



Research article

Seidel polynomials of an edge embedded t -partite Turán graphs and Seidel energy analysis for the 4-partite case

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Abstract: For a simple graph T of order n , the Seidel matrix is defined by $S(T) = J - I - 2A(T)$, where J is the all-ones matrix, I is the identity matrix, and $A(T)$ is the adjacency matrix. The characteristic polynomial of $S(T)$ is called the Seidel polynomial, and the sum of the absolute values of its eigenvalues is the Seidel energy of T . In this paper, we construct explicit Seidel polynomials for t -partite Turán graphs $T(n, t)$ after embedding an edge between two non-adjacent vertices. Next, we analyze the effect on the Seidel energy of the 4-partite Turán graph $T(n, 4)$, extending the previous results for bipartite and tripartite Turán graphs. We prove that for $n \geq 12$, the Seidel energy strictly increases under edge embedding in the 4-partite case. Meanwhile, we also prove that for $t \geq 3$, the Seidel energy of t -partite Turán graph $T(2t, t) \cong K_{2,2,\dots,2}$ strictly decreases under edge embedding. The proofs rely on block determinant techniques, coefficient relations of characteristic polynomials, and explicit root bounds obtained via the intermediate value theorem. These results contribute to the broader understanding of extremal properties of Seidel energy and provide new evidence toward the general monotonicity question posed in recent literature.

Keywords: t -partite Turán graph; edge embedding; Seidel polynomial; Seidel energy

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1. Introduction

Throughout this paper, p and t are positive integers. For a simple graph $T = (V, E)$ of order n , the Seidel matrix $S(T) = [s_{ij}]$ is defined by

$$s_{ij} = \begin{cases} 0, & i = j, \\ -1, & \{i, j\} \in E, \\ +1, & \{i, j\} \notin E. \end{cases}$$

This makes adjacency encoded by -1 , non-adjacency by $+1$, and zeros on the diagonal. The characteristic polynomial $\det(\mu I - S(T))$ of the Seidel matrix $S(T)$ is called the Seidel polynomial of T , which we denote by $S_T(\mu)$. The eigenvalues (roots of $S_T(\mu)$) and the spectrum (consisting of n eigenvalues) of the Seidel matrix $S(T)$ are called the Seidel eigenvalues and the Seidel spectrum of T , respectively. Since $S(T)$ is symmetric, its eigenvalues are real [2]. We denote these eigenvalues by $\mu_1, \mu_2, \dots, \mu_n$, ordered so that $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$. Suppose that T has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ with multiplicities $r_1, r_2, \dots, r_m$, respectively. We write the spectrum of T as $\text{Spec}(T) = \{[\lambda_1]^{r_1}, [\lambda_2]^{r_2}, \dots, [\lambda_m]^{r_m}\}$. If an eigenvalue has multiplicity 1, we omit the square brackets and write it without an exponent. In [5], Haemers defined the Seidel energy of T , denoted by $\mathcal{E}(T)$, as

$$\mathcal{E}(T) = \sum_{i=1}^n |\mu_i|.$$

Let $K_{p_1, p_2, \dots, p_t}$ denote the complete t -partite graph with part sizes $p_1, p_2, \dots, p_t$, where $p_1 \geq p_2 \geq \dots \geq p_t > 0$ and $n = p_1 + p_2 + \dots + p_t$. The Turán graph $T(n, t)$ is the complete t -partite graph whose part sizes differ by at most one. More precisely, it can be written as

$$T(n, t) \cong K_{\underbrace{p+1, \dots, p+1}_{r \text{ parts}}, \underbrace{p, \dots, p}_{t-r \text{ parts}}}, \quad (1.1)$$

where the order of the graph is $n = pt + r$, $0 \leq r < t$, $p \geq 1$, and $t \geq 2$.

The Seidel energy of graphs has received increasing interest in spectral graph theory, particularly due to its extremal behavior among different graph families. Previous investigations have considered complete graphs, regular graphs, and strongly regular graphs; see, for example, [5, 8, 9]. Besides extremal problems, the variation of Seidel energy under graph operations, especially edge perturbations, has also been studied. One such result concerning the effect of deleting an edge from certain complete multipartite graphs that is stated below was obtained by Tian et al. [13].

Theorem 1. *For $p > 1$, the following statements hold.*

- (1) *For any edge e of $K_{p,p,p}$, $\mathcal{E}(K_{p,p,p}) > \mathcal{E}(K_{p,p,p} - e)$ when $p = 2$, and $\mathcal{E}(K_{p,p,p}) < \mathcal{E}(K_{p,p,p} - e)$ for all $p \geq 3$.*
- (2) *For any edge e of $K_{p,p,p+1}$, $\mathcal{E}(K_{p,p,p+1}) < \mathcal{E}(K_{p,p,p+1} - e)$ for all $p \geq 3$.*
- (3) *For any edge e of $K_{p,p,p-1}$, $\mathcal{E}(K_{p,p,p-1}) < \mathcal{E}(K_{p,p,p-1} - e)$ for all $p \geq 4$.*

The theorem above shows that deleting any edge from a tripartite Turán graph $T(n, 3)$ of order at least 9 always increases the Seidel energy. In the meanwhile, Liu and Chen [6, 7] investigated how edge deletion affects the Seidel energy of Turán graphs $T(n, 4)$ and $T(n, 5)$. They proved that

$$\mathcal{E}(T(n, 4)) < \mathcal{E}(T(n, 4) - e), \text{ for } n \geq 10,$$

and

$$\mathcal{E}(T(n, 5)) < \mathcal{E}(T(n, 5) - e), \text{ for } n \geq 5.$$

More recently, the focus has shifted to the opposite operation edge embedding, i.e., adding an edge between two non-adjacent vertices. In 2025, Tian et al. [14] studied the effects of edge embedding on the Seidel energy of bipartite Turán graphs. They proved the following result.

Theorem 2. ([14], Corollary 1). *Let $T(p + q, 2) \cong K_{p,q}$ be the Turán graph of order $n = p + q$. Let $e_1 \in E(K_{p,q})$ and $e_2 \notin E(K_{p,q})$ be arbitrary edges. Then the inequality $\mathcal{E}(T(p + q, 2)) < \mathcal{E}(T(p + q, 2) - e_1) = \mathcal{E}(T(p + q, 2) + e_2)$ holds for all $p, q \geq 2$.*

Inspired by Theorem 1, Rehman et al. [11] investigated how edge embedding influences the Seidel energy of tripartite Turán graphs. They proved that embedding an edge $e = u_1u_2$ into $T^*(8, 3) \cong K_{\{u_1, u_2\}, \{v_1, v_2, v_3\}, \{w_1, w_2, w_3\}}$ results in $\mathcal{E}(T^*(8, 3) + e) \approx 16.9282 < \mathcal{E}(T^*(8, 3)) \approx 17.2111$.

In contrast, they established that for every tripartite Turán graph $T(n, 3)$ with $n \geq 9$, the embedding of any edge strictly increases the Seidel energy. Motivated by these contrasting behaviors, they posed the following open problem.

Question 1. *Let $e \notin E(T(n, t))$ be an embedding edge of the Turán graph $T(n, t)$, where $t \geq 2$. Does there exist a sufficiently large integer n_0 such that*

$$\mathcal{E}(T(n, t)) < \mathcal{E}(T(n, t) + e)$$

for every $n \geq n_0$?

For further results on adjacency energy and distance energy of graphs, we refer readers to [1, 10, 15] for edge embedding, [3, 4, 12] for edge deletion, and the references therein.

In this paper, we consider the t -partite Turán graph $T(n, t)$ defined in (1.1) and investigate the effect of embedding an edge on its Seidel energy. Let $e \notin E(K_{\underbrace{p+1, \dots, p+1}_{r \text{ times}}, \underbrace{p, \dots, p}_{t-r \text{ times}}})$ join two vertices within the same part of $T(n, t)$ and denote the resulting graph by $T + e$.

In this study, we focus exclusively on graph $T + e$ and give a complete, self-contained derivation of its Seidel polynomials. This specialized treatment yields explicit polynomials, simplifies several arguments, and enables a detailed analysis of the Seidel energy that a general statement does not provide directly.

Question 1 is already answered for $t = 2$ [14] and $t = 3$ [11]. For general t , some roots of $S_{T+e}(\mu)$ cannot be expressed explicitly, so their location must be analyzed separately for each fixed t . We carry out this analysis in full for $t = 4$. Specifically, we establish an explicit and sharp threshold $n_0 = 12$ above beyond which the answer to Question 1 is affirmative, and we characterize the behavior at every order $n \geq 5$, including the exceptional orders $5 \leq n \leq 11$ (with the exception of $n = 8$, which is covered by Theorem 3). Table 1 situates these results within the existing literature.

Table 1. Effect of an edge operation on the Seidel energy of $T(n, t)$.

t	Operation	Reference	Effect on Seidel energy
2	embedding	[14]	all $p, q \geq 2, n = p + q$ Seidel energy increases.
3	deletion	[13]	$n \geq 9$ Seidel energy increases.
3	embedding	[11]	$n \geq 9$ Seidel energy increases (except for $T^*(8, 3)$).
4, 5	deletion	[6, 7]	$n \geq 10$ ($t = 4$), $n \geq 5$ ($t = 5$) Seidel energy increases.
4	embedding	this paper	$n \geq 12$ Seidel energy increases (for small $5 \leq n \leq 11$, see Table 2).
t	embedding	this paper	$n = 2t, t \geq 3$ Seidel energy decreases.

In particular, our contributions as follows: (i) The explicit Seidel polynomials $S_{T+e}(\mu)$ for all $t \geq 2$. This polynomials framework provides a systematic and transferable approach for higher partite cases. (ii) A self-contained proof that the Seidel energy decreases for $t \geq 3, p = 2$ (Theorem 3) and increases for $t \geq 4, p \geq 3$ (Theorem 4). (iii) A description of the smaller orders, given in Table 2 and verified by computer search. The workflow of the paper is summarized in Figure 1.

For $t = 2$, the inequality $\mathcal{E}(K_{2,2}) < \mathcal{E}(K_{2,2} + e_2)$ holds as shown in [14]. For $t \geq 3$ and $p = 2$, we establish the following result.

