 u. \end{aligned} \quad (5.4)$$

Real-world physical environments frequently experience periodic disruptions, which can have a substantial effect on a system and most likely lead to chaotic behavior. In order to realize the chaotic nature of the governing equation, we incorporate a perturbed term into the above dynamical system. The representation of the two-dimensional system (5.4) is

$$\begin{aligned} u' &= z, \\ z' &= -k_1 u^2 - k_2 u + \mathfrak{P} \cos(\sigma \xi), \end{aligned} \quad (5.5)$$

where $\mathfrak{P}$ and σ specify the dynamical system's amplitude and frequency, respectively. Additionally, we examine the chaotic nature of the proposed dynamical system and describe its phase portraits for a wide range of parameter values. By changing the frequency and intensity of the interference in Figures 4–7, one can manipulate and visualize the trajectory status. It has been established that system (5.5) displays both periodic and quasiperiodic behavior by plotting time-series plots along with 2D and 3D phase-space portraits. This research has deepened our understanding of the system's insight, which may have applications in contemporary mathematics, physics, and engineering.

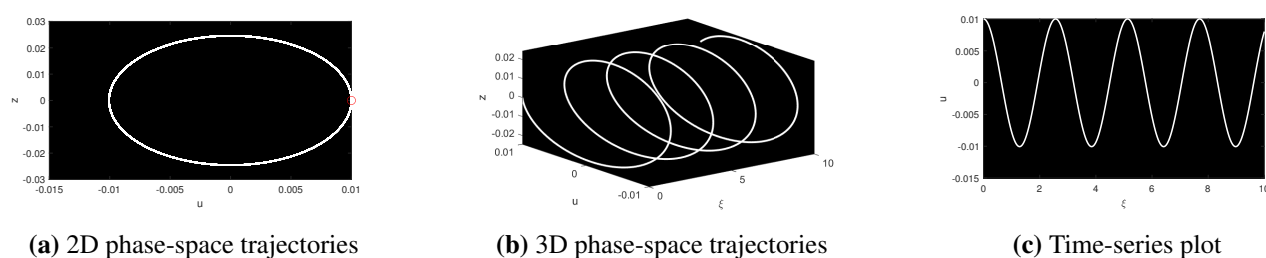


Figure 4. Phase-space portraits of system (5.5) with $k_1 = 5$, $k_2 = 6$ and $\sigma = 1.7$, $\mathfrak{P} = 0$.

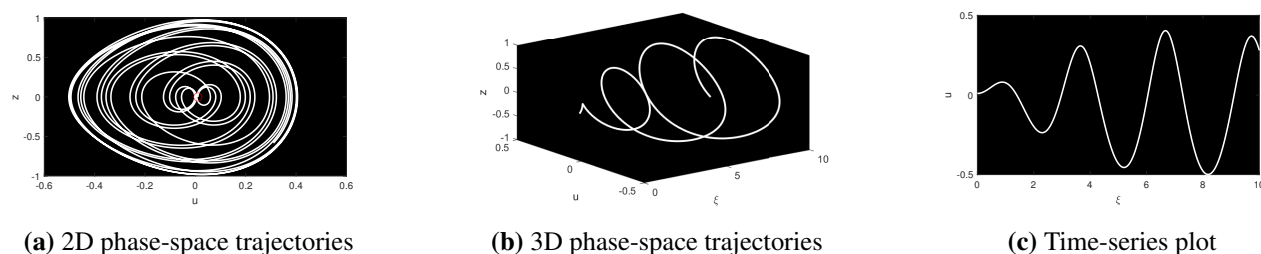


Figure 5. Phase-space portraits of system (5.5) with $k_1 = 5$, $k_2 = 6$ and $\sigma = 1.9$, $\mathfrak{P} = 0.47$.

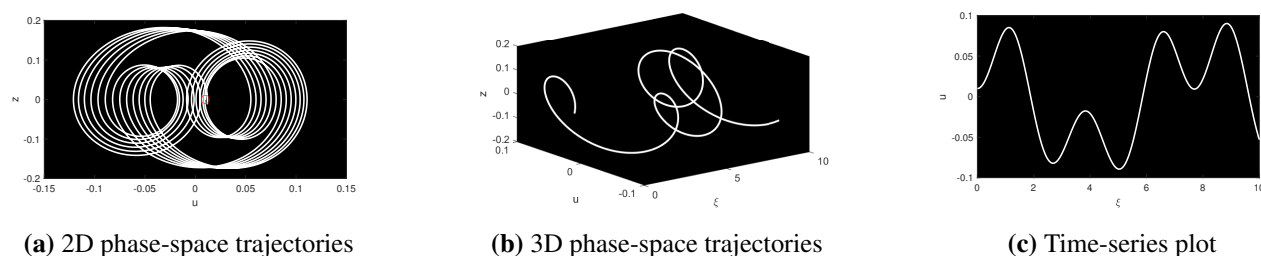


Figure 6. Phase-space portraits of system (5.5) with $k_1 = 5$, $k_2 = 6$ and $\sigma = 0.8$, $\mathfrak{P} = 0.34$.

