
*Research article***A double-inertial Z-iteration for generalized (α, β) -nonexpansive maps in Banach spaces with applications****Muhammed Mustafa Karakuş, Aynur Şahin*, Mustafa Eröz and Metin Başarır**

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Abstract: We propose a double-inertial Z-iteration to approximate fixed points of generalized (α, β) -nonexpansive maps in Banach spaces. The scheme injects a two-term inertial extrapolation, depending on three consecutive iterates x_m, x_{m-1}, x_{m-2} , into the first step of a three-layer Z-iteration. Under convergence conditions on the inertial parameters and boundedness conditions on both convex-combination parameters, we prove that the generated sequence is bounded and that $\lim_{m \rightarrow \infty} \|x_m - q\|$ exists for every fixed point (FP) q . In a uniformly convex Banach space (UCBS) satisfying Opial's property, we further establish weak convergence to an FP, and we derive a non-asymptotic bound $O(1/M)$. When the operator is in addition compact, we obtain strong convergence. As a first application, we use this strong convergence result to solve a nonlinear Hammerstein-type integral equation in $L^2([a, b], \mathbb{R})$, whose integral operator is shown to be nonexpansive and compact, and illustrate it with two examples. As a second application, we specialize the scheme to the proximal-gradient operator of the ℓ_1 -regularized least-squares model and show, on image deblurring, that it attains higher signal-to-noise ratio (SNR) and peak signal-to-noise ratio (PSNR) than the classical Z-iteration, a single-inertial Z-algorithm, and an alternative double-inertial scheme, and reaches prescribed quality targets in fewer iterations and less CPU time.

Keywords: double-inertial algorithm; generalized (α, β) -nonexpansive map; uniformly convex Banach space; nonlinear integral equation; image deblurring

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1. Introduction

Let B denote a Banach space (BS) and $S \subseteq B$ be a convex and closed set with $S \neq \emptyset$. For a map $G : S \rightarrow S$, the fixed point (FP) set is

$$F(G) = \{q \in S : Gq = q\}.$$

Iterative approximation of FPs has been the subject of sustained research since the seminal works of Mann [1], Ishikawa [2], and Noor [3]. Among the more recent multi-step methods, Thakur et al. [4] introduced a three-step algorithm that has attracted considerable attention because of its favourable rate of convergence. Subsequently, Srivastava et al. [5] proposed the following Z-iteration in a BS: For an initial point $x_1 \in S$ and parameter sequences $\{\sigma_m\}, \{\beta_m\} \subset (0, 1)$,

$$\begin{cases} z_m = G((1 - \sigma_m)x_m + \sigma_m Gx_m), \\ y_m = G((1 - \beta_m)z_m + \beta_m Gz_m), \\ x_{m+1} = Gy_m, \quad m \geq 1. \end{cases} \quad (1.1)$$

The three-layer structure of (1.1), in which every layer is wrapped by an additional application of G , leads to enhanced contractive behaviour relative to the Mann, Ishikawa, and Noor schemes.

A parallel line of research is concerned with inertial methods. Originating in the heavy-ball technique of Polyak [6] and Nesterov's optimal first-order method [7], the inertial idea was brought into the realm of monotone operator splitting by Alvarez and Attouch [8] and has since been incorporated into virtually every classical iteration (see, e.g., [9–11]). More recently, attention has turned to multi-step inertial methods, in which two or more previous iterates are recycled at every step. Iyiola and Shehu [12] gave a detailed convergence analysis for two-step inertial extrapolations, and Inkrong et al. [13] obtained a weak convergence theorem for an SP-type two-step iteration with a two-term inertial extrapolation. Given initial points $u_{-1}, u_0, u_1 \in B$, parameter sequences $\{\alpha_m\}, \{h_m\} \subset (0, 1)$, and inertial sequences $\{c_m\}, \{r_m\} \subset [0, \infty)$, their iteration reads

$$\begin{cases} w_m = (1 - \alpha_m)u_m + \alpha_m G_m(u_m + c_m(u_m - u_{m-1}) + r_m(u_{m-1} - u_{m-2})), \\ u_{m+1} = (1 - h_m)w_m + h_m G_m w_m, \quad m \geq 1, \end{cases} \quad (1.2)$$

where $\{G_m\}$ is a family of nonexpansive operators defined on a BS B . This iteration was studied in [13], where its weak convergence was established under appropriate conditions on the inertial and convex-combination parameters. We emphasise that in [13], the scheme (1.2) is set up on all of B rather than on a proper subset: The extrapolated point $u_m + c_m(u_m - u_{m-1}) + r_m(u_{m-1} - u_{m-2})$ is an affine, not a convex, combination of its three arguments, so it may leave a convex subset even when the arguments do not. We follow the same convention here.

In a complementary direction, several authors have weakened the contractive hypothesis on the operator itself. Suzuki [14] introduced the celebrated condition (C); Pant and Shukla [15] introduced the generalized α -nonexpansive map; and Pant and Pandey [16] introduced the generalized β -Reich–Suzuki nonexpansive map. Unifying these developments, Ullah et al. [17] introduced the generalized (α, β) -nonexpansive map and established its basic properties, including the demiclosedness principle. The class strictly contains the nonexpansive maps; under specific parameter choices, it recovers the condition (C), generalized α -nonexpansive and generalized β -Reich–Suzuki nonexpansive structures.

What is not yet available in the literature is the combination of these three strands: A two-term inertial extrapolation in the sense of [12, 13] inside the three-layer Z-structure of [5], driven by a single generalized (α, β) -nonexpansive operator in the sense of [17], together with its convergence theory and applications. The combination is worth making because the two inertial coefficients

are not interchangeable with genuine two-step memory. A scheme may carry two coefficients and still extrapolate along a single direction, if both are applied to the same backward difference; what distinguishes a true two-step method is that it recalls the two linearly independent differences $x_m - x_{m-1}$ and $x_{m-1} - x_{m-2}$, and it is this richer extrapolation, rather than the mere presence of a second parameter, that we isolate and test here. Since operators of this class arise both as solution operators of nonlinear integral equations of Hammerstein type [18] and as proximal-gradient operators [19] of regularized least-squares models [20], a scheme of this kind is of interest for analytical and computational fixed-point problems alike.

The study is arranged in the following way. Section 2 fixes notation and recalls the definitions and properties of the relevant map class, together with the four lemmas used in the proof. In Section 3, the inertial memory is placed in the first layer and recalls both x_{m-1} and x_{m-2} ; the precise scheme is stated as Algorithm 3.1. We analyze this scheme, which we call the double-inertial Z-iteration. Theorem 3.2 establishes that, under appropriate assumptions on the inertial coefficients, the generated sequence $\{x_m\}$ is bounded and $\lim_{m \rightarrow \infty} \|x_m - q\|$ exists for every $q \in F(G)$, and that, in a uniformly convex BS satisfying Opial's property, $\{x_m\}$ converges weakly to an FP of G . The Hilbert space case follows as an immediate corollary, and when the map G is compact, we further show that the same scheme converges strongly to an FP. Moreover, we give a non-asymptotic estimation on $h(\|Gu_m - u_m\|)$. As a first application, in Section 4 we specialize the scheme to the Hammerstein-type nonlinear integral operator and use the compactness-based strong convergence theorem to obtain the solution of the corresponding integral equation, illustrating it with two fully worked examples. As a second application, in Section 5 we specialize the scheme to the proximal-gradient operator associated with the ℓ_1 -regularized least-squares model and use it to deblur a motion-degraded image, comparing it numerically against the classical Z-iteration of Srivastava et al. [5], a single-inertial Z-algorithm, and an alternative double-inertial scheme in terms of both signal-to-noise ratio (SNR) and peak signal-to-noise ratio (PSNR). Section 6 closes with our concluding remarks.

2. Preliminaries

Throughout this paper, B denotes a BS with norm $\|\cdot\|$, $S \subseteq B$ is a set that is nonempty, convex, and closed, and I is the identity map. We use the notation $\rightarrow$ for strong convergence and $\rightharpoonup$ for weak convergence in B . The definitions and results of this section are stated, as in the literature, for a map defined on such a set S ; from Section 3, we apply them with $S = B$ because the inertial extrapolation of Algorithm 3.1 is an affine and not a convex combination of its arguments and therefore need not remain in a proper subset.

A BS B is said to be uniformly convex (UC) [21] if for every $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$ imply $\|(x + y)/2\| \leq 1 - \delta$. The L_p ($1 < p < \infty$) spaces and Hilbert spaces are UC; see [22]. Every UCBS is reflexive; in particular, every bounded sequence in such a space admits a weakly convergent subsequence; see [23].

Definition 2.1. [24] A map $G : S \rightarrow S$ is called:

- (a) *nonexpansive* if $\|Gx - Gy\| \leq \|x - y\|$ for every $x, y \in S$;
- (b) *quasi-nonexpansive* if $F(G) \neq \emptyset$ and $\|Gx - q\| \leq \|x - q\|$ for every $x \in S$ and every $q \in F(G)$.