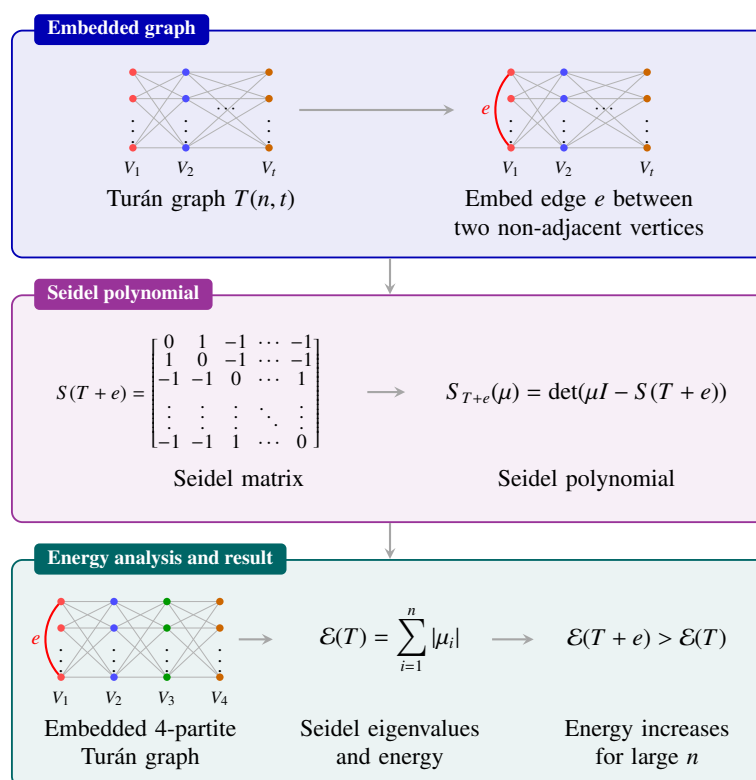
**Figure 1.** Research framework of the paper.

Table 2. Seidel energy comparison for selected small-order graphs. Here, $\Delta_1 = \mathcal{E}(T + e_{p+1}) - \mathcal{E}(T)$ and $\Delta_2 = \mathcal{E}(T + e_p) - \mathcal{E}(T)$.

n	p	T	$\mathcal{E}(T)$	$\mathcal{E}(T + e_{p+1})$	$\mathcal{E}(T + e_p)$	Δ_1	Effect of e_{p+1}	Δ_2	Effect of e_p
5	1	$K_{2,1,1,1}$	8.7446	8	—	-0.7446	↓	—	—
6	1	$K_{2,2,1,1}$	11.6569	10.9282	—	-0.7287	↓	—	—
7	1	$K_{2,2,2,1}$	14.7446	14	—	-0.7446	↓	—	—
9	2	$K_{3,2,2,2}$	20.8489	21.1485	20.0503	0.2996	↑	-0.7986	↓
10	2	$K_{3,3,2,2}$	23.7980	24.0961	22.9793	0.2981	↑	-0.8187	↓
11	2	$K_{3,3,3,2}$	26.8489	27.1388	26	0.2899	↑	-0.8489	↓

Theorem 3. Let $t \geq 3$ and let $T(2t, t) \cong K_{2,2,\dots,2}$ be the balanced t -partite Turán graph with $p = 2$. Then, for every embedding edge $e \notin E(T(2t, t))$, we have $\mathcal{E}(T(2t, t) + e_2) < \mathcal{E}(T(2t, t))$.

Theorem 4. Let $T(n, 4)$ be a 4-partite Turán graph of order $n = 4p + r$, where $p \geq 3$ and $0 \leq r \leq 3$. Suppose that

$$T(n, 4) \cong K_{\underbrace{p+1, \dots, p+1}_{r \text{ parts}}, \underbrace{p, \dots, p}_{4-r \text{ parts}}}.$$

Then, for any embedding edge $e \notin E(T(n, 4))$, we have

$$\mathcal{E}(T(n, 4) + e) > \mathcal{E}(T(n, 4)).$$

Remark 5. For $p \geq 1$, set $p' = p + 1$. Then the isomorphisms

$$K_{p+1,p,p,p} \cong K_{p',p'-1,p'-1,p'-1}, \quad K_{p+1,p+1,p,p} \cong K_{p',p',p'-1,p'-1}, \quad \text{and} \quad K_{p+1,p+1,p+1,p} \cong K_{p',p',p',p'-1}$$

hold. Hence, in the case $t = 4$, it is enough to investigate the three forms $K_{p+1,p,p,p}$, $K_{p+1,p+1,p,p}$, and $K_{p+1,p+1,p+1,p}$.

The organization of the paper is as follows. In Section 2, we derive the Seidel characteristic polynomials of the graphs $T + e \cong K_{\underbrace{p+1, \dots, p+1}_{r \text{ parts}}, \underbrace{p, \dots, p}_{t-r \text{ parts}}} + e$, which provide the necessary foundation for addressing Question 1. Sections 3–6 are devoted to analyzing the variation of Seidel energy caused by edge embedding in the four families of 4-partite Turán graphs:

$$K_{p,p,p,p}, \quad K_{p+1,p,p,p}, \quad K_{p+1,p+1,p,p}, \quad \text{and} \quad K_{p+1,p+1,p+1,p}.$$

The paper concludes in Section 7, where the principal findings are summarized and some concluding observations are presented.

2. Seidel polynomials

In this section, we first recall the Seidel polynomial of complete t -partite graphs from [16].

Theorem 6. ([16], Theorem 2.1) Let $T \cong K_{p_1, p_2, \dots, p_t}$ be a complete t -partite graph with order $n = p_1 + p_2 + \dots + p_t$, where each $p_i \geq 2$. Then the Seidel polynomial $S_T(\mu)$ is given by

$$S_T(\mu) = (\mu + 1)^{n-t} \cdot \prod_{i=1}^t (\mu - 2p_i + 1) \cdot \left[1 + \sum_{i=1}^t \frac{p_i}{\mu - 2p_i + 1} \right].$$

We now present the Seidel polynomials of graphs $T + e_p$ and $T + e_{p+1}$, where e_p and e_{p+1} denote an edge embedded in a part of size p and $p + 1$, respectively.

Theorem 7. Let $p \geq 2$, $t \geq 2$, and $T + e_p \cong K_{\underbrace{p+1, \dots, p+1}_{r \text{ times}}, \underbrace{p, \dots, p}_{t-r \text{ times}}} + e_p$, where $0 \leq r < t$. Then each of the following holds:

(1) When $1 \leq r \leq t - 2$,

$$S_{T+e_p}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt+r-t-2} \cdot (\mu - 2p + 1)^{t-r-2} \cdot (\mu - 2p - 1)^{r-1} \cdot \phi_{r,t}(\mu),$$

where

$$\begin{aligned} \phi_{r,t}(\mu) = & \mu^4 + (4 - 6p + r + pt)\mu^3 + (6 - 22p + 12p^2 + 5r - 2pr + 3pt - 4p^2t)\mu^2 \\ & + (-4 - 34p + 40p^2 - 8p^3 + 15r - 8pr + 7pt - 12p^2t + 4p^3t)\mu \\ & - 7 + 14p + 44p^2 - 24p^3 + 11r - 6pr - 11pt - 16p^2t + 12p^3t. \end{aligned} \quad (2.1)$$

(2) When $r = 0$,

$$S_{T+e_p}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt-t-2} \cdot (\mu - 2p + 1)^{t-2} \cdot \phi_t(\mu),$$

where

$$\begin{aligned} \phi_t(\mu) = \phi_{0,t}(\mu) = & \mu^3 + (5 - 4p + pt)\mu^2 + (11 - 16p + 4p^2 + 4pt - 2p^2t)\mu \\ & + 7 - 28p + 12p^2 + 11pt - 6p^2t. \end{aligned} \quad (2.2)$$

(3) When $r = t - 1$,

$$S_{T+e_p}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt-3} \cdot (\mu - 2p - 1)^{t-2} \cdot \phi_{t-1,t}(\mu),$$

where

$$\begin{aligned} \phi_{t-1,t}(\mu) = & \mu^3 + (2 - 4p + t + pt)\mu^2 + (-1 - 12p + 4p^2 + 4t + 2pt - 2p^2t)\mu \\ & - 18 - 16p + 12p^2 + 11t + 5pt - 6p^2t. \end{aligned} \quad (2.3)$$

Proof. For convenience, we set $q = p + 1$. Let $V = X \cup Y$ be the vertex set of $T + e_p$, where $X = \cup_{i=0}^{r-1} U^{(i)}$ and $Y = \cup_{j=r}^{t-1} W^{(j)}$, with $U^{(i)} = \{u_1^{(i)}, u_2^{(i)}, \dots, u_p^{(i)}\}$ and $W^{(j)} = \{w_1^{(j)}, w_2^{(j)}, \dots, w_q^{(j)}\}$. Throughout the row and column operations, all row and column labels refer to those of the current matrix.

Now, without loss of generality, we may assume that an edge e_p connects two vertices $u_1^{(0)}$ and $u_2^{(0)}$ in the part $U^{(0)}$. Then $S_{T+e_p}(\mu)$ is given by

$$S_{T+e_p}(\mu) = \det \begin{pmatrix} \Lambda_1 & -J_1 & J_2 & \cdots & J_2 & J_3 & \cdots & J_3 \\ -J_1^\top & \Lambda_2 & J_4 & \cdots & J_4 & J_5 & \cdots & J_5 \\ J_2^\top & J_4^\top & \Lambda_3 & \cdots & J_6 & J_7 & \cdots & J_7 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ J_2^\top & J_4^\top & J_6 & \cdots & \Lambda_3 & J_7 & \cdots & J_7 \\ J_3^\top & J_5^\top & J_7^\top & \cdots & J_7^\top & \Lambda_4 & \cdots & J_8 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ J_3^\top & J_5^\top & J_7^\top & \cdots & J_7^\top & J_8 & \cdots & \Lambda_4 \end{pmatrix}, \quad (2.4)$$

where $\Lambda_1, \Lambda_2, \Lambda_3$, and Λ_4 are square matrices of orders 2, $p-2$, p , and q , respectively. For $i = 2, 3, 4$, each Λ_i has diagonal entries μ and off-diagonal entries -1 , while Λ_1 has diagonal entries μ and off-diagonal entries 1. The matrices J_1 to J_8 are all-ones matrices whose sizes are listed in Table 3. The notation $(\cdot)^\top$ denotes the transpose.

Table 3. Matrix sizes of J_1 to J_8 .