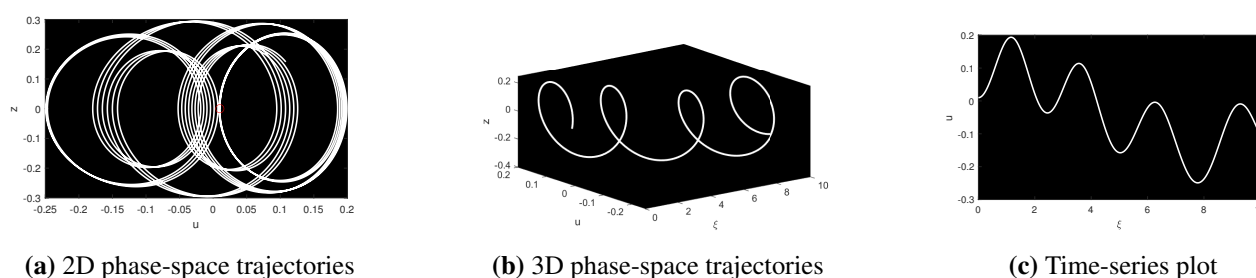


Figure 7. Phase-space portraits of system (5.5) with $k_1 = 5$, $k_2 = 6$ and $\sigma = 0.4$, $\mathfrak{P} = 0.7$.

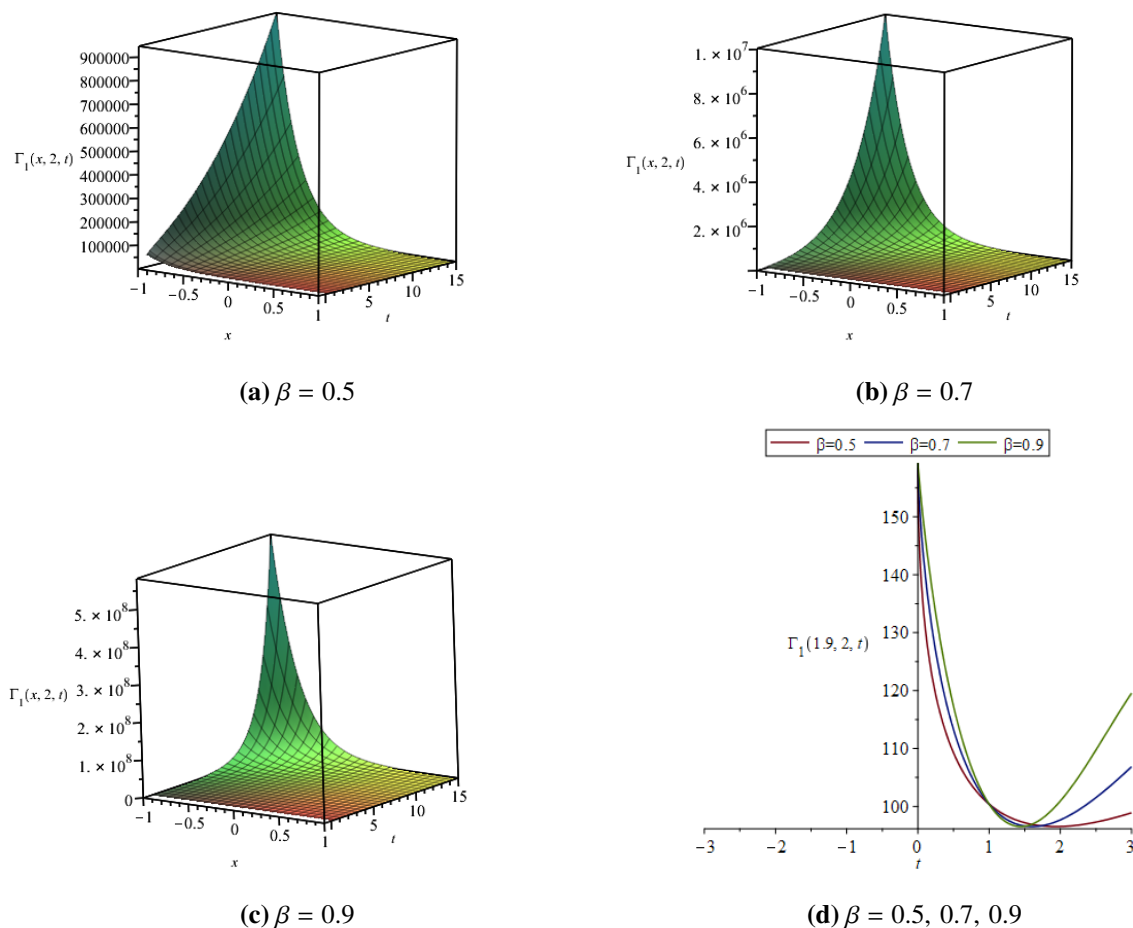
6. Impact of conformable fractional parameter on soliton solutions