The class of maps considered in this paper is the following.

Definition 2.2. [17] A map $G : S \rightarrow S$ is said to be *generalized (α, β) -nonexpansive* if there exist $\alpha, \beta \in \mathbb{R}^+$ with $\alpha + \beta < 1$ such that, for every $x, y \in S$,

$$\frac{1}{2} \|Gx - x\| \leq \|x - y\| \implies \|Gx - Gy\| \leq \alpha \|x - Gy\| + \alpha \|Gx - y\| + \beta \|x - Gx\| + \beta \|y - Gy\| + (1 - 2\alpha - 2\beta) \|x - y\|. \quad (2.1)$$

Two properties of this class, both established in [17], will be used repeatedly in the sequel.

Proposition 2.3. [17] Let G be a generalized (α, β) -nonexpansive map with $\alpha, \beta \in \mathbb{R}^+$ satisfying $\alpha + \beta < 1$ and $F(G) \neq \emptyset$. Then G is quasi-nonexpansive. Moreover, $F(G)$ is closed.

Proposition 2.4. [17] If G is a generalized (α, β) -nonexpansive self map on S with $\alpha, \beta \in \mathbb{R}^+$ satisfying $\alpha + \beta < 1$, then, for every $x, y \in S$,

$$\|x - Gy\| \leq \frac{3 + \alpha + \beta}{1 - \alpha - \beta} \|x - Gx\| + \|x - y\|. \quad (2.2)$$

Definition 2.5. [25] A BS B is said to satisfy *Opial's property* if, for every sequence $\{u_m\} \subset B$ with $u_m \rightharpoonup u$ and every $w \in B$ with $w \neq u$,

$$\liminf_{m \rightarrow \infty} \|u_m - u\| < \liminf_{m \rightarrow \infty} \|u_m - w\|.$$

The ℓ_p ($1 < p < \infty$) spaces and Hilbert spaces satisfy Opial's property, while L_p ($1 < p < \infty$, $p \neq 2$) spaces does not; see [26].

We now state the four lemmas needed in the proof.

Lemma 2.6. [27] Let $\{s_m\}, \{\delta_m\}$ and $\{\tau_m\}$ be nonnegative real sequences satisfying

$$s_{m+1} \leq (1 + \delta_m)s_m + \tau_m, \quad m \geq 1.$$

If $\sum_{m=1}^{\infty} \delta_m < \infty$ and $\sum_{m=1}^{\infty} \tau_m < \infty$, then $\lim_{m \rightarrow \infty} s_m$ exists.

Lemma 2.7. [28] Let $r > 0$ be a fixed number. Then, a BS B is UC iff there exists a strictly increasing, and continuous function $h : [0, \infty) \rightarrow [0, \infty)$ with $h(0) = 0$ such that, for every $d \in [0, 1]$ and every $a, b \in B$ with $\|a\|, \|b\| \leq r$,

$$\|d a + (1 - d) b\|^2 \leq d \|a\|^2 + (1 - d) \|b\|^2 - d(1 - d) h(\|a - b\|). \quad (2.3)$$

Lemma 2.8. [17] Let B be a BS satisfying Opial's property, $S \subseteq B$ a nonempty subset, and $G : S \rightarrow S$ a generalized (α, β) -nonexpansive map. Then $I - G$ is demiclosed at zero: Whenever $\{x_m\} \subset S$ satisfies $x_m \rightharpoonup q$ and $\|x_m - Gx_m\| \rightarrow 0$, one has $q \in F(G)$.

Lemma 2.9. [13] Let $\lambda_{-1}, \lambda_0 \geq 0$, and let $\{\lambda_m\}_{m \geq 1}$, $\{a_m\}_{m \geq 1}$, and $\{b_m\}_{m \geq 1}$ be nonnegative real sequences satisfying

$$\lambda_{m+1} \leq (1 + a_m)\lambda_m + (a_m + b_m)\lambda_{m-1} + b_m\lambda_{m-2}, \quad m \geq 1.$$

Then, with $K := \max\{\lambda_{-1}, \lambda_0, \lambda_1\}$,

$$\lambda_{m+1} \leq K \prod_{j=1}^m (1 + 2a_j + 2b_j), \quad m \geq 1.$$

In particular, if $\sum_{m=1}^{\infty} a_m < \infty$ and $\sum_{m=1}^{\infty} b_m < \infty$, then $\{\lambda_m\}$ is bounded.

3. Main results

We first state the iterative scheme as a formal algorithm.

Algorithm 3.1. (Double-inertial Z-iteration) Let $G : B \rightarrow B$ be a map. Choose initial points $x_{-1}, x_0, x_1 \in B$ and parameter sequences $\{\theta_m\}, \{t_m\} \subset [0, \infty)$ and $\{\sigma_m\}, \{\beta_m\} \subset (0, 1)$. Generate the sequence $\{x_m\}$ by

$$\begin{cases} u_m = x_m + \theta_m(x_m - x_{m-1}) + t_m(x_{m-1} - x_{m-2}), \\ z_m = G((1 - \sigma_m)u_m + \sigma_m Gu_m), \\ y_m = G((1 - \beta_m)z_m + \beta_m Gz_m), \\ x_{m+1} = Gy_m, \quad m \geq 1. \end{cases} \quad (3.1)$$

We now prove the weak convergence result of this algorithm.

Theorem 3.2. Consider a real UCBS B possessing Opial's property, and let $G : B \rightarrow B$ be a generalized (α, β) -nonexpansive map with $F(G) \neq \emptyset$. Assume that

$$(H1) \quad \sum_{m=1}^{\infty} \theta_m < \infty \text{ and } \sum_{m=1}^{\infty} t_m < \infty;$$

$$(H2) \quad 0 < \liminf_{m \rightarrow \infty} \sigma_m \leq \limsup_{m \rightarrow \infty} \sigma_m < 1;$$

$$(H3) \quad 0 < \liminf_{m \rightarrow \infty} \beta_m \leq \limsup_{m \rightarrow \infty} \beta_m < 1.$$

Let $\{x_m\}$ be the sequence produced by Algorithm 3.1. Then:

- (i) $\{x_m\}$ is bounded and $\lim_{m \rightarrow \infty} \|x_m - q\|$ exists for every $q \in F(G)$;
- (ii) $\{x_m\}$ converges weakly to a point in $F(G)$.

Proof. Fix $q \in F(G)$. Since G is generalized (α, β) -nonexpansive, Proposition 2.3 gives $\|Gx - q\| \leq \|x - q\|$ for every $x \in B$, a property that will be used throughout the proof.

(i) By the definition of u_m in (3.1) and the triangle inequality,

$$\begin{aligned} \|u_m - q\| &= \|(x_m - q) + \theta_m(x_m - x_{m-1}) + t_m(x_{m-1} - x_{m-2})\| \\ &\leq \|x_m - q\| + \theta_m \|x_m - x_{m-1}\| + t_m \|x_{m-1} - x_{m-2}\|. \end{aligned} \quad (3.2)$$

We have

$$\begin{aligned} \|z_m - q\| &= \|G((1 - \sigma_m)u_m + \sigma_m Gu_m) - q\| \\ &\leq \|(1 - \sigma_m)u_m + \sigma_m Gu_m - q\| \\ &= \|(1 - \sigma_m)(u_m - q) + \sigma_m(Gu_m - q)\| \\ &\leq (1 - \sigma_m) \|u_m - q\| + \sigma_m \|Gu_m - q\| \\ &\leq (1 - \sigma_m) \|u_m - q\| + \sigma_m \|u_m - q\| \\ &= \|u_m - q\|. \end{aligned} \quad (3.3)$$

The same chain of estimates applied to y_m gives

$$\begin{aligned}
 \|y_m - q\| &= \|G((1 - \beta_m)z_m + \beta_m Gz_m) - q\| \\
 &\leq \|(1 - \beta_m)z_m + \beta_m Gz_m - q\| \\
 &\leq (1 - \beta_m)\|z_m - q\| + \beta_m\|Gz_m - q\| \\
 &\leq (1 - \beta_m)\|z_m - q\| + \beta_m\|z_m - q\| \\
 &= \|z_m - q\|,
 \end{aligned} \tag{3.4}$$

and finally

$$\|x_{m+1} - q\| = \|Gy_m - q\| \leq \|y_m - q\|. \tag{3.5}$$

Combining (3.2)–(3.5), we obtain

$$\|x_{m+1} - q\| \leq \|x_m - q\| + \theta_m \|x_m - x_{m-1}\| + t_m \|x_{m-1} - x_{m-2}\|. \tag{3.6}$$

We next pass to a three-term recurrence in $\|x_m - q\|$. Applying the triangle inequality,

$$\|x_m - x_{m-1}\| \leq \|x_m - q\| + \|x_{m-1} - q\|, \quad \|x_{m-1} - x_{m-2}\| \leq \|x_{m-1} - q\| + \|x_{m-2} - q\|,$$

and substituting these into (3.6) gives

$$\|x_{m+1} - q\| \leq (1 + \theta_m)\|x_m - q\| + (\theta_m + t_m)\|x_{m-1} - q\| + t_m\|x_{m-2} - q\|, \quad m \geq 1. \tag{3.7}$$