Matrix	Size	Matrix	Size	Matrix	Size	Matrix	Size
J_1	$2 \times (p-2)$	J_2	$2 \times p$	J_3	$2 \times q$	J_4	$(p-2) \times p$
J_5	$(p-2) \times q$	J_6	$p \times p$	J_7	$p \times q$	J_8	$q \times q$

We first perform the column operations $C_k^{(0)} \leftarrow C_k^{(0)} - C_p^{(0)}$, $C_l^{(i)} \leftarrow C_l^{(i)} - C_p^{(i)}$, and $C_m^{(j)} \leftarrow C_m^{(j)} - C_q^{(j)}$, then perform the row operations $R_p^{(0)} \leftarrow R_p^{(0)} + R_k^{(0)}$, $R_p^{(i)} \leftarrow R_p^{(i)} + R_l^{(i)}$, and $R_q^{(j)} \leftarrow R_q^{(j)} + R_m^{(j)}$ on matrix (2.4), where $i = 1, \dots, r-1$ and $j = r, \dots, t-1$ index the parts and $k = 3, \dots, p-1$, $l = 1, \dots, p-1$, and $m = 1, \dots, q-1$ index the rows and columns within each part. After these operations, we obtain the Seidel polynomial of $T + e_p$:

$$S_{T+e_p}(\mu) = (\mu+1)^{p-3} \cdot (\mu+1)^{(p-1)(t-r-1)} \cdot (\mu+1)^{r(q-1)} \cdot D_1,$$

$$D_1 = \det \begin{pmatrix} \Lambda_1 & -J'_1 & J'_2 & J'_3 \\ -(p-2)(J'_1)^\top & \Lambda'_2 & (p-2)J'_4 & (p-2)J'_5 \\ p(J'_2)^\top & p(J'_4)^\top & \Lambda'_3 & pJ'_6 \\ q(J'_3)^\top & q(J'_5)^\top & q(J'_6)^\top & \Lambda'_4 \end{pmatrix},$$

where Λ'_2 denotes the 1×1 matrix whose only entry is $\mu - p + 3$. The matrix Λ'_3 has order $t - r - 1$, with diagonal entries $\mu - p + 1$ and identical off-diagonal entries p , while Λ'_4 is an $r \times r$ matrix having diagonal entries $\mu - q + 1$ and off-diagonal entries q . The matrices $J'_1, \dots, J'_6$ are all-one matrices, and their respective sizes are specified in Table 4.

Table 4. Matrix sizes of J'_1 to J'_6 .

Matrix	Size	Matrix	Size	Matrix	Size
J'_1	2×1	J'_2	$2 \times (t-r-1)$	J'_3	$2 \times r$
J'_4	$1 \times (t-r-1)$	J'_5	$1 \times r$	J'_6	$(t-r-1) \times r$

Next, we perform the row operations $R_{k'} \leftarrow R_{k'} - R_{t-r-1}$ and $R_{l'} \leftarrow R_{l'} - R_r$ and the column operations $C_{t-r-1} \leftarrow C_{t-r-1} + C_{k'}$ and $C_r \leftarrow C_r + C_{l'}$ on the matrix D_1 , where $k' = 1, \dots, t-r-2$ and $l' = 1, \dots, r-1$. Consequently, we obtain

$$S_{T+e_p}(\mu) = (\mu+1)^{p-3} \cdot (\mu+1)^{(p-1)(t-r-1)} \cdot (\mu+1)^{r(q-1)} \cdot (\mu-2p+1)^{t-r-2} \cdot (\mu-2q+1)^{r-1} \cdot D_2,$$

$$D_2 = \det \begin{pmatrix} \mu & 1 & -1 & t-r-1 & r \\ 1 & \mu & -1 & t-r-1 & r \\ -(p-2) & -(p-2) & \mu-p+3 & (p-2)(t-r-1) & (p-2)r \\ p & p & p & \mu-p+1+p(t-r-2) & pr \\ q & q & q & q(t-r-1) & \mu-q+1+q(r-1) \end{pmatrix}.$$

Next, applying the row operation $R'_1 \leftarrow R'_1 - R'_2$ and the column operation $C'_1 \leftarrow C'_1 + C'_2$ on matrix D_2 , we have

$$S_{T+e_p}(\mu) = (\mu - 1) \cdot (\mu + 1)^{p-3+(t-r-1)(p-1)+r(q-1)} \cdot (\mu - 2p + 1)^{t-r-2} \cdot (\mu - 2q + 1)^{r-1} \cdot D_3, \quad (2.5)$$

where

$$D_3 = \det \begin{pmatrix} \mu + 1 & -1 & t - r - 1 & r \\ -2(p - 2) & \mu - p + 3 & (p - 2)(t - r - 1) & (p - 2)r \\ 2p & p & \mu - p + 1 + p(t - r - 2) & pr \\ 2q & q & q(t - r - 1) & \mu - q + 1 + q(r - 1) \end{pmatrix}.$$

Replacing q by $p + 1$ into (2.5) and D_3 , we obtain

$$S_{T+e_p}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt+r-t-2} \cdot (\mu - 2p + 1)^{t-r-2} \cdot (\mu - 2p - 1)^{r-1} \cdot D_4, \quad (2.6)$$

$$D_4 = \det \begin{pmatrix} \mu + 1 & -1 & t - r - 1 & r \\ -2(p - 2) & \mu - p + 3 & (p - 2)(t - r - 1) & (p - 2)r \\ 2p & p & \mu - p + 1 + p(t - r - 2) & pr \\ 2(p + 1) & p + 1 & (p + 1)(t - r - 1) & \mu - p + (p + 1)(r - 1) \end{pmatrix}.$$

Set $\phi_{r,t}(\mu) = D_4$ and expand. Hence,

$$S_{T+e_p}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt+r-t-2} \cdot (\mu - 2p + 1)^{t-r-2} \cdot (\mu - 2p - 1)^{r-1} \cdot \phi_{r,t}(\mu),$$

where $\phi_{r,t}(\mu)$ is the polynomial given in (2.1).

Setting $r = 0$ into (2.6) and D_4 , we have

$$S_{T+e_p}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt-t-2} \cdot (\mu - 2p + 1)^{t-2} \cdot (\mu - 2p - 1)^{-1} \cdot D_5,$$

$$D_5 = \det \begin{pmatrix} \mu + 1 & -1 & t - 1 & 0 \\ -2(p - 2) & \mu - p + 3 & (p - 2)(t - 1) & 0 \\ 2p & p & \mu - p + 1 + p(t - 2) & 0 \\ 2(p + 1) & p + 1 & (p + 1)(t - 1) & \mu - 2p - 1 \end{pmatrix},$$

or

$$S_T(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt-t-2} \cdot (\mu - 2p + 1)^{t-2} \cdot D_6,$$

$$D_6 = \det \begin{pmatrix} \mu + 1 & -1 & t - 1 \\ -2(p - 2) & \mu - p + 3 & (p - 2)(t - 1) \\ 2p & p & \mu - p + 1 + p(t - 2) \end{pmatrix}.$$

Defining $\phi_t(\mu) = \phi_{0,t}(\mu) = D_6$, we get

$$S_{T+e_p}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt-t-2} \cdot (\mu - 2p + 1)^{t-2} \cdot \phi_t(\mu),$$

where $\phi_t(\mu)$ is the polynomial as written in (2.2).

If we set $r = t - 1$ into (2.6) and D_4 , we obtain

$$S_{T+e_p}(\mu) = (\mu - 1) \cdot (\mu + 1)^{p-3+p(t-1)} \cdot (\mu - 2p + 1)^{-1} \cdot (\mu - 2p - 1)^{t-2} \cdot D_7,$$

$$D_7 = \det \begin{pmatrix} \mu + 1 & -1 & 0 & (t-1) \\ -2(p-2) & \mu - p + 3 & 0 & (p-2)(t-1) \\ 2p & p & \mu - 2p + 1 & p(t-1) \\ 2(p+1) & p+1 & 0 & \mu - p + (p+1)(t-2) \end{pmatrix},$$

or

$$S_{T+e_p}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt-3} \cdot (\mu - 2p - 1)^{t-2} \cdot D_8,$$

$$D_8 = \det \begin{pmatrix} \mu + 1 & -1 & t-1 \\ -2(p-2) & \mu - p + 3 & (p-2)(t-1) \\ 2(p+1) & p+1 & \mu - p + (p+1)(t-2) \end{pmatrix},$$

or

$$S_{T+e_p}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt-3} \cdot (\mu - 2p - 1)^{t-2} \cdot \phi_{t-1,t}(\mu),$$

where $\phi_{t-1,t}(\mu)$ is the polynomial as in (2.3). This completes the proof. $\square$

Theorem 8. Let $p \geq 2$, $t \geq 2$, and $T + e_{p+1} \cong K_{\underbrace{p+1, \dots, p+1}_{r \text{ times}}, \underbrace{p, \dots, p}_{t-r \text{ times}}} + e_{p+1}$, where $1 \leq r \leq t-1$.

Then each of the following holds:

(1) When $2 \leq r \leq t-1$,

$$S_{T+e_{p+1}}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt-t+r-2} \cdot (\mu - 2p - 1)^{r-2} \cdot (\mu - 2p + 1)^{t-r-1} \cdot \psi_{r,t}(\mu),$$

where

$$\begin{aligned} \psi_{r,t}(\mu) = & \mu^4 + (2 - 6p + r + pt)\mu^3 + (-14p + 12p^2 + 3r - 2pr + pt - 4p^2t)\mu^2 \\ & + (-10 - 10p + 32p^2 - 8p^3 + 7r - 8pr + 3pt - 8p^2t + 4p^3t)\mu \\ & + (-9 + 14p + 20p^2 - 24p^3 + 5r - 6pr - 5pt - 4p^2t + 12p^3t). \end{aligned} \quad (2.7)$$

(2) When $r = 1$,

$$S_{T+e_{p+1}}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt-t-1} \cdot (\mu - 2p + 1)^{t-2} \cdot \psi_{1,t}(\mu),$$

where

$$\begin{aligned} \psi_{1,t}(\mu) = & \mu^3 + (4 - 4p + pt)\mu^2 + (7 - 12p + 4p^2 + 2pt - 2p^2t)\mu \\ & + (4 - 16p + 12p^2 + 5pt - 6p^2t). \end{aligned} \quad (2.8)$$

Proof. For $2 \leq r \leq t-1$, substituting $p \mapsto p+1$, $q \mapsto p$, $r \mapsto t-r$, and $t-r \mapsto r$ into (2.5) and D_3 , we obtain

$$S_{T+e_{p+1}}(\mu) = (\mu - 1) \cdot (\mu + 1)^{p-2+(r-1)p+(t-r)(p-1)} \cdot (\mu - 2p - 1)^{r-2} \cdot (\mu - 2p + 1)^{t-r-1} \cdot D_9, \quad (2.9)$$

$$D_9 = \det \begin{pmatrix} \mu + 1 & -1 & r-1 & t-r \\ -2(p-1) & \mu - p + 2 & (p-1)(r-1) & (p-1)(t-r) \\ 2(p+1) & p+1 & \mu - p + (p+1)(r-2) & (p+1)(t-r) \\ 2p & p & p(r-1) & \mu - p + 1 + p(t-r-1) \end{pmatrix}.$$

Set $\psi_{r,t}(\mu) = D_9$ and expand, we get

$$S_{T+e_{p+1}}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt-t+r-2} \cdot (\mu - 2p - 1)^{r-2} \cdot (\mu - 2p + 1)^{t-r-1} \cdot \psi_{r,t}(\mu),$$

where $\psi_{r,t}(\mu)$ is the polynomial given in (2.7).

