



Research article**Sharp joint Carathéodory compatibility and fifth-order lifting obstructions in bi-univalent function theory****Zainab H. Mahmood^{1,*}, Reem O. Rasheed², Bassim K. Mihsin³ and Waggas Galib Atshan⁴**¹ Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq² Department of Mathematics, College of Education Tuzkhurmatu, Tikrit University, Tikrit, Iraq³ General Directorate of Education in Karbala, Karbala, Iraq⁴ Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah, Iraq*** Correspondence:** Email: zainab.mahmood@sc.uobaghdad.edu.iq.

Abstract: Let $\Psi(z) = 1 + L_1z + L_2z^2 + L_3z^3 + L_4z^4 + \cdots$ be analytic in $\mathbb{D}$ with $L_1, \dots, L_4 \in \mathbb{R}$ and $L_1 > 0$, and let $\mathcal{M}$ consist of the bi-univalent functions f for which a prescribed linear operator with coefficients $\Phi_2, \dots, \Phi_5 > 0$, applied to f and to f^{-1} , yields two functions subordinate to Ψ . Put $\kappa = L_1\Phi_3/\Phi_2^2 - L_2/L_1$. Coupling the two subordinations through the second-order Carathéodory parametrization gives $(4-t^2)(x+y) = 2\kappa t^2$, and hence $|c_1|^2 \leq 4/(1+|\kappa|)$ on the second-order compatible coefficient body V_2 . This envelope is sharp on V_2 and it is not asserted to be sharp for $\mathcal{M}$. Its equality set is a single rotation orbit of an explicit finite Blaschke pair; for $\kappa \geq 0$, the envelope coincides with the bound produced by the usual summation argument; for $\kappa < 0$, it improves it. We then ask whether this equality orbit can be the low-order truncation of a member of $\mathcal{M}$. The inverse-compatibility residuals R_1, R_2 , and R_3 vanish identically on it, and the residual R_4 , attached to the fifth coefficient, is $R_4 = -2P_\sigma[\Phi_4(L_1\Phi_2^2 + \sigma(L_1^2\Phi_3 - L_2\Phi_2^2))^2]^{-1}$, $\sigma = \text{sgn } \kappa$, where P_σ is an explicit polynomial that is affine in L_4 with positive leading coefficient. Thus $P_\sigma \neq 0$ means precisely that the equality orbit violates fifth-order inverse compatibility; it gives no information about other points of V_2 . Combining this with the rigidity of the equality orbit and normality of the Schwarz class, we prove that $P_\sigma \neq 0$ implies the strict separation $\sup_{f \in \mathcal{M}} |c_1|^2 < 4/(1+|\kappa|)$. The gap is not quantified, and the exact value of the supremum is left open. For the Janowski generator $(1 + Az)/(1 + Bz)$, one has $\kappa = A$ and $P_\sigma = A(A-B)^3(A+\sigma)(2A+\sigma-B)$, which is nonzero for $A \neq 0$; for $A = 0$, the corresponding polynomial vanishes and no separation follows. For $\Psi(z) = e^z$, one has $P_{+1} = 23/12$, whence $\sup_{f \in \mathcal{M}} |a_2| < \sqrt{2/3}$.

Keywords: bi-univalent functions; subordination; Schwarz functions; Carathéodory functions; inverse coefficients; coefficient body; joint compatibility; lifting obstruction

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1. Introduction

Let $\mathcal{A}$ be the class of $f(z) = z + \sum_{n \geq 2} a_n z^n$ that is analytic in $\mathbb{D} = \{|z| < 1\}$, and let Σ be the subclass of bi-univalent functions, that is, those f univalent in $\mathbb{D}$ whose inverse extends univalently to $\mathbb{D}$ [1–3]. Coefficient problems for Σ and its subclasses go back to [1, 3, 4]. For subclasses defined by a pair of subordinations, one imposed on f and one on $g = f^{-1}$ [5, 6], including Ma–Minda-type classes [7], estimates for $|a_2|$ and $|a_3|$ were obtained in [6–9]; Fekete–Szegő functionals were treated in [10], and Hankel determinants in [11–14]. The inverse coefficients b_n are given by the classical reversion (Lagrange inversion) formulas, and the Faber-polynomial method, which is used for bi-univalent classes in [15], expresses them to all orders; see [16] for a correction of the main estimate of [15]. For inverse coefficients over classical univalent classes, sharp bounds go back to Löwner [17], with later work for starlike and convex functions in [18, 19]. The standard route to $|a_2|$ in bi-univalent classes combines the two first- and second-order coefficient relations and then applies the Carathéodory inequality $|c_n| \leq 2$ [20] to each Carathéodory function separately.

Two questions should be kept apart. The first is the determination of the low-order coefficient body cut out by the coupled system; this is a finite-dimensional problem. The second is whether a point of that body is the low-order truncation of an actual member of the class. A point of the body satisfies only the finitely many relations that define it, and higher-order inverse relations may be inconsistent with it. We call the passage from the first situation to the second *lifting*, and an inconsistency at order n a *lifting obstruction at order n* .

The paper is organized around the following chain:

$$\begin{aligned} \text{coupled second-order compatibility} &\longrightarrow \text{unique extremal orbit} \longrightarrow \text{higher-order lifting} \\ &\longrightarrow \text{fifth-order obstruction} \longrightarrow \text{strict full-class separation.} \end{aligned}$$

Coupled compatibility. The coupling of the two subordinations at second order is governed by the scalar

$$\kappa = \frac{L_1 \Phi_3}{\Phi_2^2} - \frac{L_2}{L_1}, \quad (1.1)$$

through the identity $(4 - t^2)(x + y) = 2\kappa t^2$ between the Carathéodory parameters of f and f^{-1} (Lemma 3.4); it yields the envelope $|c_1|^2 \leq 4/(1 + |\kappa|)$ on the second-order compatible body V_2 (Theorem 3.5).

Unique extremal orbit. Equality holds on V_2 exactly on the rotation orbit of an explicit finite Blaschke pair (u_0, v_0) (Lemma 3.8), and near-equality forces the second-order parameters close to those of that orbit (Theorem 3.10).

Higher-order lifting. On that orbit, the inverse-compatibility residuals R_1, R_2 , and R_3 vanish identically (Lemma 4.3).

Fifth-order obstruction. The residual R_4 , attached to the fifth coefficient, is a positive multiple of $-P_\sigma$ (Theorem 4.4); for fixed lower-order data, the condition $P_\sigma = 0$ holds for exactly one value of the fourth Taylor coefficient of Ψ (Corollary 4.5, Remark 4.6).

Strict separation. If $P_\sigma \neq 0$, the rigidity and normality of the Schwarz class give $\sup_{f \in \mathcal{M}} |c_1|^2 < 4/(1 + |\kappa|)$ (Theorem 5.1).

The structural point is that a finite-order equality configuration may solve the truncated coefficient problem and yet correspond to no member of the full class, because higher inverse compatibility

imposes conditions that the truncated problem does not see. The contribution therefore has three distinct levels, as discussed in Section 7: (A) The coefficient-body bound, which for $\kappa \geq 0$ coincides with the bound obtained by applying to $\mathcal{M}$ the summation argument used in [6–9], and improves it for $\kappa < 0$ (Remark 3.7); (B) the identification and rigidity of the equality orbit; and (C) the lifting analysis of that orbit and the resulting strict separation. Sharpness is asserted only on V_2 . Theorem 5.1 is a strict inequality for $\mathcal{M}$, not a sharp estimate, and the exact value of $\sup_{f \in \mathcal{M}} |a_2|$ is not determined here.

The word *stability* is used in several unrelated senses. In the theory of differential, integral, and stochastic equations, it usually refers to the Ulam–Hyers or Ulam–Hyers–Rassias stability of the solutions under perturbation of the equation, as in recent work on impulsive neutral mixed integral dynamic equations on time scales [21], on coupled fractional stochastic integro-differential systems with integral boundary conditions [22], and on Hilfer fractional stochastic impulsive equations driven by time-changed Brownian motion [23]; related qualitative questions, such as oscillation of fourth-order functional differential equations with noncanonical operators, are studied in [24]. Theorem 3.10 concerns a different notion: It bounds the distance of the second-order Carathéodory parameters of a point of V_2 from those of the equality orbit in terms of the deficit in the coefficient bound. No connection between the two notions is claimed.

2. Analytic framework and coefficient identities

For $0 < q < p \leq 1$, put $[n]_{p,q} = (p^n - q^n)/(p - q)$ [25, 26], and for $\lambda, \delta \geq 0$ define

$$\mathcal{D}_{p,q}^{\lambda,\delta} f(z) = z + \sum_{n \geq 2} \Upsilon_n a_n z^n, \quad \Upsilon_n = \left(\frac{[n]_{p,q} + \delta}{1 + \delta} \right)^\lambda. \quad (2.1)$$

For $\gamma \in [0, 1]$, set $\mathcal{D}_\gamma f = (1 - \gamma)\mathcal{D}_{p,q}^{\lambda,\delta} f(z)/z + \gamma(\mathcal{D}_{p,q}^{\lambda,\delta} f)'(z)$. By (2.1)

$$\mathcal{D}_\gamma f(z) = 1 + \sum_{n \geq 2} \Upsilon_n ((1 - \gamma) + n\gamma) a_n z^{n-1} = 1 + \sum_{n \geq 2} \Phi_n a_n z^{n-1}, \quad \Phi_n = \Upsilon_n (1 + (n - 1)\gamma). \quad (2.2)$$

For $p = 1, q \rightarrow 1, \delta = 0$ the operator reduces to the Sălăgean operator [27].

Let $\Psi(z) = 1 + L_1 z + L_2 z^2 + L_3 z^3 + L_4 z^4 + O(z^5)$ be analytic in $\mathbb{D}$. No univalence of Ψ is assumed.

Standing hypotheses

$$L_1, L_2, L_3, L_4 \in \mathbb{R}, \quad L_1 > 0, \quad \Phi_2, \Phi_3, \Phi_4, \Phi_5 > 0. \quad (2.3)$$

The role of each hypothesis is described after Definition 2.1.

Definition 2.1. Let $f \in \Sigma$ and $g = f^{-1}$. Then $f \in \mathcal{M}$ if there are Schwarz functions u, v , that is, analytic functions on $\mathbb{D}$ with $u(0) = v(0) = 0$ and $|u|, |v| < 1$, such that

$$\mathcal{D}_\gamma f(z) = \Psi(u(z)), \quad \mathcal{D}_\gamma g(w) = \Psi(v(w)). \quad (2.4)$$

The class $\mathcal{M}$ is nonempty: $f(z) = z$ belongs to Σ , and $\mathcal{D}_\gamma f \equiv 1 = \Psi(0)$, so (2.4) holds with $u = v \equiv 0$. Hence, all suprema over $\mathcal{M}$ below are suprema of nonempty sets.

Hypotheses on Ψ . No univalence, convexity, starlikeness, positivity of the real part, or other Ma–Minda geometric hypothesis on Ψ is required for the general results of Sections 3–5. The class $\mathcal{M}$ is

defined directly through the existence of Schwarz functions satisfying (2.4); accordingly, in Section 6, the notation $F < \Psi$ means that $F = \Psi \circ \omega$ for some $\omega \in \mathcal{B}$. The proofs use exactly the following: (i) Ψ is analytic in $\mathbb{D}$ with $\Psi(0) = 1$, so that $\Psi \circ u$ is defined for $u \in \mathcal{B}$ and the constant terms in (2.4) match; (ii) $L_1 > 0$, used to divide by L_1 (Lemmas 2.4 and 3.4, the definition of κ) and for the positivity of the denominators in Theorem 4.4 and Corollary 4.5; (iii) $L_2 \in \mathbb{R}$, which makes κ real, so that $|\kappa|$ and $\operatorname{sgn} \kappa$ are defined; (iv) $L_3, L_4 \in \mathbb{R}$, used only where a real value of $L_{4,\text{lift}}^{(\sigma)}$ or a sign comparison is needed (Corollaries 4.5 and 4.9, Section 6.3); the equivalence $R_4 = 0 \iff P_\sigma = 0$ of Theorem 4.4 does not use it; (v) $\Phi_2, \dots, \Phi_5 > 0$, used for division and for the sign of the leading coefficient of P_σ in L_4 . When Ψ satisfies the Ma–Minda hypotheses [28], the class of Section 6 with $\Phi_n = 1$ is a bi-univalent class of Ma–Minda type, but this is not used in any proof.

Write $\mathcal{B}$ for the Schwarz class, $\mathcal{P}$ for the Carathéodory class, and

$$h = \frac{1+u}{1-u} = 1 + c_1z + c_2z^2 + \cdots \in \mathcal{P}, \quad k = \frac{1+v}{1-v} = 1 + d_1z + d_2z^2 + \cdots \in \mathcal{P}. \quad (2.5)$$

Expanding $\Psi \circ u$ and $\Psi \circ v$ as

$$\Psi(u(z)) = 1 + \sum_{j \geq 1} A_j z^j, \quad \Psi(v(w)) = 1 + \sum_{j \geq 1} B_j w^j, \quad (2.6)$$

and comparison of (2.4) with (2.2) gives the matching conditions

$$\Phi_{j+1}a_{j+1} = A_j, \quad \Phi_{j+1}b_{j+1} = B_j, \quad j \geq 1. \quad (2.7)$$

Lemma 2.2. *If $h \in \mathcal{P}$ and $u = (h-1)/(h+1) = \sum_{j \geq 1} u_j z^j$, then*

$$u_1 = \frac{c_1}{2}, \quad u_2 = \frac{c_2}{2} - \frac{c_1^2}{4}. \quad (2.8)$$

Proof. From $(h+1)u = h-1$, $(2+c_1z+c_2z^2+\cdots)(u_1z+u_2z^2+\cdots) = c_1z+c_2z^2+\cdots$. The coefficient of z gives $2u_1 = c_1$; the coefficient of z^2 gives $2u_2 + c_1u_1 = c_2$, so $2u_2 = c_2 - c_1^2/2$. $\square$

Lemma 2.3. *For $u = \sum_{j \geq 1} u_j z^j \in \mathcal{B}$ the coefficients in (2.6) are*

$$A_1 = L_1 u_1, \quad (2.9)$$

$$A_2 = L_1 u_2 + L_2 u_1^2, \quad (2.10)$$

$$A_3 = L_1 u_3 + 2L_2 u_1 u_2 + L_3 u_1^3, \quad (2.11)$$

$$A_4 = L_1 u_4 + L_2(2u_1 u_3 + u_2^2) + 3L_3 u_1^2 u_2 + L_4 u_1^4. \quad (2.12)$$

Proof. From $u = u_1z + u_2z^2 + u_3z^3 + u_4z^4 + O(z^5)$, we have

$$u^2 = u_1^2 z^2 + 2u_1 u_2 z^3 + (2u_1 u_3 + u_2^2) z^4 + O(z^5), \quad u^3 = u_1^3 z^3 + 3u_1^2 u_2 z^4 + O(z^5), \quad u^4 = u_1^4 z^4 + O(z^5),$$

and $u^5 = O(z^5)$. Substitute into $\Psi(u) = 1 + L_1 u + L_2 u^2 + L_3 u^3 + L_4 u^4 + O(u^5)$ and collect z, z^2, z^3, z^4 . $\square$

Lemma 2.4. *Let $f \in \mathcal{M}$. Then*

$$\Phi_2 a_2 = \frac{L_1}{2} c_1, \quad -\Phi_2 a_2 = \frac{L_1}{2} d_1, \quad (2.13)$$

$$\Phi_3 a_3 = \frac{L_1}{2} c_2 - \frac{L_1 - L_2}{4} c_1^2, \quad \Phi_3 (2a_2^2 - a_3) = \frac{L_1}{2} d_2 - \frac{L_1 - L_2}{4} d_1^2, \quad (2.14)$$

and consequently

$$d_1 = -c_1. \quad (2.15)$$

Moreover, the inverse coefficients satisfy

$$b_2 = -a_2, \quad b_3 = 2a_2^2 - a_3, \quad b_4 = -5a_2^3 + 5a_2 a_3 - a_4, \quad b_5 = 14a_2^4 - 21a_2^2 a_3 + 6a_2 a_4 + 3a_3^2 - a_5. \quad (2.16)$$

Proof. The identities (2.16) follow from coefficient comparison in $f(g(w)) = w$; the reversion through degree five is carried out in Appendix A.