The recurrence (3.7) fits the framework of Lemma 2.9 with $\lambda_m := \|x_m - q\|$, $a_m := \theta_m$ and $b_m := t_m$. All three sequences are nonnegative, as Lemma 2.9 requires: $\lambda_m = \|x_m - q\| \geq 0$ because it is a norm, while $\theta_m, t_m \geq 0$ by the standing assumption $\{\theta_m\}, \{t_m\} \subset [0, \infty)$ in Algorithm 3.1. This nonnegativity is what allows the substitution of the two triangle inequalities into (3.6) to preserve the direction of the inequality, and it matches the coefficients of Lemma 2.9 exactly: The coefficient of λ_m is $1 + \theta_m = 1 + a_m$, that of λ_{m-1} is $\theta_m + t_m = a_m + b_m$, and that of λ_{m-2} is $t_m = b_m$. By hypothesis (H1), we have $\sum_{m=1}^{\infty} a_m < \infty$ and $\sum_{m=1}^{\infty} b_m < \infty$; therefore Lemma 2.9 yields

$$\|x_{m+1} - q\| \leq K \prod_{j=1}^m (1 + 2\theta_j + 2t_j), \quad K := \max\{\|x_{-1} - q\|, \|x_0 - q\|, \|x_1 - q\|\},$$

and hence $\{x_m\}$ is bounded. Consequently,

$$\sum_{m=1}^{\infty} \theta_m \|x_m - x_{m-1}\| < \infty, \quad \sum_{m=1}^{\infty} t_m \|x_{m-1} - x_{m-2}\| < \infty. \tag{3.8}$$

We now apply Lemma 2.6 to inequality (3.6) with

$$s_m := \|x_m - q\|, \quad \delta_m := 0, \quad \tau_m := \theta_m \|x_m - x_{m-1}\| + t_m \|x_{m-1} - x_{m-2}\|.$$

By (3.8), we have $\sum_{m=1}^{\infty} \tau_m < \infty$, so Lemma 2.6 guarantees that

$$\lim_{m \rightarrow \infty} \|x_m - q\| \quad \text{exists.} \tag{3.9}$$

(ii) From (3.1),

$$\begin{aligned}\|u_m - q\|^2 &\leq (\|x_m - q\| + \theta_m \|x_m - x_{m-1}\| + t_m \|x_{m-1} - x_{m-2}\|)^2 \\ &= \|x_m - q\|^2 + \theta_m^2 \|x_m - x_{m-1}\|^2 + t_m^2 \|x_{m-1} - x_{m-2}\|^2 \\ &\quad + 2\theta_m \|x_m - q\| \|x_m - x_{m-1}\| + 2t_m \|x_m - q\| \|x_{m-1} - x_{m-2}\| \\ &\quad + 2\theta_m t_m \|x_m - x_{m-1}\| \|x_{m-1} - x_{m-2}\|. \end{aligned} \quad (3.10)$$

From Lemma 2.7, there exists a strictly increasing and continuous function $h : [0, \infty) \rightarrow [0, \infty)$ with $h(0) = 0$ such that

$$\begin{aligned}\|z_m - q\|^2 &= \|G((1 - \sigma_m)u_m + \sigma_m Gu_m) - q\|^2 \\ &\leq \|(1 - \sigma_m)u_m + \sigma_m Gu_m - q\|^2 \\ &= \|(1 - \sigma_m)(u_m - q) + \sigma_m(Gu_m - q)\|^2 \\ &\leq (1 - \sigma_m) \|u_m - q\|^2 + \sigma_m \|Gu_m - q\|^2 - (1 - \sigma_m)\sigma_m h(\|Gu_m - u_m\|) \\ &\leq (1 - \sigma_m) \|u_m - q\|^2 + \sigma_m \|u_m - q\|^2 - (1 - \sigma_m)\sigma_m h(\|Gu_m - u_m\|) \\ &= \|u_m - q\|^2 - (1 - \sigma_m)\sigma_m h(\|Gu_m - u_m\|). \end{aligned} \quad (3.11)$$

Substituting (3.10) into (3.11),

$$\begin{aligned}\|z_m - q\|^2 &\leq \|x_m - q\|^2 + \theta_m^2 \|x_m - x_{m-1}\|^2 + t_m^2 \|x_{m-1} - x_{m-2}\|^2 \\ &\quad + 2\theta_m \|x_m - q\| \|x_m - x_{m-1}\| + 2t_m \|x_m - q\| \|x_{m-1} - x_{m-2}\| \\ &\quad + 2\theta_m t_m \|x_m - x_{m-1}\| \|x_{m-1} - x_{m-2}\| - (1 - \sigma_m)\sigma_m h(\|Gu_m - u_m\|). \end{aligned} \quad (3.12)$$

The same argument is now applied to y_m and hence

$$\begin{aligned}\|y_m - q\|^2 &= \|G((1 - \beta_m)z_m + \beta_m Gz_m) - q\|^2 \\ &\leq \|(1 - \beta_m)z_m + \beta_m Gz_m - q\|^2 \\ &= \|(1 - \beta_m)(z_m - q) + \beta_m(Gz_m - q)\|^2 \\ &\leq (1 - \beta_m) \|z_m - q\|^2 + \beta_m \|Gz_m - q\|^2 - (1 - \beta_m)\beta_m h(\|Gz_m - z_m\|) \\ &\leq \|z_m - q\|^2 - (1 - \beta_m)\beta_m h(\|Gz_m - z_m\|). \end{aligned} \quad (3.13)$$

Combining (3.13) and (3.12), we obtain

$$\begin{aligned}\|x_{m+1} - q\|^2 &= \|Gy_m - q\|^2 \\ &\leq \|y_m - q\|^2 \\ &\leq \|z_m - q\|^2 - (1 - \beta_m)\beta_m h(\|Gz_m - z_m\|) \\ &\leq \|x_m - q\|^2 + \theta_m^2 \|x_m - x_{m-1}\|^2 + t_m^2 \|x_{m-1} - x_{m-2}\|^2 \\ &\quad + 2\theta_m \|x_m - q\| \|x_m - x_{m-1}\| + 2t_m \|x_m - q\| \|x_{m-1} - x_{m-2}\| \\ &\quad + 2\theta_m t_m \|x_m - x_{m-1}\| \|x_{m-1} - x_{m-2}\| \\ &\quad - (1 - \sigma_m)\sigma_m h(\|Gu_m - u_m\|) - (1 - \beta_m)\beta_m h(\|Gz_m - z_m\|). \end{aligned} \quad (3.14)$$

Rearranging (3.14) and writing

$$\Lambda_m := \theta_m^2 \|x_m - x_{m-1}\|^2 + t_m^2 \|x_{m-1} - x_{m-2}\|^2 + 2\theta_m \|x_m - q\| \|x_m - x_{m-1}\|$$

$$+2t_m \|x_m - q\| \|x_{m-1} - x_{m-2}\| + 2\theta_m t_m \|x_m - x_{m-1}\| \|x_{m-1} - x_{m-2}\|$$

for the collected inertial remainder, we get

$$(1 - \sigma_m)\sigma_m h(\|Gu_m - u_m\|) + (1 - \beta_m)\beta_m h(\|Gz_m - z_m\|) \leq \|x_m - q\|^2 - \|x_{m+1} - q\|^2 + \Lambda_m. \quad (3.15)$$

Since $\lim_{m \rightarrow \infty} \|x_m - q\|$ exists by part (i), $\{x_m\}$ is bounded, and $\theta_m, t_m \rightarrow 0$ by (H1), we have $\Lambda_m \rightarrow 0$, and hence the right-hand side of (3.15) tends to zero. Both terms on the left are nonnegative, so each of them tends to zero separately:

$$\lim_{m \rightarrow \infty} (1 - \sigma_m)\sigma_m h(\|Gu_m - u_m\|) = 0, \quad \lim_{m \rightarrow \infty} (1 - \beta_m)\beta_m h(\|Gz_m - z_m\|) = 0.$$

By (H2), the factor $(1 - \sigma_m)\sigma_m$ is bounded away from zero, and by (H3), so is $(1 - \beta_m)\beta_m$; together with the continuity and strict monotonicity of h (with $h(0) = 0$), this gives

$$\lim_{m \rightarrow \infty} \|Gu_m - u_m\| = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} \|Gz_m - z_m\| = 0. \quad (3.16)$$

From (3.1), $\theta_m, t_m \rightarrow 0$, and the boundedness of $\{x_m\}$,

$$\|u_m - x_m\| \leq \theta_m \|x_m - x_{m-1}\| + t_m \|x_{m-1} - x_{m-2}\| \rightarrow 0. \quad (3.17)$$

By Proposition 2.4 applied with $x = u_m$ and $y = x_m$,

$$\|u_m - Gx_m\| \leq \frac{3 + \alpha + \beta}{1 - \alpha - \beta} \|u_m - Gu_m\| + \|u_m - x_m\|. \quad (3.18)$$

Using (3.18) and the triangle inequality,

$$\begin{aligned} \|Gx_m - x_m\| &\leq \|Gx_m - u_m\| + \|u_m - x_m\| \\ &\leq \frac{3 + \alpha + \beta}{1 - \alpha - \beta} \|Gu_m - u_m\| + 2\|u_m - x_m\|. \end{aligned} \quad (3.19)$$

Combining (3.19) with (3.16) and (3.17),

$$\lim_{m \rightarrow \infty} \|Gx_m - x_m\| = 0. \quad (3.20)$$

Since B is a UCBS, it is reflexive, so the bounded sequence $\{x_m\}$ admits a weakly convergent subsequence; moreover, B satisfies Opial's property by hypothesis. We may therefore apply the demiclosedness principle: By (3.20) and Lemma 2.8, every weak cluster point of $\{x_m\}$ belongs to $F(G)$.