If we set $r = 1$ into (2.9) and D_9 , then we have

$$S_{T+e_{p+1}}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt-t-1} \cdot (\mu - 2p - 1)^{-1} \cdot (\mu - 2p + 1)^{t-2} \cdot D_{10},$$

$$D_{10} = \det \begin{pmatrix} \mu + 1 & -1 & 0 & t - 1 \\ -2(p - 1) & \mu - p + 2 & 0 & (p - 1)(t - 1) \\ 2(p + 1) & p + 1 & \mu - 2p - 1 & (p + 1)(t - 1) \\ 2p & p & 0 & \mu - p + 1 + p(t - 2) \end{pmatrix},$$

or

$$S_{T+e_{p+1}}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt-t-1} \cdot (\mu - 2p + 1)^{t-2} \cdot D_{11},$$

$$D_{11} = \det \begin{pmatrix} \mu + 1 & -1 & t - 1 \\ -2(p - 1) & \mu - p + 2 & (p - 1)(t - 1) \\ 2p & p & \mu - p + 1 + p(t - 2) \end{pmatrix}.$$

Set $\psi_{1,t}(\mu) = D_{11}$ and expand, we have

$$S_{T+e_{p+1}}(\mu) = (\mu - 1) \cdot (\mu + 1)^{pt-t-1} \cdot (\mu - 2p + 1)^{t-2} \cdot \psi_{1,t}(\mu),$$

where $\psi_{1,t}(\mu)$ is the polynomial as in (2.8). □

3. Seidel energy change in $K_{p,p,p,p}$ under edge embedding

The graph $K_{p,p,p,p} + e_p$ is obtained from $K_{p,p,p,p}$ by embedding an edge e between two vertices x_i and x_j belonging to a same part, where $e \notin E(K_{p,p,p,p})$ and $1 \leq i < j \leq p$.

Corollary 9. Let $T \cong K_{p,p,p,p}$ be the complete 4-partite graph of order $4p$ with $p \geq 2$. Then the Seidel spectrum of T is $\{-2p - 1, [-1]^{4p-4}, [2p - 1]^3\}$.

Proof. Applying Theorem 6 to the complete multipartite graph $K_{p,p,p,p}$ gives the stated Seidel spectrum. □

Corollary 10. Let $T + e_p \cong K_{p,p,p,p} + e_p$ with $p \geq 2$. Then the Seidel spectrum of $T + e_p$ is $\{[-1]^{4p-6}, 1, [2p - 1]^2, \mu_1, \mu_2, \mu_3\}$, where μ_1, μ_2 , and μ_3 are the three roots of $\phi_4(\mu) = \mu^3 + 5\mu^2 + (11 - 4p^2)\mu + 7 + 16p - 12p^2$.

Proof. Setting $r = 0$ and $t = 4$ in Theorem 7, we obtain the desired result. □

Remark 11. Denote the three roots of $\phi_4(\mu)$ by μ_1, μ_2 , and μ_3 . For $p \geq 4$, the roots can be ordered as $\mu_3 < \mu_2 < 0 < \mu_1$, and moreover they satisfy the bounds $-3 < \mu_2 < -2$ and $-2p - 2 < \mu_3 < -2p - 1$.

Proof. Since μ_1, μ_2 , and μ_3 are the roots of $\phi_4(\mu)$, the relations between the root and coefficient yield

$$\mu_1 + \mu_2 + \mu_3 = -5, \quad \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 = 11 - 4p^2, \quad \text{and} \quad \mu_1\mu_2\mu_3 = 12p^2 - 16p - 7.$$

For $p \geq 4$, the product of the roots satisfies $\mu_1\mu_2\mu_3 = 12p^2 - 16p - 7 = 4p(3p - 4) - 7 > 0$, while their sum is negative, namely $\mu_1 + \mu_2 + \mu_3 = -5 < 0$. Hence, $\phi_4(\mu)$ has one positive root and two negative roots, that is, $\mu_3 \leq \mu_2 < 0 < \mu_1$. Bound the negative roots using the intermediate value theorem:

- $\phi_4(-3) = 8(2p - 1) > 0$ and $\phi_4(-2) = -4p(p - 4) - 3 < 0$, so $-3 < \mu_2 < -2$.
- $\phi_4(-2p - 1) = 8p > 0$ and $\phi_4(-2p - 2) = -2p(4p - 5) - 3 < 0$, so $-2p - 2 < \mu_3 < -2p - 1$.

Since the intervals $(-3, -2)$ and $(-2p - 2, -2p - 1)$ are disjoint for $p \geq 4$, we conclude that $\mu_3 < \mu_2 < 0 < \mu_1$, $-3 < \mu_2 < -2$, and $-2p - 2 < \mu_3 < -2p - 1$. $\square$

Since $\mathcal{E}(K_{2,2} + e_2) = 2 + 2\sqrt{5} > 6 = \mathcal{E}(K_{2,2})$, this holds for $p = 2$ and $t = 2$. For $p = 2$ and $t \geq 3$, we now prove Theorem 3.

Proof of Theorem 3. Let $T \cong K_{2,2,\dots,2}$, then by Theorem 7, we have $S_{T+e_2}(\mu) = (\mu-1) \cdot (\mu+1)^{t-2} \cdot (\mu-3)^{t-2} \cdot \phi_t(\mu)$, where $\phi_t(\mu) = \mu^3 + (2t-3)\mu^2 - 5\mu - (2t-1)$. We factor $\phi_t(\mu)$ as follows:

$$\begin{aligned} \phi_t(\mu) &= \mu^3 + (2t-3)\mu^2 - 5\mu - (2t-1) \\ &= \mu^3 + (2t-4)\mu^2 - (2t+1)\mu + \mu^2 + (2t-4)\mu - (2t+1) \\ &= \mu[\mu^2 + (2t-4)\mu - (2t+1)] + 1[\mu^2 + (2t-4)\mu - (2t+1)] \\ &= (\mu+1)[\mu^2 + (2t-4)\mu - (2t+1)]. \end{aligned}$$

Hence, the Seidel spectrum of $T + e_2$ is

$$\text{Spec}(T + e_2) = \{[-1]^{t-1}, 1, [3]^{t-2}, 2-t+\sqrt{t^2-2t+5}, 2-t-\sqrt{t^2-2t+5}\}.$$

Let

$$\mu_+ = 2-t+\sqrt{t^2-2t+5} \text{ and } \mu_- = 2-t-\sqrt{t^2-2t+5}.$$

One can verify that $\mu_+ > 0$ and $\mu_- < 0$ for $t \geq 3$. Therefore,

$$|\mu_+| = 2-t+\sqrt{t^2-2t+5} \text{ and } |\mu_-| = t-2+\sqrt{t^2-2t+5}.$$

By the definition of Seidel energy,

$$\begin{aligned} \mathcal{E}(T + e_2) &= (t-1) + 1 + 3 \cdot (t-2) + (2-t+\sqrt{t^2-2t+5}) + (t-2+\sqrt{t^2-2t+5}) \\ &= 4t-6+2\sqrt{t^2-2t+5}. \end{aligned}$$

By applying Theorem 6 together with the definition of Seidel energy, we obtain $\mathcal{E}(T) = 6t-6$. We now compute

$$\mathcal{E}(T) - \mathcal{E}(T + e_2) = 6t-6 - (4t-6+2\sqrt{t^2-2t+5}) = 2t-2\sqrt{t^2-2t+5}.$$

Since $t \geq 3$ implies $t^2 > t^2 - 2t + 5$, we have $\sqrt{t^2 - 2t + 5} < t$, which concludes that $2t - 2\sqrt{t^2 - 2t + 5} > 0$. Therefore, $\mathcal{E}(T) > \mathcal{E}(T + e_2)$. This completes the proof. $\square$

We turn to the case $p \geq 3$, $r = 0$, and $t = 4$ and prove the following result.

Lemma 12. *Let $T \cong K_{p,p,p,p}$ and $e_p \notin E(T)$, where $p \geq 3$. Then we have $\mathcal{E}(T + e_p) > \mathcal{E}(T)$.*

Proof. By Corollary 9, the Seidel energy of T is

$$\mathcal{E}(T) = (2p + 1) + (4p - 4) + 3 \cdot (2p - 1) = 12p - 6$$

for $p \geq 3$.

We first consider the case $p = 3$. A direct computer computation gives

$$\text{Spec}(T + e_3) = \{-7.4088, -1.7289, [-1]^6, 1, 4.1377, [5]^2\}.$$

Hence,

$$\mathcal{E}(T + e_3) = 7.4088 + 1.7289 + 6 + 1 + 4.1377 + 5 \cdot 2 = 30.2754,$$

and therefore

$$\mathcal{E}(T + e_3) - \mathcal{E}(T) = 30.2754 - 30 = 0.2754 > 0.$$

Now assume that $p \geq 4$. Applying Corollary 10 together with the definition of Seidel energy yields

$$\mathcal{E}(T + e_p) = (4p - 6) + 1 + 2 \cdot (2p - 1) + |\mu_1| + |\mu_2| + |\mu_3| = 8p - 7 + |\mu_1| + |\mu_2| + |\mu_3|.$$

From Remark 11, we have $\mu_3 < \mu_2 < 0 < \mu_1$ and $\mu_1 + \mu_2 + \mu_3 = -5$, so $|\mu_i| = -\mu_i$ for $i = 2, 3$ and $|\mu_1| = \mu_1 = -5 - \mu_2 - \mu_3$. Substituting these into the above expression yields

$$\mathcal{E}(T + e_p) = 8p - 7 + (-5 - \mu_2 - \mu_3) + (-\mu_2) + (-\mu_3) = 8p - 12 - 2(\mu_2 + \mu_3).$$

We then compute

$$\mathcal{E}(T + e_p) - \mathcal{E}(T) = 8p - 12 - 2(\mu_2 + \mu_3) - (12p - 6) = -4p - 6 - 2(\mu_2 + \mu_3).$$

Again, by Remark 11, $-3 < \mu_2 < -2$ and $-2p - 2 < \mu_3 < -2p - 1$ for $p \geq 4$, which imply that $\mu_2 + \mu_3 < -2p - 3$. Consequently,

$$\mathcal{E}(T + e_p) - \mathcal{E}(T) > -4p - 6 - 2(-2p - 3) = -4p - 6 + 4p + 6 = 0.$$

Hence,

$$\mathcal{E}(T + e_p) > \mathcal{E}(T).$$

This completes the proof. $\square$

4. Seidel energy change in $K_{p+1,p,p,p}$ under edge embedding

In this section, we investigate how the Seidel energy of the complete 4-partite Turán graph $K_{p+1,p,p,p}$ of order $n = 4p + 1$ changes after embedding an edge.

4.1. Edge embedding in a part of size p

Let $K_{p+1,p,p,p} + e_p$ denote the graph obtained from $K_{p+1,p,p,p}$ by embedding an edge e_p between two vertices x_i and x_j belonging to a same part of size p , where $e_p \notin E(K_{p+1,p,p,p})$ and $1 \leq i < j \leq p$.

Corollary 13. For $p \geq 2$, let $T \cong K_{p+1,p,p,p}$. Then the Seidel spectrum of T is $\{[-1]^{4p-3}, [2p-1]^2, \frac{-1+\sqrt{16p^2+16p+1}}{2}, \frac{-1-\sqrt{16p^2+16p+1}}{2}\}$.