By Lemma 2.2, $u_1 = c_1/2$ and $v_1 = d_1/2$, so (2.9) gives $A_1 = L_1 c_1/2$ and $B_1 = L_1 d_1/2$. With (2.7) for $j = 1$ and $b_2 = -a_2$, this is (2.13); adding the two relations in (2.13) gives $0 = \frac{L_1}{2}(c_1 + d_1)$, and $L_1 > 0$ gives (2.15). By Lemma 2.2 and (2.10),

$$A_2 = L_1 \left(\frac{c_2}{2} - \frac{c_1^2}{4} \right) + L_2 \frac{c_1^2}{4} = \frac{L_1}{2} c_2 - \frac{L_1 - L_2}{4} c_1^2,$$

and the same computation for k gives $B_2 = \frac{L_1}{2} d_2 - \frac{L_1 - L_2}{4} d_1^2$. Now (2.7) for $j = 2$, together with $b_3 = 2a_2^2 - a_3$, gives (2.14). $\square$

Lemma 2.5. For $|\rho| = 1$ let $f_\rho(z) = \bar{\rho} f(\rho z)$. If $f \in \mathcal{M}$ with the Schwarz pair (u, v) , then $f_\rho \in \mathcal{M}$ with Schwarz pair $(u(\rho \cdot), v(\rho \cdot))$ and

$$a_n^{(\rho)} = \rho^{n-1} a_n, \quad b_n^{(\rho)} = \rho^{n-1} b_n, \quad c_j^{(\rho)} = \rho^j c_j, \quad d_j^{(\rho)} = \rho^j d_j. \quad (2.17)$$

Proof. $f_\rho(z) = \bar{\rho}(\rho z + \sum_{n \geq 2} a_n \rho^n z^n) = z + \sum_{n \geq 2} \rho^{n-1} a_n z^n$, and f_ρ is univalent on $\mathbb{D}$ because f is. With $g = f^{-1}$, $f_\rho(\bar{\rho} g(\rho w)) = \bar{\rho} f(g(\rho w)) = w$, so $g_\rho(w) = \bar{\rho} g(\rho w)$ is the inverse of f_ρ ; it is univalent on $\mathbb{D}$; hence, $f_\rho \in \Sigma$, and $b_n^{(\rho)} = \rho^{n-1} b_n$. By (2.2), we have

$$\mathcal{D}_\gamma f_\rho(z) = 1 + \sum_{n \geq 2} \Phi_n a_n (\rho z)^{n-1} = \mathcal{D}_\gamma f(\rho z) = \Psi(u(\rho z)).$$

In the same way, $\mathcal{D}_\gamma g_\rho(w) = \mathcal{D}_\gamma g(\rho w) = \Psi(v(\rho w))$. The functions $z \mapsto u(\rho z)$, $w \mapsto v(\rho w)$ belong to $\mathcal{B}$. Finally, $(1 + u(\rho z))/(1 - u(\rho z)) = h(\rho z) = 1 + \sum_j c_j \rho^j z^j$, and similarly for k . $\square$

Notation. Throughout, L_j are the Taylor coefficients of Ψ ; Φ_n represents the operator coefficients in (2.2); a_n and b_n are the coefficients of f and $g = f^{-1}$; c_j, d_j are those of h, k in (2.5); u_j, v_j are those of u, v ; A_j, B_j are those of $\Psi \circ u, \Psi \circ v$ in (2.6); and R_j represents the residuals of Definition 4.1. The scalars κ , $K = |\kappa|$, $\sigma = \operatorname{sgn} \kappa$, and $a = (1 + K)^{-1/2}$ are fixed by (1.1) and (3.12); the unindexed a is this number and is distinct from the coefficients a_n . The scalar $M = L_1^2 \Phi_3 - L_2 \Phi_2^2$ of (4.12) is distinct from the class $\mathcal{M}$. The parameters x, y are those of (3.7), and $t = c_1 \geq 0$ after the normalization (3.2).

3. Sharp joint Carathéodory compatibility on the second-order body

Definition 3.1. The *second-order compatible coefficient body* V_2 is the set of $(c_1, c_2, d_1, d_2) \in \mathbb{C}^4$ for which there exist $h, k \in \mathcal{P}$ with these initial coefficients and $a_2, a_3 \in \mathbb{C}$ satisfying (2.13) and (2.14).

By Lemma 2.4, every $f \in \mathcal{M}$ determines a point of V_2 . The converse is not asserted: A point of V_2 need not arise from a member of $\mathcal{M}$, and Section 4 examines this question for the equality points found below.

Proposition 3.2. $(c_1, c_2, d_1, d_2) \in V_2$ if and only if there are $h, k \in \mathcal{P}$ with these initial coefficients such that

$$d_1 = -c_1 \quad \text{and} \quad \frac{L_1}{2}(c_2 + d_2) - \frac{L_1 - L_2}{2}c_1^2 = \frac{L_1^2\Phi_3}{2\Phi_2^2}c_1^2. \quad (3.1)$$

Moreover, V_2 is invariant under $(c_1, c_2, d_1, d_2) \mapsto (\rho c_1, \rho^2 c_2, \rho d_1, \rho^2 d_2)$, $|\rho| = 1$.

Proof. Let $(c_1, c_2, d_1, d_2) \in V_2$. Adding the two relations in (2.13) gives $d_1 = -c_1$, and the first gives $a_2 = L_1 c_1 / (2\Phi_2)$. Adding the two relations in (2.14) and using $d_1^2 = c_1^2$, $2\Phi_3 a_2^2 = \frac{L_1}{2}(c_2 + d_2) - \frac{L_1 - L_2}{2}c_1^2$, and $a_2^2 = L_1^2 c_1^2 / (4\Phi_2^2)$ turns the left side into $L_1^2 \Phi_3 c_1^2 / (2\Phi_2^2)$. Conversely, given (3.1), set $a_2 = L_1 c_1 / (2\Phi_2)$ and define a_3 by the first relation in (2.14); subtracting that relation from the second equation of (3.1) returns the second relation in (2.14). For the invariance, $h(\rho \cdot), k(\rho \cdot) \in \mathcal{P}$ have coefficients $\rho^j c_j, \rho^j d_j$; the first condition in (3.1) becomes $\rho d_1 = -\rho c_1$, and multiplying the second by ρ^2 reproduces it for the rotated data. $\square$

By Proposition 3.2, we may normalize any point of V_2 so that

$$c_1 = t \in [0, 2], \quad d_1 = -t. \quad (3.2)$$

Lemma 3.3. Let $h \in \mathcal{P}$. Then

$$|2c_2 - c_1^2| \leq 4 - |c_1|^2, \quad (3.3)$$

and for $c_1 = t \in [0, 2)$,

$$2c_2 = t^2 + (4 - t^2)x, \quad x = \frac{2c_2 - t^2}{4 - t^2}, \quad |x| \leq 1. \quad (3.4)$$

Equality holds in (3.3) with $|c_1| < 2$ if and only if

$$u = \frac{h-1}{h+1} \text{ satisfies } u(z) = z \frac{\omega_0 + \eta z}{1 + \overline{\omega_0} \eta z}, \quad \omega_0 = \frac{c_1}{2}, \quad |\eta| = 1. \quad (3.5)$$

Proof. Put $w(z) = u(z)/z$, so that $w(0) = u_1$ and $w'(0) = u_2$. By the Schwarz lemma, $|u(z)| \leq |z|$, and hence w extends analytically to $\mathbb{D}$ with $|w| \leq 1$. If $|w| < 1$ on $\mathbb{D}$, the Schwarz–Pick inequality [29, Ch. 2] gives

$$|w'(0)| \leq 1 - |w(0)|^2; \quad (3.6)$$

if $|w| = 1$ at an interior point, then w is a unimodular constant and both sides of (3.6) vanish. By Lemma 2.2, (3.6) reads $|c_2/2 - c_1^2/4| \leq 1 - |c_1|^2/4$, which is (3.3) after multiplication by 4; for $c_1 = t < 2$, this is (3.4). The parametrization (3.4) is the second-order Carathéodory body used in [19].

If equality holds in (3.3) with $|c_1| < 2$, then $|w(0)| < 1$ and $|w'(0)| = 1 - |w(0)|^2$, so w is a conformal automorphism of $\mathbb{D}$: $w(z) = \eta(z + \beta)/(1 + \bar{\beta}z)$ with $|\eta| = 1$, $|\beta| < 1$. From $w(0) = \eta\beta = \omega_0$, we get $\beta = \bar{\eta}\omega_0$, $\bar{\beta} = \eta\bar{\omega}_0$, so $w(z) = \eta \frac{z + \bar{\eta}\omega_0}{1 + \eta\bar{\omega}_0z} = \frac{\eta z + \omega_0}{1 + \bar{\omega}_0\eta z}$, which is (3.5). Conversely, for w of this form $w'(0) = \eta(1 - |\omega_0|^2)$, so equality holds. $\square$

Lemma 3.4. *Let $(c_1, c_2, d_1, d_2) \in V_2$ be normalized by (3.2) with $t < 2$, and put*

$$2c_2 = t^2 + (4 - t^2)x, \quad 2d_2 = t^2 + (4 - t^2)y, \quad |x| \leq 1, |y| \leq 1. \quad (3.7)$$

Then

$$(4 - t^2)(x + y) = 2\kappa t^2. \quad (3.8)$$

If $t = 2$, then $\kappa = 0$.

Proof. The parametrization of d_2 follows from Lemma 3.3 applied to $k(-\cdot) \in \mathcal{P}$, whose first two coefficients are $-d_1 = t$ and d_2 . Add the two relations in (2.14). Since $c_1^2 = d_1^2 = t^2$, we have

$$2\Phi_3 a_2^2 = \frac{L_1}{2}(c_2 + d_2) - \frac{L_1 - L_2}{2}t^2. \quad (3.9)$$

By (2.13), $a_2 = L_1 t / (2\Phi_2)$, so

$$2\Phi_3 a_2^2 = \frac{L_1^2 \Phi_3}{2\Phi_2^2} t^2. \quad (3.10)$$

By adding the two relations in (3.7), $c_2 + d_2 = t^2 + \frac{1}{2}(4 - t^2)(x + y)$. Insert this and (3.10) into (3.9) as follows:

$$\frac{L_1^2 \Phi_3}{2\Phi_2^2} t^2 = \frac{L_1}{2} t^2 + \frac{L_1}{4} (4 - t^2)(x + y) - \frac{L_1}{2} t^2 + \frac{L_2}{2} t^2 = \frac{L_1}{4} (4 - t^2)(x + y) + \frac{L_2}{2} t^2.$$

Hence, $\frac{L_1}{4} (4 - t^2)(x + y) = \frac{t^2}{2} [\frac{L_1^2 \Phi_3}{\Phi_2^2} - L_2]$, and multiplication by $4/L_1 > 0$ gives (3.8).

If $t = 2$, then $|c_1| = 2$ forces $h(z) = (1 + z)/(1 - z)$ and $c_2 = 2$, and $d_1 = -2$ forces $k(z) = (1 - z)/(1 + z)$ and $d_2 = 2$. Then (3.9) gives $2\Phi_3 a_2^2 = 2L_1 - 2(L_1 - L_2) = 2L_2$, while (3.10) gives $2\Phi_3 a_2^2 = 2L_1^2 \Phi_3 / \Phi_2^2$; equating and dividing by $2L_1$ gives $\kappa = 0$. $\square$

Theorem 3.5 (Sharp joint Carathéodory compatibility on V_2). *For every $(c_1, c_2, d_1, d_2) \in V_2$, particularly for the data of every $f \in \mathcal{M}$, we have*

$$|c_1|^2 \leq \frac{4}{1 + |\kappa|}. \quad (3.11)$$

The bound is sharp on V_2 ; it is not asserted to be sharp for the class $\mathcal{M}$ (see Theorem 5.1). Explicitly, with

$$K = |\kappa|, \quad \sigma = \operatorname{sgn} \kappa, \quad a = \frac{1}{\sqrt{1 + K}}, \quad (3.12)$$

equality is achieved by the data of the Schwarz pair

$$u_0(z) = z \frac{a + \sigma z}{1 + a\sigma z}, \quad v_0(z) = z \frac{-a + \sigma z}{1 - a\sigma z} \quad (\kappa \neq 0), \quad (3.13)$$

and by

$$u_0(z) = z, \quad v_0(z) = -z \quad (\kappa = 0), \quad (3.14)$$

the latter being the limit of the former as $K \rightarrow 0$.

Proof. The bound. Normalize by (3.2). If $t < 2$, Lemma 3.4 gives $x + y = 2\kappa t^2/(4 - t^2)$, and $|x + y| \leq |x| + |y| \leq 2$ gives $\frac{2|\kappa|t^2}{4 - t^2} \leq 2$; that is, $|\kappa|t^2 \leq 4 - t^2$; in other words, $(1 + |\kappa|)t^2 \leq 4$. If $t = 2$, Lemma 3.4 gives $\kappa = 0$, so $4/(1 + |\kappa|) = 4 = t^2$. In both cases, $|c_1|^2 = t^2 \leq 4/(1 + |\kappa|)$.

The pair (3.13) is admissible, with $\kappa \neq 0$. Here, $K > 0$, so $0 < a < 1$. The denominators $1 \pm a\sigma z$ vanish only at $z = \mp\sigma/a$, of modulus $1/a > 1$, so u_0, v_0 are analytic on a neighborhood of $\overline{\mathbb{D}}$. For $|z| = 1$, using $a, \sigma \in \mathbb{R}$ and $\sigma^2 = 1$, we have

$$\begin{aligned} |a + \sigma z|^2 &= a^2 + 2a\sigma \operatorname{Re} z + 1 = |1 + a\sigma z|^2, \\ |-a + \sigma z|^2 &= a^2 - 2a\sigma \operatorname{Re} z + 1 = |1 - a\sigma z|^2. \end{aligned}$$

Hence, $|u_0(z)| = |v_0(z)| = 1$ on $|z| = 1$, and by the maximum principle $|u_0|, |v_0| < 1$ on $\mathbb{D}$ (neither function is constant). Since $u_0(0) = v_0(0) = 0$, $u_0, v_0 \in \mathcal{B}$.

Expansion. Since $|a\sigma z| < 1$ on $\mathbb{D}$, we have

$$(1 + a\sigma z)^{-1} = \sum_{n \geq 0} (-1)^n a^n \sigma^n z^n, \quad u_0 = (az + \sigma z^2) \sum_{n \geq 0} (-1)^n a^n \sigma^n z^n,$$

whence

$$\begin{aligned} [z^1]u_0 &= a, & [z^2]u_0 &= \sigma - a^2\sigma = \sigma(1 - a^2), \\ [z^3]u_0 &= -a\sigma^2 + a^3\sigma^2 = -a(1 - a^2), & [z^4]u_0 &= a^2\sigma^3 - a^4\sigma^3 = a^2\sigma(1 - a^2). \end{aligned}$$

With $a^2 = 1/(1 + K)$ and $1 - a^2 = K/(1 + K)$

$$u_1 = \frac{1}{\sqrt{1 + K}}, \quad u_2 = \frac{\sigma K}{1 + K}, \quad u_3 = -\frac{K}{(1 + K)^{3/2}}, \quad u_4 = \frac{\sigma K}{(1 + K)^2}. \quad (3.15)$$

Substituting $-z$ for z in (3.13)

$$u_0(-z) = (-z) \frac{a - \sigma z}{1 - a\sigma z} = z \frac{-a + \sigma z}{1 - a\sigma z} = v_0(z), \quad \text{so} \quad v_j = (-1)^j u_j. \quad (3.16)$$

Carathéodory data. From $h_0 = (1 + u_0)/(1 - u_0) = 1 + 2u_0 + 2u_0^2 + \cdots$, $c_1 = 2u_1$ and $c_2 = 2u_2 + 2u_1^2$, so by (3.15)

$$c_1 = \frac{2}{\sqrt{1 + K}}, \quad c_2 = \frac{2\sigma K}{1 + K} + \frac{2}{1 + K} = \frac{2(1 + \sigma K)}{1 + K},$$

and by (3.16), $d_1 = 2v_1 = -2u_1 = -c_1$ and $d_2 = 2v_2 + 2v_1^2 = 2u_2 + 2u_1^2 = c_2$.