Visual representation is one of the best ways to understand the physical dynamics of systems found in the real world. To understand the role of the fractional parameter, we examine its effect on several soliton solutions by changing β and presenting the outcomes using 2D and 3D graphs with carefully chosen parameters: $\beta = 0.5, 0.7, 0.9$; $r = 1.4$; $c = 0.8$; $\gamma = 2.5$; $k = 1$; $\Delta = -2.9$; $\Upsilon = 2$; $\Omega = -0.9$; $s = 1.2$; $l = 0.9$; $\delta = 8$; $y = 2$; $m = 2.5$; $x = 1.9$. in the generalized Sardar sub-equation approach and $K = 2.5$; $\delta = 1.2$; $\beta = 0.35, 0.55, 0.85$; $\Theta = 2$; $E = 1$; $r = -1.35$; $a = 2.1$; $\Lambda = 3$; $b = 1.1$; $C = 1$; $\gamma = -3.1$; $x = 1.2$; $y = 4.4$. in the modified exp-function technique. Figures 8–11 depict the influence of β on the exponential solitary wave, 1-singular soliton, and multi-singular soliton solutions in 2D and 3D plots for $\beta = 0.5, 0.7, 0.9$. Figures 12–15 illustrate the impact of β on the multi-singular soliton and exponential solitary wave through 3D and 2D plots for $\beta = 0.35, 55, 85$.

The exponential solitary wave is shown in Figure 8. The dispersion and amplitude characteristics of the exponential solitary wave are significantly influenced by the parameter β . This behavior can be clearly observed from both the 3D and 2D plots, where the exponential solitary wave has decreasing dispersion and increasing amplitude, resulting in a more localized wave profile, and the wave profile remains structurally stable with increasing values of β over time. Figure 9 displays bright solitons. As β increases, the dispersion decreases sharply, indicating that the parameter β primarily influences the wave's dispersive behavior over time without changing its amplitude or intensity. It further indicates the stable behavior during propagation. Figure 10 illustrates the multi-singular solution. The parameter β significantly influences the number of singular poles and the order of the singularity over time. This shows that varying β leads to an increasing number of singular poles while there is small variation in the order of singularity poles of the multi-singular solutions. Figure 11 displays a 1-singular soliton. As β increases, there is position shifting of the singular poles and the order of the singularity first increases and then decreases over time.

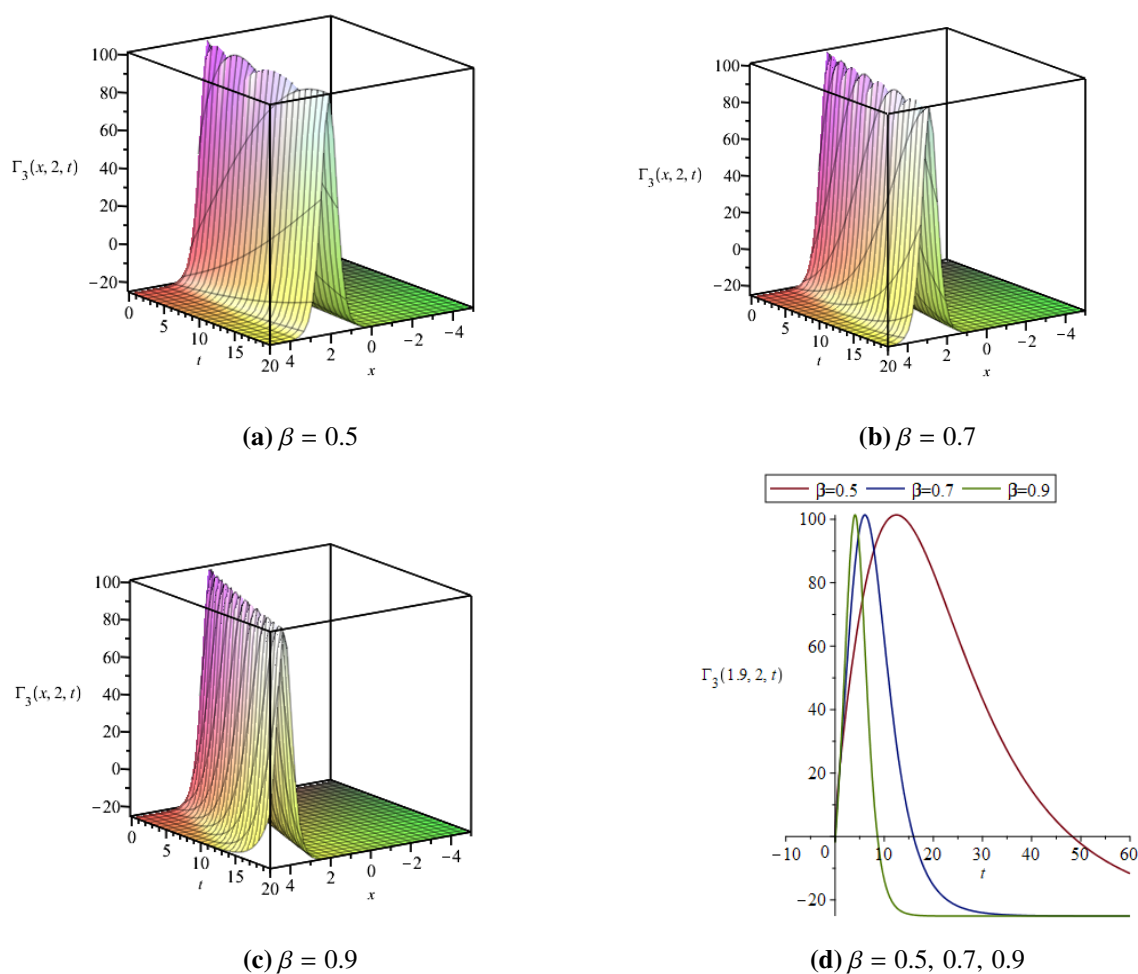
The multi-singular solution is shown in Figure 12. The varying β affects both the number of singular poles and the order of singularity. As β increases, the number of singularity poles increases whereas the order of singularity increases and decreases over time. Figure 13 illustrates the multi-singular solution. The number of singularity poles decreases whereas the order of singularity increases over time as β increases. Figures 14 and 15 display the exponential solitary wave. Variations in β influence both the dispersion and amplitude of the solitary wave. In Figure 14, as β increases, both the dispersion and amplitude increase over time. Whereas, in Figure 15, dispersion decreases gradually while amplitude increases over time as β increases. The wave structure remains stable during propagation. Here, we