Proof. Setting $t = 4$, $p_1 = p + 1$, and $p_2 = p_3 = p_4 = p$ in Theorem 6, we obtain $S_T(\mu)$ in the following form:

$$\begin{aligned} S_T(\mu) &= (\mu + 1)^{4p-3} \cdot (\mu - 2p + 1)^2 \cdot [(\mu - 2p - 1)(\mu - 2p + 1) + (p + 1)(\mu - 2p + 1) \\ &\quad + 3p(\mu - 2p - 1)] \\ &= (\mu + 1)^{4p-3} \cdot (\mu - 2p + 1)^2 \cdot (\mu^2 + \mu - 4p - 4p^2) \\ &= (\mu + 1)^{4p-3} \cdot (\mu - 2p + 1)^2 \cdot \left(\mu + \frac{1 + \sqrt{16p^2 + 16p + 1}}{2}\right) \cdot \left(\mu + \frac{1 - \sqrt{16p^2 + 16p + 1}}{2}\right). \end{aligned}$$

Hence,

$$\text{Spec}(T) = \{[-1]^{4p-3}, [2p-1]^2, \frac{-1 + \sqrt{16p^2 + 16p + 1}}{2}, \frac{-1 - \sqrt{16p^2 + 16p + 1}}{2}\}.$$

□

Corollary 14. Let $T + e_p \cong K_{p+1,p,p,p} + e_p$, where $p \geq 2$. Then the Seidel spectrum of $T + e_p$ is $\{[-1]^{4p-5}, 1, 2p-1, \mu_i\}$, where μ_i ($i = 1, \dots, 4$) are the roots of the polynomial $\phi_{1,4}(\mu) = \mu^4 + (5-2p)\mu^3 + (11-12p-4p^2)\mu^2 + (11-14p-8p^2+8p^3)\mu + 4-36p-20p^2+24p^3$.

Proof. The result follows by setting $t = 4$ and $r = 1$ in Theorem 7. □

Remark 15. Denote the roots of $\phi_{1,4}(\mu)$ by μ_1, μ_2, μ_3 , and μ_4 . When $p \geq 4$, the following hold.

- (i) $\mu_4 < \mu_3 < 0$ and $\mu_1 \geq \mu_2 > 0$.
- (ii) $-3 < \mu_3 < -2$ and $-2p-2 < \mu_4 < -2p-\frac{3}{2}$.
- (iii) $\mu_3 + \mu_4 < -2p - \frac{7}{2} < \frac{-5-\sqrt{16p^2+16p+1}}{2}$.

Proof. Given that μ_1, μ_2, μ_3 , and μ_4 are the roots of $\phi_{1,4}(\mu)$, now by the relation of the root and coefficient, we have $\sum_{i=1}^4 \mu_i = 2p-5 > 0$ and $\prod_{i=1}^4 \mu_i = 24p^3 - 20p^2 - 36p + 4 > 0$ for $p \geq 4$. Hence, either all four roots are positive, or exactly two roots are positive and two negative. However, we evaluate $\phi_{1,4}(\mu)$ at specific points. We find that $\phi_{1,4}(-2) = 4p[p(2p-5)-10] + 2 > 0$ and $\phi_{1,4}(-3) = -16(2p^2+3p-1) < 0$ for $p \geq 4$, so by the intermediate value theorem, $-3 < \mu_3 < -2$. Similarly, $\phi_{1,4}(-2p-2) = 16p^3-20p^2-30p+2 > 0$ and $\phi_{1,4}(-2p-\frac{3}{2}) = -26p^2-\frac{127}{4}p+\frac{7}{16} < 0$, which yield $-2p-2 < \mu_4 < -2p-\frac{3}{2}$. This proves (i) and (ii).

From (ii), we have $\mu_3 + \mu_4 < -2p - \frac{7}{2}$. The inequality $-2p - \frac{7}{2} < \frac{-5-\sqrt{16p^2+16p+1}}{2}$ holds since

$$-4p-7 < -5 - \sqrt{16p^2 + 16p + 1} \implies 4p+2 > \sqrt{16p^2 + 16p + 1},$$

which is readily verified for all $p \geq 4$. The proof is completed. □

Lemma 16. Consider $T \cong K_{p+1,p,p,p}$ and let $e_p \notin E(T)$ be an embedding edge, where $p \geq 3$. Then Seidel energy satisfies $\mathcal{E}(T + e_p) > \mathcal{E}(T)$.

Proof. By Corollary 13,

$$\mathcal{E}(T) = (4p - 3) + 2 \cdot (2p - 1) + |\mu_1| + |\mu_2| = 8p - 5 + |\mu_1| + |\mu_2|,$$

where $\mu_1 = \frac{-1 + \sqrt{16p^2 + 16p + 1}}{2}$ and $\mu_2 = \frac{-1 - \sqrt{16p^2 + 16p + 1}}{2}$. Since $\mu_1 > 0 > \mu_2$, $|\mu_1| + |\mu_2| = \sqrt{16p^2 + 16p + 1}$, so

$$\mathcal{E}(T) = 8p - 5 + \sqrt{16p^2 + 16p + 1}.$$

When $p = 3$, a direct computation gives

$$\mathcal{E}(T + e_3) - \mathcal{E}(T) = 33.1623 - 32.8924 = 0.2699 \implies \mathcal{E}(T + e_3) > \mathcal{E}(T).$$

Assume $p \geq 4$. By Corollary 14 together with the definition of Seidel energy, we obtain

$$\mathcal{E}(T + e_p) = (4p - 5) + 1 + (2p - 1) + \sum_{i=1}^4 |\mu_i| = 6p - 5 + \sum_{i=1}^4 |\mu_i|.$$

By Remark 15 (i), $|\mu_i| = \mu_i$, $i = 1, 2$ and $|\mu_i| = -\mu_i$, $i = 3, 4$. Since $\sum_{i=1}^4 \mu_i = 2p - 5$, we can write $|\mu_1| + |\mu_2| = \mu_1 + \mu_2 = 2p - 5 - \mu_3 - \mu_4$. Therefore,

$$\mathcal{E}(T + e_p) = 6p - 5 + (2p - 5 - \mu_3 - \mu_4) + (-\mu_3) + (-\mu_4) = 8p - 10 - 2(\mu_3 + \mu_4).$$

Hence,

$$\begin{aligned} \mathcal{E}(T + e_p) - \mathcal{E}(T) &= 8p - 10 - 2(\mu_3 + \mu_4) - (8p - 5 + \sqrt{16p^2 + 16p + 1}) \\ &= -5 - \sqrt{16p^2 + 16p + 1} - 2(\mu_3 + \mu_4). \end{aligned}$$

Remark 15 (iii) yields $\mu_3 + \mu_4 < \frac{-5 - \sqrt{16p^2 + 16p + 1}}{2}$ or $2(\mu_3 + \mu_4) < -5 - \sqrt{16p^2 + 16p + 1}$. Therefore,

$$\mathcal{E}(T + e_p) - \mathcal{E}(T) > 0 \implies \mathcal{E}(T + e_p) > \mathcal{E}(T),$$

which completes the proof. $\square$

4.2. Edge embedding in the part of size $p + 1$

$K_{p+1,p,p,p} + e_{p+1}$ denotes the graph obtained from $K_{p+1,p,p,p}$ by embedding an edge e_{p+1} between two vertices x_i and x_j in the part of size $p + 1$ with $1 \leq i < j \leq p + 1$.

Corollary 17. Let $T + e_{p+1} \cong K_{p+1,p,p,p} + e_{p+1}$ for $p \geq 2$. Then the Seidel spectrum of $T + e_{p+1}$ is $\{-1\}^{4p-5}, 1, [2p-1]^2, \mu_1, \mu_2, \mu_3\}$, where μ_1, μ_2 , and μ_3 are the roots of the polynomial $\psi_{1,4}(\mu) = \mu^3 + 4\mu^2 + (7 - 4p - 4p^2)\mu + 4 + 4p - 12p^2$.

Proof. We can obtain the desired result by setting $t = 4$ and $r = 1$ in Theorem 8. $\square$

Remark 18. Denote the three roots of $\psi_{1,4}(\mu)$ by μ_1, μ_2 , and μ_3 . Then, for $p \geq 3$, the following hold.

- (i) $\mu_3 < \mu_2 < 0 < \mu_1$.
- (ii) $-3 < \mu_2 < -2$ and $-2p - 2 < \mu_3 < -2p - \frac{3}{2}$.
- (iii) $\mu_2 + \mu_3 < -2p - \frac{7}{2} < \frac{-5 - \sqrt{16p^2 + 16p + 1}}{2}$.

Proof. We may assume that $\mu_3 \leq \mu_2 \leq \mu_1$. Then by the relation of the root and coefficient, one gets

$$\mu_1 + \mu_2 + \mu_3 = -4, \quad \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 = 7 - 4p - 4p^2, \quad \text{and} \quad \mu_1\mu_2\mu_3 = 12p^2 - 4p - 4.$$

Since $\mu_1\mu_2\mu_3 = 4p(3p - 1) - 4 > 0$ for $p \geq 1$ and $\mu_1 + \mu_2 + \mu_3 = -4 < 0$, it follows that $\psi_{1,4}(\mu)$ has one positive and two negative roots. Thus, $\mu_3 \leq \mu_2 < 0 < \mu_1$. For $p \geq 3$, we evaluate $\psi_{1,4}(\mu)$ at suitable points:

- $\psi_{1,4}(-2) = -4p(p - 3) - 2 < 0$ and $\psi_{1,4}(-3) = 8(2p - 1) > 0$.
- $\psi_{1,4}(-2p - 2) = -2p(2p - 3) - 2 < 0$ and $\psi_{1,4}(-2p - \frac{3}{2}) = \frac{1}{8}(52p - 7) > 0$.

Thus, statement (ii) holds from the intermediate value theorem. From (ii), the intervals $(-3, -2)$ and $(-2p - 2, -2p - \frac{3}{2})$ are disjoint for all $p \geq 3$, and the ordering in (i) follows immediately.

By (ii), we have $\mu_2 + \mu_3 < -2 - 2p - \frac{3}{2} = -2p - \frac{7}{2}$. Moreover, $-2p - \frac{7}{2} < \frac{-5 - \sqrt{16p^2 + 16p + 1}}{2}$ since

$$-4p - 7 < -5 - \sqrt{16p^2 + 16p + 1} \implies 4p + 2 > \sqrt{16p^2 + 16p + 1},$$

which is true for all $p \geq 3$. This completes the proof. $\square$

Lemma 19. Let $T \cong K_{p+1,p,p,p}$ and let $e_{p+1} \notin E(T)$ be an embedding edge, where $p \geq 3$. Then $\mathcal{E}(T + e_{p+1}) > \mathcal{E}(T)$ holds.