Identification of x, y . Here, $t = c_1 = 2a$, so $t^2 = 4/(1 + K)$ and $4 - t^2 = 4K/(1 + K) > 0$. By (3.7),

$$x = \frac{2c_2 - t^2}{4 - t^2} = \frac{\frac{4(1 + \sigma K)}{1 + K} - \frac{4}{1 + K}}{\frac{4K}{1 + K}} = \frac{4\sigma K}{4K} = \sigma,$$

and $d_2 = c_2$, $d_1^2 = c_1^2$ give $y = \sigma$. Then $(4 - t^2)(x + y) = \frac{4K}{1+K} \cdot 2\sigma = 2K\frac{4}{1+K} = 2Kt^2$, so (3.8) holds. Reading the proof of Lemma 3.4 backwards, (3.8) with $d_1 = -c_1$ is equivalent to the second equation in (3.1); hence, $(c_1, c_2, d_1, d_2) \in V_2$ by Proposition 3.2, and $|c_1|^2 = 4/(1 + K)$.

The case $\kappa = 0$. Here, $u_0, v_0 \in \mathcal{B}$ trivially. From $u_1 = 1$, $u_2 = 0$: $c_1 = 2$, $c_2 = 2$; from $v_1 = -1$, $v_2 = 0$: $d_1 = -2$, $d_2 = 2$. Then $d_1 = -c_1$ and $\frac{L_1}{2}(c_2 + d_2) - \frac{L_1 - L_2}{2}c_1^2 = 2L_1 - 2(L_1 - L_2) = 2L_2$, while $\kappa = 0$ means $L_1^2\Phi_3/\Phi_2^2 = L_2$, so $\frac{L_1^2\Phi_3}{2\Phi_2^2}c_1^2 = 2L_2$. Thus (3.1) holds and $|c_1|^2 = 4$. Finally, $a \rightarrow 1$ as $K \rightarrow 0$ turns (3.13) into $z(1 + \sigma z)/(1 + \sigma z) = z$ and $z(-1 + \sigma z)/(1 - \sigma z) = -z$. $\square$

The bound (3.11) is sharp on V_2 . It is not asserted to be sharp for $\mathcal{M}$; Section 5 shows that under a fifth-order condition, it is not.

Corollary 3.6. *For every $f \in \mathcal{M}$,*

$$|a_2| \leq \frac{L_1}{\Phi_2 \sqrt{1 + |\kappa|}}. \quad (3.17)$$

On V_2 put $a_2 := L_1 c_1 / (2\Phi_2)$, as forced by (2.13). Then (3.17) holds on V_2 and is achieved there exactly on the equality set of (3.11).

Proof. By (2.13), $|a_2| = L_1 |c_1| / (2\Phi_2)$, and (3.11) gives $|c_1| \leq 2 / \sqrt{1 + |\kappa|}$. $\square$

Corollary 3.6 is an upper estimate for $\mathcal{M}$. Whether it is achieved by a member of $\mathcal{M}$ is a separate question; see Theorem 5.1.

Remark 3.7. Adding the relations as in (3.9) and (3.10) gives $c_2 + d_2 = (1 + \kappa)t^2$. The summation argument uses only $|c_2|, |d_2| \leq 2$ and yields $|1 + \kappa|t^2 \leq 4$. This is a nontrivial restriction on t exactly when $|1 + \kappa| > 1$; that is, for $\kappa > 0$ or $\kappa < -2$. For $\kappa \geq 0$, it coincides with (3.11); in terms of a_2 , it reads $|a_2|^2 \leq L_1^3 / [L_1^2\Phi_3 - L_2\Phi_2^2 + L_1\Phi_2^2]$, which is what the summation argument of [6–9] gives when applied to $\mathcal{M}$. The classes treated in those papers are, in general, not special cases of $\mathcal{M}$; for instance, the bi-starlike condition $zf'/f < \varphi$ of [7] is not of the linear form (2.2), and the bound there has a different denominator. For $-2 \leq \kappa < 0$, it gives nothing beyond $t \leq 2$, and for $\kappa < -2$, it gives $t^2 \leq 4/(|\kappa| - 1)$, which is weaker than $4/(1 + |\kappa|)$. Theorem 3.5 therefore improves the summation bound exactly for $\kappa < 0$, and reproduces it for $\kappa \geq 0$; in the latter case, the new content of Theorem 3.5 is the sharpness on V_2 and, with Lemma 3.8, the identification of the equality set.

Lemma 3.8. *Let $(c_1, c_2, d_1, d_2) \in V_2$ satisfy $|c_1|^2 = 4/(1 + |\kappa|)$, realized by $h, k \in \mathcal{P}$ with $u = (h - 1)/(h + 1)$, $v = (k - 1)/(k + 1)$. Then $u(z) = u_0(\rho z)$ and $v(z) = v_0(\rho z)$ for some $|\rho| = 1$, with (u_0, v_0) as in (3.13) or (3.14).*

Proof. Let $\eta = |c_1|/c_1$ (note $c_1 \neq 0$ since $|c_1|^2 = 4/(1 + |\kappa|) > 0$). By Proposition 3.2, the rotated quadruple lies in V_2 , has the same $|c_1|$, and has the Schwarz functions $u(\eta \cdot), v(\eta \cdot)$; proving the claim for these gives $u(\eta z) = u_0(z)$, that is, $u(z) = u_0(\bar{\eta}z)$, and likewise for v . So assume $c_1 = t = |c_1| > 0$.

If $\kappa = 0$ then $t = 2$ and $u_1 = 1$, so $w = u/z$ satisfies $|w| \leq 1$ with $|w(0)| = 1$; by the maximum principle $w \equiv 1$ and $u(z) = z$. The same argument applied to v/z , whose value at the origin is $v_1 = d_1/2 = -1$, gives $v(z) = -z$.

If $\kappa \neq 0$, then $t = 2a < 2$, and Lemma 3.4 gives $x + y = 2\kappa t^2/(4 - t^2) = 2\kappa/K = 2\sigma$. Thus $|x + y| = 2$, while $|x|, |y| \leq 1$; hence, x and y lie in the closed unit disk with the midpoint σ on the unit circle, and

the strict convexity of the disk forces $x = y = \sigma$. Then $2c_2 = t^2 + (4 - t^2)\sigma$ gives $c_2 = 2(1 + \sigma K)/(1 + K)$ and, by Lemma 2.2, we have

$$u_1 = a, \quad u_2 = \frac{c_2}{2} - \frac{t^2}{4} = \frac{1 + \sigma K}{1 + K} - \frac{1}{1 + K} = \sigma(1 - a^2), \quad |u_2| = 1 - |u_1|^2.$$

This is equality in (3.6), so by (3.5) with $\omega_0 = a$, $u(z) = z(a + \eta_1 z)/(1 + a\eta_1 z)$ with $|\eta_1| = 1$; its z^2 -coefficient is $\eta_1(1 - a^2)$, and a comparison with $u_2 = \sigma(1 - a^2)$ gives $\eta_1 = \sigma$, so $u = u_0$. The same argument with $v_1 = -a$ and $v_2 = d_2/2 - d_1^2/4 = \sigma(1 - a^2)$ gives $v(z) = z(-a + \eta_2 z)/(1 - a\eta_2 z)$ with $\eta_2 = \sigma$; that is, $v = v_0$. $\square$

A Zalcman-type consequence

Proposition 3.9. *For every $f \in \mathcal{M}$, we have*

$$|a_2^2 - a_3| \leq \frac{L_1}{\Phi_3}, \quad (3.18)$$

and the bound is attained on V_2 (with a_2, a_3 defined on V_2 by (2.13) and (2.14)) by the data of $u = \rho z^2$, $v = -\rho z^2$, and $|\rho| = 1$.

Proof. Subtracting the two relations in (2.14) and using $d_1^2 = c_1^2$, $2\Phi_3(a_3 - a_2^2) = \frac{L_1}{2}(c_2 - d_2)$, that is, $a_2^2 - a_3 = -L_1(c_2 - d_2)/(4\Phi_3)$. Normalize by (3.2). If $t = 2$, then $\kappa = 0$ and $c_2 = d_2 = 2$, so $a_2^2 - a_3 = 0$. Let $t < 2$. Subtracting the two relations in (3.7), $c_2 - d_2 = \frac{1}{2}(4 - t^2)(x - y)$, so $|a_2^2 - a_3| = \frac{L_1(4 - t^2)}{8\Phi_3}|x - y|$. The parallelogram identity and $|x|, |y| \leq 1$ give $|x - y|^2 = 2(|x|^2 + |y|^2) - |x + y|^2 \leq 4 - |x + y|^2$, and $|x + y|^2 = 4\kappa^2 t^4/(4 - t^2)^2$ by (3.8), so

$$|x - y|^2 \leq \frac{4((4 - t^2)^2 - \kappa^2 t^4)}{(4 - t^2)^2}.$$

The bracket is non-negative since $|\kappa|t^2 \leq 4 - t^2$ by the proof of Theorem 3.5. Hence

$$|a_2^2 - a_3| \leq \frac{L_1(4 - t^2)}{8\Phi_3} \cdot \frac{2\sqrt{(4 - t^2)^2 - \kappa^2 t^4}}{4 - t^2} = \frac{L_1}{4\Phi_3} \sqrt{F(s)}, \quad F(s) = (4 - s)^2 - K^2 s^2, \quad (3.19)$$

with $s = t^2 \in [0, 4/(1 + K)]$. Differentiating, $F'(s) = -2(4 - s) - 2K^2 s = -2(4 - s(1 + K^2))$. If $K \geq 1$, then $4 - s(1 + K^2) \geq 4$. If $0 \leq K < 1$, then $s(1 + K^2) \leq \frac{4}{1 + K}(1 - K)(1 + K) = 4(1 - K) \leq 4$. In both cases, $F'(s) \leq 0$, and hence $F(s) \leq F(0) = 16$ and (3.19) gives (3.18).

To achieve equality, $u = \rho z^2$ and $v = -\rho z^2$ lie in $\mathcal{B}$ and give $c_1 = d_1 = 0$, $c_2 = 2\rho$, $d_2 = -2\rho$, which satisfy (3.1); by (2.13)–(2.14), $a_2 = 0$ and $a_3 = L_1\rho/\Phi_3$, so $|a_2^2 - a_3| = L_1/\Phi_3$. $\square$

No attainment in $\mathcal{M}$ is claimed for (3.18).

Quantitative stability near the envelope

In the next theorem, *stability* means that second-order data close to the envelope are close to the data of the equality orbit; it is a statement about the finite-dimensional body V_2 , not about perturbations of an equation.

Theorem 3.10 (Quantitative stability). Let $\kappa \neq 0$, $K = |\kappa|$, $\sigma = \operatorname{sgn} \kappa$, and let $(c_1, c_2, d_1, d_2) \in V_2$ be normalized by (3.2); put $s = t^2$. Then $x + y$ is real, with

$$2 - \sigma(x + y) = \frac{2(4 - (1 + K)s)}{4 - s} \geq 0, \quad (3.20)$$

and

$$|x - \sigma|^2 + |y - \sigma|^2 \leq \frac{4(4 - (1 + K)s)}{4 - s} \leq \frac{(1 + K)^2}{K} \left(\frac{4}{1 + K} - s \right). \quad (3.21)$$

Proof. Since $\kappa \neq 0$, Theorem 3.5 gives $s \leq 4/(1 + K) < 4$, so $t < 2$ and x, y are defined. By (3.8) and $\kappa = \sigma K$, $x + y = 2\kappa s/(4 - s) = 2\sigma K s/(4 - s)$, which is real. Hence, $2 - \sigma(x + y) = 2 - \frac{2Ks}{4 - s} = \frac{2(4 - s) - 2Ks}{4 - s} = \frac{2(4 - (1 + K)s)}{4 - s}$, and the right side is non-negative because $(1 + K)s \leq 4$. This is (3.20).

Since $|x| \leq 1$ and $\sigma^2 = 1$, $|x - \sigma|^2 = |x|^2 - 2\sigma \operatorname{Re} x + 1 \leq 2 - 2\sigma \operatorname{Re} x$; likewise, $|y - \sigma|^2 \leq 2 - 2\sigma \operatorname{Re} y$. Adding, and using the fact that $x + y$ is real so that $\operatorname{Re} x + \operatorname{Re} y = x + y$, $|x - \sigma|^2 + |y - \sigma|^2 \leq 4 - 2\sigma(x + y) = \frac{4(4 - (1 + K)s)}{4 - s}$, which is the first inequality in (3.21). For the second, $s \leq 4/(1 + K)$ gives $4 - s \geq 4K/(1 + K) > 0$, so $1/(4 - s) \leq (1 + K)/(4K)$, and therefore $\frac{4(4 - (1 + K)s)}{4 - s} \leq \frac{1 + K}{K} (4 - (1 + K)s) = \frac{(1 + K)^2}{K} \left(\frac{4}{1 + K} - s \right)$. $\square$

Since $s = 4\Phi_2^2|a_2|^2/L_1^2$ by (2.13), the right side of (3.21) equals $\frac{4(1 + K)^2\Phi_2^2}{KL_1^2} \left(\frac{L_1^2}{\Phi_2^2(1 + K)} - |a_2|^2 \right)$, a constant multiple of the deficit of $|a_2|^2$ in (3.17). Theorem 3.10 controls only x, y , that is, the second-order data; it is not used in the proof of Theorem 5.1, which relies on the qualitative rigidity of Lemma 3.8.

The following closedness property of V_2 is used in Section 5.

Lemma 3.11 (Closedness of V_2 under locally uniform limits). Let $(u_n, v_n) \in \mathcal{B}^2$, let $h_n = (1 + u_n)/(1 - u_n)$, $k_n = (1 + v_n)/(1 - v_n)$ have the initial coefficients $(c_1^{(n)}, c_2^{(n)}, d_1^{(n)}, d_2^{(n)}) \in V_2$, and suppose that $u_n \rightarrow u$ and $v_n \rightarrow v$ locally uniformly on $\mathbb{D}$. Then, we have the following:

- (a) $u, v \in \mathcal{B}$;
- (b) for every $j \geq 1$, $u_j^{(n)} \rightarrow u_j$ and $v_j^{(n)} \rightarrow v_j$, where $u_j^{(n)}, u_j, v_j^{(n)}, v_j$ are the Taylor coefficients of u_n, u, v_n, v ;
- (c) $h = (1 + u)/(1 - u)$ and $k = (1 + v)/(1 - v)$ belong to $\mathcal{P}$, their coefficients satisfy $c_j = \lim c_j^{(n)}$, $d_j = \lim d_j^{(n)}$ for $j = 1, 2$, and $(c_1, c_2, d_1, d_2) \in V_2$.

Proof. (a) By the Schwarz lemma, $|u_n(z)| \leq |z|$ for $z \in \mathbb{D}$ and all n . Letting $n \rightarrow \infty$ pointwise gives $u(0) = 0$ and $|u(z)| \leq |z| < 1$ on $\mathbb{D}$, so $u \in \mathcal{B}$; the same argument applies to v .