examine the graphical behavior of selected soliton solutions. The remaining solutions can also be explored visually in a similar way.



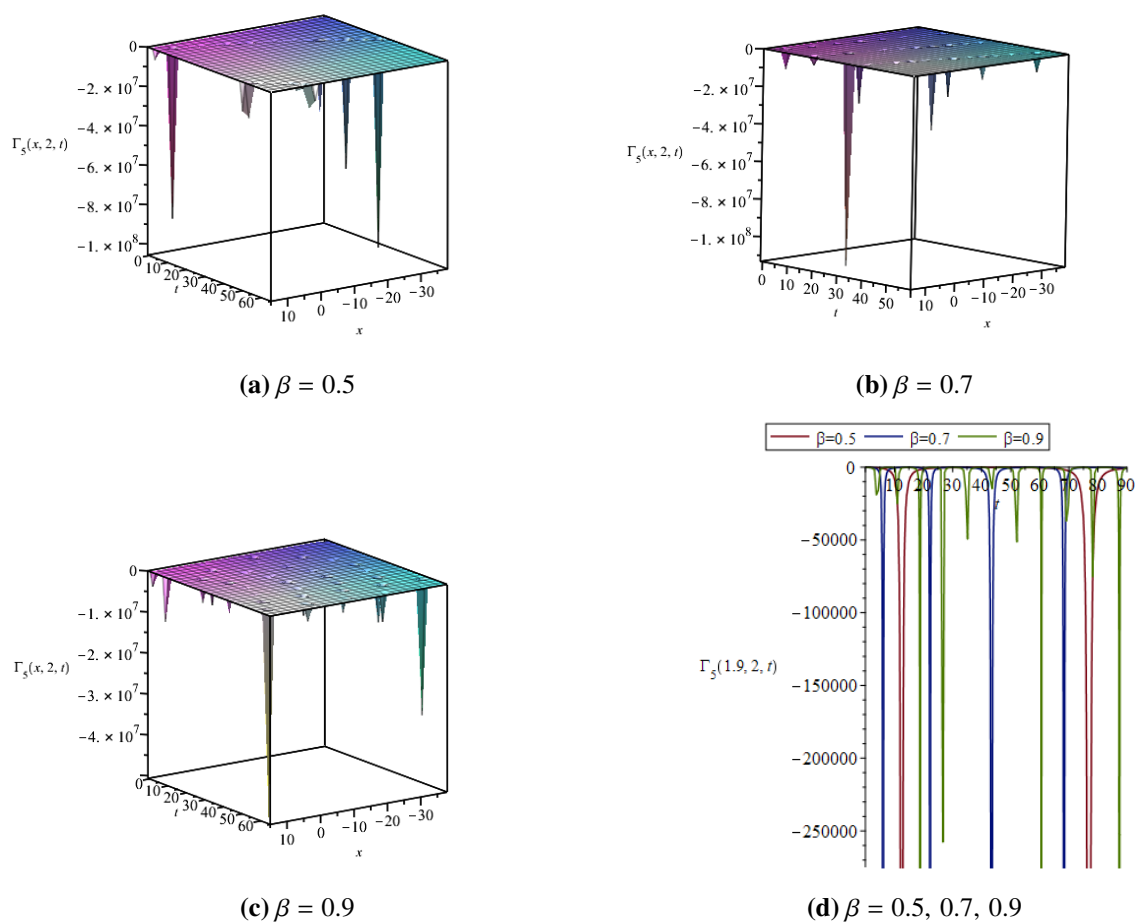
Note: As the value of β increases, there is less dispersion and high amplitude in the exponential solitary wave over time.