Proof. By the definition of Seidel energy and Corollary 17,

$$\mathcal{E}(T + e_{p+1}) = (4p - 5) + 1 + 2 \cdot (2p - 1) + |\mu_1| + |\mu_2| + |\mu_3| = 8p - 6 + |\mu_1| + |\mu_2| + |\mu_3|.$$

From Remark 18 (i), we have $|\mu_1| = \mu_1$ and $|\mu_i| = -\mu_i$ for $i = 2, 3$. Since $\mu_1 + \mu_2 + \mu_3 = -4$, it follows that $\mu_1 = -4 - \mu_2 - \mu_3$. Therefore,

$$\mathcal{E}(T + e_{p+1}) = 8p - 6 + (-4 - \mu_2 - \mu_3) + (-\mu_2) + (-\mu_3) = 8p - 10 - 2(\mu_2 + \mu_3).$$

Compute the difference

$$\begin{aligned} \mathcal{E}(T + e_{p+1}) - \mathcal{E}(T) &= 8p - 10 - 2(\mu_2 + \mu_3) - (8p - 5 + \sqrt{16p^2 + 16p + 1}) \\ &= -5 - \sqrt{16p^2 + 16p + 1} - 2(\mu_2 + \mu_3). \end{aligned}$$

By Remark 18 (iii), we have $\mu_2 + \mu_3 < \frac{-5 - \sqrt{16p^2 + 16p + 1}}{2}$ or $2(\mu_2 + \mu_3) < -5 - \sqrt{16p^2 + 16p + 1}$. So, $\mathcal{E}(T + e_{p+1}) - \mathcal{E}(T) > 0$. Hence, $\mathcal{E}(T + e_{p+1}) > \mathcal{E}(T)$. $\square$

5. Seidel energy change in $K_{p+1,p+1,p,p}$ under edge embedding

In this section, we study the effect of embedding an edge on the Seidel energy of the complete 4-partite graph $K_{p+1,p+1,p,p}$, which has order $n = 4p + 2$.

5.1. Edge embedding in a part of size p

Consider the complete 4-partite graph $T \cong K_{p+1,p+1,p,p}$ with $n = 4p + 2$ vertices. Choose two vertices x_i and x_j from the same part of cardinality p , where $1 \leq i < j \leq p$. Since these vertices are non-adjacent in T , the embedded edge $e_p = x_i x_j$ produces a new graph, denoted by $K_{p+1,p,p,p} + e_p$, where $e_p \notin E(T)$.

Corollary 20. For $T \cong K_{p+1,p+1,p,p}$ with $p \geq 2$, the Seidel spectrum of T is given by

$$\text{Spec}(T) = \{[-1]^{4p-2}, 2p-1, 2p+1, -1+2\sqrt{p^2+p}, -1-2\sqrt{p^2+p}\}.$$

Proof. Setting $t = 4$, $p_1 = p_2 = p + 1$, and $p_3 = p_4 = p$ in Theorem 6, we obtain

$$\begin{aligned} S_T(\mu) &= (\mu + 1)^{4p-2} \cdot (\mu - 2p + 1) \cdot (\mu - 2p - 1) \cdot [(\mu - 2p + 1)(\mu - 2p - 1) + 2p(\mu - 2p - 1) \\ &\quad + 2(p + 1)(\mu - 2p + 1)] \\ &= (\mu + 1)^{4p-2} \cdot (\mu - 2p + 1) \cdot (\mu - 2p - 1) \cdot (\mu^2 + 2\mu + 1 - 4p - 4p^2) \\ &= (\mu + 1)^{4p-2} \cdot (\mu - 2p + 1) \cdot (\mu - 2p - 1) \cdot (\mu + 1 - 2\sqrt{p^2 + p}) \cdot (\mu + 1 + 2\sqrt{p^2 + p}). \end{aligned}$$

Thus,

$$\text{Spec}(T) = \{[-1]^{4p-2}, 2p-1, 2p+1, -1+2\sqrt{p^2+p}, -1-2\sqrt{p^2+p}\}.$$

□

Corollary 21. Let $T + e_p \cong K_{p+1,p+1,p,p} + e_p$, where $p \geq 2$. Then the Seidel spectrum of $T + e_p$ is $\{[-1]^{4p-4}, 1, 2p+1, \mu_i\}$, where μ_i ($i = 1, \dots, 4$) are the roots of the polynomial $\phi_{2,4}(\mu) = \mu^4 + (6 - 2p)\mu^3 + (16 - 14p - 4p^2)\mu^2 + (26 - 22p - 8p^2 + 8p^3)\mu + 15 - 42p - 20p^2 + 24p^3$.

Proof. By setting $t = 4$ and $r = 2$ in Theorem 7, we obtain the desired result. □

Remark 22. Suppose that μ_1, μ_2, μ_3 , and μ_4 are the zeros of the polynomial $\phi_{2,4}(\mu)$. When $p \geq 4$, the ordering and bounds of these zeros are described as follows:

- (i) $\mu_4 < \mu_3 < \mu_2 \leq \mu_1$.
- (ii) $-3 < \mu_3 < -2$ and $-2 - 2\sqrt{p^2 + p + 1} < \mu_4 < -1 - 2\sqrt{p^2 + p + 1}$.
- (iii) $\mu_3 + \mu_4 < -3 - 2\sqrt{p^2 + p + 1} < -3 - 2\sqrt{p^2 + p}$.

Proof. We assume that given roots are ordered as $\mu_4 \leq \mu_3 \leq \mu_2 \leq \mu_1$. Using the relations between the root and coefficient of $\phi_{2,4}(\mu)$, we obtain

$$\begin{aligned} \sum_{i=1}^4 \mu_i &= 2p - 6, & \sum_{i < j} \mu_i \mu_j &= 16 - 14p - 4p^2, \\ \sum_{i < j < k} \mu_i \mu_j \mu_k &= -8p^3 + 8p^2 + 22p - 26, & \prod_{i=1}^4 \mu_i &= 24p^3 - 20p^2 - 42p + 15. \end{aligned}$$

We note that $\prod_{i=1}^4 \mu_i = 24p^3 - 20p^2 - 42p + 15 = 4p^2(6p - 5) - 3(14p - 5) > 0$ and $\sum_{i=1}^4 \mu_i = 2p - 6 > 0$ for $p \geq 4$. It follows that the polynomial $\phi_{2,4}(\mu)$ has either four positive roots, or exactly two positive and two negative. Next, we evaluate $\phi_{2,4}(\mu)$ at specific points and find that

- $\phi_{2,4}(-2) = 8p^3 - 20p^2 - 38p - 5 > 0$ and $\phi_{2,4}(-3) = -16p(2p + 3) < 0$;
- $\phi_{2,4}(-1 - 2\sqrt{p^2 + p + 1}) = -16(p - 2 + 2\sqrt{p^2 + p + 1}) < 0$ and $\phi_{2,4}(-2 - 2\sqrt{p^2 + p + 1}) = 16p^3 + 20p^2 + 2p + 27 + (16p^2 + 12p + 12)\sqrt{p^2 + p + 1} > 0$

for all $p \geq 4$. Then by the intermediate value theorem, we obtain that $-3 < \mu_3 < -2$ and $-2 - 2\sqrt{p^2 + p + 1} < \mu_4 < -1 - 2\sqrt{p^2 + p + 1}$. Therefore, (i) and (ii) are true.

Adding the upper bounds $\mu_3 < -2$ and $\mu_4 < -1 - 2\sqrt{p^2 + p + 1}$ from (ii) gives $\mu_3 + \mu_4 < -3 - 2\sqrt{p^2 + p + 1}$. Since $p^2 + p + 1 > p + 1$, $\sqrt{p^2 + p + 1} > \sqrt{p^2 + p}$, hence $-3 - 2\sqrt{p^2 + p + 1} < -3 - 2\sqrt{p^2 + p}$, proving (iii). $\square$

Lemma 23. Let $T \cong K_{p+1,p+1,p,p}$ and e_p be an embedding edge, where $p \geq 3$. Then Seidel energy satisfies $\mathcal{E}(T + e_p) > \mathcal{E}(T)$.

Proof. By Corollary 20,

$$\begin{aligned}\mathcal{E}(T) &= (4p - 2) + (2p - 1) + (2p + 1) + |2\sqrt{p^2 + p} - 1| + |-2\sqrt{p^2 + p} - 1| \\ &= 8p - 2 + 2\sqrt{p^2 + p} - 1 + 2\sqrt{p^2 + p} + 1 \\ &= 8p - 2 + 4\sqrt{p^2 + p}.\end{aligned}$$

For $p = 3$, a computer search gives $\mathcal{E}(T + e_3) - \mathcal{E}(T) = 36.1174 - 35.8564 = 0.2612$, this implies that $\mathcal{E}(T + e_3) > \mathcal{E}(T)$.

Now consider $p \geq 4$. By the definition of Seidel energy and Corollary 21, we get

$$\mathcal{E}(T + e_p) = (4p - 4) + 1 + (2p + 1) + \sum_{i=1}^4 |\mu_i| = 6p - 2 + \sum_{i=1}^4 |\mu_i|.$$

Remark 22 (i), we have $|\mu_i| = \mu_i$, $i = 1, 2$ and $|\mu_i| = -\mu_i$, $i = 3, 4$. Now, can write the relation $\sum \mu_i = 2p - 6$ as $|\mu_1| + |\mu_2| = \mu_1 + \mu_2 = 2p - 6 - \mu_3 - \mu_4$. Hence,

$$\mathcal{E}(T + e_p) = 6p - 2 + (-\mu_3) + (-\mu_4) + (2p - 6 - \mu_3 - \mu_4) = 8p - 8 - 2(\mu_3 + \mu_4).$$

Compute the difference

$$\begin{aligned}\mathcal{E}(T + e_p) - \mathcal{E}(T) &= 8p - 8 - 2(\mu_3 + \mu_4) - (8p - 2 + 4\sqrt{p^2 + p}) \\ &= -6 - 4\sqrt{p^2 + p} - 2(\mu_3 + \mu_4).\end{aligned}$$

By Remark 22 (iii), $\mu_3 + \mu_4 < -3 - 2\sqrt{p^2 + p}$ or $2(\mu_3 + \mu_4) < -6 - 4\sqrt{p^2 + p}$. Therefore, $\mathcal{E}(T + e_p) - \mathcal{E}(T) > 0$. Hence, $\mathcal{E}(T + e_p) > \mathcal{E}(T)$. $\square$

5.2. Edge embedding in a part of size $p + 1$

Let $K_{p+1,p+1,p,p}$ denote the complete 4-partite graph of order $n = 4p + 2$. Within one of the parts of size $p + 1$, select two non-adjacent vertices x_i, x_j (with $1 \leq i < j \leq p + 1$) and embed an edge $e_{p+1} = x_i x_j$ between them. The resulting graph is denoted $K_{p+1,p+1,p,p} + e_{p+1}$.

