



Research article

Geometric properties of harmonic functions associated with the normalized Mittag–Leffler function

Mohammad Faisal Khan^{1,*}, Muqrin A. Almuqrin^{2,*} and Mohammed AbaOud³

¹ Department of Basic Sciences, College of Science and Theoretical Studies, Prince Musaid bin Abdulrahman University, Riyadh 11673, Saudi Arabia; f.khan@seu.edu.sa

² Department of Mathematics, College of Science in Zulfi, Majmaah University, Al-Majmaah 11952, Saudi Arabia; m.almuqrin@mu.edu.sa

³ Department of Mathematics and Statistics, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11564, Saudia Arabia; maabaoud@imamu.edu.sa

* **Correspondence:** Email: f.khan@seu.edu.sa, m.almuqrin@mu.edu.sa.

Abstract: In this paper, we introduced and investigated a new subclass of complex-valued harmonic functions associated with the normalized two-parameter Mittag–Leffler function. The proposed class was defined by a second-order differential inequality involving the normalized Mittag–Leffler kernel, thereby establishing a direct connection between harmonic mapping theory and Mittag–Leffler-type special functions. An analytic characterization of the proposed harmonic class was derived through a family of analytic functions $\Upsilon_{\varepsilon} = \eta + \varepsilon\xi$, where $|\varepsilon| = 1$. Furthermore, inclusion relations, parameter-dependent coefficient estimates, growth estimates, and closure under convex combinations were obtained. A Fekete–Szegő type inequality was also derived for the associated family of analytic functions. Several examples and graphical illustrations are included to demonstrate the geometric behavior of the proposed class. This paper unifies several known harmonic classes via differential inequalities and introduces a new framework for harmonic mappings with Mittag–Leffler and related functions.

Keywords: univalent functions; harmonic mappings; Mittag–Leffler function; differential inequality; starlike function; close-to-convex function; coefficient estimates; growth theorem; convex combinations; Fekete–Szegő inequality

Mathematics Subject Classification: Primary 30C45, 30C50, Secondary 05A30, 33E12

1. Introduction

Harmonic mappings constitute an important area of research in geometric function theory (GFT) and complex analysis (CA) owing to their rich analytical structure and broad range of applications. They arise naturally as solutions of Laplace's equation and play a significant role in several branches of mathematics and applied sciences, including potential theory, fluid mechanics, elasticity, electrostatics, and conformal mapping theory. Their smoothness, mean-value property, and close relationship with analytic functions (AFs) have made harmonic mappings (HPs) an active topic of investigation over the past several decades. Besides the classical theory, considerable attention has recently been devoted to subclasses of harmonic mappings characterized by differential operators and differential inequalities. Such additional constraints provide an effective framework for studying geometric properties that are not immediately accessible through the standard harmonic setting. In particular, second-order differential inequalities have proved to be a useful tool for constructing new harmonic subclasses and deriving coefficient estimates, growth theorems, inclusion relations, distortion bounds, and close-to-convexity criteria. Harmonic functions (HFs) have a central role in mathematical analysis and applied mathematics because of their remarkable analytical properties and numerous applications in physics, engineering, fluid dynamics, and potential theory.

Let $\mathcal{A}$ denote the class of analytic function in the open unit disk

$$\mathbb{D} = \{v \in \mathbb{C} : |v| < 1\}.$$

Thus, every $\xi \in \mathcal{A}$ has the Taylor series expansion of the form

$$\xi(v) = v + \sum_{s=2}^{\infty} a_s v^s \quad (1.1)$$

and are normalized by

$$\xi(0) = 0, \quad \xi'(0) = 1.$$

Definition 1.1 (Starlike functions of order τ). Let $0 \leq \tau < 1$. An analytic function $\xi \in \mathcal{A}$ is said to be starlike of order τ in $\mathbb{D}$ if

$$\Re \left\{ \frac{v \xi'(v)}{\xi(v)} \right\} > \tau, \quad v \in \mathbb{D}.$$

The class of all such functions is denoted by $\mathcal{S}^*(\tau)$. In particular, for $\tau = 0$, we obtain the usual class of starlike functions

$$\mathcal{S}^* = \mathcal{S}^*(0).$$

Definition 1.2 (Close-to-convex functions of order τ). Let $0 \leq \tau < 1$. An analytic function $\xi \in \mathcal{A}$ is said to be close-to-convex of order τ in $\mathbb{D}$ if there exists a function $g \in \mathcal{S}^*$ such that

$$\Re \left\{ \frac{v \xi'(v)}{g(v)} \right\} > \tau, \quad v \in \mathbb{D}.$$

The class of all such functions is denoted by $\mathcal{K}(\tau)$. For $\tau = 0$, this reduces to the usual class of close-to-convex functions.

A complex-valued harmonic functions in the open unit disk $\mathbb{D}$ can be written in the form

$$f(v) = \eta(v) + \overline{\xi(v)},$$

where η and ξ are analytic in $\mathbb{D}$. Let $\mathcal{H}$ denote the class of harmonic functions normalized by

$$f(0) = 0, \quad f'(0) = 1,$$

and let

$$\mathcal{H}^0 = \{f \in \mathcal{H} : \xi'(0) = 0\}.$$

Every function $f \in \mathcal{H}^0$ admits the representation

$$\eta(v) = v + \sum_{s=2}^{\infty} a_s v^s, \quad \xi(v) = \sum_{s=2}^{\infty} b_s v^s.$$

Moreover,

$$f = \eta + \bar{\xi}$$

is locally univalent and sense-preserving (see [1, 2]) in $\mathbb{D}$, whenever

$$|\xi'(v)| < |\eta'(v)|, \quad v \in \mathbb{D}.$$

The geometric theory of harmonic mappings was developed extensively through the pioneering work of Clunie and Sheil-Small [1]. Since then, numerous subclasses of HPs associated with convexity, starlikeness, close-to-convexity, coefficient inequalities, convolution operators, and differential operators have been introduced and investigated. Moreover, particular attention has been devoted to harmonic classes characterized by differential inequalities involving the analytic and co-analytic parts. Such classes have proved to be an effective tool for establishing coefficient estimates, growth and distortion theorems, inclusion relations, and closure properties. One of the earliest harmonic subclasses defined through a differential inequality was introduced by Ponnusamy et al. [3]. They considered the class

$$\mathcal{P}_H^0 = \{f = \eta + \bar{\xi} \in \mathcal{H}^0 : \Re(\eta'(v)) > |\xi'(v)|\}$$

and proved that every function belonging to $\mathcal{P}_{\mathcal{H}^0}$ is harmonic close-to-convex. Their work laid the foundation for the subsequent development of several harmonic subclasses generated by first and higher-order differential inequalities. Later, Ghosh and Vasudevarao [4] considered harmonic functions satisfying

$$\Re(\eta'(v) + \gamma v \eta''(v)) > |\xi'(v) + \gamma v \xi''(v)|$$

and found coefficient estimates, growth results, convolution properties, and radius problems. Subsequently, Breaz et al. [5] investigated a generalized class involving the parameter λ , while Nagpal and Ravichandran [6] studied the harmonic analogue of Chichra's analytic class and established several geometric properties. Other related harmonic subclasses generated by differential operators were investigated by Mishra et al. [7], Çakmak et al. [8], Breaz et al. [9], and many others. Moreover, Yalçın et al. [10] introduced a new subclass of harmonic functions defined by a second-order

differential inequality. Specifically, they considered the class $\mathcal{BH}^0(\gamma, \delta)$, consisting of functions $f = \eta + \bar{\xi} \in \mathcal{H}^0$ satisfying

$$\Re \left\{ \gamma \frac{\eta(v)}{v} + \delta \eta'(v) + \left(\delta - \frac{\gamma}{2} \right) v \eta''(v) \right\} > \left| \gamma \frac{\xi(v)}{v} + \delta \xi'(v) + \left(\delta - \frac{\gamma}{2} \right) v \xi''(v) \right|, \quad (1.2)$$

where

$$\delta \geq \gamma \geq 0, v \in \mathbb{D}.$$

They also considered the associated analytic subclass $\mathcal{B}_1(\gamma, \delta, 0; 0)$, defined by $\eta \in \mathcal{A}$ and satisfying

$$\Re \left\{ \gamma \frac{\eta(v)}{v} + \delta \eta'(v) + \left(\delta - \frac{\gamma}{2} \right) v \eta''(v) \right\} > 0, \quad \delta \geq \gamma \geq 0, \quad v \in \mathbb{D}. \quad (1.3)$$

The inclusion properties of the analytic class $\mathcal{B}_1(\gamma, \delta, 0; 0)$ were investigated by Al-Refai [11].