To complete the proof, it suffices to show that $\{x_m\}$ has a unique weak cluster point. Suppose, for the sake of contradiction, that q_1 and q_2 are two weak cluster points of $\{x_m\}$ in $F(G)$ with $q_1 \neq q_2$, and let $\{x_{m_i}\}, \{x_{m_j}\}$ be subsequences with $x_{m_i} \rightharpoonup q_1$ and $x_{m_j} \rightharpoonup q_2$. By part (i), $\lim_{m \rightarrow \infty} \|x_m - q_1\|$ and $\lim_{m \rightarrow \infty} \|x_m - q_2\|$ exist. Applying Opial's property,

$$\begin{aligned} \lim_{m \rightarrow \infty} \|x_m - q_1\| &= \liminf_{i \rightarrow \infty} \|x_{m_i} - q_1\| < \liminf_{i \rightarrow \infty} \|x_{m_i} - q_2\| \\ &= \lim_{m \rightarrow \infty} \|x_m - q_2\| = \liminf_{j \rightarrow \infty} \|x_{m_j} - q_2\| \\ &< \liminf_{j \rightarrow \infty} \|x_{m_j} - q_1\| = \lim_{m \rightarrow \infty} \|x_m - q_1\|, \end{aligned}$$

we get a contradiction. Therefore $q_1 = q_2$, and $\{x_m\}$ converges weakly to a point of $F(G)$. $\square$

In a real Hilbert space, uniform convexity and Opial's property hold automatically, so the hypotheses of Theorem 3.2 reduce to those on the parameter sequences.

Corollary 3.3. *Assume that H is a real Hilbert space, and let $G : H \rightarrow H$ be a generalized (α, β) -nonexpansive map with $F(G) \neq \emptyset$. If hypotheses (H1)–(H3) of Theorem 3.2 hold, then the sequence $\{x_m\}$ produced by Algorithm 3.1 converges weakly to a point of $F(G)$.*

The argument just given yields a non-asymptotic estimate as follows:

Proposition 3.4. *Under the hypotheses of Theorem 3.2, suppose in addition that $\sigma_m \in [\underline{\sigma}, \overline{\sigma}] \subset (0, 1)$ for all $m \geq 1$, and set $c := \underline{\sigma}(1 - \overline{\sigma}) > 0$. Then, for every $q \in F(G)$ and every $M \geq 1$,*

$$\min_{1 \leq m \leq M} h(\|Gu_m - u_m\|) \leq \frac{1}{cM} \left(\|x_1 - q\|^2 + \sum_{m=1}^{\infty} \Lambda_m \right) = O\left(\frac{1}{M}\right),$$

where Λ_m is the inertial remainder defined in the proof of Theorem 3.2 and $\sum_{m=1}^{\infty} \Lambda_m < \infty$. In a real Hilbert space, one may take $h(\varrho) = \varrho^2$ in Lemma 2.7, and the estimate becomes

$$\min_{1 \leq m \leq M} \|Gu_m - u_m\| = O(M^{-1/2}).$$

Proof. Discarding the nonnegative term including β_m in (3.15) and summing the resulting inequality over $m = 1, \dots, M$, the differences $\|x_m - q\|^2 - \|x_{m+1} - q\|^2$ telescope, so

$$\sum_{m=1}^M (1 - \sigma_m) \sigma_m h(\|Gu_m - u_m\|) \leq \|x_1 - q\|^2 - \|x_{M+1} - q\|^2 + \sum_{m=1}^M \Lambda_m \leq \|x_1 - q\|^2 + \sum_{m=1}^{\infty} \Lambda_m.$$

The series $\sum_{m=1}^{\infty} \Lambda_m$ converges: Each of its five groups of terms is dominated, using the boundedness of $\{x_m\}$ established in part (i), by a constant multiple of θ_m , t_m , θ_m^2 , t_m^2 or $\theta_m t_m$, all of which are summable by (H1). Since $(1 - \sigma_m) \sigma_m \geq c$ for every m , the left-hand side is at least $cM \min_{1 \leq m \leq M} h(\|Gu_m - u_m\|)$, and the claim follows. The Hilbert space case is immediate from $h(\varrho) = \varrho^2$. $\square$

Remark 3.5. Algorithm 3.1 admits two natural variants that we use as comparators in Section 5. Taking $t_m \equiv 0$ collapses the two-term inertial step into a single one and yields the single-inertial Z-iteration

$$\begin{cases} w_m = x_m + \theta_m(x_m - x_{m-1}), \\ z_m = G((1 - \sigma_m)w_m + \sigma_m Gw_m), \\ y_m = G((1 - \beta_m)z_m + \beta_m Gz_m), \\ x_{m+1} = Gy_m, \end{cases} \quad (3.21)$$

to which Theorem 3.2 applies verbatim. A different way of introducing two inertial coefficients is to split the inertial information across two independent extrapolations that share the same one-step backward difference,

$$\begin{cases} u_m = x_m + \theta_m(x_m - x_{m-1}), \\ v_m = x_m + t_m(x_m - x_{m-1}), \\ z_m = G((1 - \sigma_m)v_m + \sigma_m Gu_m), \\ y_m = G((1 - \beta_m)z_m + \beta_m Gz_m), \\ x_{m+1} = Gy_m. \end{cases} \quad (3.22)$$

Because both u_m and v_m involve only the single backward difference $x_m - x_{m-1}$, scheme (3.22) carries essentially one effective inertial direction, in contrast to the genuine two-step memory of Algorithm 3.1, which recalls the two linearly independent differences $x_m - x_{m-1}$ and $x_{m-1} - x_{m-2}$. The numerical comparison in Section 5 quantifies the resulting difference in performance.

Finally, we establish the following strong convergence result, which is used in the application to nonlinear integral equations in Section 4.

Theorem 3.6. *Let B be a real UCBS possessing Opial's property, and let $G : B \rightarrow B$ be a generalized (α, β) -nonexpansive map with $F(G) \neq \emptyset$, satisfying hypotheses (H1)–(H3) of Theorem 3.2. Assume in addition that G is compact, in the sense that it maps bounded subsets of B onto relatively compact sets. Then the sequence $\{x_m\}$ produced by Algorithm 3.1 converges strongly to an FP of G .*

Proof. All hypotheses of Theorem 3.2 hold, so (3.9) and (3.20) are available; in particular, $\{x_m\}$ is bounded and $\lim_{m \rightarrow \infty} \|x_m - Gx_m\| = 0$.

We first check that $\{x_m\}$ lies in a relatively compact set. Fix $q \in F(G)$. By (3.2)–(3.4),

$$\|y_m - q\| \leq \|z_m - q\| \leq \|u_m - q\| \leq \|x_m - q\| + \theta_m \|x_m - x_{m-1}\| + t_m \|x_{m-1} - x_{m-2}\|,$$

and the right-hand side is bounded, since $\{x_m\}$ is bounded and $\theta_m, t_m \rightarrow 0$ by (H1). Hence $\{y_m\}$ is a bounded subset of B , and since $x_{m+1} = Gy_m$, $\{x_{m+1}\}$ is contained in the image under G of a bounded set, which is relatively compact by hypothesis.

Consequently $\{x_m\}$ admits a strongly convergent subsequence: There exist $\{x_{m_k}\}$ and $q_0 \in B$ with $x_{m_k} \rightarrow q_0$ as $k \rightarrow \infty$. Applying Proposition 2.4 with $x = x_{m_k}$ and $y = q_0$, we obtain

$$\|x_{m_k} - Gq_0\| \leq \frac{3 + \alpha + \beta}{1 - \alpha - \beta} \|x_{m_k} - Gx_{m_k}\| + \|x_{m_k} - q_0\|,$$

whence $\lim_{k \rightarrow \infty} \|x_{m_k} - Gq_0\| = 0$. Since the subsequence $\{x_{m_k}\}$ converges to both q_0 and Gq_0 , the uniqueness of limits yields $Gq_0 = q_0$, that is, $q_0 \in F(G)$. Finally, (3.9) guarantees that $\lim_{m \rightarrow \infty} \|x_m - q_0\|$ exists; as one subsequence converges to q_0 , this limit must be zero. Consequently, the sequence $\{x_m\}$ converges strongly to the FP q_0 . $\square$

4. Application to a nonlinear integral equation

We demonstrate an application of Theorem 3.6 to the following nonlinear Hammerstein-type integral equation

$$x(t) = g(t) + \gamma \int_a^b M(t, s)h(s, x(s)) ds, \quad t \in [a, b], \quad (4.1)$$

where the kernel $M : [a, b] \times [a, b] \rightarrow \mathbb{R}^+$ and $g : [a, b] \rightarrow \mathbb{R}$ are given continuous functions, $\gamma \in (0, 1]$, and the known function $h : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and nonlinear with respect to x ; see [18].