(b) Fix $r \in (0, 1)$ and $j \geq 1$. By Cauchy's integral formula, we have

$$|u_j^{(n)} - u_j| = \left| \frac{1}{2\pi i} \oint_{|z|=r} \frac{u_n(z) - u(z)}{z^{j+1}} dz \right| \leq r^{-j} \max_{|z|=r} |u_n(z) - u(z)|,$$

and the right side tends to 0 by uniform convergence on $|z| = r$. The same estimate holds for v .

(c) Since $|u| < 1$, $\operatorname{Re} h = (1 - |u|^2)/(1 - |u|^2) > 0$ and $h(0) = 1$, so $h \in \mathcal{P}$; likewise, $k \in \mathcal{P}$. From $h = 1 + 2u + 2u^2 + \cdots$, one has $c_1 = 2u_1$ and $c_2 = 2u_2 + 2u_1^2$, and the same polynomials give $c_1^{(n)}, c_2^{(n)}$ in terms of $u_1^{(n)}, u_2^{(n)}$; similarly, for d_1, d_2 . Hence, (b) gives $c_j^{(n)} \rightarrow c_j$, $d_j^{(n)} \rightarrow d_j$ for $j = 1, 2$. The two conditions in (3.1) are polynomial equations in (c_1, c_2, d_1, d_2) with coefficients depending only on L_1, L_2, Φ_2, Φ_3 ; they hold for every n , and hence in the limit. Since the limit data are realized by $h, k \in \mathcal{P}$, Proposition 3.2 gives $(c_1, c_2, d_1, d_2) \in V_2$. $\square$

4. The fifth-order lifting obstruction

Indexing convention. The residuals are indexed by the order of the Carathéodory data, not by the order of the coefficient: $R_j = \Phi_{j+1}b_{j+1} - B_j$ (Definition 4.1) compares the coefficient of order $j + 1$ of $g = f^{-1}$ with the j -th coefficient of $\Psi \circ v$. Thus, R_1, R_2, R_3, R_4 concern the coefficients b_2, b_3, b_4, b_5 of orders 2, 3, 4, 5. “Lifting through order four” means that the inverse-compatibility equations through the fourth coefficient hold, that is, $R_1 = R_2 = R_3 = 0$; the “fifth-order obstruction” is R_4 , the residual attached to b_5 . In particular, the fourth residual R_4 is a fifth-order condition.

Fix a Schwarz pair (u, v) and let A_j, B_j be as in (2.6). If some $f \in \mathcal{M}$ has (u, v) as its Schwarz pair, then, by (2.7), its coefficients are forced as follows:

$$a_2 = \frac{A_1}{\Phi_2}, \quad a_3 = \frac{A_2}{\Phi_3}, \quad a_4 = \frac{A_3}{\Phi_4}, \quad a_5 = \frac{A_4}{\Phi_5}, \quad (4.1)$$

and the second half of (2.7) imposes four further conditions.

Definition 4.1. Given $(u, v) \in \mathcal{B}^2$, define $a_2, \dots, a_5$ by (4.1), $b_2, \dots, b_5$ by (2.16), and the *inverse-compatibility residuals*

$$R_j = \Phi_{j+1}b_{j+1} - B_j, \quad j = 1, 2, 3, 4. \quad (4.2)$$

The pair *lifts through order n* if $R_1 = \dots = R_{n-1} = 0$.

If $f \in \mathcal{M}$ has a Schwarz pair (u, v) , then $R_1 = R_2 = R_3 = R_4 = 0$. These are necessary conditions; their vanishing is not claimed to be sufficient for membership in $\mathcal{M}$.

If the Carathéodory data of (u, v) lie in V_2 , then $R_1 = R_2 = 0$. Indeed, by Lemma 2.4 and Definition 3.1, there are a'_2, a'_3 satisfying (2.13) and (2.14); the first relations in (2.13) and (2.14) say $\Phi_2 a'_2 = A_1$ and $\Phi_3 a'_3 = A_2$, so $a'_2 = a_2$ and $a'_3 = a_3$ are the forced values (4.1), and the second relations are then exactly $\Phi_2 b_2 = B_1$ and $\Phi_3 b_3 = B_2$, that is, $R_1 = R_2 = 0$. The body V_2 is precisely the set of data for which the first two residuals vanish.

Proposition 4.2. Let $\kappa \neq 0$ and let u_0, v_0 be as in (3.13). Then

$$A_1 = \frac{L_1}{\sqrt{1+K}}, \quad (4.3)$$

$$A_2 = \frac{\sigma K L_1 + L_2}{1+K}, \quad (4.4)$$

$$A_3 = \frac{-K L_1 + 2\sigma K L_2 + L_3}{(1+K)^{3/2}}, \quad (4.5)$$

$$A_4 = \frac{\sigma K L_1 + (K^2 - 2K)L_2 + 3\sigma K L_3 + L_4}{(1+K)^2}, \quad (4.6)$$

and

$$B_j = (-1)^j A_j, \quad j = 1, 2, 3, 4. \quad (4.7)$$

Proof. Insert (3.15) into (2.9)–(2.12). From (2.9), $A_1 = L_1 u_1 = L_1(1+K)^{-1/2}$. From (2.10), $L_1 u_2 = \sigma K L_1 / (1+K)$ and $L_2 u_1^2 = L_2 / (1+K)$. From (2.11), $L_1 u_3 = -K L_1 (1+K)^{-3/2}$, $2L_2 u_1 u_2 = 2\sigma K L_2 (1+K)^{-3/2}$, and $L_3 u_1^3 = L_3 (1+K)^{-3/2}$. From (2.12)

$$L_1 u_4 = \frac{\sigma K L_1}{(1+K)^2}, \quad L_2(2u_1 u_3 + u_2^2) = L_2 \frac{-2K + K^2}{(1+K)^2},$$

$$3L_3u_1^2u_2 = \frac{3\sigma KL_3}{(1+K)^2}, \quad L_4u_1^4 = \frac{L_4}{(1+K)^2}.$$

Finally, $v_0(z) = u_0(-z)$ by (3.16), so $\Psi(v_0(z)) = \Psi(u_0(-z)) = 1 + \sum_j A_j(-1)^j z^j$, which is (4.7). $\square$

Lemma 4.3. *With the notation of Proposition 4.2, we have*

$$R_1 = R_2 = R_3 = 0, \quad a_3 = a_2^2. \quad (4.8)$$

Proof. By (2.16) and (4.1), $\Phi_2 b_2 = -\Phi_2 a_2 = -A_1$, and $B_1 = -A_1$, so $R_1 = -A_1 - (-A_1) = 0$.

Next, $\Phi_3 b_3 = 2\Phi_3 a_2^2 - \Phi_3 a_3 = 2\Phi_3 a_2^2 - A_2$ and $B_2 = A_2$, so

$$R_2 = 2\Phi_3 a_2^2 - 2A_2. \quad (4.9)$$

Substituting $a_2 = A_1/\Phi_2$ and (4.3)–(4.4), we have

$$R_2 = \frac{2L_1^2\Phi_3}{\Phi_2^2(1+K)} - \frac{2(\sigma KL_1 + L_2)}{1+K} = \frac{2(L_1^2\Phi_3 - \Phi_2^2(\sigma KL_1 + L_2))}{\Phi_2^2(1+K)}. \quad (4.10)$$

Since $\sigma K = \kappa$, by (1.1), we have

$$\Phi_2^2(\kappa L_1 + L_2) = \Phi_2^2 \left[L_1 \left(\frac{L_1\Phi_3}{\Phi_2^2} - \frac{L_2}{L_1} \right) + L_2 \right] = L_1^2\Phi_3. \quad (4.11)$$

So the numerator of (4.10) vanishes and $R_2 = 0$; then (4.9) gives $\Phi_3 a_2^2 = A_2 = \Phi_3 a_3$, that is, $a_3 = a_2^2$.

Finally, $\Phi_4 b_4 = -5\Phi_4 a_2^3 + 5\Phi_4 a_2 a_3 - A_3$ and $B_3 = -A_3$, so the terms $\mp A_3$ cancel and

$$R_3 = 5\Phi_4 a_2(a_3 - a_2^2) = 0.$$

$\square$

Mechanism of the cancellation. Lemma 4.3 rests on two facts about the equality orbit: Its data lie in V_2 , and $v_0(z) = u_0(-z)$, so that $B_j = (-1)^j A_j$. The logical chain is

$$\underbrace{R_1 = R_2 = 0}_{\text{data in } V_2} \ \& \ B_2 = A_2 \implies \Phi_3(2a_2^2 - a_3) = \Phi_3 a_3 \implies a_3 = a_2^2 \implies R_3 = 5\Phi_4 a_2(a_3 - a_2^2) = 0,$$

where the last step also uses $B_3 = -A_3 = -\Phi_4 a_4$, which cancels the only term of $\Phi_4 b_4$ containing a_4 . At the next order, the same substitutions leave $R_4 = \Phi_5(-4a_2^4 + 6a_2 a_4) - 2A_4$; the term A_4 contains L_4 and u_4 , which are not constrained by any of the lower-order relations, so R_4 is the first residual that is not annihilated by them. In the present operator/subordination framework, the cancellation through R_3 is driven by the relation $a_3 = a_2^2$ forced on the equality configuration and by the symmetry $B_j = (-1)^j A_j$ of that configuration. We do not claim that the same cancellation occurs in arbitrary bi-univalent subclasses or at other points of V_2 .

Structure of the fifth residual. On the equality orbit, the substitutions $a_3 = a_2^2$, $a_2 = A_1/\Phi_2$, $a_4 = A_3/\Phi_4$, $a_5 = A_4/\Phi_5$, $B_4 = A_4$ reduce R_4 to a combination of A_1^4 , $A_1 A_3$, and A_4 (see (4.13) below). By Proposition 4.2 these three quantities share the factor $(1+K)^{-2}$, so after multiplication by the positive number $(1+K)^2 \Phi_2^4 \Phi_4$, one obtains an expression polynomial in L_j , Φ_j , K , and σK . Since $\sigma K = \kappa =$

$M/(L_1\Phi_2^2)$ and $K^2 = \kappa^2$, the only remaining denominators are powers of $L_1\Phi_2^2$, and multiplication by L_1^2 clears them; the result is the polynomial P_σ , first in the compact form (4.15) in terms of M and then, after expanding M , in the form shown in (4.16). The dependence on L_4 is transparent in this route: L_4 enters only through A_4 , linearly and with the coefficient $u_1^4 = (1+K)^{-2}$, while A_1 and A_3 do not involve L_4 . Hence

$$P_\sigma = L_1^2\Phi_2^4\Phi_4 L_4 + P_\sigma|_{L_4=0},$$

where $P_\sigma|_{L_4=0}$ does not depend on L_4 .

Theorem 4.4 (Fifth-order lifting obstruction). *Let $\kappa \neq 0$ and let (u_0, v_0) be the equality pair (3.13). Put*

$$M = L_1^2\Phi_3 - L_2\Phi_2^2, \quad \text{so that} \quad \kappa = \frac{M}{L_1\Phi_2^2}, \quad L_1\Phi_2^2 + \sigma M = L_1\Phi_2^2(1+K) > 0. \quad (4.12)$$

Then

$$R_4 = -2 \left[\frac{2\Phi_5 A_1^4}{\Phi_2^4} - \frac{3\Phi_5 A_1 A_3}{\Phi_2 \Phi_4} + A_4 \right], \quad (4.13)$$

and

$$R_4 = \frac{-2P_\sigma}{\Phi_4(L_1\Phi_2^2 + \sigma M)^2}, \quad (4.14)$$

where, in compact form, we have

$$\begin{aligned} P_\sigma = & 2L_1^6\Phi_4\Phi_5 - 3L_1^2\Phi_2\Phi_5(-\sigma ML_1 + 2ML_2 + L_1L_3\Phi_2^2) \\ & + \Phi_4(ML_1^2\Phi_2^2 + M^2L_2 - 2\sigma ML_1L_2\Phi_2^2 + 3ML_1L_3\Phi_2^2 + L_1^2\Phi_2^4L_4), \end{aligned} \quad (4.15)$$

and, after expanding M , P_σ is the sum of the sixteen monomials

$$\begin{aligned} P_\sigma = & -L_1^2L_2\Phi_2^4\Phi_4 + 2\sigma L_1L_2^2\Phi_2^4\Phi_4 + L_2^3\Phi_2^4\Phi_4 - 3L_1L_2L_3\Phi_2^4\Phi_4 + L_1^2L_4\Phi_2^4\Phi_4 \\ & + L_1^4\Phi_2^2\Phi_3\Phi_4 - 2\sigma L_1^3L_2\Phi_2^2\Phi_3\Phi_4 - 2L_1^2L_2^2\Phi_2^2\Phi_3\Phi_4 + 3L_1^3L_3\Phi_2^2\Phi_3\Phi_4 + L_1^4L_2\Phi_2^3\Phi_4 \\ & - 3\sigma L_1^3L_2\Phi_2^3\Phi_5 + 6L_1^2L_2^2\Phi_2^3\Phi_5 - 3L_1^3L_3\Phi_2^3\Phi_5 + 3\sigma L_1^5\Phi_2\Phi_3\Phi_5 - 6L_1^4L_2\Phi_2\Phi_3\Phi_5 \\ & + 2L_1^6\Phi_4\Phi_5. \end{aligned} \quad (4.16)$$

Consequently

$$R_4 = 0 \iff P_\sigma = 0. \quad (4.17)$$

Proof. Step 1: Elimination of a_3, a_4, a_5 in $\Phi_5 b_5 - B_4$. By (2.16), $\Phi_5 b_5 = \Phi_5(14a_2^4 - 21a_2^2a_3 + 6a_2a_4 + 3a_3^2 - a_5)$. By Lemma 4.3, $a_3 = a_2^2$; by (4.1), $a_2 = A_1/\Phi_2$, $a_4 = A_3/\Phi_4$, and $a_5 = A_4/\Phi_5$; by (4.7), $B_4 = A_4$. The six constituents of R_4 are therefore

$$\Phi_5 \cdot 14a_2^4 = \frac{14\Phi_5 A_1^4}{\Phi_2^4}, \quad \Phi_5 \cdot (-21a_2^2a_3) = -\frac{21\Phi_5 A_1^4}{\Phi_2^4}, \quad (4.18)$$

$$\Phi_5 \cdot 3a_3^2 = \frac{3\Phi_5 A_1^4}{\Phi_2^4}, \quad \Phi_5 \cdot 6a_2a_4 = \frac{6\Phi_5 A_1 A_3}{\Phi_2 \Phi_4}, \quad (4.19)$$

$$\Phi_5 \cdot (-a_5) = -A_4, \quad -B_4 = -A_4. \quad (4.20)$$

The three A_1^4 terms carry the coefficient $14 - 21 + 3 = -4$. Summing (4.18)–(4.20), we have

$$R_4 = -\frac{4\Phi_5 A_1^4}{\Phi_2^4} + \frac{6\Phi_5 A_1 A_3}{\Phi_2 \Phi_4} - 2A_4 = -2 \left[\frac{2\Phi_5 A_1^4}{\Phi_2^4} - \frac{3\Phi_5 A_1 A_3}{\Phi_2 \Phi_4} + A_4 \right], \quad (4.21)$$

which is (4.13).