Developments in geometric function theory have studied the harmonic and analytic function classes through differential operators and geometric properties. The Janowski-type p -harmonic functions associated with the generalized Sălăgean operator were investigated in [12], where several coefficient estimates and geometric properties were investigated. Furthermore, the application of the q -derivative operator to harmonic functions with positive real part was explored in [13], leading to new inclusion relationships and distortion results. In addition, geometric properties of Bazilevič harmonic functions were studied in [14], providing further insights into starlikeness and convexity. Along this direction, Çakmak and Yalçın [15] introduced the harmonic class $\mathcal{KH}^0(k, \gamma)$, which extends the analytic class $\mathcal{K}(k, \gamma)$ by incorporating the co-analytic part into a differential inequality. Class $\mathcal{KH}^0(k, \gamma)$ is defined as follows:

Let

$$\phi(v) = v + \sum_{s=2}^{\infty} c_s v^s \in \mathcal{S}^* \left(\frac{k-1}{k} \right),$$

where $k \in \mathbb{N}$, and define

$$\phi_k(v) = \prod_{j=0}^{k-1} \mu^{-j} \phi(\mu^j v), \quad \mu = e^{2\pi i/k}, \mu^k = 1. \quad (1.4)$$

Then, the class $\mathcal{KH}^0(k, \gamma)$ consists of harmonic mappings

$$f = \eta + \bar{\xi} \in \mathcal{SH}^0,$$

satisfying

$$\Re \left\{ \frac{v^k \eta'(v)}{\phi_k(v)} - \gamma \right\} > \left| \frac{v^k \xi'(v)}{\phi_k(v)} \right|, \quad 0 \leq \gamma < 1, v \in \mathbb{D}. \quad (1.5)$$

This formulation considerably enlarges the corresponding analytic family while preserving many of its geometric features. In particular, when co-analytic part is zero, then class $\mathcal{KH}^0(k, \gamma)$ reduces to analytic class $\mathcal{K}(k, \gamma)$. Several subclasses of harmonic mappings and their geometric properties have been investigated in [16–18]. Further developments concerning close-to-convex harmonic mappings can be found in [19–21]. More recent contributions to related harmonic mapping problems and their applications are presented in [22]. Consequently, $\mathcal{KH}^0(k, \gamma)$ provides a unified setting for studying a wide variety of harmonic close-to-convex mappings.

1.1. Mittag–Leffler functions in geometric function theory

The Mittag–Leffler functions (MLFs) are some of the most important special functions arising in fractional calculus, integral equations, mathematical physics, viscoelasticity, anomalous diffusion, and related areas. The one-parameter Mittag–Leffler function was introduced by Mittag–Leffler [23] and later generalized by Wiman [24], who defined the classical two-parameter Mittag–Leffler function as

$$E_{d_1, d_2}(v) = \sum_{n=0}^{\infty} \frac{v^n}{\Gamma(d_1 n + d_2)}, \quad d_1, d_2 > 0.$$

This function has been extensively studied in fractional calculus and related differential equations, provides a natural generalization of the exponential functions and plays a fundamental role in the theory of fractional differential equations [25, 26]. In geometric function theory, normalized Mittag–Leffler functions have attracted considerable attention because of their rich coefficient structure and their ability to generate subclasses of analytic and univalent functions. Various geometric properties, including starlikeness, coefficient estimates, convexity, close-to-convexity, differential subordinations, and partial sums, have been investigated for MLFs and their generalizations [27–29].

The Mittag–Leffler functions have also become important special functions in geometric function theory, where its normalized forms and associated operators provide useful tools for constructing and investigating subclasses of analytic and univalent functions. In particular, coefficient estimates, starlikeness, convexity, close-to-convexity, and related geometric properties of Mittag–Leffler-type functions have attracted considerable attention. The flexibility of its parameters makes the Mittag–Leffler functions particularly remarkable for generating new families of analytic functions and for defining connections between special functions and classical geometric function theory. These developments motivate the study of a normalized Mittag–Leffler kernel into harmonic differential inequalities, leading to the harmonic subclass considered in this work. For more studies related to harmonic mappings, differential inequalities, and Mittag–Leffler functions, (see [10, 30, 31]). Among the most studied forms is the normalized two-parameter Mittag–Leffler function introduced by Bansal and Orhan [27]. Since the function E_{d_1, d_2} does not belong to the standard class $\mathcal{A}$ of normalized analytic functions, it is natural to consider the normalization

$$\mathcal{E}_{d_1, d_2}(v) = \Gamma(d_2) v E_{d_1, d_2}(v) = \sum_{s=0}^{\infty} \frac{\Gamma(d_2)}{\Gamma(d_1 s + d_2)} v^{s+1} = v + \sum_{s=2}^{\infty} A_s v^s, \quad (1.6)$$

where

$$(v, d_1, d_2 \in \mathbb{C}; \Re(d_1) > 0, d_2 \neq 0, -1, \dots).$$

The coefficients A_s are given by

$$A_s = \frac{\Gamma(d_2)}{\Gamma(d_1(s-1) + d_2)}, \quad s \geq 2. \quad (1.7)$$

Formula (1.6) is valid for complex-valued d_1 , d_2 , and $v \in \mathbb{C}$. Throughout this paper, we restrict our attention to the case where $d_1, d_2 \in \mathbb{R}$ and $v \in \mathbb{D}$. Function $\mathcal{E}_{d_1, d_2}$ contains many well-known functions

as special cases, including

$$\begin{cases} \mathcal{E}_{0,1}(v) = \frac{v}{1-v}, & \mathcal{E}_{1,1}(v) = ve^v, & \mathcal{E}_{2,1}(v) = v \cosh(\sqrt{v}), \\ \mathcal{E}_{1,2}(v) = e^v - 1, & \mathcal{E}_{1,3}(v) = \frac{2(e^v - v - 1)}{v}, & \mathcal{E}_{1,4}(v) = \frac{6(e^v - 1 - v) - 3v^2}{v^2}, \\ \mathcal{E}_{2,2}(v) = \sqrt{v} \sinh(\sqrt{v}), & \mathcal{E}_{3,1}(v) = \frac{1}{2} \left(e^{v^{1/3}} + 2e^{-v^{1/3}/2} \cos\left(\frac{\sqrt{3}}{2}v^{1/3}\right) \right). \end{cases}$$

For convenience, we define the normalized Mittag–Leffler kernel

$$\mathfrak{E}_{d_1,d_2}(v) = \frac{\mathcal{E}_{d_1,d_2}(v)}{v} = 1 + \sum_{s=1}^{\infty} A_{s+1} v^s. \quad (1.8)$$

Equivalently,

$$\mathfrak{E}_{d_1,d_2}(v) = \sum_{s=0}^{\infty} \frac{\Gamma(d_2)}{\Gamma(d_1 s + d_2)} v^s. \quad (1.9)$$

Equations (1.8) and (1.9) show that the normalized Mittag–Leffler kernel inherits the analytic structure of the classical Mittag–Leffler function while belonging to the standard normalized class of analytic functions.

Remark 1.1 (Kernel positivity and admissible parameter ranges). *The coefficient-sum condition*

$$\sum_{s=1}^{\infty} \left| \frac{\Gamma(d_2)}{\Gamma(d_1 s + d_2)} \right| \leq 1 \quad (1.10)$$

is a sufficient condition for the positivity of the real part of the normalized Mittag–Leffler kernel. Indeed, by Proposition 1.1, condition (1.10) is satisfied, in particular, for the nontrivial parameter ranges

$$d_1 \geq 1, d_2 \geq 2,$$

and

$$d_1 \geq 2, 1 \leq d_2 \leq 2.$$

For these parameter ranges,

$$\Re \{ \mathfrak{E}_{d_1,d_2}(v) \} > 0, \quad v \in \mathbb{D},$$

and hence

$$\mathfrak{E}_{d_1,d_2}(v) \neq 0, \quad v \in \mathbb{D}.$$

Thus, all quotients involving $\mathfrak{E}_{d_1,d_2}$ in the subsequent results are well defined. Condition (1.10) is used as a convenient sufficient criterion and is not intended to characterize all parameter values for which the normalized Mittag–Leffler kernel have positive real part.