We take the Hilbert space

$$B = L^2([a, b], \mathbb{R}), \quad \|x\| = \left(\int_a^b |x(t)|^2 dt \right)^{1/2},$$

which is automatically a UCBS satisfying Opial's property. We retain the hypothesis that M and g are continuous on $[a, b] \times [a, b]$ and $[a, b]$, respectively, so that M and g are in particular bounded, say $|M(t, s)| \leq M_0$ and $|g(t)| \leq g_0$ for all $t, s \in [a, b]$.

Theorem 4.1. Define the operator $G : B \rightarrow B$ by

$$(Gx)(t) = g(t) + \gamma \int_a^b M(t, s)h(s, x(s)) ds, \quad t \in [a, b].$$

Assume that

(E1) There exists a constant $\nu \geq 0$ such that

$$|h(s, x(s)) - h(s, y(s))| \leq \nu |x(s) - y(s)|$$

for every $s \in [a, b]$ and every $x(s), y(s) \in \mathbb{R}$.

(E2) $\gamma \nu M_0(b - a) \leq 1$.

(E3) There exists a constant $h_0 \geq 0$ such that $|h(s, 0)| \leq h_0$ for every $s \in [a, b]$.

If the integral equation (4.1) has a solution, then the sequence produced by Algorithm 3.1, run with this G from arbitrary initial points $x_{-1}, x_0, x_1 \in B$ and with parameters satisfying (H1)–(H3), converges strongly to a solution of (4.1).

Proof. We first show that G is nonexpansive. Let $x, y \in B$ be arbitrary, with no order relation assumed between them. By the Cauchy–Schwarz inequality and (E1), for every $t \in [a, b]$,

$$\begin{aligned} |(Gy)(t) - (Gx)(t)| &= \left| \gamma \int_a^b M(t, s)(h(s, y(s)) - h(s, x(s))) ds \right| \\ &\leq \gamma \nu M_0 \int_a^b |y(s) - x(s)| ds \\ &\leq \gamma \nu M_0 \sqrt{b - a} \|y - x\|. \end{aligned}$$

Squaring and integrating in t over $[a, b]$,

$$\begin{aligned} \|Gy - Gx\|^2 &= \int_a^b |(Gy)(t) - (Gx)(t)|^2 dt \leq (b - a)(\gamma \nu M_0 \sqrt{b - a})^2 \|y - x\|^2 \\ &= (\gamma \nu M_0(b - a))^2 \|y - x\|^2, \end{aligned}$$

so that $\|Gy - Gx\| \leq \gamma \nu M_0(b - a) \|y - x\|$. By (E2), the constant is at most 1, hence $\|Gy - Gx\| \leq \|y - x\|$ for every $x, y \in B$: G is nonexpansive on B . By [17, Remark 1 and Proposition 1], every nonexpansive map is generalized $(0, 0)$ -nonexpansive; this holds for G on B , as required by Theorem 3.6.

It remains to prove that G is compact. Let $D \subset B$ be bounded, say $\|x\| \leq \rho$ for every $x \in D$. Using (E1) and (E3), $|h(s, x(s))| \leq \nu |x(s)| + h_0$, so by Cauchy–Schwarz, for every $x \in D$ and $t \in [a, b]$,

$$|(Gx)(t)| \leq g_0 + \gamma M_0 \int_a^b (\nu |x(s)| + h_0) ds \leq g_0 + \gamma M_0(\nu \sqrt{b - a} \rho + h_0(b - a)) =: \Lambda_\rho,$$

so $\{Gx : x \in D\}$ is uniformly bounded on $[a, b]$. Moreover, for $x \in D$ and $t_1, t_2 \in [a, b]$,

$$|(Gx)(t_1) - (Gx)(t_2)| \leq |g(t_1) - g(t_2)| + \gamma \left(\sup_s |M(t_1, s) - M(t_2, s)| \right) \int_a^b (\nu |x(s)| + h_0) ds.$$

The last integral is bounded by $\nu \sqrt{b-a} \rho + h_0(b-a)$, uniformly in $x \in D$. Since g is continuous on the compact interval $[a, b]$ and M is continuous on the compact square $[a, b] \times [a, b]$, both are uniformly continuous there, so $|g(t_1) - g(t_2)| \rightarrow 0$ and $\sup_s |M(t_1, s) - M(t_2, s)| \rightarrow 0$ as $|t_1 - t_2| \rightarrow 0$. Hence $\{Gx : x \in D\}$ is equicontinuous, and by the Arzelà–Ascoli theorem [22], it has compact closure in $(C([a, b], \mathbb{R}), \|\cdot\|_\infty)$. Since the natural embedding $I : C([a, b], \mathbb{R}) \rightarrow L^2([a, b], \mathbb{R}), I(x) = x$ is continuous, indeed $\|x\| \leq \sqrt{b-a} \|x\|_\infty$, the set $\overline{G(D)}$ is compact in B as well. Thus G maps bounded sets onto relatively compact sets.

Finally, if (4.1) has a solution x^* , then $Gx^* = x^*$, so $F(G) \neq \emptyset$ and all hypotheses of Theorem 3.6 are satisfied. Algorithm 3.1 therefore converges strongly to some $\tilde{x} \in F(G)$, and by the definition of G ,

$$\tilde{x}(t) = g(t) + \gamma \int_a^b M(t, s) h(s, \tilde{x}(s)) ds \quad \text{for a.e. } t \in [a, b],$$

so $\tilde{x}$ is a solution of the nonlinear integral equation (4.1). $\square$

Example 4.2. Let $a = 0$, $b = 1$, and $\gamma = \frac{1}{3}$. Define

$$M(t, s) \equiv t + s, \quad h(s, x(s)) = x(s), \quad g(t) = \frac{5}{6}t - \frac{1}{9}, \quad t, s \in [0, 1],$$

and work in $B = L^2([0, 1], \mathbb{R})$. We check that hypotheses (E1)–(E3) of Theorem 4.1 hold.

(E1) Since $h(s, x(s)) = x(s)$,

$$|h(s, x(s)) - h(s, y(s))| = |x(s) - y(s)|, \quad \forall s \in [0, 1], \forall x(s), y(s) \in \mathbb{R},$$

so (E1) holds with $\nu = 1$.

(E2) Since $M(t, s) = t + s \geq 0$ on $[0, 1] \times [0, 1]$ attains its maximum 2 at $t = s = 1$, we have $M_0 = 2$. Hence

$$\gamma \nu M_0 (b - a) = \frac{1}{3} \cdot 1 \cdot 2 \cdot 1 = \frac{2}{3} < 1,$$

so (E2) holds.

(E3) $h(s, 0) = 0$, so (E3) holds with $h_0 = 0$.

Consequently, by the first part of the proof of Theorem 4.1, the linear operator

$$(Gx)(t) = \frac{5}{6}t - \frac{1}{9} + \frac{1}{3} \int_0^1 (t + s)x(s) ds, \quad t \in [0, 1],$$

is nonexpansive, with Lipschitz constant $\frac{2}{3}$, on $B = L^2([0, 1], \mathbb{R})$; a direct computation confirms this operator-norm bound, since for the linear integral operator $(Kx)(t) = \int_0^1 (t + s)x(s) ds$, the Cauchy–Schwarz estimate used there gives $\|K\| \leq M_0 \sqrt{b-a} \cdot \sqrt{b-a} = M_0(b-a) = 2$, so $\|\gamma K\| \leq \frac{1}{3} \cdot 2 = \frac{2}{3}$, matching the bound above. Theorem 4.1 therefore applies. Direct substitution shows that $x^*(t) = t$ satisfies

$$\frac{5}{6}t - \frac{1}{9} + \frac{1}{3} \int_0^1 (t + s)s ds = \frac{5}{6}t - \frac{1}{9} + \frac{1}{3} \left(\frac{t}{2} + \frac{1}{3} \right) = \frac{5}{6}t - \frac{1}{9} + \frac{t}{6} + \frac{1}{9} = t,$$

so $x^*(t) = t$ is an exact solution and $F(G) \neq \emptyset$. All hypotheses of Theorem 4.1 are therefore satisfied, and we conclude that the linear Hammerstein integral equation

$$x(t) = \frac{5}{6}t - \frac{1}{9} + \frac{1}{3} \int_0^1 (t+s)x(s) ds, \quad t \in [0, 1]$$

possesses the exact solution $x^*(t) = t$, obtainable as the strong limit of the sequence $\{x_m\}$ produced by Algorithm 3.1 applied to G from arbitrary initial points of B .