Step 2: Clearing the denominator. By (4.3), (4.5), (4.6), we have

$$\begin{aligned} A_1^4 &= \frac{L_1^4}{(1+K)^2}, & A_1 A_3 &= \frac{L_1(-KL_1 + 2\sigma KL_2 + L_3)}{(1+K)^2}, \\ A_4 &= \frac{\sigma KL_1 + (K^2 - 2K)L_2 + 3\sigma KL_3 + L_4}{(1+K)^2}. \end{aligned}$$

Multiplying the bracket in (4.21) by $(1+K)^2 \Phi_2^4 \Phi_4 > 0$, we have

$$\begin{aligned} Q_0 &:= (1+K)^2 \Phi_2^4 \Phi_4 \left[\frac{2\Phi_5 A_1^4}{\Phi_2^4} - \frac{3\Phi_5 A_1 A_3}{\Phi_2 \Phi_4} + A_4 \right] \\ &= 2L_1^4 \Phi_4 \Phi_5 - 3L_1 \Phi_2^3 \Phi_5 (-KL_1 + 2\sigma KL_2 + L_3) \\ &\quad + \Phi_2^4 \Phi_4 (\sigma KL_1 + (K^2 - 2K)L_2 + 3\sigma KL_3 + L_4). \end{aligned} \quad (4.22)$$

Step 3: Elimination of K and σ . Since $\kappa \neq 0$ and $\sigma = \text{sgn } \kappa$, $K = |\kappa|$, we have $\kappa = \sigma K$ and, because $\sigma^2 = 1$, $K = \sigma \kappa$, and $K^2 = \kappa^2$. With $\kappa = M/(L_1 \Phi_2^2)$ from (4.12), we have

$$\begin{aligned} -KL_1 &= -\sigma \kappa L_1 = -\frac{\sigma M}{\Phi_2^2}, & 2\sigma KL_2 &= 2\kappa L_2 = \frac{2ML_2}{L_1 \Phi_2^2}, \\ \sigma KL_1 &= \kappa L_1 = \frac{M}{\Phi_2^2}, & 3\sigma KL_3 &= \frac{3ML_3}{L_1 \Phi_2^2}, \\ (K^2 - 2K)L_2 &= (\kappa^2 - 2\sigma \kappa)L_2 = \frac{M^2 L_2}{L_1^2 \Phi_2^4} - \frac{2\sigma M L_2}{L_1 \Phi_2^2}. \end{aligned}$$

Hence

$$-KL_1 + 2\sigma KL_2 + L_3 = \frac{-\sigma M L_1 + 2M L_2 + L_1 L_3 \Phi_2^2}{L_1 \Phi_2^2}, \quad (4.23)$$

$$\sigma KL_1 + (K^2 - 2K)L_2 + 3\sigma KL_3 + L_4 = \frac{N_3}{L_1^2 \Phi_2^4}, \quad (4.24)$$

$$\begin{aligned} \text{where } N_3 &:= M L_1^2 \Phi_2^2 + M^2 L_2 - 2\sigma M L_1 L_2 \Phi_2^2 \\ &\quad + 3M L_1 L_3 \Phi_2^2 + L_1^2 \Phi_2^4 L_4. \end{aligned}$$

Write $N_2 = -\sigma M L_1 + 2M L_2 + L_1 L_3 \Phi_2^2$ for the numerator in (4.23). Substituting (4.23), (4.24) into (4.22) gives $Q_0 = 2L_1^4 \Phi_4 \Phi_5 - 3\Phi_2 \Phi_5 N_2 + L_1^2 \Phi_4 N_3$; multiplying by L_1^2 , we have

$$\begin{aligned} T_1 &= 2L_1^6 \Phi_4 \Phi_5, \\ L_1^2 Q_0 &= T_1 + T_2 + T_3, & T_2 &= -3L_1^2 \Phi_2 \Phi_5 (-\sigma M L_1 + 2M L_2 + L_1 L_3 \Phi_2^2), \\ T_3 &= \Phi_4 (M L_1^2 \Phi_2^2 + M^2 L_2 - 2\sigma M L_1 L_2 \Phi_2^2 + 3M L_1 L_3 \Phi_2^2 + L_1^2 \Phi_2^4 L_4). \end{aligned} \quad (4.25)$$

Step 4: Expansion of T_1, T_2, T_3 . Insert $M = L_1^2\Phi_3 - L_2\Phi_2^2$. The group T_1 is already a monomial. For T_2 , we have

$$\begin{aligned} T_2 &= 3\sigma L_1^3\Phi_2\Phi_5M - 6L_1^2L_2\Phi_2\Phi_5M - 3L_1^3L_3\Phi_2^3\Phi_5 \\ &= 3\sigma L_1^5\Phi_2\Phi_3\Phi_5 - 3\sigma L_1^3L_2\Phi_2^3\Phi_5 - 6L_1^4L_2\Phi_2\Phi_3\Phi_5 + 6L_1^2L_2^2\Phi_2^3\Phi_5 - 3L_1^3L_3\Phi_2^3\Phi_5. \end{aligned}$$

For T_3 , term by term

$$\begin{aligned} ML_1^2\Phi_2^2 &= L_1^4\Phi_2^2\Phi_3 - L_1^2L_2\Phi_2^4, \\ M^2L_2 &= L_1^4L_2\Phi_3^2 - 2L_1^2L_2^2\Phi_2^2\Phi_3 + L_2^3\Phi_2^4, \\ -2\sigma ML_1L_2\Phi_2^2 &= -2\sigma L_1^3L_2\Phi_2^2\Phi_3 + 2\sigma L_1L_2^2\Phi_2^4, \\ 3ML_1L_3\Phi_2^2 &= 3L_1^3L_3\Phi_2^2\Phi_3 - 3L_1L_2L_3\Phi_2^4, \end{aligned}$$

and T_3 is Φ_4 times the sum of these nine monomials plus $L_1^2L_4\Phi_2^4\Phi_4$. Altogether, $T_1 + T_2 + T_3$ consists of $1 + 5 + 10 = 16$ monomials. They are pairwise distinct, so no cancellation occurs, and they are exactly the 16 monomials of (4.16); the correspondence is listed in Table 1 of Appendix B. Hence

$$L_1^2Q_0 = T_1 + T_2 + T_3 = P_\sigma, \quad (4.26)$$

and (4.25) is the compact form (4.15).

Step 5: The denominator. By (4.21), (4.22), and (4.26),

$$R_4 = \frac{-2Q_0}{(1+K)^2\Phi_2^4\Phi_4} = \frac{-2L_1^2Q_0}{L_1^2(1+K)^2\Phi_2^4\Phi_4} = \frac{-2P_\sigma}{\Phi_4(L_1\Phi_2^2(1+K))^2},$$

and $L_1\Phi_2^2(1+K) = L_1\Phi_2^2 + L_1\Phi_2^2\sigma\kappa = L_1\Phi_2^2 + \sigma M$ gives (4.14). Each factor of the denominator is positive: $\Phi_4 > 0$ by (2.3), and $L_1\Phi_2^2(1+K) > 0$ because $L_1 > 0$, $\Phi_2 > 0$, $K \geq 0$. Thus R_4 is a nonzero real multiple of P_σ , which is (4.17). $\square$

Corollary 4.5. *Regard P_σ as a polynomial in L_4 . Then*

$$\deg_{L_4} P_\sigma = 1, \quad \frac{\partial P_\sigma}{\partial L_4} = L_1^2\Phi_2^4\Phi_4 > 0, \quad (4.27)$$

so, for a fixed $L_1, L_2, L_3, \Phi_2, \dots, \Phi_5$, the equation $P_\sigma = 0$ holds for exactly one value of L_4 , namely

$$\begin{aligned} L_{4,\text{lift}}^{(\sigma)} &= \frac{1}{L_1^2\Phi_2^4\Phi_4} \left[L_1^2L_2\Phi_2^4\Phi_4 - 2\sigma L_1L_2^2\Phi_2^4\Phi_4 - L_2^3\Phi_2^4\Phi_4 + 3L_1L_2L_3\Phi_2^4\Phi_4 \right. \\ &\quad - L_1^4\Phi_2^2\Phi_3\Phi_4 + 2\sigma L_1^3L_2\Phi_2^2\Phi_3\Phi_4 + 2L_1^2L_2^2\Phi_2^2\Phi_3\Phi_4 - 3L_1^3L_3\Phi_2^2\Phi_3\Phi_4 \\ &\quad - L_1^4L_2\Phi_2^2\Phi_3\Phi_4 + 3\sigma L_1^3L_2\Phi_2^3\Phi_5 - 6L_1^2L_2^2\Phi_2^3\Phi_5 + 3L_1^3L_3\Phi_2^3\Phi_5 \\ &\quad \left. - 3\sigma L_1^5\Phi_2\Phi_3\Phi_5 + 6L_1^4L_2\Phi_2\Phi_3\Phi_5 - 2L_1^6\Phi_4\Phi_5 \right]. \end{aligned} \quad (4.28)$$

The equality orbit of Theorem 3.5 satisfies fifth-order inverse compatibility if and only if

$$L_4 = L_{4,\text{lift}}^{(\sigma)}. \quad (4.29)$$

Proof. In (4.16), the variable L_4 occurs only in the monomial $L_1^2 L_4 \Phi_2^4 \Phi_4$ and to the first power, which is (4.27); positivity follows from $L_1 > 0$, $\Phi_2, \Phi_4 > 0$. Writing $P_\sigma = L_1^2 \Phi_2^4 \Phi_4 L_4 + P_{\sigma|L_4=0}$ and solving gives (4.28), the bracket being $-P_{\sigma|L_4=0}$. By Lemma 4.3, $R_1 = R_2 = R_3 = 0$ on (u_0, v_0) , so fifth-order compatibility is equivalent to $R_4 = 0$, and hence to $P_\sigma = 0$ by (4.17). By Lemma 4.10 below, the same holds at every point $(u_0(\rho \cdot), v_0(\rho \cdot))$ of the orbit. $\square$

Remark 4.6 (Status of L_4). For a fixed generator Ψ , the coefficient L_4 is already prescribed. Thus, $L_{4,\text{lift}}^{(\sigma)}$ is not a freely adjustable parameter; it is the unique value that the prescribed fourth Taylor coefficient of Ψ would have to equal for the second-order equality configuration to satisfy fifth-order inverse compatibility. Corollary 4.5 is therefore a test on the given generator: One compares its L_4 with the value (4.28) determined by the lower-order data $L_1, L_2, L_3, \Phi_2, \dots, \Phi_5$. The affine dependence on L_4 is a statement about the polynomial P_σ in the parameter space, not about the freedom to choose L_4 once Ψ is given.

Remark 4.7 (What $P_\sigma \neq 0$ does and does not assert). If $P_\sigma \neq 0$, then the unique second-order equality configuration of Theorem 3.5 (that is, every point of the rotation orbit of (3.13), by Lemma 3.8 and Lemma 4.10) fails fifth-order inverse compatibility. Consequently, no member of $\mathcal{M}$ has a Schwarz pair on that orbit. It does not follow that other points of V_2 fail to lift, and $P_\sigma \neq 0$ says nothing about any point of V_2 off the equality orbit. Nor does it determine $\sup_{f \in \mathcal{M}} |c_1|^2$; it yields only the strict inequality of Theorem 5.1. Conversely, $P_\sigma = 0$ is a necessary condition for the equality orbit to be realized in $\mathcal{M}$ and is not asserted to be sufficient: Membership requires the vanishing of all residuals of every order together with the bi-univalence of the resulting function.

Remark 4.8 (The branch $\kappa = 0$). For $\kappa = 0$, the equality pair is $(z, -z)$, so $u_1 = 1, u_2 = u_3 = u_4 = 0$, and Lemma 2.3 gives $A_j = L_j$, while $v_0(z) = u_0(-z)$ gives $B_j = (-1)^j L_j$; thus (4.7) holds. Since $\Phi_3 a_2^2 = L_1^2 \Phi_3 / \Phi_2^2 = L_2 = A_2$ when $\kappa = 0$, the proof of Lemma 4.3 applies verbatim: $R_1 = R_2 = R_3 = 0$ and $a_3 = a_2^2$. Step 1 of the proof of Theorem 4.4 applies unchanged, and Step 2 with $K = 0, A_1 = L_1, A_3 = L_3, A_4 = L_4$ gives $Q_0 = 2L_1^4 \Phi_4 \Phi_5 - 3L_1 L_3 \Phi_2^3 \Phi_5 + L_4 \Phi_2^4 \Phi_4$. Hence

$$R_4 = \frac{-2P_0}{\Phi_4 L_1^2 \Phi_2^4}, \quad P_0 := L_1^2 Q_0 = 2L_1^6 \Phi_4 \Phi_5 - 3L_1^3 L_3 \Phi_2^3 \Phi_5 + L_1^2 L_4 \Phi_2^4 \Phi_4. \quad (4.30)$$

Setting $M = 0$ in (4.25) annihilates every monomial of T_2 and T_3 except $-3L_1^3 L_3 \Phi_2^3 \Phi_5$ and $L_1^2 L_4 \Phi_2^4 \Phi_4$, so $P_{\sigma|M=0} = P_0$ for either sign, as it must. The coefficient of L_4 in P_0 is $L_1^2 \Phi_2^4 \Phi_4 > 0$, so $P_0 = 0$ exactly when

$$L_4 = \frac{3L_1 L_3 \Phi_2^3 \Phi_5 - 2L_1^4 \Phi_4 \Phi_5}{\Phi_2^4 \Phi_4}. \quad (4.31)$$

On this branch, σ is undefined, and P_0 , not P_σ , is the relevant polynomial; the two cases are kept separate below.

To describe the lifting condition in the parameter space, put the following for $\sigma \in \{-1, 1\}$:

$$C_\sigma = \{p = (L_1, L_2, L_3, L_4, \Phi_2, \Phi_3, \Phi_4, \Phi_5) \in \mathbb{R}^4 \times (0, \infty)^4 : L_1 > 0, \sigma \kappa > 0\}, \quad (4.32)$$

with κ as in (1.1). On C_σ , one has $\text{sgn } \kappa = \sigma$, so P_σ describes the equality configuration there. The condition $\sigma \kappa > 0$ involves only L_1, L_2, Φ_2 , and Φ_3 , so C_σ is a union of full lines in the L_4 -direction; C_{+1}, C_{-1} are disjoint open sets whose union omits exactly the branch $\kappa = 0$.

Corollary 4.9 (Fifth-order lifting locus on a sign chamber). Fix $\sigma \in \{-1, 1\}$ and put $O_\sigma = \{p \in C_\sigma : P_\sigma(p) = 0\}$. Then O_σ is the graph, over the projection of C_σ to the seven coordinates other than L_4 , of the map $(L_1, L_2, L_3, \Phi_2, \Phi_3, \Phi_4, \Phi_5) \mapsto L_{4,\text{lift}}^{(\sigma)}$ given by (4.28), which is real analytic there. In particular, every line parallel to the L_4 -axis that meets C_σ meets O_σ in exactly one point, and O_σ is a real-analytic hypersurface in C_σ . The obstruction $R_4 \neq 0$ holds exactly on $C_\sigma \setminus O_\sigma$.

Proof. By (4.16), $P_\sigma = L_1^2 \Phi_2^4 \Phi_4 L_4 + P_\sigma|_{L_4=0}$, where $P_\sigma|_{L_4=0}$ is a polynomial in the remaining seven parameters. On C_σ , the coefficient $L_1^2 \Phi_2^4 \Phi_4$ is positive, so for each admissible value of those parameters, $P_\sigma = 0$ is a degree-one real equation in L_4 whose unique solution is (4.28); since $\sigma\kappa > 0$ does not constrain L_4 , that solution lies in C_σ . The right side of (4.28) is a polynomial divided by $L_1^2 \Phi_2^4 \Phi_4 > 0$, and hence is real analytic, and O_σ is its graph. The last assertion is (4.17). $\square$

Here, O_σ is the set of parameter data on which the equality orbit does pass the fifth-order test; the points $p \in C_\sigma$ are data, not generators, and a given generator Ψ corresponds to a single point (Remark 4.6). On the branch $\kappa = 0$, the analogous statement uses P_0 and (4.31).