Figure 8. Exponential solitary waves of $\Gamma_1(x, y, t)$ with the parameter values $\beta = 0.5, 0.7, 0.9$, $r = 1.4$, $c = 0.8$, $\gamma = 2.5$, $k = 1$, $\Delta = -2.9$, $\Upsilon = 2$, $\Omega = -0.9$, $s = 1.2$, $l = 0.9$, $\delta = 8$, $y = 2$, $m = 2.5$, $x = 1.9$.



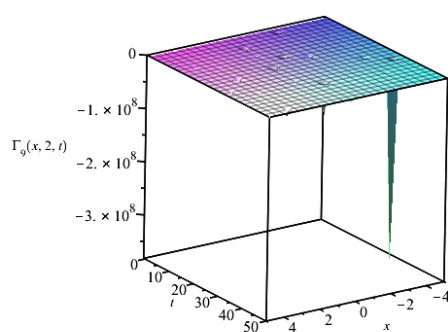
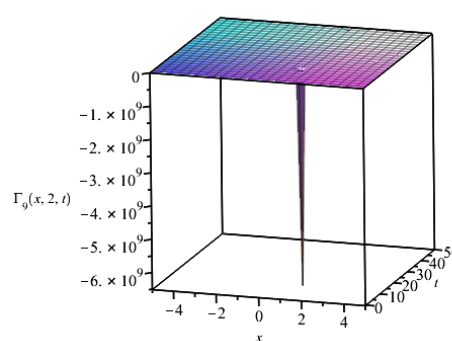
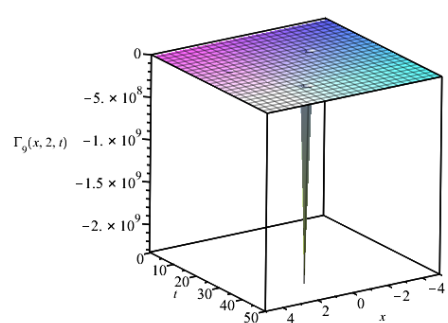
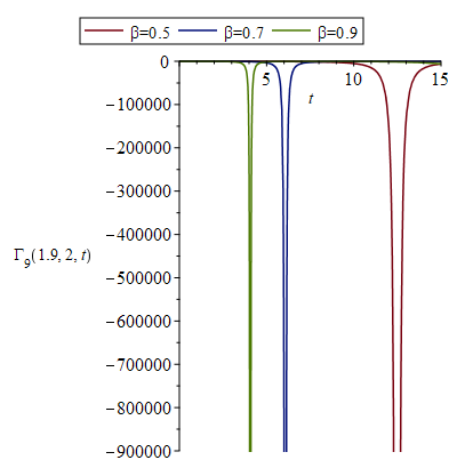
Note: The bright soliton's dispersion decreases sharply, without changing its amplitude or intensity as β increases with time.

Figure 9. Bright solitons of $\Gamma_3(x, y, t)$ with the parameter values $\beta = 0.5, 0.7, 0.9, r = 1.4, c = 0.8, \gamma = 2.5, k = 1, \Delta = -2.9, \Upsilon = 2, \Omega = -0.9, s = 1.2, l = 0.9, \delta = 8, y = 2, m = 2.5, x = 1.9$.



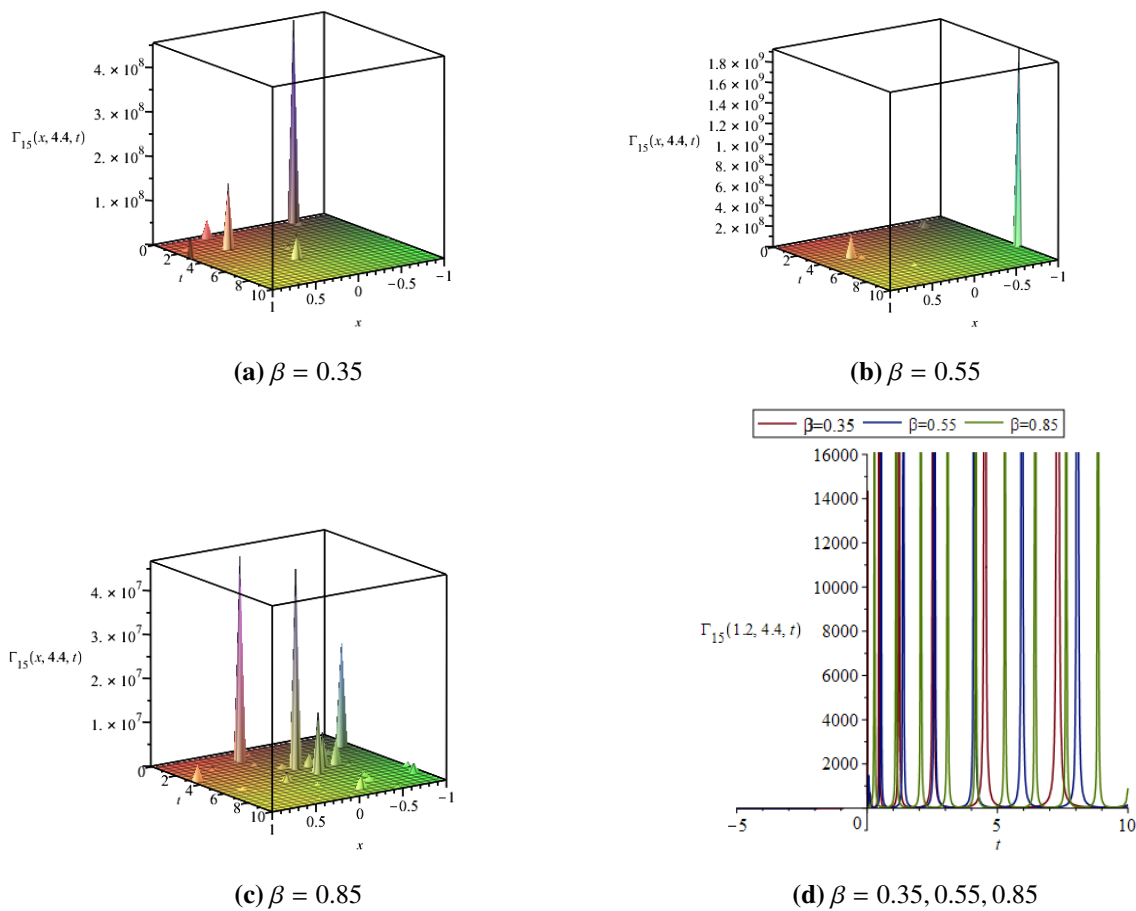
Note: As β increases, the number of singular poles increases, while their order of singularity exhibits only slight variations over time.