Proposition 1.1. *Let*

$$\mathfrak{E}_{d_1,d_2}(v) = \sum_{s=0}^{\infty} \frac{\Gamma(d_2)}{\Gamma(d_1 s + d_2)} v^s, \quad v \in \mathbb{D}.$$

The coefficient-sum condition

$$\sum_{s=1}^{\infty} \left| \frac{\Gamma(d_2)}{\Gamma(d_1 s + d_2)} \right| \leq 1 \quad (1.11)$$

holds, in particular, in each of the following nontrivial parameter ranges:

$$d_1 \geq 1, d_2 \geq 2,$$

and

$$d_1 \geq 2, 1 \leq d_2 \leq 2.$$

Consequently,

$$\Re \{ \mathfrak{E}_{d_1, d_2}(v) \} > 0, \quad v \in \mathbb{D},$$

and hence

$$\mathfrak{E}_{d_1, d_2}(v) \neq 0, \quad v \in \mathbb{D}.$$

Proof. We first consider the case

$$d_1 \geq 1, d_2 \geq 2.$$

Since $d_1 s + d_2 \geq s + d_2 \geq 3$ for $s \geq 1$ and the gamma function is increasing on $[2, \infty)$, we have

$$\Gamma(d_1 s + d_2) \geq \Gamma(s + d_2).$$

Moreover,

$$\frac{\Gamma(d_2)}{\Gamma(s + d_2)} = \frac{1}{(d_2)_s},$$

where $(d_2)_s$ denotes the Pochhammer symbol. Since $d_2 \geq 2$,

$$(d_2)_s = d_2(d_2 + 1) \cdots (d_2 + s - 1) \geq 2 \cdot 3 \cdots (s + 1) = (s + 1)!.$$

Therefore,

$$\frac{\Gamma(d_2)}{\Gamma(d_1 s + d_2)} \leq \frac{1}{(s + 1)!}.$$

Consequently,

$$\begin{aligned} \sum_{s=1}^{\infty} \left| \frac{\Gamma(d_2)}{\Gamma(d_1 s + d_2)} \right| &\leq \sum_{s=1}^{\infty} \frac{1}{(s + 1)!} \\ &= e - 2 < 1. \end{aligned}$$

Thus, condition (1.11) is satisfied.

Next, suppose that

$$d_1 \geq 2, 1 \leq d_2 \leq 2.$$

For $1 \leq d_2 \leq 2$, we have

$$\Gamma(d_2) \leq 1.$$

Furthermore,

$$d_1 s + d_2 \geq 2s + 1, \quad s \geq 1.$$

Since $2s + 1 \geq 3$ and the gamma function is increasing on $[2, \infty)$,

$$\Gamma(d_1 s + d_2) \geq \Gamma(2s + 1) = (2s)!.$$

Hence,

$$\frac{\Gamma(d_2)}{\Gamma(d_1 s + d_2)} \leq \frac{1}{(2s)!},$$

and therefore

$$\begin{aligned} \sum_{s=1}^{\infty} \left| \frac{\Gamma(d_2)}{\Gamma(d_1 s + d_2)} \right| &\leq \sum_{s=1}^{\infty} \frac{1}{(2s)!} \\ &= \cosh(1) - 1 \\ &< 1. \end{aligned}$$

Thus, condition (1.11) also holds in this parameter range.

Finally, let

$$a_s = \frac{\Gamma(d_2)}{\Gamma(d_1 s + d_2)}.$$

Under either of the above parameter restrictions,

$$\sum_{s=1}^{\infty} |a_s| < 1.$$

For $|v| < 1$, we consequently obtain

$$\Re \{ \mathfrak{E}_{d_1, d_2}(v) \} = \Re \left\{ 1 + \sum_{s=1}^{\infty} a_s v^s \right\} \geq 1 - \sum_{s=1}^{\infty} |a_s| |v|^s > 1 - \sum_{s=1}^{\infty} |a_s| > 0.$$

Therefore,

$$\mathfrak{E}_{d_1, d_2}(v) \neq 0, \quad v \in \mathbb{D}.$$

This completes the proof. $\square$

Definition 1.3. Let $0 \leq \tau < 1$, and $\varkappa \geq 0$. An analytic function $\Upsilon(v)$ is said to belong to the class $\mathcal{E}_{1, \varkappa, \tau}(d_1, d_2)$, if

$$\Re \left\{ \frac{\Upsilon'(v) + \varkappa v \Upsilon''(v)}{\mathfrak{E}_{d_1, d_2}(v)} \right\} > \tau, \quad v \in \mathbb{D}. \quad (1.12)$$

Definition 1.4. Let $0 \leq \tau < 1$, $\varkappa \geq 0$, and $k \in \mathbb{N}$. A harmonic mapping $f = \eta + \bar{\xi} \in \mathcal{H}^0$, where

$$\eta(v) = v + \sum_{s=2}^{\infty} a_s v^s, \quad \xi(v) = \sum_{s=2}^{\infty} b_s v^s,$$

is said to belong to the class $\mathcal{HE}_{1, \varkappa, \tau}^0(d_1, d_2)$, if

$$\Re \left\{ \frac{\eta'(v) + \varkappa v \eta''(v)}{\mathfrak{E}_{d_1, d_2}(v)} - \tau \right\} > \left| \frac{\xi'(v) + \varkappa v \xi''(v)}{\mathfrak{E}_{d_1, d_2}(v)} \right|, \quad v \in \mathbb{D}. \quad (1.13)$$

Table 1 show the relations between the new class and previous known classes.

Table 1. Relations between the proposed class and selected previously studied harmonic differential inequalities.

Previously studied classes	Specialization in the present framework	Resulting condition
Second-order harmonic differential structure [4]	$\mathfrak{E}_{d_1,d_2}(v) \equiv 1, \tau = 0, \kappa = \gamma$	$\Re \{ \eta'(v) + \gamma v \eta''(v) \} > \xi'(v) + \gamma v \xi''(v) $
Associated second-order differential structure [6]	$\mathfrak{E}_{d_1,d_2}(v) \equiv 1, \tau = 0, \kappa = 1$	$\Re \{ \eta'(v) + v \eta''(v) \} > \xi'(v) + v \xi''(v) $
Second-order differential inequality of Yalçın et al. [10]	$\mathfrak{E}_{d_1,d_2}(v) \equiv 1, k = 1$	$\Re \{ \eta'(v) + \kappa v \eta''(v) - \tau \} > \xi'(v) + \kappa v \xi''(v) $
First-order harmonic differential structure [10, 15]	$\mathfrak{E}_{d_1,d_2}(v) \equiv 1, \kappa = 0$	$\Re \{ \eta'(v) - \tau \} > \xi'(v) $
Present Mittag–Leffler-based class	$\mathfrak{E}_{d_1,d_2}(v)$ retained, $0 \leq \tau < 1, \kappa \geq 0$	$\Re \left\{ \frac{\eta'(v) + \kappa v \eta''(v)}{\mathfrak{E}_{d_1,d_2}(v)} - \tau \right\} > \left \frac{\xi'(v) + \kappa v \xi''(v)}{\mathfrak{E}_{d_1,d_2}(v)} \right $