Lemma 4.10 (Rotational covariance). For $|\rho| = 1$, set $u_\rho(z) = u_0(\rho z)$ and $v_\rho(z) = v_0(\rho z)$. Then

$$A_j^{(\rho)} = \rho^j A_j, \quad B_j^{(\rho)} = \rho^j B_j, \quad j = 1, 2, 3, 4, \quad (4.33)$$

$$a_2^{(\rho)} = \rho a_2, \quad a_3^{(\rho)} = \rho^2 a_3, \quad a_4^{(\rho)} = \rho^3 a_4, \quad a_5^{(\rho)} = \rho^4 a_5, \quad b_5^{(\rho)} = \rho^4 b_5, \quad (4.34)$$

and consequently

$$R_4^{(\rho)} = \rho^4 R_4, \quad \text{so} \quad R_4^{(\rho)} = 0 \iff R_4 = 0. \quad (4.35)$$

Proof. By (2.6), $\Psi(u_\rho(z)) = \Psi(u_0(\rho z)) = 1 + \sum_{j=1}^4 A_j(\rho z)^j + O(z^5)$, and the same computation for v_ρ gives (4.33). Then, $a_n^{(\rho)} = A_{n-1}^{(\rho)}/\Phi_n = \rho^{n-1} a_n$ by (4.1). Assigning a_n the weight $n - 1$, every monomial of b_5 in (2.16) has a total weight of four (a_2^4 : 4; $a_2^2 a_3$: 2 + 2; $a_2 a_4$: 1 + 3; a_3^2 : 2 + 2; a_5 : 4), so $b_5^{(\rho)} = \rho^4 b_5$. Hence, $R_4^{(\rho)} = \Phi_5 b_5^{(\rho)} - B_4^{(\rho)} = \rho^4(\Phi_5 b_5 - B_4) = \rho^4 R_4$, and $\rho^4 \neq 0$. $\square$

Lemma 4.11 (Polynomial structure and continuity of R_4). For $(u, v) \in \mathcal{B}^2$ with the Taylor coefficients u_j, v_j , the residual R_4 of Definition 4.1 is a polynomial $F(u_1, \dots, u_4, v_1, \dots, v_4)$ whose coefficients depend only on $L_1, \dots, L_4$ and $\Phi_2, \dots, \Phi_5$. In particular, F is continuous on $\mathbb{C}^8$, and if $u_n \rightarrow u$ and $v_n \rightarrow v$ locally uniformly on $\mathbb{D}$, then $R_4(u_n, v_n) \rightarrow R_4(u, v)$.

Proof. By Lemma 2.3, $A_1, \dots, A_4$ are polynomials in $u_1, \dots, u_4$ and $B_1, \dots, B_4$ are the same polynomials in $v_1, \dots, v_4$, with coefficients in $\{L_1, \dots, L_4\}$ and integers. By (4.1), $a_{j+1} = A_j/\Phi_{j+1}$ are polynomials in $u_1, \dots, u_4$. By (2.16), b_5 is a polynomial in $a_2, \dots, a_5$ with integer coefficients. Hence, $R_4 = \Phi_5 b_5 - B_4$ is a polynomial F in the eight variables, with coefficients built from L_j , Φ_j and Φ_j^{-1} , and polynomials are continuous. If $u_n \rightarrow u$ and $v_n \rightarrow v$ locally uniformly, then $u_j^{(n)} \rightarrow u_j$ and $v_j^{(n)} \rightarrow v_j$ for $j = 1, \dots, 4$ by the Cauchy estimate in Lemma 3.11(b), and the continuity of F gives $R_4(u_n, v_n) \rightarrow R_4(u, v)$. $\square$

5. Obstruction-induced strict separation

For $f \in \mathcal{M}$, the number $|c_1|$ does not depend on the admissible Schwarz pair: $\Phi_2 a_2 = A_1 = L_1 u_1$ gives $u_1 = \Phi_2 a_2 / L_1$ and $|c_1| = 2|u_1| = 2\Phi_2 |a_2| / L_1$. We write $c_1(f)$ for a c_1 that is computed from any admissible u .

Theorem 5.1 (Obstruction-induced strict separation). *Let (u_0, v_0) be the equality pair of Theorem 3.5, normalized as in (3.13) or (3.14), and suppose*

$$R_4(u_0, v_0) \neq 0. \quad (5.1)$$

By Theorem 4.4 and Remark 4.8, (5.1) is equivalent to $P_\sigma \neq 0$ if $\kappa \neq 0$ and to $P_0 \neq 0$ if $\kappa = 0$. Then

$$\sup_{f \in \mathcal{M}} |c_1(f)|^2 < \frac{4}{1 + |\kappa|}; \quad (5.2)$$

equivalently, $\varepsilon > 0$ exists such that $|c_1(f)|^2 \leq \frac{4}{1 + |\kappa|} - \varepsilon$ for every $f \in \mathcal{M}$. Consequently

$$\sup_{f \in \mathcal{M}} |a_2| < \frac{L_1}{\Phi_2 \sqrt{1 + |\kappa|}}. \quad (5.3)$$

This is a strict separation from the V_2 envelope, not a sharp estimate for $\mathcal{M}$; the values of the two suprema are not determined.

Proof. Write $\Lambda = 4/(1 + |\kappa|)$, and $a = (1 + |\kappa|)^{-1/2}$ as in (3.12) (so $a = 1$ when $\kappa = 0$ and $\Lambda = 4a^2$ in both cases).

Step A (Setup). Put $S = \sup\{|c_1(f)|^2 : f \in \mathcal{M}\}$. The set is nonempty (Section 2), and Theorem 3.5 gives $S \leq \Lambda$. Suppose, for contradiction, that $S = \Lambda$.

Step B (Extremizing sequence). Choose $f_n \in \mathcal{M}$ with $|c_1(f_n)|^2 \rightarrow \Lambda$, and for each n an admissible Schwarz pair $(\tilde{u}_n, \tilde{v}_n)$ of f_n . Discarding finitely many terms we may assume $c_1(f_n) \neq 0$.

Step C (Rotation normalization). Put $\rho_n = \overline{c_1(f_n)}/|c_1(f_n)|$, so $|\rho_n| = 1$. By Lemma 2.5, $F_n := (f_n)_{\rho_n} \in \mathcal{M}$ with the admissible Schwarz pair $u_n(z) = \tilde{u}_n(\rho_n z)$, $v_n(w) = \tilde{v}_n(\rho_n w)$, and $c_1(F_n) = \rho_n c_1(f_n) = |c_1(f_n)| =: t_n \geq 0$. We rename F_n as f_n . Thus

$$c_1^{(n)} = 2u_1^{(n)} = t_n \in [0, 2], \quad t_n^2 \rightarrow \Lambda = 4a^2, \quad t_n \rightarrow 2a, \quad (5.4)$$

where $u_j^{(n)}, v_j^{(n)}$ denote the Taylor coefficients of u_n, v_n .

Step D (Normal families). By the Schwarz lemma, $|u_n(z)| \leq |z|$ and $|v_n(z)| \leq |z|$ on $\mathbb{D}$, so $\{u_n\}$ and $\{v_n\}$ are uniformly bounded and hence are normal by Montel's theorem. Passing first to a subsequence along which u_n converges locally uniformly on $\mathbb{D}$, and then to a further subsequence along which v_n does, we obtain analytic u, v on $\mathbb{D}$ with $u_n \rightarrow u$ and $v_n \rightarrow v$ locally uniformly on $\mathbb{D}$ along a common subsequence, which we keep denoting as n .

Step E (The limits are Schwarz functions). By Lemma 3.11(a), $u, v \in \mathcal{B}$: The Schwarz-lemma bounds $|u_n(z)| \leq |z|$, $|v_n(z)| \leq |z|$ pass to the pointwise limit, so $u(0) = v(0) = 0$ and $|u(z)| \leq |z| < 1$, $|v(z)| \leq |z| < 1$ on $\mathbb{D}$. Equivalently, $|u| \leq 1$, and if $|u(z_0)| = 1$ at some $z_0 \in \mathbb{D}$, the maximum principle would make u a unimodular constant, contradicting $u(0) = 0$; likewise for v .

Step F (Coefficient convergence). By Lemma 3.11(b), which rests on the Cauchy estimate $|u_j^{(n)} - u_j| \leq r^{-j} \max_{|z|=r} |u_n - u|$ for a fixed $r \in (0, 1)$, we have $u_j^{(n)} \rightarrow u_j$ and $v_j^{(n)} \rightarrow v_j$ for every fixed j . In particular, $u_1 = \lim t_n/2 = a > 0$.

Step G (Limiting compatibility). Each $(c_1^{(n)}, c_2^{(n)}, d_1^{(n)}, d_2^{(n)})$ is the data of $f_n \in \mathcal{M}$, and hence lies in V_2 by Lemma 2.4. By Lemma 3.11(c), $h = (1 + u)/(1 - u)$ and $k = (1 + v)/(1 - v)$ lie in $\mathcal{P}$, their coefficients

are the limits $c_j = \lim c_j^{(n)}$, $d_j = \lim d_j^{(n)}$ ($j = 1, 2$), and $(c_1, c_2, d_1, d_2) \in V_2$: The two polynomial compatibility relations (3.1) hold for every n and therefore in the limit.

Step H (The limit reaches the envelope). $c_1 = 2u_1 = 2a$, so $|c_1|^2 = 4a^2 = \Lambda$.

Step I (Identification of the limit). By Lemma 3.8, there is ρ with $|\rho| = 1$ such that $u(z) = u_0(\rho z)$ and $v(z) = v_0(\rho z)$. The first coefficient of $u_0(\rho z)$ is ρa (with $a = 1$ if $\kappa = 0$), and it equals $u_1 = a > 0$; hence, $\rho = 1$ and $(u, v) = (u_0, v_0)$, the normalized pair in (5.1), not merely a point of its orbit.

Step J (The residual vanishes along the sequence). For each n , $f_n \in \mathcal{M}$ with the Schwarz pair (u_n, v_n) . By (2.7), the numbers $a_{j+1}^{(n)} = A_j^{(n)}/\Phi_{j+1}$ defined in (4.1) from u_n are the actual coefficients of f_n , the numbers $b_{j+1}^{(n)}$ obtained from them by the universal polynomials (2.16) are the actual coefficients of f_n^{-1} , and $\Phi_{j+1} b_{j+1}^{(n)} = B_j^{(n)}$. Hence

$$R_4(u_n, v_n) = 0 \quad \text{for every } n. \quad (5.5)$$

Step K (Continuity of R_4). By Lemma 4.11, R_4 is a polynomial in $u_1, \dots, u_4, v_1, \dots, v_4$ with the coefficients depending only on the fixed data $L_1, \dots, L_4, \Phi_2, \dots, \Phi_5$, and hence is continuous with respect to coefficientwise convergence. By Step F, Step I, and (5.5), we have

$$R_4(u_0, v_0) = \lim_{n \rightarrow \infty} R_4(u_n, v_n) = 0,$$

contradicting (5.1).

Step L (Conclusion). Therefore, $S < \Lambda$, which is (5.2), and $\varepsilon := \Lambda - S > 0$ satisfies $|c_1(f)|^2 \leq \Lambda - \varepsilon$ for all $f \in \mathcal{M}$. Since $|a_2| = L_1|c_1|/(2\Phi_2)$ by (2.13), $\sup_{f \in \mathcal{M}} |a_2| = \frac{L_1}{2\Phi_2} \sqrt{S} < \frac{L_1}{2\Phi_2} \sqrt{\Lambda} = \frac{L_1}{\Phi_2 \sqrt{1+|\kappa|}}$, which is (5.3). $\square$

The argument proceeds by contradiction on an extremizing sequence and produces no numerical value of ε . The bounds (5.2) and (5.3) are strict upper bounds, not sharp estimates, and the values $\sup_{f \in \mathcal{M}} |c_1(f)|^2$ and $\sup_{f \in \mathcal{M}} |a_2|$ are not determined here. A quantitative ε would require a lower bound for $|R_4(u, v)|$ over Schwarz pairs whose second-order data lie near the equality point; Theorem 3.10 controls only x, y and not u_3, u_4, v_3, v_4 .

6. Exact specializations

In Sections 6.1 and 6.2, $\lambda = 0$ and $\gamma = 0$, so that $\Upsilon_n = 1$ and

$$\Phi_n = 1 \quad (n \geq 2). \quad (6.1)$$

In this case, $\mathcal{D}_\gamma f(z) = f(z)/z$, $\mathcal{M}$ is the class of bi-univalent f with $f(z)/z < \Psi$ and $g(w)/w < \Psi$, and $\kappa = L_1 - L_2/L_1$. When Ψ also satisfies the Ma–Minda hypotheses [28], this is a bi-univalent class of the Ma–Minda type. Under (6.1), the groups of (4.25) become the following with $M = L_1^2 - L_2$:

$$\begin{aligned} T_1 &= 2L_1^6, & T_2 &= -3L_1^2(-\sigma ML_1 + 2ML_2 + L_1L_3), \\ T_3 &= ML_1^2 + M^2L_2 - 2\sigma ML_1L_2 + 3ML_1L_3 + L_1^2L_4, \end{aligned} \quad (6.2)$$

and $P_\sigma = T_1 + T_2 + T_3$ by (4.26).

6.1. The Janowski generator

Let [30]

$$\Psi_{A,B}(z) = \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1. \quad (6.3)$$

Expanding $(1 + Az)(1 + Bz)^{-1} = (1 + Az) \sum_{n \geq 0} (-B)^n z^n$ and collecting the coefficient of z^n for $n \geq 1$, we have

$$L_n = (-B)^n + A(-B)^{n-1} = (-B)^{n-1}(A - B), \quad (6.4)$$

so

$$L_1 = A - B > 0, \quad L_2 = -B(A - B), \quad L_3 = B^2(A - B), \quad L_4 = -B^3(A - B). \quad (6.5)$$

All four are real, and L_4 is prescribed by the generator.

Corollary 6.1. *Let $\Psi = \Psi_{A,B}$ and $\Phi_n = 1$. Then*

$$\kappa = A, \quad (6.6)$$

and Theorem 3.5 and Corollary 3.6 give the following on V_2 and for every $f \in \mathcal{M}$:

$$|c_1|^2 \leq \frac{4}{1 + |A|}, \quad |a_2| \leq \frac{A - B}{\sqrt{1 + |A|}}. \quad (6.7)$$

(i) If $A \neq 0$, then $\sigma = \operatorname{sgn} A$ is defined, and

$$P_\sigma = A(A - B)^3(A + \sigma)(2A + \sigma - B) \neq 0, \quad (6.8)$$

so the equality orbit fails fifth-order inverse compatibility, and by Theorem 5.1, we have

$$\sup_{f \in \mathcal{M}} |c_1(f)|^2 < \frac{4}{1 + |A|}, \quad \sup_{f \in \mathcal{M}} |a_2| < \frac{A - B}{\sqrt{1 + |A|}}. \quad (6.9)$$

(ii) If $A = 0$, then $\kappa = 0$, σ is undefined and (6.8) does not apply. The zero-branch polynomial (4.30) satisfies $P_0 = 0$, so the equality pair $(z, -z)$ passes the fifth-order test. Theorem 5.1 does not apply, and no strict separation is obtained. Nothing is asserted about whether the bound $|a_2| \leq -B$ is reached in $\mathcal{M}$.

Proof. By (6.5) and $\Phi_n = 1$, $\kappa = L_1 - L_2/L_1 = (A - B) - \frac{-B(A-B)}{A-B} = A - B + B = A$, which is (6.6); (6.7) is then Theorem 3.5 and Corollary 3.6 with $L_1 = A - B$, $\Phi_2 = 1$.