Figure 10. Multi-singular solitary wave of $\Gamma_5(x, y, t)$ with the parameter values $\beta = 0.5, 0.7, 0.9$, $r = 1.4$, $c = 0.8$, $\gamma = 2.5$, $k = 1$, $\Delta = -2.9$, $\Upsilon = 2$, $\Omega = -0.9$, $s = 1.2$, $l = 0.9$, $\delta = 8$, $y = 2$, $m = 2.5$, $x = 1.9$.

(a) $\beta = 0.5$ (b) $\beta = 0.7$ (c) $\beta = 0.9$ (d) $\beta = 0.5, 0.7, 0.9$

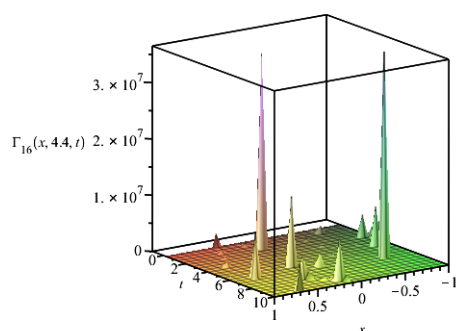
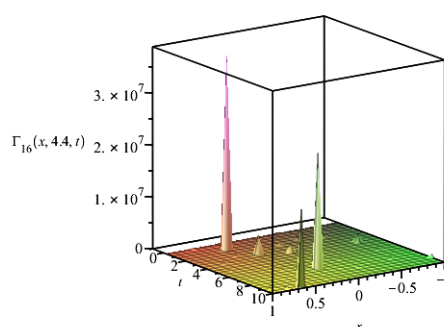
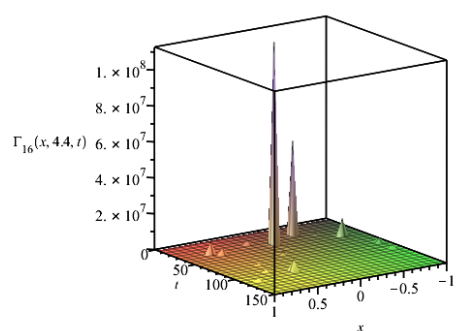
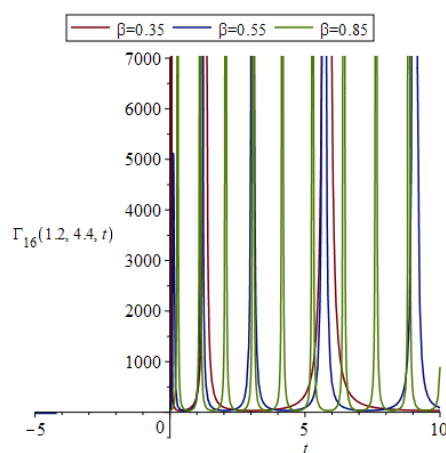
Note: The singular poles undergo positional shifts, while their order of singularity initially increases and subsequently decreases with time as β increases.

Figure 11. 1-singular solitary wave of $\Gamma_g(x, y, t)$ with the parameter values $\beta = 0.5, 0.7, 0.9$, $r = 1.4, c = 0.8, \gamma = 2.5, k = 1, \Delta = -2.9, \Upsilon = 2, \Omega = -0.9, s = 1.2, l = 0.9, \delta = 8, y = 2, m = 2.5, x = 1.9$.