Example 1.1. Let $d_1 = d_2 = 1$. Then the normalized Mittag–Leffler kernel reduces to $\mathfrak{E}_{1,1}(v) = e^v$. Although this parameter choice does not satisfy the sufficient coefficient-sum condition (1.10), the required non-vanishing property can be verified directly. Indeed, for $|v| < 1$,

$$\Re \{ \mathfrak{E}_{1,1}(v) \} = \Re(e^v) = e^{\Re(v)} \cos(\Im(v)).$$

Since $|v| < 1$, we have

$$|\Im(v)| < 1 < \frac{\pi}{2},$$

and hence

$$\cos(\Im(v)) > 0.$$

Therefore,

$$\Re \{ \mathfrak{E}_{1,1}(v) \} > 0, \quad v \in \mathbb{D},$$

which implies

$$\mathfrak{E}_{1,1}(v) \neq 0, \quad v \in \mathbb{D}.$$

Now consider

$$f(v) = \eta(v) + \xi(v),$$

where

$$\eta(v) = v + \frac{1}{5}v^2, \quad \xi(v) = \frac{1}{20}v^2.$$

Choose

$$\kappa = 0.8, \quad \tau = \frac{1}{4}.$$

Since

$$\eta'(v) = 1 + \frac{2}{5}v, \quad \eta''(v) = \frac{2}{5},$$

and

$$\xi'(v) = \frac{1}{10}v, \quad \xi''(v) = \frac{1}{10},$$

the defining condition (1.13) becomes

$$\Re \left\{ \frac{1 + \frac{18}{25}v}{e^v} - \frac{1}{4} \right\} > \left| \frac{\frac{9}{50}v}{e^v} \right|, \quad v \in \mathbb{D}.$$

The corresponding residual function is

$$\mathcal{R}(v) = \Re \left\{ \frac{1 + \frac{18}{25}v}{e^v} - \frac{1}{4} \right\} - \left| \frac{\frac{9}{50}v}{e^v} \right|.$$

Numerical evaluation of $\mathcal{R}(v)$ on a fine grid in the unit disk indicates positive values throughout the sampled region. This example is therefore presented only as a numerical illustration of the defining inequality and does not constitute a rigorous proof of class membership.

Example 1.2. Let $d_1 = 2$ and $d_2 = 1$, so that

$$\mathfrak{G}_{2,1}(v) = \cosh(\sqrt{v}).$$

Consider

$$f(v) = \eta(v) + \overline{\xi(v)},$$

where

$$\eta(v) = v, \quad \xi(v) = \frac{1}{500}v^2.$$

Choose

$$\kappa = \frac{1}{5}, \quad \tau = \frac{1}{4}.$$

Then

$$\eta'(v) = 1, \quad \eta''(v) = 0,$$

and

$$\xi'(v) = \frac{1}{250}v, \quad \xi''(v) = \frac{1}{250}.$$

Therefore, condition (1.13) becomes

$$\Re \left\{ \frac{1}{\cosh(\sqrt{v})} - \frac{1}{4} \right\} > \left| \frac{\frac{3}{625}v}{\cosh(\sqrt{v})} \right|, \quad v \in \mathbb{D}.$$

Define the residual function

$$\mathcal{R}(v) = \Re \left\{ \frac{1}{\cosh(\sqrt{v})} - \frac{1}{4} \right\} - \left| \frac{\frac{3}{625}v}{\cosh(\sqrt{v})} \right|.$$

Numerical evaluation of $\mathcal{R}(v)$ on a fine grid in the unit disk indicates positive values throughout the sampled region. The example is included as a numerical illustration of the proposed Mittag–Leffler-based condition and is not intended as a rigorous verification of class membership. Below Figures 1–4 show the image of unit disk under the different analytic functions.

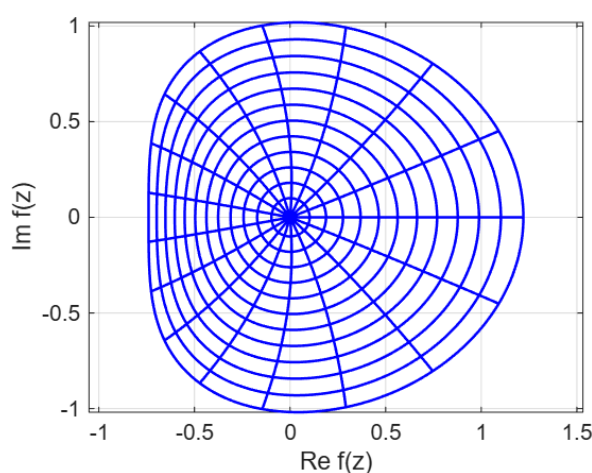


Figure 1. Image of the unit disk under the harmonic mapping $f(v) = v + \frac{1}{5}v^2 + \overline{\left(\frac{1}{20}v^2\right)}$, related to the normalized Mittag–Leffler kernel for $d_1 = d_2 = 1, \kappa = 0.8, \tau = \frac{1}{4}$.

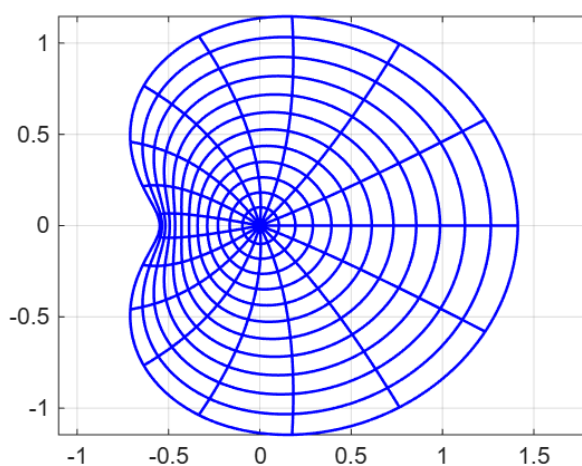


Figure 2. Image of the unit disk under the harmonic mapping $f(v) = v + \frac{1}{4}v^2 + \overline{\left(\frac{1}{15}v^2\right)}$, associated with the normalized Mittag–Leffler kernel for $d_1 = 2, d_2 = 1, \kappa = 0, \tau = \frac{1}{3}$.

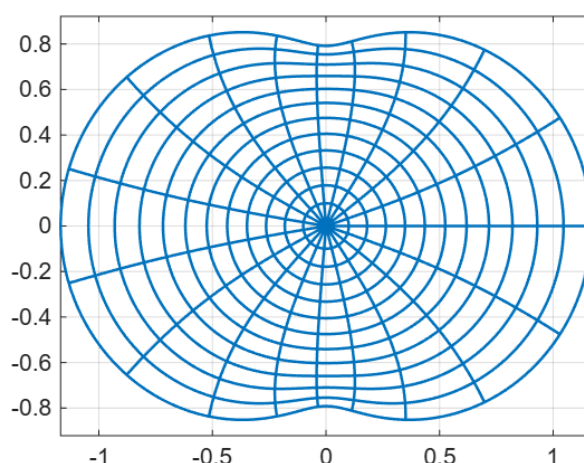


Figure 3. Image of the unit disk under the analytic function $f(v) = v + \frac{1}{5}v^3$, corresponding to the parameters $d_1 = 2$, $d_2 = 1$, $\tau = \frac{1}{1000}$, and $\varkappa = 0.1$. The figure shows the geometric transformation of concentric circles and radial segments of the unit disk under the mapping f .

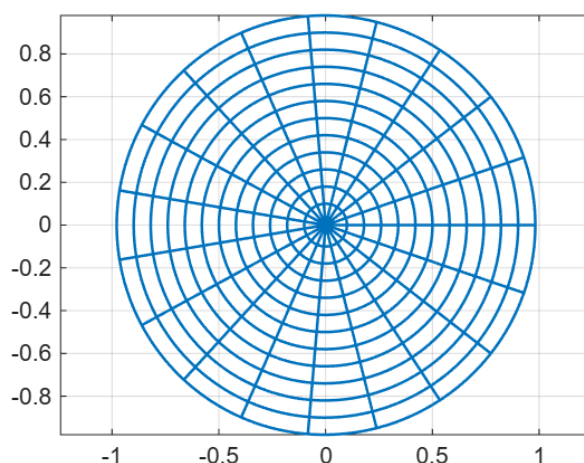


Figure 4. Image of the unit disk under the analytic function $f(v) = v + \frac{1}{500}v^2$, associated with the parameters $d_1 = 2$, $d_2 = 1$, $\tau = \frac{1}{4}$, and $\varkappa = \frac{1}{5}$.