Example 4.3. Let $B = L^2([0, 1], \mathbb{R})$, $\gamma = 1$, and consider the nonlinear Hammerstein integral equation

$$x(t) = 1 + t + \int_0^1 \frac{1}{4} (\tanh(x(s)) - \tanh(1+s)) ds, \quad t \in [0, 1]. \quad (4.2)$$

Define

$$g(t) = 1 + t, \quad M(t, s) \equiv 1,$$

and

$$h(s, x(s)) = \frac{1}{4} (\tanh(x(s)) - \tanh(1+s)).$$

Then, $G : B \rightarrow B$ is given by

$$(Gx)(t) = 1 + t + \int_0^1 \frac{1}{4} (\tanh(x(s)) - \tanh(1+s)) ds.$$

(E1) Since

$$\left| \frac{d}{dx} \tanh(x(s)) \right| = \operatorname{sech}^2(x(s)) \leq 1,$$

the mean value theorem implies that

$$|h(s, x(s)) - h(s, y(s))| = \frac{1}{4} |\tanh(x(s)) - \tanh(y(s))| \leq \frac{1}{4} |x(s) - y(s)|$$

for all $x(s), y(s) \in \mathbb{R}$ and all $s \in [0, 1]$. Hence (E1) holds with the constant $\nu = \frac{1}{4}$.

(E2) Here $M_0 = 1$ and $b - a = 1$, so

$$\gamma \nu M_0 (b - a) = 1 \cdot \frac{1}{4} \cdot 1 \cdot 1 = \frac{1}{4} < 1,$$

thus (E2) holds.

(E3) Since $|\tanh(x(s))| \leq 1$,

$$|h(s, 0)| = \frac{1}{4} |\tanh(0) - \tanh(1+s)| = \frac{1}{4} \tanh(1+s) \leq \frac{1}{4},$$

so (E3) holds with $h_0 = \frac{1}{4}$. Therefore, all hypotheses of Theorem 4.1 are satisfied.

Now consider the function

$$x^*(t) = 1 + t, \quad t \in [0, 1].$$

Then

$$h(s, x^*(s)) = \frac{1}{4} (\tanh(1+s) - \tanh(1+s)) = 0,$$

which yields

$$\begin{aligned}(Gx^*)(t) &= 1 + t + \int_0^1 0 \, ds \\ &= 1 + t = x^*(t).\end{aligned}$$

Hence, x^* is an exact solution of (4.2).

Consequently, by Theorem 4.1, the nonlinear Hammerstein integral equation (4.2) possesses the solution

$$x^*(t) = 1 + t, \quad t \in [0, 1],$$

obtainable as the strong limit of the sequence $\{x_m\}$ produced by Algorithm 3.1 applied to G from arbitrary initial points of B .

5. Application to image deblurring

As outlined in the introduction, image restoration can be cast as a linear inverse problem. The degradation of an image by a linear blur operator together with additive noise is modelled as

$$v = Dx + b, \tag{5.1}$$

where $v \in \mathbb{R}^m$ is the observed blurred image (vectorized), $x \in \mathbb{R}^n$ is the unknown sharp image, $D \in \mathbb{R}^{m \times n}$ is the blur operator, and $b \in \mathbb{R}^m$ is additive noise. Sparse solutions of (5.1) can be recovered by solving the ℓ_1 -regularized least-squares problem [20]

$$\xi^* \in \arg \min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|Dx - v\|_2^2 + \tau \|x\|_1 \right\}, \quad \tau > 0. \tag{5.2}$$

Setting $h(x) := \frac{1}{2} \|Dx - v\|_2^2$ and $g(x) := \tau \|x\|_1$, the gradient $\nabla h(x) = D^\top(Dx - v)$ is L -Lipschitz with $L = \lambda_{\max}(D^\top D)$. For a step size $\lambda \in (0, 2/L)$, the proximal-gradient (forward–backward) operator

$$G(x) := \text{prox}_{\lambda g}(x - \lambda \nabla h(x)) \tag{5.3}$$

is nonexpansive with $F(G) = \arg \min\{h + g\}$, and is evaluated explicitly through the soft-thresholding operator associated with g ; see [19]. The function h is convex and differentiable with L -Lipschitz ∇h , and g is proper, convex, and lower semicontinuous. Under these hypotheses, the gradient-descent step $I - \lambda \nabla h$ is nonexpansive for every $\lambda \in (0, 2/L]$ while $\text{prox}_{\lambda g}$ is firmly nonexpansive, hence nonexpansive, for every $\lambda > 0$ because g is proper, convex, and lower semicontinuous. The composition $G = \text{prox}_{\lambda g} \circ (I - \lambda \nabla h)$ of two nonexpansive maps is therefore nonexpansive on all of $\mathbb{R}^n$ for every $\lambda \in (0, 2/L]$, and its FP set coincides with $\arg \min\{h + g\}$. Our experiments use $\lambda = 1/L$, which lies in this range. By [17, Remark 1 and Proposition 1], every nonexpansive map is generalized $(0, 0)$ -nonexpansive. Hence Theorem 3.2 applies whenever $F(G) \neq \emptyset$, which holds here because (5.2) is a coercive convex problem and therefore has a minimizer. The proposed deblurring method is therefore Algorithm 3.1 driven by the operator (5.3), and the generated sequence converges weakly to a solution of (5.2).

The reconstruction quality is measured by the SNR and the PSNR, defined for the original image x and the m -th iterate x_m by

$$\text{SNR}(x, x_m) = 20 \log_{10} \frac{\|x\|_2}{\|x - x_m\|_2}, \quad \text{PSNR}(x, x_m) = 10 \log_{10} \frac{1}{\text{MSE}(x, x_m)},$$

where $\text{MSE}(x, x_m) = \|x - x_m\|_2^2 / N$ for an N -pixel image with values in $[0, 1]$ (see [29]); higher values indicate better quality.

In our experiment, the test image is a 256×256 grayscale photograph, degraded by motion blur of length 15 at orientation 60° and corrupted by Gaussian noise. The blur operator D of (5.1) is realized as a discrete convolution with the corresponding 15×15 point-spread function (PSF), periodic boundary conditions being used, and its adjoint $D^\top$ as the correlation with the same PSF; the Lipschitz constant L is obtained from the squared modulus of the discrete Fourier transform of the PSF. We compare four schemes, each run for 100 iterations: the classical Z-iteration (1.1), the single-inertial Z-iteration (3.21), the alternative double-inertial scheme (3.22), and the proposed double-inertial Z-iteration (3.1). The proximal-gradient operator uses the constant step size $\lambda = 1/L$ and regularization parameter $\tau = 10^{-5}$, and the convex-combination parameters are fixed at $\sigma_m = \beta_m = \frac{1}{2}$. The inertial parameters of the three inertial schemes follow the adaptive rule

$$\theta_m = \min \left\{ 0.45, \frac{10^{20}}{m^2 \|x_m - x_{m-1}\|} \right\}, \quad t_m = \min \left\{ 0.45, \frac{10^{18}}{m^2 \|x_{m-1} - x_{m-2}\|} \right\}, \quad (5.4)$$

with the convention that a coefficient equals 0.45 when the corresponding backward difference vanishes; the single-inertial scheme uses only θ_m , and in the alternative scheme, both extrapolations share the difference $x_m - x_{m-1}$. Two ingredients of the equations in (5.4) play distinct roles. The cap 0.45 bounds the extrapolation and is chosen below $\frac{1}{2}$ so that the extrapolated point stays close to the current iterate. The decaying factor $1/m^2$, together with the numerators, controls the products in which the inertial terms actually enter the analysis: By construction,

$$\theta_m \|x_m - x_{m-1}\| \leq \frac{10^{20}}{m^2}, \quad t_m \|x_{m-1} - x_{m-2}\| \leq \frac{10^{18}}{m^2},$$

so both are summable, which is precisely the form used in (3.8). Since (5.4) forces $\theta_m, t_m = O(1/m^2)$, the hypothesis (H1) of Theorem 3.2 holds, while (H2) and (H3) hold trivially because σ_m and β_m are fixed at $\frac{1}{2}$ in $(0, 1)$.

All experiments were implemented in Python 3.12.13 using NumPy 2.0.2, SciPy 1.16.3, and Matplotlib 3.10.0 for visualization. The computations were carried out on Google Colab (CPU runtime, no GPU acceleration), equipped with an Intel(R) Xeon(R) CPU @ 2.20GHz and approximately 12 GB of RAM.

Tables 1 and 2 report the SNR and PSNR values attained by the four schemes at selected iteration counts, while Figures 1 and 2 display the corresponding convergence curves and Figure 3 provides a visual comparison of the final reconstructed images. The results are discussed after the tables and figures.