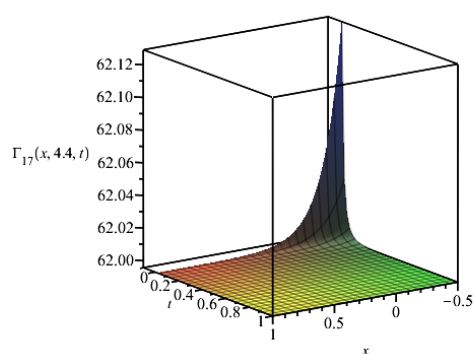
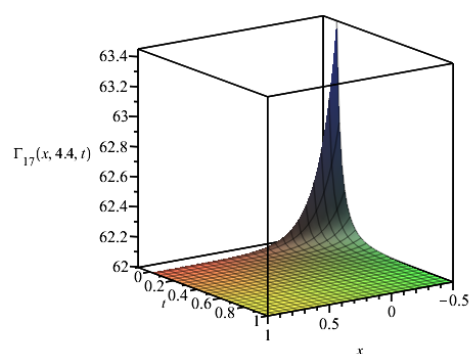
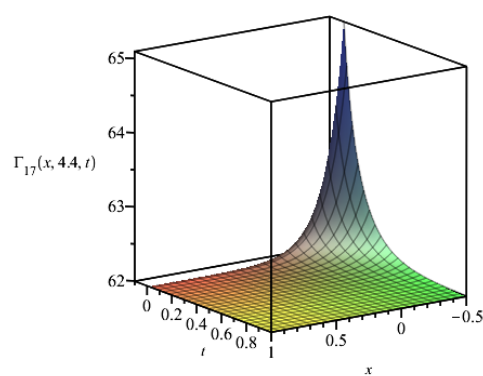
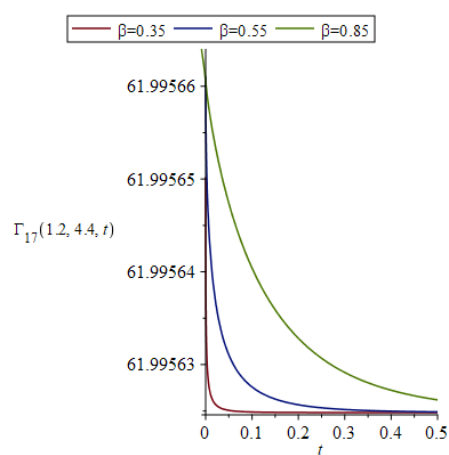
Note: As β increases, more singular poles emerge, while their order of singularity changes over time, initially increasing and then gradually decreasing.

Figure 12. Multi-singular solitary wave of $\Gamma_{15}(x, y, t)$ with the parameter values $K = 2.5$, $\delta = 1.2$, $\beta = 0.35, 0.55, 0.85$, $\Theta = 2$, $E = 1$, $r = -1.35$, $a = 2.1$, $\Lambda = 3$, $b = 1.1$, $C = 1$, $\gamma = -3.1$, $x = 1.2$, $y = 4.4$.

(a) $\beta = 0.35$ (b) $\beta = 0.55$ (c) $\beta = 0.85$ (d) $\beta = 0.35, 0.55, 0.85$

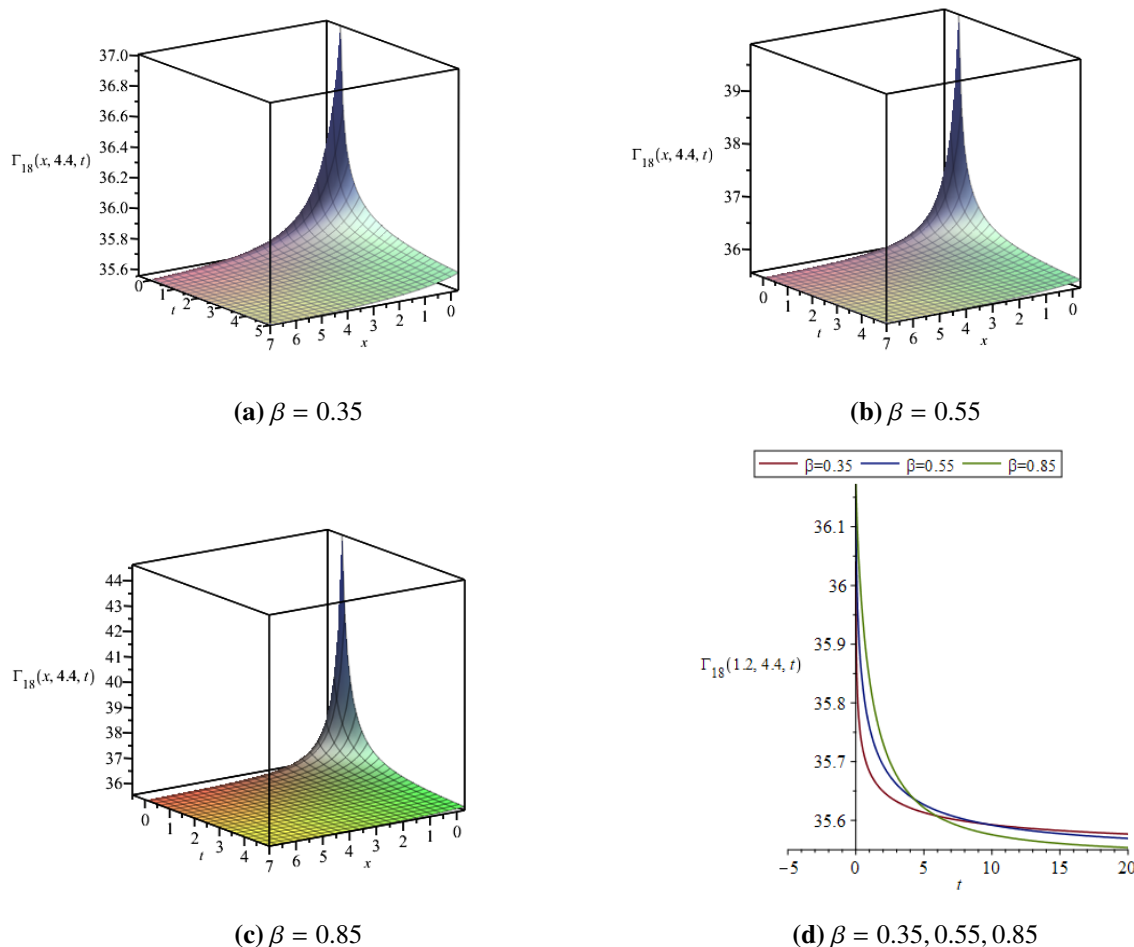
Note: The number of singular poles decreases, while their order of singularity progressively increases over time, as β increases.