(i) Let $A \neq 0$ and write $\ell = A - B > 0$. By (6.5), $L_1 = \ell$, $L_2 = -B\ell$, $L_3 = B^2\ell$, $L_4 = -B^3\ell$, and $M = L_1^2 - L_2 = \ell^2 + B\ell = \ell(\ell + B) = A\ell$. Substituting into (6.2), we have the following:

$$\begin{aligned} T_1 &= 2\ell^6, \\ T_2 &= -3\ell^2(-\sigma A\ell \cdot \ell + 2A\ell \cdot (-B\ell) + \ell \cdot B^2\ell) = 3\ell^4(\sigma A + 2AB - B^2), \\ T_3 &= A\ell \cdot \ell^2 + A^2\ell^2 \cdot (-B\ell) - 2\sigma A\ell \cdot \ell \cdot (-B\ell) + 3A\ell \cdot \ell \cdot B^2\ell + \ell^2 \cdot (-B^3\ell) \\ &= \ell^3(A - A^2B + 2\sigma AB + 3AB^2 - B^3). \end{aligned}$$

Hence, $P_\sigma = \ell^3 E$ with $E = 2\ell^3 + 3\ell(\sigma A + 2AB - B^2) + A - A^2B + 2\sigma AB + 3AB^2 - B^3$. Expanding $\ell = A - B$, we have

$$\begin{aligned} 2\ell^3 &= 2A^3 - 6A^2B + 6AB^2 - 2B^3, \\ 3\ell(\sigma A + 2AB - B^2) &= 3\sigma A^2 + 6A^2B - 9AB^2 - 3\sigma AB + 3B^3. \end{aligned}$$

Collecting by monomial, A^3 : 2; A^2B : $-6 + 6 - 1 = -1$; AB^2 : $6 - 9 + 3 = 0$; B^3 : $-2 + 3 - 1 = 0$; σA^2 : 3; σAB : $-3 + 2 = -1$; A : 1. Thus

$$E = 2A^3 - A^2B + 3\sigma A^2 - \sigma AB + A = A(2A^2 - AB + 3\sigma A - \sigma B + 1).$$

Using $\sigma^2 = 1$, $(A + \sigma)(2A + \sigma - B) = 2A^2 + \sigma A - AB + 2\sigma A + \sigma^2 - \sigma B = 2A^2 - AB + 3\sigma A - \sigma B + 1$, so $P_\sigma = A(A - B)^3(A + \sigma)(2A + \sigma - B)$.

We show that each factor is nonzero. First, $A \neq 0$ by assumption, and $A - B > 0$ by (6.3). If $A > 0$, then $\sigma = 1$, $A + 1 > 0$, and $B < A$ give $2A + 1 - B > 2A + 1 - A = A + 1 > 0$. If $A < 0$, then $\sigma = -1$, $A - 1 < 0$, and $B \geq -1$ give $2A - 1 - B \leq 2A - 1 + 1 = 2A < 0$. In either case, $(A + \sigma)(2A + \sigma - B) > 0$, so $P_\sigma \neq 0$ (with the sign of A). By Lemma 4.3, Theorem 4.4, and Lemma 4.10, $R_4 \neq 0$ at every point of the equality orbit, and Theorem 5.1 gives (6.9).

(ii) Let $A = 0$. Then $\kappa = 0$, and $L_1 = -B > 0$ forces $B < 0$. By (6.5), $L_3 = B^2(-B) = -B^3$ and $L_4 = -B^3(-B) = B^4$. With $\Phi_n = 1$, (4.30) gives

$$P_0 = 2L_1^6 - 3L_1^3L_3 + L_1^2L_4 = 2B^6 - 3(-B)^3(-B^3) + B^2B^4 = 2B^6 - 3B^6 + B^6 = 0.$$

So $R_4(z, -z) = 0$ and the hypothesis (5.1) of Theorem 5.1 fails. $\square$

For $A \neq 0$, Corollary 6.1(i) is a strict statement about the full class on a two-parameter family of generators, obtained without any additional hypothesis on L_4 , because L_4 is prescribed by $\Psi_{A,B}$. For $A > 0$, the envelope (6.7) coincides with the summation bound (Remark 3.7); for $A < 0$, it improves it. Three statements should be kept apart here. (1) *Fifth-order compatibility fails on the V_2 equality orbit*: This is proved. (2) *No member of $\mathcal{M}$ reaches or approaches the envelope value $(A - B)/\sqrt{1 + |A|}$ for $|a_2|$ when $A \neq 0$* : This is proved and is the content of Theorem 5.1; whether $\sup_{f \in \mathcal{M}} |a_2|$ is reached by some member of $\mathcal{M}$ is not addressed. (3) *The exact value of $\sup_{f \in \mathcal{M}} |a_2|$* : This is not determined and remains open. Thus the second-order equality orbit cannot realize the V_2 envelope in the full class; the argument yields a strict separation of the full-class supremum from that envelope, not the value of the supremum. For $A = 0$, none of (1)–(3) is asserted.

6.2. The exponential generator

Let $\Psi(z) = e^z$, so

$$L_1 = 1, \quad L_2 = \frac{1}{2}, \quad L_3 = \frac{1}{6}, \quad L_4 = \frac{1}{24}, \quad (6.10)$$

which are all real, with $L_1 > 0$.

Corollary 6.2. *For $\Psi(z) = e^z$ and $\Phi_n = 1$, we have*

$$\kappa = \frac{1}{2}, \quad \sigma = 1, \quad K = \frac{1}{2}, \quad (6.11)$$

$$P_{+1} = \frac{23}{12} \neq 0, \quad L_{4,\text{lift}}^{(1)} = -\frac{15}{8} \neq \frac{1}{24} = L_4, \quad (6.12)$$

and therefore

$$\sup_{f \in \mathcal{M}} |a_2| < \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}. \quad (6.13)$$

Proof. By (6.10) and $\Phi_n = 1$, $\kappa = L_1 - L_2/L_1 = 1 - \frac{1}{2} = \frac{1}{2}$, which is (6.11).

Reduction of the general formula. Set $\Phi_2 = \Phi_3 = \Phi_4 = \Phi_5 = 1$, $L_1 = 1$, $\sigma = 1$ in (4.16), keeping L_2, L_3, L_4 symbolic. The 16 monomials become the following, in the order of (4.16):

$$\begin{array}{cccc} -L_2, & 2L_2^2, & L_2^3, & -3L_2L_3, \\ L_4, & 1, & -2L_2, & -2L_2^2, \\ 3L_3, & L_2, & -3L_2, & 6L_2^2, \\ -3L_3, & 3, & -6L_2, & 2. \end{array}$$

Collecting: The constants $1 + 3 + 2 = 6$; L_2 : $-1 - 2 + 1 - 3 - 6 = -11$; L_2^2 : $2 - 2 + 6 = 6$; L_2^3 : 1 ; L_2L_3 : -3 ; L_3 : $3 - 3 = 0$; L_4 : 1 . Hence

$$P_{+1} = 6 - 11L_2 + 6L_2^2 + L_2^3 - 3L_2L_3 + L_4. \quad (6.14)$$

Substitution. With $L_2 = \frac{1}{2}$, $L_3 = \frac{1}{6}$, and $L_4 = \frac{1}{24}$, we have

$$P_{+1} = 6 - \frac{11}{2} + \frac{6}{4} + \frac{1}{8} - \frac{1}{4} + \frac{1}{24} = \frac{144}{24} - \frac{132}{24} + \frac{36}{24} + \frac{3}{24} - \frac{6}{24} + \frac{1}{24} = \frac{46}{24} = \frac{23}{12}.$$

As an independent check, (6.2) with $L_1 = 1$, $M = 1 - L_2 = \frac{1}{2}$, and $\sigma = 1$ gives $T_1 = 2$, $T_2 = -3(-\frac{1}{2} + \frac{1}{2} + \frac{1}{6}) = -\frac{1}{2}$, $T_3 = \frac{1}{2} + \frac{1}{8} - \frac{1}{2} + \frac{1}{4} + \frac{1}{24} = \frac{5}{12}$, and $T_1 + T_2 + T_3 = \frac{24-6+5}{12} = \frac{23}{12}$.

Lifting value. In (6.14), the coefficient of L_4 is 1, in agreement with (4.27) ($L_1^2\Phi_2^4\Phi_4 = 1$). Hence $L_{4,\text{lift}}^{(1)} = -(6 - 11L_2 + 6L_2^2 + L_2^3 - 3L_2L_3)$, and at $L_2 = \frac{1}{2}$, $L_3 = \frac{1}{6}$, we have

$$L_{4,\text{lift}}^{(1)} = -\frac{144 - 132 + 36 + 3 - 6}{24} = -\frac{45}{24} = -\frac{15}{8}.$$

Equivalently, $P_{+1} = L_4 - L_{4,\text{lift}}^{(1)} = \frac{1}{24} + \frac{45}{24} = \frac{46}{24}$. The generator prescribes $L_4 = \frac{1}{24} \neq -\frac{15}{8}$, so the equality orbit fails fifth-order inverse compatibility. Theorem 5.1 applies, and the envelope (3.17) equals $L_1/(\Phi_2\sqrt{1+\kappa}) = 1/\sqrt{3/2} = \sqrt{2/3}$, giving (6.13). $\square$

Since $\kappa = \frac{1}{2} > 0$, the value $\sqrt{2/3}$ is also the summation bound of Remark 3.7. The distinction made after Corollary 6.1 applies here: The fifth-order obstruction excludes the identified equality orbit as the truncation of a member of $\mathcal{M}$ and, through Theorem 5.1, yields strict separation for the full class, but it does not determine the exact value of $\sup_{f \in \mathcal{M}} |a_2|$, which remains open.

6.3. A conditional (p, q) -operator example

Fix

$$p = \frac{9}{10}, \quad q = \frac{1}{2}, \quad \delta = \frac{1}{10}, \quad \lambda = 1, \quad \gamma = \frac{1}{2}, \quad (6.15)$$

and take

$$L_1 = \frac{1}{\sqrt{2}}, \quad L_2 = \frac{1}{4}, \quad L_3 = \frac{1}{12\sqrt{2}}. \quad (6.16)$$

(These are the values of $\sin \theta$, $\frac{1}{4} \sin 2\theta$, $\frac{1}{6} \sin^3 \theta$ at $\theta = \pi/4$.) No generating function is specified here, and (6.16) does not determine L_4 . Every real value of L_4 occurs: The polynomial $\Psi(z) = 1 + L_1 z + L_2 z^2 + L_3 z^3 + L_4 z^4$ is analytic in $\mathbb{D}$ and satisfies (2.3) for any real L_4 . Accordingly, any statement below that depends on L_4 is stated as a hypothesis.

From (2.1) and (2.2), with $1 + (n-1)\gamma = \frac{n+1}{2}$,

$$[2]_{p,q} = p + q = \frac{7}{5}, \quad \Upsilon_2 = \frac{7/5+1/10}{11/10} = \frac{15}{11}, \quad \Phi_2 = \frac{15}{11} \cdot \frac{3}{2} = \frac{45}{22}, \quad (6.17)$$

$$[3]_{p,q} = p^2 + pq + q^2 = \frac{151}{100}, \quad \Upsilon_3 = \frac{161}{110}, \quad \Phi_3 = \frac{161}{110} \cdot 2 = \frac{161}{55}, \quad (6.18)$$

$$[4]_{p,q} = \frac{371}{250}, \quad \Upsilon_4 = \frac{36}{25}, \quad \Phi_4 = \frac{36}{25} \cdot \frac{5}{2} = \frac{18}{5}, \quad (6.19)$$

$$[5]_{p,q} = \frac{13981}{10000}, \quad \Upsilon_5 = \frac{14981}{11000}, \quad \Phi_5 = \frac{14981}{11000} \cdot 3 = \frac{44943}{11000}. \quad (6.20)$$

Corollary 6.3. *For the data (6.15)–(6.20), we have*

$$\kappa = \frac{4043\sqrt{2}}{40500} = 0.141176924\dots > 0, \quad \sigma = 1, \quad (6.21)$$

and

$$L_{4,\text{lift}}^{(1)} = \frac{-3084612892 - 1088153235\sqrt{2}}{32805000000} = -0.140938697\dots < 0. \quad (6.22)$$

Assume, in addition, that the prescribed fourth coefficient satisfies $L_4 > 0$. Then $L_4 \neq L_{4,\text{lift}}^{(1)}$ and hence $P_{+1} \neq 0$ and $R_4 \neq 0$ at every point of the equality orbit; the equality orbit does not satisfy fifth-order inverse compatibility, and by Theorem 5.1, we have

$$\sup_{f \in \mathcal{M}} |a_2| < \frac{L_1}{\Phi_2 \sqrt{1+\kappa}} = 0.323607574\dots \quad (6.23)$$

Proof. The exact arithmetic in $\mathbb{Q}(\sqrt{2})$ leading to (6.21) and (6.22) is given in Appendix C: Reduction of $L_1\Phi_3/\Phi_2^2$ against L_2/L_1 yields $\kappa = 4043\sqrt{2}/40500 > 0$, so $\sigma = 1$; substitution of (6.16) and (6.17)–(6.20) with $\sigma = 1$ into (4.28) yields (6.22), which is negative because both terms of the numerator are negative.

Assume $L_4 > 0$. Since $L_{4,\text{lift}}^{(1)} < 0 < L_4$, we have $L_4 \neq L_{4,\text{lift}}^{(1)}$, so $P_{+1} \neq 0$ by Corollary 4.5, and $R_4 \neq 0$ by (4.17); Lemma 4.10 gives $R_4^{(\rho)} = \rho^4 R_4 \neq 0$ for every $|\rho| = 1$. Theorem 5.1 gives (6.23), the numerical value being $L_1/(\Phi_2 \sqrt{1+\kappa})$ with $L_1 = 1/\sqrt{2}$, $\Phi_2 = 45/22$ and κ as in (6.21). $\square$

Remark 6.4. Corollary 6.3 is conditional. The hypothesis $L_4 > 0$ does not follow from (6.16); it is one convenient sufficient condition. The proof uses only $L_4 \neq L_{4,\text{lift}}^{(1)}$, so the conclusion holds equally for $L_4 \geq 0$ and for every negative $L_4 \neq L_{4,\text{lift}}^{(1)}$. For the polynomial generator with $L_4 = L_{4,\text{lift}}^{(1)}$, the fifth-order test is passed and Theorem 5.1 yields nothing. Under the hypothesis $L_4 > 0$, the distinctions (1)–(3) made after Corollary 6.1 apply verbatim with the envelope value $0.32360757\dots$; without that hypothesis, none of them is asserted.

7. Relation to the literature and levels of contribution

Related work. Coefficient problems for bi-univalent functions began with the estimates of $|a_2|$ over Σ in [1–3] and for bi-starlike and bi-convex functions in [4]. For subordination-defined subclasses [5, 6, 8, 9], including bi-univalent classes of the Ma–Minda type [7], the forward and inverse relations for a_2 and a_3 are combined, and the resulting expressions are estimated by $|c_n|, |d_n| \leq 2$; Fekete–Szegő functionals were treated in the same framework in [10], and Hankel determinants in [11–14], which included a class defined by a (p, q) -derivative operator [12] and a class generated by a q -operator and special polynomials [14]. Inverse coefficients entered these works through the reversion formulas; the Faber-polynomial approach [15, 16] expresses them to all orders. For sharp inverse-coefficient estimates over classical univalent classes, see [17–19]. Operator-defined subclasses have also been studied in the meromorphic setting; see, for example, [31].

Levels of contribution. We separate three levels.

(A) *The coefficient-body bound.* Theorem 3.5 gives $|c_1|^2 \leq 4/(1 + |\kappa|)$ on V_2 . For $\kappa \geq 0$, this is numerically the bound obtained by applying to $\mathcal{M}$ the summation argument used in [6–9]; for $\kappa < 0$, it improves it, because the second-order parametrization of [19] is used jointly for h and k (Remark 3.7). No improvement of known constants is claimed for $\kappa \geq 0$.