1.2. Motivation, contributions, and novelty

Although harmonic subclasses defined by differential inequalities and the geometric properties of Mittag–Leffler functions have been extensively investigated separately, the interaction between these two directions has received comparatively limited attention. In particular, harmonic subclasses generated directly through normalized Mittag–Leffler kernels have not been systematically investigated. Motivated by these developments and the literature, we introduce the normalized Mittag–Leffler kernel

$$\mathfrak{E}_{d_1, d_2}(v) = \frac{\mathcal{E}_{d_1, d_2}(v)}{v},$$

and use it to define the new harmonic subclass

$$\mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2), \quad 0 \leq \tau < 1, \kappa \geq 0.$$

The main novelty of this work lies in integrating the normalized two-parameter Mittag–Leffler kernel directly into a second-order harmonic differential inequality. This construction provides a natural connection between harmonic mapping theory and Mittag–Leffler-type special functions and enables the effect of parameters d_1 and d_2 to be reflected in the resulting geometric properties.

Our primary objective of this paper is to develop a systematic geometric theory for the newly introduced class $\mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2)$. Especially, we establish an analytic properties through the related analytic family, derive close-to-convexity criteria, and obtain coefficient and growth estimates. We further investigate inclusion relations and closure under convex combinations, together with a Fekete–Szegő type coefficient inequality for the associated analytic combinations. These results establish the normalized two-parameter Mittag–Leffler kernel provides an effective and flexible framework for determining and studying new harmonic subclasses and their geometric behavior. The identity kernel also clarifies the connection of the proposed class with previously studied harmonic differential inequalities.

2. Set of lemmas

Lemma 2.1. [32]. *Let*

$$f(v) = v + \sum_{s=2}^{\infty} a_s v^s.$$

If

$$\sum_{s=2}^{\infty} (s - \tau) |a_s| < 1 - \tau,$$

then

$$f \in \mathcal{S}^*(\tau).$$

Lemma 2.2 (Clunie–Sheil–Small [1]). *Let η and ξ be analytic functions in the unit disk $\mathbb{D}$ such that*

$$|\xi'(0)| < |\eta'(0)|.$$

Suppose that, for each ε with $|\varepsilon| = 1$, the analytic function

$$\Upsilon_{\varepsilon}(v) = \eta(v) + \varepsilon \xi(v)$$

is close-to-convex in $\mathbb{D}$. Then the harmonic function

$$f(v) = \eta(v) + \overline{\xi(v)}$$

is close-to-convex in $\mathbb{D}$.

Lemma 2.3 ([33, 34]). *Let*

$$q(v) \in \mathcal{P},$$

where

$$\mathcal{P} = \left\{ q(v) = 1 + \sum_{s=1}^{\infty} q_s v^s : \Re\{q(v)\} > 0, v \in \mathbb{D} \right\}.$$

Then

$$|q_s| \leq 2, \quad s \geq 1.$$

Moreover, for every $\alpha \in \mathbb{C}$,

$$|q_2 - \alpha q_1^2| \leq 2 \max\{1, |2\alpha - 1|\}.$$

3. Main results

3.1. Analytic characterization

The result we present below establishes a connection between the harmonic function class $\mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2)$ and the analytic function class $\mathcal{E}_{1,\kappa,\tau}(d_1, d_2)$.

Theorem 3.1. *Let $f = \eta + \bar{\xi} \in \mathcal{H}^0$. Then*

$$f \in \mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2)$$

if and only if

$$\Upsilon_{\varepsilon}(v) = \eta(v) + \varepsilon \xi(v) \in \mathcal{E}_{1,\kappa,\tau}(d_1, d_2)$$

for every ε with $|\varepsilon| = 1$.

Proof. Since

$$\mathfrak{E}_{d_1, d_2}(v) \neq 0, v \in \mathbb{D},$$

the functions

$$A(v) = \frac{\eta'(v) + \kappa v \eta''(v)}{\mathfrak{E}_{d_1, d_2}(v)}$$

and

$$B(v) = \frac{\xi'(v) + \kappa v \xi''(v)}{\mathfrak{E}_{d_1, d_2}(v)}$$

are well defined in $\mathbb{D}$. Suppose that

$$f \in \mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2).$$

Then, by the definition of $\mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2)$, we have

$$\Re A(v) - \tau > |B(v)|, \quad v \in \mathbb{D}.$$

Let ε be any complex number with $|\varepsilon| = 1$. Then

$$\Re\{A(v) + \varepsilon B(v)\} = \Re A(v) + \Re\{\varepsilon B(v)\}.$$

Since

$$\Re\{\varepsilon B(v)\} \geq -|B(v)|,$$

we obtain

$$\Re\{A(v) + \varepsilon B(v)\} \geq \Re A(v) - |B(v)| > \tau.$$

Moreover,

$$\Upsilon'_\varepsilon(v) + \varkappa v \Upsilon''_\varepsilon(v) = \eta'(v) + \varkappa v \eta''(v) + \varepsilon(\xi'(v) + \varkappa v \xi''(v)).$$

Therefore,

$$\Re\left\{\frac{\Upsilon'_\varepsilon(v) + \varkappa v \Upsilon''_\varepsilon(v)}{\mathfrak{E}_{d_1, d_2}(v)}\right\} = \Re\{A(v) + \varepsilon B(v)\} > \tau.$$

Hence,

$$\Upsilon_\varepsilon \in \mathcal{E}_{1, \varkappa, \tau}(d_1, d_2)$$

for every $|\varepsilon| = 1$.

Conversely, suppose that

$$\Upsilon_\varepsilon(v) = \eta(v) + \varepsilon \xi(v) \in \mathcal{E}_{1, \varkappa, \tau}(d_1, d_2)$$

for every ε with $|\varepsilon| = 1$. Then

$$\Re\{A(v) + \varepsilon B(v)\} > \tau, \quad v \in \mathbb{D}.$$

If $B(v) = 0$, then

$$\Re A(v) > \tau,$$

and hence

$$\Re A(v) - \tau > |B(v)|.$$

If $B(v) \neq 0$, choose

$$\varepsilon = -\frac{\overline{B(v)}}{|B(v)|}.$$

Then $|\varepsilon| = 1$ and

$$\varepsilon B(v) = -|B(v)|.$$

Thus,

$$\tau < \Re\{A(v) + \varepsilon B(v)\} = \Re A(v) - |B(v)|.$$

Therefore,

$$\Re A(v) - \tau > |B(v)|.$$

Hence, by the definition of $\mathcal{HE}_{1, \varkappa, \tau}^0(d_1, d_2)$, we get

$$f \in \mathcal{HE}_{1, \varkappa, \tau}^0(d_1, d_2).$$

This completes the proof. □

3.2. Starlikeness of the normalized Mittag–Leffler function

Theorem 3.2. Let $d_1 > 0$ and $d_2 > 0$. Suppose that

$$\sum_{s=2}^{\infty} (s - \tau) \left| \frac{\Gamma(d_2)}{\Gamma(d_1(s-1) + d_2)} \right| \leq 1 - \tau, \quad 0 \leq \tau < 1.$$

Then the normalized Mittag–Leffler function $\mathcal{E}_{d_1, d_2}$ belongs to $\mathcal{S}^*(\tau)$.

Proof. The normalized Mittag–Leffler function has the expansion

$$\mathcal{E}_{d_1, d_2}(v) = v + \sum_{s=2}^{\infty} A_s v^s,$$

where

$$A_s = \frac{\Gamma(d_2)}{\Gamma(d_1(s-1) + d_2)}.$$

Hence, the assumed condition is precisely

$$\sum_{s=2}^{\infty} (s - \tau) |A_s| \leq 1 - \tau.$$

Therefore, Lemma 2.1 gives

$$\mathcal{E}_{d_1, d_2} \in \mathcal{S}^*(\tau).$$

This completes the proof. □

Corollary 3.1. If $\tau = 0$ and

$$\sum_{s=2}^{\infty} s \left| \frac{\Gamma(d_2)}{\Gamma(d_1(s-1) + d_2)} \right| \leq 1,$$

then

$$\mathcal{E}_{d_1, d_2} \in \mathcal{S}^*.$$

3.3. Close-to-convexity for the analytic κ -transform

Theorem 3.3. Let $0 \leq \tau < 1$, $\kappa \geq 0$, and suppose that

$$\mathcal{E}_{d_1, d_2} \in \mathcal{S}^*.$$

If

$$\Upsilon \in \mathcal{E}_{1, \kappa, \tau}(d_1, d_2),$$

then the function

$$L(v) = (1 - \kappa)\Upsilon(v) + \kappa v \Upsilon'(v)$$

is close-to-convex of order τ in $\mathbb{D}$.