Corollary 24. Let $T + e_{p+1} \cong K_{p+1,p+1,p,p} + e_{p+1}$, where $p \geq 2$. Then the Seidel spectrum of $T + e_{p+1}$ is $\{[-1]^{4p-4}, 1, 2p - 1, \mu_i\}$, where μ_i (for $i = 1, 2, 3, 4$) are the roots of the polynomial $\psi_{2,4}(\mu) = \mu^4 + (4 - 2p)\mu^3 + (6 - 14p - 4p^2)\mu^2 + (4 - 14p + 8p^3)\mu + 1 - 18p + 4p^2 + 24p^3$.

Proof. Setting $t = 4$ and $r = 2$ in Theorem 8, we have the desired result. $\square$

Remark 25. Suppose that $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4$ are the zeros of the polynomial $\psi_{2,4}(\mu)$. When $p \geq 3$, the ordering and bounds of these zeros are described as follows:

- (i) $\mu_4 < \mu_3 < 0 < \mu_2 \leq \mu_1$.
- (ii) $-3 < \mu_3 < -2$ and $-2 - 2\sqrt{p^2 + p + 1} < \mu_4 < -1 - 2\sqrt{p^2 + p + 1}$.
- (iii) $\mu_3 + \mu_4 < -3 - 2\sqrt{p^2 + p + 1} < -3 - 2\sqrt{p^2 + p}$.

Proof. Assume that roots are ordered as $\mu_4 \leq \mu_3 \leq \mu_2 \leq \mu_1$. The relations between the root and coefficient yield

$$\begin{aligned} \sum_{i=1}^4 \mu_i &= 2p - 4, & \sum_{i < j} \mu_i \mu_j &= 6 - 16 - p^2 - 4p^2, \\ \sum_{i < j < k} \mu_i \mu_j \mu_k &= -4 + 14p - 8p^2, & \prod_{i=1}^4 \mu_i &= 24p^3 + 4p^2 - 18p + 1. \end{aligned}$$

For all $p \geq 1$, $\prod_{i=1}^4 \mu_i = 24p^3 + 4p^2 - 18p + 1 > 0$, so the number of negative roots is 0, 2, or 4. Four negative roots would force $\sum_i \mu_i < 0$, contradicting $\sum_{i=1}^4 \mu_i = 2p - 4 > 0$ for $p \geq 3$. No negative roots would force $\sum_{i < j} \mu_i \mu_j \geq 0$, contradicting $\sum_{i < j} \mu_i \mu_j = 6 - 16 - p^2 - 4p^2 < 0$ for $p \geq 3$. Hence, exactly two roots are negative and two positive. For the negative roots, note

- $\psi_{2,4}(-3) = -16(2p^2 + 3p - 1) < 0$, $\psi_{2,4}(-2) = 4p^2(2p - 3) - 30p + 1 > 0$;
- $\psi_{2,4}(-2 - 2\sqrt{p^2 + p + 1}) = 16p^3 + 36p^2 + 18p + 41 + (16p^2 + 12p + 40)\sqrt{p^2 + p + 1} > 0$,
 $\psi_{2,4}(-1 - 2\sqrt{p^2 + p + 1}) = 16(1 - p) < 0$

for $p \geq 3$. Then by the intermediate value theorem, we have $-3 < \mu < -2$ and $-2 - 2\sqrt{p^2 + p + 1} < \mu_4 < -1 - 2\sqrt{p^2 + p + 1}$. Since $\sqrt{p^2 + p + 1} > 1$ for all $p \geq 3$, we have $-1 - 2\sqrt{p^2 + p + 1} < -3$, so such two intervals are disjoint. (i) and (ii) are proved.

Part (iii) can be proved similar to Remark 22 (iii), so we omit its detail. $\square$

Lemma 26. Let $T \cong K_{p+1,p+1,p,p}$ and let e_{p+1} be an embedded edge such that $e_{p+1} \notin E(T)$, where $p \geq 3$. Then Seidel energy satisfies $\mathcal{E}(K_{p+1,p+1,p,p} + e_{p+1}) > \mathcal{E}(K_{p+1,p+1,p,p})$.

Proof. By the definition of Seidel energy and Corollary 24, we have

$$\mathcal{E}(T + e_{p+1}) = (4p - 4) + 1 + (2p - 1) + \sum_{i=1}^4 |\mu_i| = 6p - 4 + \sum_{i=1}^4 |\mu_i|.$$

By Remark 25 (i), we get $|\mu_i| = \mu_i$ for $i = 1, 2$ and $|\mu_i| = -\mu_i$ for $i = 3, 4$. From $\sum_{i=1}^4 \mu_i = 2p - 4$, we can write $|\mu_1| + |\mu_2| = \mu_1 + \mu_2 = 2p - 4 - \mu_3 - \mu_4$. Therefore,

$$\mathcal{E}(T + e_{p+1}) = 6p - 4 + (2p - 4 - \mu_3 - \mu_4) + (-\mu_3) + (-\mu_4) = 8p - 8 - 2(\mu_3 + \mu_4).$$

The difference is

$$\mathcal{E}(T + e_{p+1}) - \mathcal{E}(T) = 8p - 8 - 2(\mu_3 + \mu_4) - (8p - 2 + 4\sqrt{p^2 + p}) = -6 - 4\sqrt{p^2 + p} - 2(\mu_3 + \mu_4).$$

Now by Remark 25 (iii), $\mu_3 + \mu_4 < -3 - 2\sqrt{p^2 + p}$. Substituting this bound above yields $\mathcal{E}(T + e_{p+1}) - \mathcal{E}(T) > 0$, i.e., $\mathcal{E}(T + e_{p+1}) > \mathcal{E}(T)$, completing the proof. $\square$

6. Seidel energy change in $K_{p+1,p+1,p+1,p}$ under edge embedding

This section turns to the effect that edge embedding has on the Seidel energy of the complete 4-partite graph $K_{p+1,p+1,p+1,p}$ of order $n = 4p + 3$.

6.1. Edge embedding in the part of size p

Within the part of size p in $K_{p+1,p+1,p+1,p}$, choose two non-adjacent vertices x_i and x_j , where $1 \leq i < j \leq p$, and insert an edge $e_p = x_i x_j$ between them, with $e_p \notin E(K_{p+1,p+1,p+1,p})$. This produces the graph $K_{p+1,p+1,p+1,p} + e_p$.

Corollary 27. *Let $T \cong K_{p+1,p+1,p+1,p}$ for $p \geq 2$. Then the Seidel spectrum of T is given by $\{-1\}^{4p-1}, [2p+1]^2, \frac{-3+\sqrt{16p^2+16p+1}}{2}, \frac{-3-\sqrt{16p^2+16p+1}}{2}\}$.*

Proof. Setting $p_1 = p_2 = p_3 = p + 1$ and $p_4 = p$ in Theorem 6, we get

$$\begin{aligned} S_T(\mu) &= (\mu + 1)^{4p-1} \cdot (\mu - 2p - 1)^2 \cdot [(\mu - 2p + 1)(\mu - 2p - 1) + 3(p + 1)(\mu - 2p + 1) \\ &\quad + p(\mu - 2p - 1)] \\ &= (\mu + 1)^{4p-1} \cdot (\mu - 2p - 1)^2 \cdot (\mu^2 + 3\mu - 4p^2 - 4p + 2) \\ &= (\mu + 1)^{4p-1} \cdot (\mu - 2p - 1)^2 \cdot \left(\mu + \frac{3 + \sqrt{16p^2 + 16p + 1}}{2}\right) \cdot \left(\mu + \frac{3 - \sqrt{16p^2 + 16p + 1}}{2}\right). \end{aligned}$$

Hence,

$$\text{Spec}(T) = \{-1\}^{4p-1}, [2p+1]^2, \frac{-3 + \sqrt{16p^2 + 16p + 1}}{2}, \frac{-3 - \sqrt{16p^2 + 16p + 1}}{2}\}.$$

□

Corollary 28. *Let $T + e_p \cong K_{p+1,p+1,p+1,p} + e_p$, where $p \geq 2$. Then the Seidel spectrum of $T + e_p$ is $\{-1\}^{4p-3}, 1, [2p+1]^2, \mu_i\}$, where μ_i for $i = 1, 2, 3$ are the roots of the polynomial $\phi_{3,4}(\mu) = \mu^3 + 6\mu^2 + (15 - 4p - 4p^2)\mu + 26 + 4p - 12p^2$.*

Proof. The desired result follows by setting $t = 4$ and $r = t - 1 = 3$ in Theorem 7. □

Remark 29. *Suppose that $\mu_3 \leq \mu_2 \leq \mu_1$ are the roots of $\phi_{3,4}(\mu)$. Then for $p \geq 4$, each of the following holds:*

- (i) $\mu_3 < \mu_2 < 0 < \mu_1$.
- (ii) $-3 < \mu_2 < -2$ and $-2p - 3 < \mu_3 < -2p - \frac{5}{2}$.
- (iii) $\mu_2 + \mu_3 < -2p - \frac{9}{2} < \frac{-7 - \sqrt{16p^2 + 16p + 1}}{2}$.

Proof. Since μ_1, μ_2 , and μ_3 are the roots of $\phi_{3,4}(\mu)$, by the relation of the root and coefficient, we get

$$\mu_1 + \mu_2 + \mu_3 = -6, \quad \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 = 15 - 4p - 4p^2, \quad \text{and} \quad \mu_1\mu_2\mu_3 = 12p^2 - 4p - 26.$$

We know that $\mu_1\mu_2\mu_3 = 4p(3p - 1) - 26 > 0$ for $p \geq 4$ and $\mu_1 + \mu_2 + \mu_3 < -6$. This implies that exactly two roots are negative and one is positive, that is, $\mu_3 \leq \mu_2 < 0 < \mu_1$. To determine the bounds in (ii), we evaluate $\phi_{3,4}(\mu)$ at specific points. We find that

- $\phi_{3,4}(-2) = -4p(p-3) + 12 < 0$ and $\phi_{3,4}(-3) = 16p + 8 > 0$;
- $\phi_{3,4}(-2p-3) = -4p(p-1) + 8 < 0$ and $\phi_{3,4}(-2p-\frac{5}{2}) = \frac{13}{2}p + \frac{83}{8} > 0$

for all $p \geq 4$. Applying the intermediate value theorem, we have $-3 < \mu_2 < -2$ and $-2p-3 < \mu_3 < -2p-\frac{5}{2}$. Notice that the intervals $(-3, -2)$ and $(-2p-3, -2p-\frac{5}{2})$ are disjoint; this means $\mu_3 < \mu_2$. Statements (i) and (ii) are true.

By (ii), we have $\mu_2 + \mu_3 < -2p - \frac{9}{2}$. It remains to show that $-2p - \frac{9}{2} < \frac{-7 - \sqrt{16p^2 + 16p + 1}}{2}$. This follows since $(4p+2)^2 = 16p^2 + 16p + 4 > 16p^2 + 16 + 1$, so $4p+2 > \sqrt{16p^2 + 16p + 1}$, which yields $-4p-9 < -7 - \sqrt{16p^2 + 16p + 1}$, as required. The desired result is completed. $\square$

Lemma 30. Let $T \cong K_{p+1, p+1, p+1, p}$ and let e_p be an embedded edge such that $e_p \notin E(T)$, where $p \geq 3$. Then Seidel energy satisfies $\mathcal{E}(T + e_p) > \mathcal{E}(T)$.