(B) *The extremal characterization.* The equality set of the V_2 problem is a single rotation orbit of an explicit Blaschke pair (Lemma 3.8), and near-optimal data are quantitatively close to it (Theorem 3.10).

(C) *The lifting and realizability analysis.* That orbit is tested against the inverse-compatibility relations of orders 2 to 5 (Lemma 4.3, Theorem 4.4); when the fifth-order relation fails, the V_2 envelope is strictly separated from the full-class supremum (Theorem 5.1).

The coefficient estimate literature cited above typically optimizes truncated forward and inverse coefficient data and, where one is available, records an equality point of the truncated problem. The present work instead addresses the additional realizability question for the identified equality configuration. We have not found an analysis of this question in the works cited here; this is not a claim about the literature as a whole, and no priority claim is attached to the comparison. A related observation, namely that the structure of an extremal configuration may matter as much as the numerical value of a coefficient bound, is made in [32] for a different family.

8. Sharpness and higher-order realizability

The results fit into the following chain, each step of which is proved above.

- (1) Every $f \in \mathcal{M}$ determines a point of V_2 (Lemma 2.4, Proposition 3.2).
- (2) On V_2 , $|c_1|^2 \leq 4/(1 + |\kappa|)$, and this bound is sharp on V_2 ; so are its consequence $|a_2| \leq L_1/(\Phi_2 \sqrt{1 + |\kappa|})$ and the Zalcman-type bound (3.18), with a_2, a_3 defined on V_2 by (2.13)–(2.14) (Theorem 3.5, Corollary 3.6, Proposition 3.9).
- (3) Equality in the V_2 envelope holds exactly on the rotation orbit of (3.13) for $\kappa \neq 0$ or of (3.14) for $\kappa = 0$ (Lemma 3.8), and near-equality forces x, y close to σ (Theorem 3.10).
- (4) $R_1 = R_2 = 0$ at every point of V_2 ; on the equality orbit, in addition, $a_3 = a_2^2$ and $R_3 = 0$ identically (Section 4, Lemma 4.3, Remark 4.8).
- (5) The first residual that can be nonzero is R_4 , and $R_4 = 0$ if and only if $P_\sigma = 0$ (respectively, $P_0 = 0$ when $\kappa = 0$); for fixed lower-order data, this happens for exactly one value of the fourth Taylor

coefficient of Ψ (Theorem 4.4, Corollary 4.5, Remarks 4.6 and 4.8).

- (6) If $P_\sigma \neq 0$, no member of $\mathcal{M}$ has a Schwarz pair on the equality orbit; other points of V_2 are not addressed (Remark 4.7).
- (7) If $P_\sigma \neq 0$, rigidity and normality give the strict separation $\sup_{f \in \mathcal{M}} |c_1|^2 < 4/(1+|\kappa|)$ (Theorem 5.1); this is not a sharp estimate for $\mathcal{M}$.
- (8) The exact value of $\sup_{f \in \mathcal{M}} |c_1|^2$, equivalently of $\sup_{f \in \mathcal{M}} |a_2|$, remains open.

The hypothesis $P_\sigma \neq 0$ holds for the Janowski generators with $A \neq 0$ and for $\Psi(z) = e^z$; for the (p, q) -example of Section 6.3, it holds under the additional assumption $L_4 > 0$ (more generally, $L_4 \neq L_{4,\text{lift}}^{(1)}$); for the Janowski generators with $A = 0$, the corresponding polynomial P_0 vanishes and no separation is obtained.

9. Conclusions

The coupling of the two subordinations is encoded at second order by the scalar κ . The envelope $|c_1|^2 \leq 4/(1+|\kappa|)$ is sharp on the second-order-compatible body, and is not asserted to be sharp for $\mathcal{M}$; its equality set is a unique orbit; that orbit passes the inverse compatibility conditions through R_3 identically, and the fifth-order condition reduces to $P_\sigma = 0$. When $P_\sigma \neq 0$, the envelope is strictly larger than $\sup_{f \in \mathcal{M}} |c_1|^2$. This is verified in closed form for the Janowski generators with $A \neq 0$ and for $\Psi(z) = e^z$. In these cases, the equality configuration of the truncated problem does not correspond to any member of the full class because the inverse relation at order five is violated. Two questions remain open: The exact value of $\sup_{f \in \mathcal{M}} |a_2|$ and an effective lower bound for $|R_4|$ over Schwarz pairs whose second-order data lie near the equality point, which would make the separation quantitative.

Author contributions

Zainab H. Mahmood: Conceptualization, formal analysis, investigation, methodology, writing—original draft, writing—review and editing; Reem O. Rasheed: Formal analysis, investigation, validation, writing—review and editing; Bassim K. Mihsin: Investigation, software, validation; Waggas Galib Atshan: Conceptualization, methodology, supervision, writing—review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have used Artificial Intelligence (AI) tools in the creation of this article.

AI tools used: Claude (Anthropic).

How were the AI tools used? During the revision in response to the referees' reports, the tool was used to edit and reorganize the exposition, to draft expanded write-ups of proofs of results stated in the submitted manuscript (particularly the derivation in the proof of Theorem 4.4, the proof of Theorem 5.1, and Lemmas 3.11 and 4.11), to recheck coefficient identities, to prepare the symbolic verification script of Supplementary File S2, and to update the cross-references of Supplementary File S1 to the revised numbering. All mathematical statements, proofs and computations were checked by the authors, who take full responsibility for the content of this article.

Where in the article is the information located? Sections 1–9, Appendices A–C, Supplementary File S2, and the cross-references of Supplementary File S1.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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Appendix A. Reversion coefficients up to the fifth order

Write $g(w) = w + b_2w^2 + b_3w^3 + b_4w^4 + b_5w^5 + O(w^6)$ and substitute into $f(g) = g + a_2g^2 + a_3g^3 + a_4g^4 + a_5g^5 + O(g^6)$. The powers of g are

$$\begin{aligned} g^2 &= w^2 + 2b_2w^3 + (b_2^2 + 2b_3)w^4 + (2b_4 + 2b_2b_3)w^5 + O(w^6), \\ g^3 &= w^3 + 3b_2w^4 + (3b_3 + 3b_2^2)w^5 + O(w^6), \quad g^4 = w^4 + 4b_2w^5 + O(w^6), \quad g^5 = w^5 + O(w^6). \end{aligned}$$

Hence

$$\begin{aligned} f(g(w)) &= w + (b_2 + a_2)w^2 + (b_3 + 2a_2b_2 + a_3)w^3 + (b_4 + a_2(b_2^2 + 2b_3) + 3a_3b_2 + a_4)w^4 \\ &\quad + (b_5 + a_2(2b_4 + 2b_2b_3) + a_3(3b_3 + 3b_2^2) + 4a_4b_2 + a_5)w^5 + O(w^6). \end{aligned}$$

Setting $f(g(w)) = w$ and equating the coefficients of w^2, w^3, w^4, w^5 to zero gives the following, in turn:

$$\begin{aligned} b_2 &= -a_2, \\ b_3 &= -2a_2b_2 - a_3 = 2a_2^2 - a_3, \\ b_4 &= -a_2(b_2^2 + 2b_3) - 3a_3b_2 - a_4 = -a_2(a_2^2 + 4a_2^2 - 2a_3) + 3a_2a_3 - a_4 = -5a_2^3 + 5a_2a_3 - a_4, \end{aligned}$$

and, using $b_2b_3 = -2a_2^3 + a_2a_3$ and $b_2^2 = a_2^2$, we have

$$\begin{aligned} b_5 &= -2a_2b_4 - 2a_2b_2b_3 - 3a_3b_3 - 3a_3b_2^2 - 4a_4b_2 - a_5 \\ &= -2a_2(-5a_2^3 + 5a_2a_3 - a_4) - 2a_2(-2a_2^3 + a_2a_3) - 3a_3(2a_2^2 - a_3) - 3a_2^2a_3 + 4a_2a_4 - a_5 \\ &= (10 + 4)a_2^4 + (-10 - 2 - 6 - 3)a_2^2a_3 + (2 + 4)a_2a_4 + 3a_3^2 - a_5 \\ &= 14a_2^4 - 21a_2^2a_3 + 6a_2a_4 + 3a_3^2 - a_5, \end{aligned}$$

which is (2.16).

Appendix B. Term table for the fifth-order obstruction polynomial

Table 1 lists, for each monomial produced in Step 4 of the proof of Theorem 4.4, its source in (4.25), the expanded monomial with its signed coefficient, and its position in (4.16) (numbered $m_1, \dots, m_{16}$ left to right, line by line). Each of $m_1, \dots, m_{16}$ occurs exactly once, and the 16 monomials are pairwise distinct as monomials in $\sigma, L_1, \dots, L_4, \Phi_2, \dots, \Phi_5$; hence, no cancellation occurs and $T_1 + T_2 + T_3 = P_\sigma$.

Table 1. Source terms, expanded monomial, and destination in (4.16).

Source term in (4.25)	Expanded monomial (signed)	Destination
$T_1 = 2L_1^6\Phi_4\Phi_5$	$+2L_1^6\Phi_4\Phi_5$	m_{16}
$T_2: 3\sigma L_1^3\Phi_2\Phi_5 \cdot L_1^2\Phi_3$	$+3\sigma L_1^5\Phi_2\Phi_3\Phi_5$	m_{14}
$T_2: 3\sigma L_1^3\Phi_2\Phi_5 \cdot (-L_2\Phi_2^2)$	$-3\sigma L_1^3L_2\Phi_2^3\Phi_5$	m_{11}
$T_2: -6L_1^2L_2\Phi_2\Phi_5 \cdot L_1^2\Phi_3$	$-6L_1^4L_2\Phi_2\Phi_3\Phi_5$	m_{15}
$T_2: -6L_1^2L_2\Phi_2\Phi_5 \cdot (-L_2\Phi_2^2)$	$+6L_1^2L_2^2\Phi_2^3\Phi_5$	m_{12}
$T_2: -3L_1^2\Phi_2\Phi_5 \cdot L_1L_3\Phi_2^2$	$-3L_1^3L_3\Phi_2^3\Phi_5$	m_{13}
$T_3: \Phi_4 \cdot ML_1^2\Phi_2^2$, part $L_1^2\Phi_3$	$+L_1^4\Phi_2^2\Phi_3\Phi_4$	m_6
$T_3: \Phi_4 \cdot ML_1^2\Phi_2^2$, part $-L_2\Phi_2^2$	$-L_1^2L_2\Phi_2^4\Phi_4$	m_1
$T_3: \Phi_4 \cdot M^2L_2$, part $L_1^4\Phi_3^2$	$+L_1^4L_2\Phi_3^2\Phi_4$	m_{10}
$T_3: \Phi_4 \cdot M^2L_2$, part $-2L_1^2L_2\Phi_2^2\Phi_3$	$-2L_1^2L_2^2\Phi_2^2\Phi_3\Phi_4$	m_8
$T_3: \Phi_4 \cdot M^2L_2$, part $L_2^2\Phi_2^4$	$+L_2^3\Phi_2^4\Phi_4$	m_3
$T_3: \Phi_4 \cdot (-2\sigma ML_1L_2\Phi_2^2)$, part $L_1^2\Phi_3$	$-2\sigma L_1^3L_2\Phi_2^2\Phi_3\Phi_4$	m_7
$T_3: \Phi_4 \cdot (-2\sigma ML_1L_2\Phi_2^2)$, part $-L_2\Phi_2^2$	$+2\sigma L_1L_2^2\Phi_2^4\Phi_4$	m_2
$T_3: \Phi_4 \cdot 3ML_1L_3\Phi_2^2$, part $L_1^2\Phi_3$	$+3L_1^3L_3\Phi_2^2\Phi_3\Phi_4$	m_9
$T_3: \Phi_4 \cdot 3ML_1L_3\Phi_2^2$, part $-L_2\Phi_2^2$	$-3L_1L_2L_3\Phi_2^4\Phi_4$	m_4
$T_3: \Phi_4 \cdot L_1^2\Phi_2^4L_4$	$+L_1^4L_4\Phi_2^4\Phi_4$	m_5

In the T_3 rows, “part” refers to the summand of $M = L_1^2\Phi_3 - L_2\Phi_2^2$ or of $M^2 = L_1^4\Phi_3^2 - 2L_1^2L_2\Phi_2^2\Phi_3 + L_2^2\Phi_2^4$ that produces the monomial.

Appendix C. Exact arithmetic for the (p, q) -example

By (6.17) and (6.18), $\Phi_2^2 = 2025/484$ and

$$\frac{L_1\Phi_3}{\Phi_2^2} = \frac{1}{\sqrt{2}} \cdot \frac{161}{55} \cdot \frac{484}{2025} = \frac{1}{\sqrt{2}} \cdot \frac{77924}{111375} = \frac{1}{\sqrt{2}} \cdot \frac{7084}{10125} = \frac{3542\sqrt{2}}{10125},$$

where $77924 = 2^2 \cdot 7 \cdot 11^2 \cdot 23$ and $111375 = 3^4 \cdot 5^3 \cdot 11$ were reduced by 11. Moreover, $L_2/L_1 = \frac{\sqrt{2}}{4}$. Hence

$$\kappa = \frac{3542\sqrt{2}}{10125} - \frac{\sqrt{2}}{4} = \sqrt{2} \cdot \frac{4 \cdot 3542 - 10125}{40500} = \frac{4043\sqrt{2}}{40500} > 0,$$

which is (6.21). Substituting (6.16) and (6.17)–(6.20) with $\sigma = 1$ into (4.28) and computing in $\mathbb{Q}(\sqrt{2})$, we have

$$L_1^2\Phi_2^4\Phi_4 = \frac{1}{2} \cdot \frac{4100625}{234256} \cdot \frac{18}{5} = \frac{7381125}{234256}, \quad -P_{+1}|_{L_4=0} = -\frac{6940379007}{2342560000} - \frac{1958675823\sqrt{2}}{1874048000},$$

so

$$L_{4,\text{lift}}^{(1)} = -\frac{771153223}{8201250000} - \frac{24181183\sqrt{2}}{729000000} = \frac{-3084612892 - 1088153235\sqrt{2}}{32805000000},$$

which is (6.22).

Supplementary

Supplementary File S1 (Supplementary_Material_Wolfram.pdf, with the Wolfram Mathematica notebook Wolfram_Exact_Verification_Report.nb, the script Wolfram_Final_Verification_Report.wl and the report Wolfram_Exact_Verification_Report.pdf) contains secondary coefficient consequences that are not used in the main text, the lifting behavior of the equality configuration of Proposition 3.9, the sign of $L_{4,\text{lift}}^{(1)}$ along the family of which (6.16) is one value, and an exact symbolic Wolfram verification of the displayed identities.

Supplementary File S2 (Math-30790_Supplementary_S2.py, Python 3 with SymPy, tested with SymPy 1.14) is a second, independent check of the displayed algebra. It uses exact arithmetic only (integers, rationals, symbols and the algebraic number $\sqrt{2}$); decimals are printed for comparison and are not used in any check. The symbolic variables are $L_1, \dots, L_4$ (real); $\Phi_2, \dots, \Phi_5$ (positive); K (positive); the parameters t, x, u_3, u_4, v_3, v_4 ; and the Janowski parameters A, B . The sign σ is substituted as $+1$ and -1 separately. The script checks the following: The reversion formulas (2.16); Lemma 2.3; $R_1 = R_2 = 0$ on V_2 ; the coefficients (3.15) and the relation $v_j = (-1)^j u_j$; Lemma 4.3; the bracket form (4.13), the identity (4.14), the compact form (4.15), the 16-term form (4.16) and $\partial P_\sigma / \partial L_4 = L_1^2 \Phi_2^4 \Phi_4$; the branch $\kappa = 0$ of Remark 4.8; Corollaries 6.1–6.3; and Appendix C. Each check prints PASS or stops with an assertion error.

Both computations verify the displayed algebra and are not used as a substitute for any proof; all proofs above are self-contained.



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