Proof. Since

$$\Upsilon \in \mathcal{E}_{1,\kappa,\tau}(d_1, d_2),$$

we have

$$\Re \left\{ \frac{\Upsilon'(v) + \kappa v \Upsilon''(v)}{\mathfrak{E}_{d_1, d_2}(v)} \right\} > \tau, \quad v \in \mathbb{D}.$$

Using

$$\mathfrak{E}_{d_1, d_2}(v) = \frac{\mathcal{E}_{d_1, d_2}(v)}{v},$$

we obtain

$$\Re \left\{ \frac{v(\Upsilon'(v) + \kappa v \Upsilon''(v))}{\mathcal{E}_{d_1, d_2}(v)} \right\} > \tau.$$

Define

$$L(v) = (1 - \kappa)\Upsilon(v) + \kappa v \Upsilon'(v).$$

Then

$$L'(v) = (1 - \kappa)\Upsilon'(v) + \kappa(\Upsilon'(v) + v \Upsilon''(v)) = \Upsilon'(v) + \kappa v \Upsilon''(v).$$

Therefore,

$$\Re \left\{ \frac{v L'(v)}{\mathcal{E}_{d_1, d_2}(v)} \right\} > \tau, \quad v \in \mathbb{D}.$$

Since

$$\mathcal{E}_{d_1, d_2} \in \mathcal{S}^*,$$

the last inequality shows that L is close-to-convex of order τ in $\mathbb{D}$. This completes the proof. $\square$

3.4. Close-to-convexity of the harmonic κ -transform

Theorem 3.4. Let $0 \leq \tau < 1$, $\kappa \geq 0$, and suppose that

$$\mathcal{E}_{d_1, d_2} \in \mathcal{S}^*.$$

If

$$f = \eta + \bar{\xi} \in \mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2),$$

then the harmonic mapping

$$\mathcal{L}_\kappa f(v) = \mathcal{L}_\kappa \eta(v) + \overline{\mathcal{L}_\kappa \xi(v)}$$

is close-to-convex in $\mathbb{D}$, where

$$\mathcal{L}_\kappa \eta(v) = (1 - \kappa)\eta(v) + \kappa v \eta'(v),$$

and

$$\mathcal{L}_\kappa \xi(v) = (1 - \kappa)\xi(v) + \kappa v \xi'(v).$$

Proof. Let

$$f = \eta + \bar{\xi} \in \mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2).$$

Then, by Theorem 3.1, for every ε with $|\varepsilon| = 1$, the analytic function

$$\Upsilon_\varepsilon(v) = \eta(v) + \varepsilon \xi(v) \in \mathcal{E}_{1,\kappa,\tau}(d_1, d_2).$$

Hence, by Theorem 3.3, the function

$$L_\varepsilon(v) = (1 - \kappa)\Upsilon_\varepsilon(v) + \kappa v\Upsilon'_\varepsilon(v)$$

is close-to-convex of order τ in $\mathbb{D}$.

Now

$$L_\varepsilon(v) = (1 - \kappa)(\eta(v) + \varepsilon\xi(v)) + \kappa v(\eta'(v) + \varepsilon\xi'(v)).$$

Therefore,

$$L_\varepsilon(v) = \mathcal{L}_\kappa\eta(v) + \varepsilon\mathcal{L}_\kappa\xi(v).$$

Thus, for every $|\varepsilon| = 1$, the analytic function

$$\mathcal{L}_\kappa\eta(v) + \varepsilon\mathcal{L}_\kappa\xi(v)$$

is close-to-convex in $\mathbb{D}$.

Since $f \in \mathcal{H}^0$, we have

$$\eta'(0) = 1, \quad \xi'(0) = 0.$$

Moreover,

$$(\mathcal{L}_\kappa\eta)'(0) = (1 - \kappa)\eta'(0) + \kappa(\eta'(0) + 0 \cdot \eta''(0)) = 1,$$

and

$$(\mathcal{L}_\kappa\xi)'(0) = (1 - \kappa)\xi'(0) + \kappa(\xi'(0) + 0 \cdot \xi''(0)) = 0.$$

Hence,

$$|(\mathcal{L}_\kappa\xi)'(0)| < |(\mathcal{L}_\kappa\eta)'(0)|.$$

By Lemma 2.2, it follows that

$$\mathcal{L}_\kappa f(v) = \mathcal{L}_\kappa\eta(v) + \overline{\mathcal{L}_\kappa\xi(v)}$$

is close-to-convex in $\mathbb{D}$. □

Corollary 3.2. *Let*

$$\mathcal{E}_{d_1, d_2} \in \mathcal{S}^*.$$

If

$$f \in \mathcal{HE}_{1,0,\tau}^0(d_1, d_2),$$

then f is close-to-convex in $\mathbb{D}$.

Proof. Setting $\kappa = 0$ in Theorem 3.4, we obtain

$$\mathcal{L}_0 f = f.$$

Hence, the conclusion follows immediately. □

3.5. Coefficient estimates

Theorem 3.5. *Let*

$$f = \eta + \bar{\xi} \in \mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2).$$

Assume that

$$\mathcal{E}_{d_1,d_2}(v) = v + \sum_{s=2}^{\infty} A_s v^s \in \mathcal{S}^*.$$

Then, for $s \geq 2$,

$$|a_s| + |b_s| \leq \frac{\tau + s(1 - \tau)}{1 + \kappa(s - 1)}.$$

Proof. Let

$$f = \eta + \bar{\xi} \in \mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2).$$

By Theorem 3.1, for every ε with $|\varepsilon| = 1$, the analytic function

$$\Upsilon_{\varepsilon}(v) = \eta(v) + \varepsilon \xi(v) = v + \sum_{s=2}^{\infty} (a_s + \varepsilon b_s) v^s \in \mathcal{E}_{1,\kappa,\tau}(d_1, d_2).$$

Hence,

$$\Re \left\{ \frac{\Upsilon'_{\varepsilon}(v) + \kappa v \Upsilon''_{\varepsilon}(v)}{\mathfrak{E}_{d_1,d_2}(v)} \right\} > \tau, \quad v \in \mathbb{D}.$$

Since

$$\mathfrak{E}_{d_1,d_2}(v) = \frac{\mathcal{E}_{d_1,d_2}(v)}{v},$$

we obtain

$$\Re \left\{ \frac{v(\Upsilon'_{\varepsilon}(v) + \kappa v \Upsilon''_{\varepsilon}(v))}{\mathcal{E}_{d_1,d_2}(v)} \right\} > \tau.$$

Therefore, for each fixed ε with $|\varepsilon| = 1$, there exists a function

$$q_{\varepsilon}(v) = 1 + \sum_{s=1}^{\infty} q_s v^s, \quad \Re\{q_{\varepsilon}(v)\} > 0,$$

such that

$$\frac{v(\Upsilon'_{\varepsilon}(v) + \kappa v \Upsilon''_{\varepsilon}(v))}{\mathcal{E}_{d_1,d_2}(v)} = \tau + (1 - \tau)q_{\varepsilon}(v).$$

Equivalently,

$$v(\Upsilon'_{\varepsilon}(v) + \kappa v \Upsilon''_{\varepsilon}(v)) = [\tau + (1 - \tau)q_{\varepsilon}(v)]\mathcal{E}_{d_1,d_2}(v). \quad (3.1)$$

From the left-hand side of (3.1), we get

$$v(\Upsilon'_{\varepsilon}(v) + \kappa v \Upsilon''_{\varepsilon}(v)) = v + \sum_{s=2}^{\infty} s[1 + \kappa(s - 1)](a_s + \varepsilon b_s)v^s.$$

Comparing the coefficient of v^s in (3.1), we obtain

$$s[1 + \kappa(s - 1)](a_s + \varepsilon b_s) = A_s + (1 - \tau)q_{s-1} + (1 - \tau) \sum_{j=2}^{s-1} A_j q_{s-j}.$$