Figure 13. Multi-singular solitary wave of $\Gamma_{16}(x, y, t)$ with the parameter values $K = 2.5$, $\delta = 1.2$, $\beta = 0.35, 0.55, 0.85$, $\Theta = 2$, $E = 1$, $r = -1.35$, $a = 2.1$, $\Lambda = 3$, $b = 1.1$, $C = 1$, $\gamma = -3.1$, $x = 1.2$, $y = 4.4$.

(a) $\beta = 0.35$ (b) $\beta = 0.55$ (c) $\beta = 0.85$ (d) $\beta = 0.35, 0.55, 0.85$

Note: The dispersion and amplitude of the solitary wave increases over time as the value of β increases.

Figure 14. Exponential solitary wave of $\Gamma_{17}(x, y, t)$ with the parameter values $K = 2.5$, $\delta = 1.2$, $\beta = 0.35, 0.55, 0.85$, $\Theta = 2$, $E = 1$, $r = -1.35$, $a = 2.1$, $\Lambda = 3$, $b = 1.1$, $C = 1$, $\gamma = -3.1$, $x = 1.2$, $y = 4.4$.



Note: As β increases, the dispersion and amplitude of the solitary wave decreases and increases, respectively, and the wave has stable propagation with time.

Figure 15. Exponential solitary wave of $\Gamma_{18}(x, y, t)$ with the parameter values $K = 2.5$, $\delta = 1.2$, $\beta = 0.35, 0.55, 0.85$, $\Theta = 2$, $E = 1$, $r = -1.35$, $a = 2.1$, $\Lambda = 3$, $b = 1.1$, $C = 1$, $\gamma = -3.1$, $x = 1.2$, $y = 4.4$.

7. Conclusions

In this study, we explored the conformable time-fractional (2+1)-dimensional generalized Boussinesq equation describing long waves in shallow water under weak dispersion and nonlinearity, which incorporates the classical (1+1)- and (2+1)-dimensional Boussinesq and Benjamin-Ono equations. Our approach combined Lie symmetry analysis, dimensional reduction via invariance, analytical methods, and chaotic analysis. Using techniques such as the Sardar sub-equation method, modified exp-function approach, sine-cosine method, and extended tanh method, we derived a family of soliton solutions (exponential, bright, combined-singular, breather soliton, 1-singular, multi-singular) and two families of analytic solutions. With respect to wave dynamics in oceanography and atmospheric sciences, these solitons are closely linked to nonlinear wave

dynamics. The 3D and 2D graphical analyses demonstrated that the fractional parameter β significantly affects the dispersion, amplitude, singularity, and merging behavior of the obtained wave solutions. Furthermore, chaotic behavior in both fractional and integer-order equations was investigated using a planar dynamical system approach, showing a transition from periodic to chaotic behavior. This study offers deeper physical insights into geophysical fluid dynamics and provides a foundation for future studies on complex nonlinear systems.

Author contributions

Nurzhan Serikbayev: Conceptualization, investigation, methodology, formal analysis, validation. Rajesh Kumar Gupta: Supervision, investigation, validation, writing-review and editing. Gaukhar Shaikhova: Investigation, visualization, formal analysis, validation, writing-original draft. Sharmila: Investigation, visualization, formal analysis, writing-original draft. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors claim that the results of the study reported in this work were unaffected by their personal relationships or known financial conflicts.

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