Proof. Applying Corollary 27 together the definition of Seidel energy, we have

$$\begin{aligned}\mathcal{E}(T) &= (4p-1) + 2 \cdot (2p+1) + \left| \frac{-3 + \sqrt{16p^2 + 16p + 1}}{2} \right| + \left| \frac{-3 - \sqrt{16p^2 + 16p + 1}}{2} \right| \\ &= 8p + 1 + \frac{\sqrt{16p^2 + 16p + 1} - 3}{2} + \frac{\sqrt{16p^2 + 16p + 1} + 3}{2} \\ &= 8p + 1 + \sqrt{16p^2 + 16p + 1}.\end{aligned}$$

We handle the case $p = 3$. By direct calculation, this gives $\mathcal{E}(T + e_3) - \mathcal{E}(T) = 39.1413 - 38.8924 = 0.2489$, so $\mathcal{E}(T + e_3) > \mathcal{E}(T)$.

Now assume $p \geq 4$. By the definition of Seidel energy and Corollary 28, we obtain

$$\mathcal{E}(T + e_p) = (4p-3) + 1 + 2 \cdot (2p+1) + |\mu_1| + |\mu_2| + |\mu_3| = 8p + |\mu_1| + |\mu_2| + |\mu_3|.$$

Since $\mu_3 < \mu_2 < 0 < \mu_1$ by Remark 29 (i), $|\mu_1| = \mu_1$, $|\mu_2| = -\mu_2$, and $|\mu_3| = -\mu_3$. From $\mu_1 + \mu_2 + \mu_3 = -6$, it follows that $|\mu_1| = \mu_1 = -6 - \mu_2 - \mu_3$. Hence,

$$\mathcal{E}(T + e_p) = 8p + (-6 - \mu_2 - \mu_3) + (-\mu_2) + (-\mu_3) = 8p - 6 - 2(\mu_2 + \mu_3).$$

We now compute the difference

$$\begin{aligned}\mathcal{E}(T + e_p) - \mathcal{E}(T) &= 8p - 6 - 2(\mu_2 + \mu_3) - (8p + 1 + \sqrt{16p^2 + 16p + 1}) \\ &= -7 - \sqrt{16p^2 + 16p + 1} - 2(\mu_2 + \mu_3).\end{aligned}$$

By Remark 29 (iii), $\mu_2 + \mu_3 < \frac{-7 - \sqrt{16p^2 + 16p + 1}}{2}$ or $2(\mu_2 + \mu_3) < -7 - \sqrt{16p^2 + 16p + 1}$. Therefore, $\mathcal{E}(T + e_p) - \mathcal{E}(T) > 0$. Hence, $\mathcal{E}(T + e_p) > \mathcal{E}(T)$, completing the proof. $\square$

6.2. Edge embedding in a part of size $p+1$

Within a part of size $p+1$ of $K_{p+1, p+1, p+1, p}$, fix two non-adjacent vertices x_i and x_j , where $1 \leq i < j \leq p+1$, and add the edge $e_{p+1} \notin E(K_{p+1, p+1, p+1, p})$ between them. The resulting graph is denoted $K_{p+1, p+1, p+1, p} + e_{p+1}$.

Corollary 31. Let $T + e_{p+1} \cong K_{p+1,p+1,p+1,p} + e_{p+1}$, where $p \geq 2$. Then the Seidel spectrum of $T + e_{p+1}$ is $\{[-1]^{4p-3}, 1, 2p+1, \mu_i\}$, where μ_i ($i = 1, \dots, 4$) are the roots of the polynomial $\psi_{3,4}(\mu) = \mu^4 + (5-2p)\mu^3 + (9-16p-4p^2)\mu^2 + (11-22p+8p^3)\mu + 6-24p+4p^2+24p^3$.

Proof. This result follows by setting $t = 4$ and $r = 3$ in Theorem 8. $\square$

Remark 32. Let $\mu_4 \leq \mu_3 \leq \mu_2 \leq \mu_1$ be the roots of $\psi_{3,4}(\mu)$. Then, for every $p \geq 3$, the following statements hold.

- (i) $\mu_4 < \mu_3 < 0 < \mu_2 \leq \mu_1$.
- (ii) $-3 < \mu_3 < -2$ and $-2p-3 < \mu_4 < -2p-\frac{5}{2}$.
- (iii) $\mu_3 + \mu_4 < -2p - \frac{9}{2} < \frac{-7-\sqrt{16p^2+16p+1}}{2}$.

Proof. For $p \geq 3$, the relations between the root and coefficient give $\sum_{i=1}^4 \mu_i = 2p-5 > 0$ and $\prod_{i=1}^4 \mu_i = 24p^3 - 24p + 4p^2 + 6 = 24p(p^2 - 1) + 2(2p^2 + 3) > 0$. This means that we have two possibilities for the signs of roots: (1) four roots positive, and (2) exactly two positive and two negative roots. To distinguish between two possibilities, we evaluate $\psi_{3,4}(\mu)$ at specific points. We find that $\psi_{3,4}(-2) = 8p^3 - 12p^2 - 28p - 4 > 0$ and $\psi_{3,4}(-3) = -16p(2p+3) < 0$ for $p \geq 3$, so the intermediate value theorem gives $-3 < \mu_3 < -2$. Similarly, we find that $\psi_{3,4}(-2p-3) = 16p(p^2-1) > 0$ and $\psi_{3,4}(-2p-\frac{5}{2}) = -26p^2 - \frac{129}{4}p - \frac{69}{16} < 0$, therefore $-2p-3 < \mu_4 < -2p-\frac{5}{2}$. Since $-2p-\frac{5}{2} < -3$ for $p \geq 3$, both intervals $(-3, -2)$ and $(-2p-3, -2p-\frac{5}{2})$ are disjoint. Statements (i) and (ii) have been proved. The proof of (iii) follows by the same argument as Remark 29 (iii), and the details are omitted. $\square$

Lemma 33. Let $T \cong K_{p+1,p+1,p+1,p}$ and let e_{p+1} be an embedded edge such that $e_{p+1} \notin E(T)$, where $p \geq 3$. Then Seidel energy satisfies $\mathcal{E}(T + e_{p+1}) > \mathcal{E}(T)$.

Proof. By the definition of Seidel energy and Corollary 31, we have

$$\mathcal{E}(T + e_{p+1}) = (4p-3) + 1 + (2p+1) + \sum_{i=1}^4 |\mu_i| = 6p-1 + \sum_{i=1}^4 |\mu_i|.$$

Remark 32 (i) gives $|\mu_i| = \mu_i$ for $i = 1, 2$ and $|\mu_i| = -\mu_i$ for $i = 3, 4$. From $\sum_{i=1}^4 \mu_i = 2p-5$, we can write $|\mu_1| + |\mu_2| = \mu_1 + \mu_2 = 2p-5 - \mu_3 - \mu_4$. Therefore,

$$\mathcal{E}(T + e_{p+1}) = 6p-1 + (2p-5 - \mu_3 - \mu_4) + (-\mu_3) + (-\mu_4) = 8p-6 - 2(\mu_3 + \mu_4).$$

The difference is

$$\begin{aligned} \mathcal{E}(T + e_{p+1}) - \mathcal{E}(T) &= 8p-6 - 2(\mu_3 + \mu_4) - (8p+1 + \sqrt{16p^2+16p+1}) \\ &= -7 - \sqrt{16p^2+16p+1} - 2(\mu_3 + \mu_4). \end{aligned}$$

By Remark 32 (iii), we have $\mu_3 + \mu_4 < \frac{-7-\sqrt{16p^2+16p+1}}{2}$. Substituting this bound into the expression above and simplifying gives $\mathcal{E}(T + e_{p+1}) - \mathcal{E}(T) > 0$. Hence, $\mathcal{E}(T + e_{p+1}) > \mathcal{E}(T)$. $\square$

7. Conclusions

In this paper, we have constructed Seidel polynomials $S_{T+e}(\mu)$ for the t -partite Turán graph after embedding an edge, expressed in terms of the general parameters p , r , and t . Unlike the Seidel polynomials known in the literature for bipartite [14] and tripartite [11] Turán graphs, which are confined to the fixed cases $t = 2$ and $t = 3$, respectively, the polynomials obtained here are valid for general edge embedded t -partite Turán graphs for every positive integer t . This generality is precisely what makes the polynomial important: It provides a unified expression from which Question 1 can be analyzed for arbitrary t . It therefore represents a strict generalization of the existing results.

Next, we showed that the Seidel energy can either increase or decrease under edge embedding, depending on the order and the part in which the edge is embedded. Theorem 3 shows that embedding an edge into the balanced t -partite Turán graph $T(2t, t) \cong K_{2,2,\dots,2}$ always decreases the Seidel energy; for every $t \geq 3$, the inequality reverses, as noted in [14] for $t = 2$. In contrast, Theorem 4 establishes that for $t = 4$ and $p \geq 3$ (i.e., $n \geq 12$), embedding an edge into 4-partite Turán graph $T(n, 4)$ always increases the Seidel energy, and its proof draws on the case-by-case lemmas developed above.

Proof of Theorem 4. We consider each value of r separately.

- (1) If $r = 0$, this is precisely the statement of Lemma 12.
- (2) If $r = 1$, the claim is established by Lemmas 16 and 19.
- (3) If $r = 2$, Lemmas 23 and 26 give the desired inequality.
- (4) If $r = 3$, combining Lemmas 30 and 33 yields the result.

Since these four cases cover all admissible values $0 \leq r \leq 3$, the proof is complete. $\square$

Theorem 4 gives a complete picture for $t = 4$: The threshold $n \geq 12$ in Theorem 4 is sharp, since some embedding edges decrease the Seidel energy at every order $5 \leq n \leq 11$ (Table 2), while Theorem 3 parallels the exception at $T^*(8, 3)$ for $t = 3$ [11].

The methodological contribution of this paper lies in the systematic use of block matrix determinants and coefficient relations to derive root bounds, ensuring rigorous justification of energy comparisons. These techniques can be adapted to higher values of t , although the complexity of the polynomials increases significantly. The findings strengthen the evidence for the conjecture (Question 1) that Seidel energy is monotone under edge embedding in Turán graphs for large order n .

Author contributions

M. U. Rehman: Supervision of the project and conceptualization; A. Yang: Led the mathematical investigation, performed all computational work and writing the original draft; M. Ajmal: Developed the theoretical framework, implemented the algorithm, development of the proofs techniques and correspondence; B. G. Rodrigues: Verified all proofs, performed computational checks and revised the manuscript for clarity; A. D. Maden: Visualization and review; G. Cai: Formal analysis and developed the spectral method. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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