Since $\mathcal{E}_{d_1, d_2} \in \mathcal{S}^*$, we have

$$|A_j| \leq j, \quad j \geq 2.$$

Also, Lemma 2.3 gives

$$|q_j| \leq 2, \quad j \geq 1.$$

Therefore,

$$\begin{aligned} & s[1 + \kappa(s-1)]|a_s + \varepsilon b_s| \\ & \leq |A_s| + (1-\tau)|q_{s-1}| + (1-\tau) \sum_{j=2}^{s-1} |A_j||q_{s-j}| \\ & \leq s + 2(1-\tau) + 2(1-\tau) \sum_{j=2}^{s-1} j. \end{aligned}$$

Hence,

$$\begin{aligned} s[1 + \kappa(s-1)]|a_s + \varepsilon b_s| & \leq s + 2(1-\tau) + 2(1-\tau) \left(\frac{s(s-1)}{2} - 1 \right) \\ & = s + (1-\tau)s(s-1) \\ & = s[\tau + s(1-\tau)]. \end{aligned}$$

Thus,

$$|a_s + \varepsilon b_s| \leq \frac{\tau + s(1-\tau)}{1 + \kappa(s-1)}.$$

Choosing ε such that

$$|a_s + \varepsilon b_s| = |a_s| + |b_s|,$$

we obtain

$$|a_s| + |b_s| \leq \frac{\tau + s(1-\tau)}{1 + \kappa(s-1)}.$$

This completes the proof. □

3.6. Growth estimates

Theorem 3.6 (Growth estimate). *Let*

$$f = \eta + \bar{\xi} \in \mathcal{HE}_{1, \kappa, \tau}^0(d_1, d_2),$$

and suppose that

$$\mathcal{E}_{d_1, d_2} \in \mathcal{S}^*.$$

For $|v| = r$, where $0 \leq r < 1$, define

$$M(r) = \sum_{s=2}^{\infty} \frac{\tau + s(1-\tau)}{1 + \kappa(s-1)} r^s.$$

Then $M(r)$ converges for every $0 \leq r < 1$, and

$$|f(v)| \leq r + M(r), \quad |v| = r < 1,$$

while

$$|f(v)| \geq r - M(r), \quad |v| = r < 1.$$

In particular, whenever

$$M(r) < r,$$

the lower estimate is positive and hence nontrivial, so that

$$r - M(r) \leq |f(v)| \leq r + M(r).$$

Proof. Let $|v| = r$, where $0 \leq r < 1$. We have

$$f(v) = v + \sum_{s=2}^{\infty} a_s v^s + \overline{\sum_{s=2}^{\infty} b_s v^s}.$$

Therefore, by the triangle inequality,

$$|f(v)| \leq |v| + \sum_{s=2}^{\infty} |a_s| |v|^s + \sum_{s=2}^{\infty} |b_s| |v|^s.$$

Hence,

$$|f(v)| \leq r + \sum_{s=2}^{\infty} (|a_s| + |b_s|) r^s.$$

Using Theorem 3.5, we obtain

$$|a_s| + |b_s| \leq \frac{\tau + s(1 - \tau)}{1 + \kappa(s - 1)}.$$

Thus,

$$|f(v)| \leq r + \sum_{s=2}^{\infty} \frac{\tau + s(1 - \tau)}{1 + \kappa(s - 1)} r^s = r + M(r).$$

Moreover,

$$\frac{\tau + s(1 - \tau)}{1 + \kappa(s - 1)} = O(1) \quad (s \rightarrow \infty),$$

for $\kappa \geq 0$. Consequently, the series defining $M(r)$ converges for every $0 \leq r < 1$.

For the lower bound, using the reverse triangle inequality, we get

$$|f(v)| \geq |v| - \left| \sum_{s=2}^{\infty} a_s v^s + \sum_{s=2}^{\infty} b_s v^s \right|.$$

Therefore,

$$|f(v)| \geq r - \sum_{s=2}^{\infty} |a_s| r^s - \sum_{s=2}^{\infty} |b_s| r^s.$$

That is,

$$|f(v)| \geq r - \sum_{s=2}^{\infty} (|a_s| + |b_s|) r^s.$$

Applying Theorem 3.5 once more, we obtain

$$|f(v)| \geq r - \sum_{s=2}^{\infty} \frac{\tau + s(1-\tau)}{1 + \kappa(s-1)} r^s = r - M(r).$$

Hence,

$$|f(v)| \geq r - M(r).$$

In particular, if $M(r) < r$, then

$$|f(v)| \geq r - M(r) > 0,$$

and the lower estimate is nontrivial. This completes the proof. $\square$

3.7. Closure under convex combinations

Theorem 3.7. *Let*

$$f_j = \eta_j + \overline{\xi_j} \in \mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2), \quad j = 1, 2, \dots, N,$$

and let $\lambda_j \geq 0$ with

$$\sum_{j=1}^N \lambda_j = 1.$$

Then

$$L(v) = \sum_{j=1}^N \lambda_j f_j(v)$$

also belongs to

$$\mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2).$$

Proof. For each $j = 1, 2, \dots, N$, write

$$f_j(v) = \eta_j(v) + \overline{\xi_j(v)}.$$

Define

$$L(v) = \sum_{j=1}^N \lambda_j f_j(v).$$

Then

$$L(v) = \sum_{j=1}^N \lambda_j \eta_j(v) + \overline{\sum_{j=1}^N \lambda_j \xi_j(v)}.$$

Put

$$\Upsilon(v) = \sum_{j=1}^N \lambda_j \eta_j(v), \quad \Lambda(v) = \sum_{j=1}^N \lambda_j \xi_j(v).$$

Thus,

$$L(v) = \Upsilon(v) + \overline{\Lambda(v)}.$$

Since

$$f_j \in \mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2),$$

we have

$$\Re \left\{ \frac{\eta'_j(v) + \kappa v \eta''_j(v)}{\mathfrak{E}_{d_1, d_2}(v)} - \tau \right\} > \left| \frac{\xi'_j(v) + \kappa v \xi''_j(v)}{\mathfrak{E}_{d_1, d_2}(v)} \right|, \quad v \in \mathbb{D}.$$

Now

$$\Upsilon'(v) + \kappa v \Upsilon''(v) = \sum_{j=1}^N \lambda_j (\eta'_j(v) + \kappa v \eta''_j(v)),$$

and

$$\Lambda'(v) + \kappa v \Lambda''(v) = \sum_{j=1}^N \lambda_j (\xi'_j(v) + \kappa v \xi''_j(v)).$$

Therefore,

$$\begin{aligned} & \Re \left\{ \frac{\Upsilon'(v) + \kappa v \Upsilon''(v)}{\mathfrak{E}_{d_1, d_2}(v)} - \tau \right\} \\ &= \Re \left\{ \sum_{j=1}^N \lambda_j \frac{\eta'_j(v) + \kappa v \eta''_j(v)}{\mathfrak{E}_{d_1, d_2}(v)} - \tau \sum_{j=1}^N \lambda_j \right\} \\ &= \sum_{j=1}^N \lambda_j \Re \left\{ \frac{\eta'_j(v) + \kappa v \eta''_j(v)}{\mathfrak{E}_{d_1, d_2}(v)} - \tau \right\} \\ &> \sum_{j=1}^N \lambda_j \left| \frac{\xi'_j(v) + \kappa v \xi''_j(v)}{\mathfrak{E}_{d_1, d_2}(v)} \right|. \end{aligned}$$

Using the triangle inequality, we obtain

$$\sum_{j=1}^N \lambda_j \left| \frac{\xi'_j(v) + \kappa v \xi''_j(v)}{\mathfrak{E}_{d_1, d_2}(v)} \right| \geq \left| \sum_{j=1}^N \lambda_j \frac{\xi'_j(v) + \kappa v \xi''_j(v)}{\mathfrak{E}_{d_1, d_2}(v)} \right|.$$

Hence,

$$\Re \left\{ \frac{\Upsilon'(v) + \kappa v \Upsilon''(v)}{\mathfrak{E}_{d_1, d_2}(v)} - \tau \right\} > \left| \frac{\Lambda'(v) + \kappa v \Lambda''(v)}{\mathfrak{E}_{d_1, d_2}(v)} \right|.$$