Table 1. SNR (dB) of the four schemes versus the iteration count m on the motion-deblurring task.

m	Z-iteration	Single-inertial Z	Alternative double-inertial	Algorithm 3.1
1	20.5568	20.5568	19.6923	20.0430
5	22.7738	23.7066	23.3131	23.1788
10	24.1573	25.3497	25.1648	25.8529
20	25.6684	26.9834	26.8937	28.4607
30	26.5887	27.9228	27.8647	30.1945
40	27.2462	28.5771	28.5343	31.4474
50	27.7540	29.0789	29.0450	32.3198
60	28.1661	29.4871	29.4589	32.9756
70	28.5124	29.8321	29.8080	33.5036
80	28.8108	30.1318	30.1106	33.9421
90	29.0733	30.3972	30.3783	34.3130
100	29.3076	30.6358	30.6187	34.6334

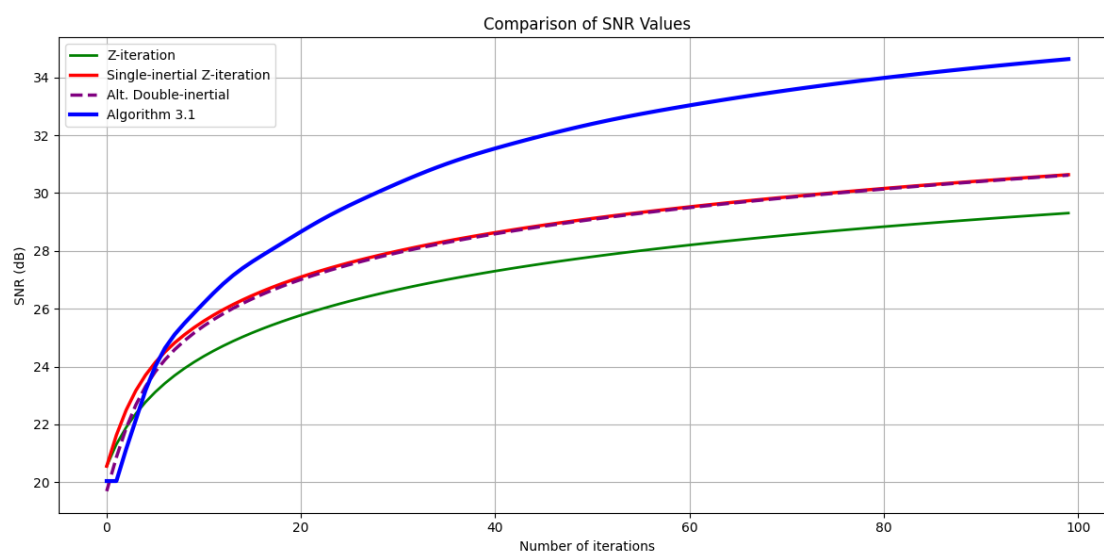


Figure 1. SNR (dB) versus iteration count for the four schemes on the motion-deblurring task.

Table 2. PSNR (dB) of the four schemes versus the iteration count m on the motion-deblurring task.

m	Z-iteration	Single-inertial Z	Alternative double-inertial	Algorithm 3.1
1	24.5111	24.5111	23.6466	23.9972
5	26.7280	27.6608	27.2673	27.1330
10	28.1115	29.3039	29.1191	29.8072
20	29.6226	30.9377	30.8480	32.4149
30	30.5430	31.8771	31.8189	34.1487
40	31.2004	32.5313	32.4885	35.4017
50	31.7083	33.0332	32.9993	36.2741
60	32.1204	33.4413	33.4132	36.9298
70	32.4666	33.7864	33.7622	37.4579
80	32.7651	34.0860	34.0648	37.8964
90	33.0275	34.3514	34.3325	38.2673
100	33.2619	34.5900	34.5729	38.5876

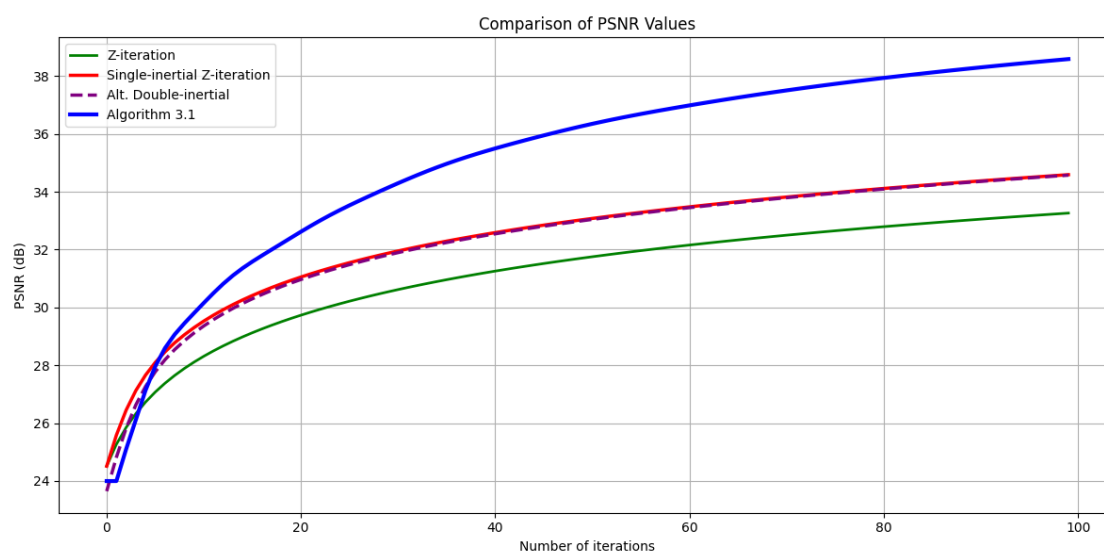


Figure 2. PSNR (dB) versus iteration count for the four schemes on the motion-deblurring task.



Figure 3. Visual comparison of the deblurred images on the test photograph at iteration 100. From left to right: Noisy and blurred input, the Z-iteration, the single-inertial Z-iteration, the alternative double-inertial scheme, the proposed double-inertial Z-iteration (3.1), and the original ground-truth image.

Table 3. Final SNR and PSNR (dB) at iteration 100 for the four schemes on the motion-deblurring task. Best values are in bold.

Method	SNR (dB)	PSNR (dB)
Z-iteration	29.3076	33.2619
Single-inertial Z	30.6358	34.5900
Alternative double-inertial	30.6187	34.5729
Algorithm 3.1	34.6334	38.5876

The proposed double-inertial Z-iteration (3.1) attains the highest SNR and PSNR after the fifth iteration. Its advantage becomes pronounced by iteration 20 and grows steadily thereafter: At iteration 100, it reaches SNR = 34.6334 dB and PSNR = 38.5876 dB, compared with 30.6358 dB and 34.5900 dB for the single-inertial scheme (3.21), 30.6187 dB and 34.5729 dB for the alternative double-inertial scheme (3.22), and only 29.3076 dB and 33.2619 dB for the plain Z-iteration.

Two observations stand out. First, the gap between the proposed scheme and the single-inertial Z-iteration (3.21) widens with the iteration count, reaching about 4 dB in both SNR and PSNR by iteration 100; the additional inertial term $t_m(x_{m-1} - x_{m-2})$, absent from (3.21), provides a genuine second-order memory effect that accelerates the reconstruction. Second, the alternative double-inertial

scheme (3.22) tracks the single-inertial scheme almost exactly, since both rely on the single backward difference $x_m - x_{m-1}$ and thus carry only one effective inertial direction. The two-step memory of (3.1), which recalls the two linearly independent differences $x_m - x_{m-1}$ and $x_{m-1} - x_{m-2}$, yields a qualitatively richer extrapolation, also visible in the sharper reconstructions of Figure 3. Since the analysis carries over verbatim to any generalized (α, β) -nonexpansive operator (Definition 2.2), the scheme applies unchanged to sparse recovery, monotone inclusions, and variational inequalities whose solution operators belong to this class.

The fixed-budget protocol above describes the quality reached after a prescribed amount of work; the converse measurement, in the spirit of the comparative study of Inkrong et al. [13], fixes the quality and reports the work needed to attain it. We now adopt this latter protocol: Each scheme is run twice, once until the SNR of the iterate first reaches the target value of 30 dB and once until the PSNR of the iterate first reaches the same target value of 30 dB, subject in each case to an upper bound of 2000 iterations, and we record the number of iterations and the CPU time required to reach the corresponding target. Figures 4 and 5 display the resulting SNR and PSNR convergence curves under this target-based stopping rule, and Table 4 collects the numerical comparison in the format adopted by [13].

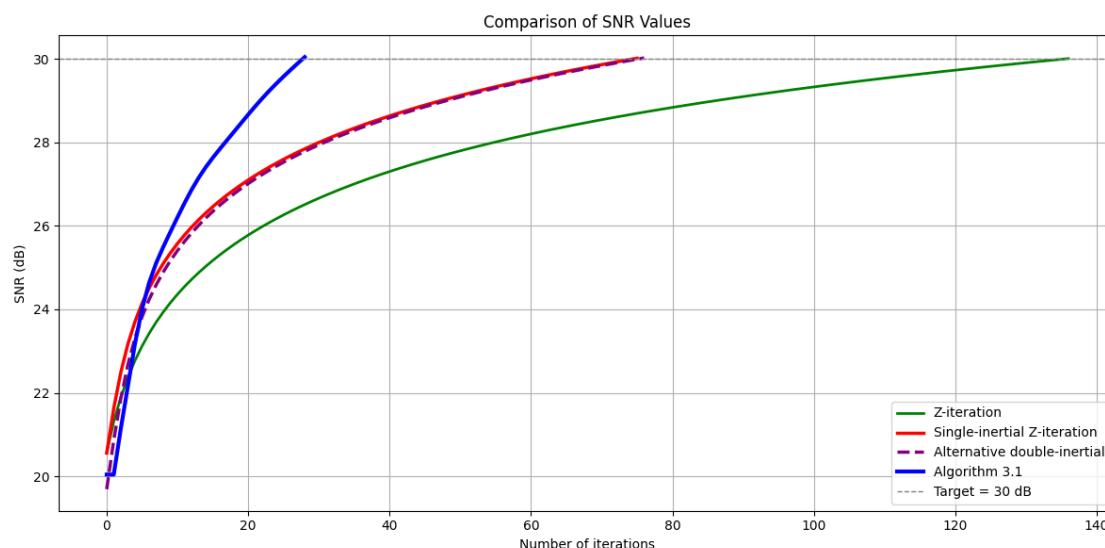


Figure 4. SNR (dB) versus iteration count for the four schemes, each run until the SNR first reaches the target value of 30 dB on the motion-deblurring task.

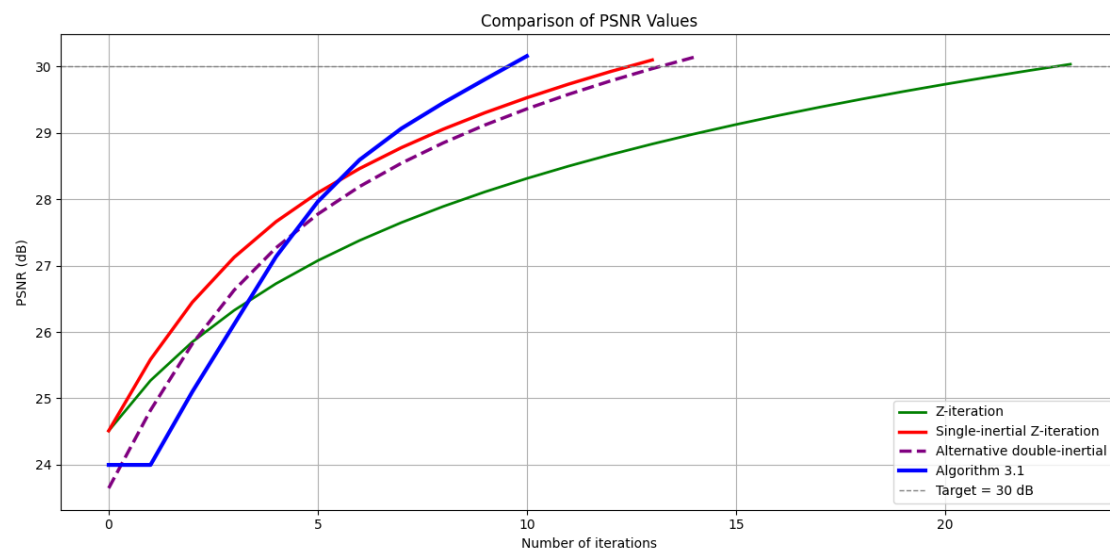


Figure 5. PSNR (dB) versus iteration count for the four schemes, each run until the target value of 30 dB is first reached on the motion-deblurring task.

Table 4. Numerical comparison of each scheme by the number of iterations and the CPU time required to cross the two reconstruction-quality milestones, SNR = 30 dB and PSNR = 30 dB, on the motion-deblurring task.

Methods	SNR = 30		PSNR = 30	
	No. Iter	CPU time (s)	No. Iter	CPU time (s)
Z-iteration	137	96.4132	24	21.3627
Single-inertial Z	76	50.1140	14	9.4403
Alternative double-inertial	77	50.9123	15	9.4306
Algorithm 3.1	29	17.3483	11	5.2931

Under the target-based protocol, the stopping value attained by every scheme is close to the corresponding threshold by construction, so the schemes are compared primarily on the computational effort required to reach it. As reported in Table 4, for the SNR = 30 dB milestone, the proposed double-inertial Z-iteration (3.1) needs only 29 iterations and 17.3483 seconds, against 76 iterations and 50.1140 seconds for the single-inertial scheme (3.21), 77 iterations and 50.9123 seconds for the alternative double-inertial scheme (3.22), and 137 iterations and 96.4132 seconds for the classical Z-iteration (1.1). Relative to the classical Z-iteration, the proposed scheme is thus about 4.7 times faster in iteration count and roughly 5.6 times faster in CPU time, and relative to either single-inertial scheme, it is about 2.6–2.7 times faster in iteration count and roughly 2.9 times faster in CPU time. For the PSNR = 30 dB milestone, the same ordering holds but with a narrower margin: The proposed scheme needs only 11 iterations and 5.2931 seconds, against 14 iterations and 9.4403 seconds for the single-inertial scheme, 15 iterations and 9.4306 seconds for the alternative double-inertial scheme, and 24 iterations and 21.3627 seconds for the classical Z-iteration. Relative to the classical Z-iteration, the proposed scheme is about twice as fast in iteration count and roughly 4 times faster in CPU time, while relative to either single-inertial scheme it is about 1.3 times faster in iteration count

and roughly 1.8 times faster in CPU time. Figures 4 and 5 confirm that the curve of the proposed scheme crosses the target threshold before those of the competing methods at both milestones. The alternative double-inertial scheme continues to track the single-inertial scheme closely. This behaviour reflects, in terms of computational cost, the same single-effective-direction phenomenon already noted under the fixed-budget protocol. Since the per-iteration cost is dominated by the common proximal-gradient evaluation, the CPU time ordering across schemes closely mirrors the iteration-count ordering at both milestones.

The two protocols yield a consistent picture. At a common iteration budget, the proposed scheme delivers the highest SNR and PSNR (Tables 1–3), and at a common target quality, it requires the fewest iterations and the lowest CPU time (Table 4); in the iteration count, the gap to the single-inertial schemes ranges from about 1.3 times (PSNR milestone) to roughly 2.7 times (SNR milestone), and the gap to the classical Z-iteration ranges from about 2 times (PSNR milestone) to nearly 4.7 times (SNR milestone), with the corresponding CPU time gaps somewhat larger at both milestones. The genuine two-step memory of Algorithm 3.1, which recalls the two linearly independent backward differences $x_m - x_{m-1}$ and $x_{m-1} - x_{m-2}$, is thus the source of acceleration in both directions.

6. Conclusions

We introduced a double-inertial Z-iteration for generalized (α, β) -nonexpansive maps in real BSs, augmenting the three-layer Z-iteration with a two-term inertial extrapolation that recalls three consecutive iterates. Under mild control conditions on the parameters, we proved that the generated sequence is bounded with $\lim_{m \rightarrow \infty} \|x_m - q\|$ existing for every FP q , that it converges weakly to an FP in UCSs enjoying Opial's property, with the Hilbert space case following as a direct corollary, and that it converges strongly whenever the operator is compact. We also established a non-asymptotic bound $O(1/M)$. We illustrated these outcomes with two applications: A theorem for the solution of a nonlinear Hammerstein-type integral equation, confirmed on two examples, and a deblurring method built on the proximal-gradient operator of the ℓ_1 -regularized least-squares model, which outperformed the classical Z-iteration, a single-inertial algorithm, and an alternative double-inertial scheme in both reconstruction quality and computational cost. Together, these findings indicate that the genuine two-step memory of Algorithm 3.1 accounts for the improved performance observed numerically, and that the scheme is a flexible tool for both analytical and computational FP problems.

This research opens the way to several possible extensions. The inertial parameters (θ_m, t_m) could be tuned automatically, rather than fixed in advance, following the adaptive restart strategies developed for accelerated gradient methods [30]. Also, the same framework applies to split feasibility problems [31, 32]. Finally, following the monotone-inclusion and variational-inequality treatments already available for related nonexpansive-type maps [10, 12], it is expected to extend similarly to such problems when the underlying operator is generalized (α, β) -nonexpansive.

Author contributions

Muhammed Mustafa Karakuş: Conceptualization, methodology, software, formal analysis, writing—original draft; Aynur Şahin: Conceptualization, methodology, supervision, writing—review & editing; Mustafa Eröz: Visualization, formal analysis, writing—review & editing; Metin Başarır: Project

administration, validation, writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

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