Therefore,

$$L = \Upsilon + \overline{\Lambda} \in \mathcal{HE}_{1, \kappa, \tau}^0(d_1, d_2).$$

This completes the proof. $\square$

3.8. Inclusion relation

Theorem 3.8. *Let*

$$0 \leq \tau_1 \leq \tau_2 < 1.$$

Then

$$\mathcal{HE}_{1, \kappa, \tau_2}^0(d_1, d_2) \subset \mathcal{HE}_{1, \kappa, \tau_1}^0(d_1, d_2).$$

Proof. Let

$$f = \eta + \bar{\xi} \in \mathcal{HE}_{1,\kappa,\tau_2}^0(d_1, d_2).$$

Then

$$\Re \left\{ \frac{\eta'(v) + \kappa v \eta''(v)}{\mathfrak{E}_{d_1, d_2}(v)} - \tau_2 \right\} > \left| \frac{\xi'(v) + \kappa v \xi''(v)}{\mathfrak{E}_{d_1, d_2}(v)} \right|.$$

Since

$$\tau_1 \leq \tau_2,$$

we have

$$\Re \left\{ \frac{\eta'(v) + \kappa v \eta''(v)}{\mathfrak{E}_{d_1, d_2}(v)} - \tau_1 \right\} = \Re \left\{ \frac{\eta'(v) + \kappa v \eta''(v)}{\mathfrak{E}_{d_1, d_2}(v)} - \tau_2 \right\} + (\tau_2 - \tau_1).$$

Hence,

$$\Re \left\{ \frac{\eta'(v) + \kappa v \eta''(v)}{\mathfrak{E}_{d_1, d_2}(v)} - \tau_1 \right\} > \left| \frac{\xi'(v) + \kappa v \xi''(v)}{\mathfrak{E}_{d_1, d_2}(v)} \right|.$$

Therefore,

$$f \in \mathcal{HE}_{1,\kappa,\tau_1}^0(d_1, d_2).$$

This proves the inclusion. $\square$

3.9. Initial coefficient estimates

Theorem 3.9. *Let*

$$f = \eta + \bar{\xi} \in \mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2).$$

Then

$$|a_2| + |b_2| \leq \frac{|A_2| + 2(1 - \tau)}{2(1 + \kappa)},$$

and

$$|a_3| + |b_3| \leq \frac{|A_3| + 2(1 - \tau)|A_2| + 2(1 - \tau)}{3(1 + 2\kappa)},$$

where

$$A_2 = \frac{\Gamma(d_2)}{\Gamma(d_1 + d_2)}, \quad A_3 = \frac{\Gamma(d_2)}{\Gamma(2d_1 + d_2)}.$$

Proof. By Theorem 3.1, for every ε with $|\varepsilon| = 1$, the analytic function

$$\Upsilon_\varepsilon(v) = \eta(v) + \varepsilon \xi(v) \in \mathcal{E}_{1,\kappa,\tau}(d_1, d_2).$$

Write

$$\Upsilon_\varepsilon(v) = v + c_2 v^2 + c_3 v^3 + \cdots,$$

where

$$c_2 = a_2 + \varepsilon b_2, \quad c_3 = a_3 + \varepsilon b_3.$$

Since

$$\Upsilon_\varepsilon \in \mathcal{E}_{1,\kappa,\tau}(d_1, d_2),$$

there exists a function

$$q_\varepsilon(v) = 1 + q_1 v + q_2 v^2 + \cdots, \quad \Re\{q_\varepsilon(v)\} > 0,$$

such that

$$\frac{\Upsilon'_\varepsilon(v) + \kappa v \Upsilon''_\varepsilon(v)}{\mathfrak{E}_{d_1, d_2}(v)} = \tau + (1 - \tau)q_\varepsilon(v).$$

Equivalently,

$$\Upsilon'_\varepsilon(v) + \kappa v \Upsilon''_\varepsilon(v) = [\tau + (1 - \tau)q_\varepsilon(v)]\mathfrak{E}_{d_1, d_2}(v). \quad (3.2)$$

Now, from the left-hand side of (3.2), we have

$$\Upsilon'_\varepsilon(v) + \kappa v \Upsilon''_\varepsilon(v) = 1 + 2(1 + \kappa)c_2v + 3(1 + 2\kappa)c_3v^2 + \cdots. \quad (3.3)$$

Also,

$$\mathfrak{E}_{d_1, d_2}(v) = 1 + A_2v + A_3v^2 + \cdots,$$

where

$$A_2 = \frac{\Gamma(d_2)}{\Gamma(d_1 + d_2)}, \quad A_3 = \frac{\Gamma(d_2)}{\Gamma(2d_1 + d_2)}.$$

Hence, from the right-hand side of (3.2), we have

$$[\tau + (1 - \tau)q(v)]\mathfrak{E}_{d_1, d_2}(v) = 1 + [A_2 + (1 - \tau)q_1]v + [A_3 + (1 - \tau)A_2q_1 + (1 - \tau)q_2]v^2 + \cdots. \quad (3.4)$$

Comparing the coefficients of v in (3.3) and (3.4), we obtain

$$2(1 + \kappa)c_2 = A_2 + (1 - \tau)q_1. \quad (3.5)$$

Thus,

$$2(1 + \kappa)|c_2| \leq |A_2| + (1 - \tau)|q_1|.$$

Lemma 2.3 gives

$$|c_2| \leq \frac{|A_2| + 2(1 - \tau)}{2(1 + \kappa)}.$$

Hence,

$$|a_2| + |b_2| \leq \frac{|A_2| + 2(1 - \tau)}{2(1 + \kappa)}.$$

Again, comparing the coefficients of v^2 in (3.3) and (3.4), we obtain

$$3(1 + 2\kappa)c_3 = A_3 + (1 - \tau)A_2q_1 + (1 - \tau)q_2. \quad (3.6)$$

Hence,

$$3(1 + 2\kappa)|c_3| \leq |A_3| + (1 - \tau)|A_2||q_1| + (1 - \tau)|q_2|.$$

Again, by Lemma 2.3, we obtain

$$|c_3| \leq \frac{|A_3| + 2(1 - \tau)|A_2| + 2(1 - \tau)}{3(1 + 2\kappa)}.$$

Hence,

$$|a_3| + |b_3| \leq \frac{|A_3| + 2(1 - \tau)|A_2| + 2(1 - \tau)}{3(1 + 2\kappa)}.$$

This completes the proof. $\square$

3.10. Fekete–Szegő inequality for functions in the family $\mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2)$

Theorem 3.10. *Let*

$$f = \eta + \bar{\xi} \in \mathcal{HE}_{1,\kappa,\tau}^0(d_1, d_2).$$

For each fixed ε with $|\varepsilon| = 1$, define the analytic combination

$$\Upsilon_\varepsilon(v) = \eta(v) + \varepsilon\xi(v) = v + \sum_{s=2}^{\infty} c_s v^s,$$

where

$$c_s = a_s + \varepsilon b_s, \quad s \geq 2.$$

For any $\mu \in \mathbb{C}$, consider the associated Fekete–Szegő functional

$$\Phi_{\mu,\varepsilon}(f) = c_3 - \mu c_2^2 = (a_3 + \varepsilon b_3) - \mu(a_2 + \varepsilon b_2)^2.$$

Then

$$\begin{aligned} |\Phi_{\mu,\varepsilon}(f)| \leq & \left| \frac{A_3}{3(1+2\kappa)} - \frac{\mu A_2^2}{4(1+\kappa)^2} \right| \\ & + 2(1-\tau)|A_2| \left| \frac{1}{3(1+2\kappa)} - \frac{\mu}{2(1+\kappa)^2} \right| \\ & + \frac{2(1-\tau)}{3(1+2\kappa)} \max \left\{ 1, \left| \frac{3\mu(1-\tau)(1+2\kappa)}{2(1+\kappa)^2} - 1 \right| \right\}, \end{aligned}$$

where

$$A_2 = \frac{\Gamma(d_2)}{\Gamma(d_1 + d_2)}, \quad A_3 = \frac{\Gamma(d_2)}{\Gamma(2d_1 + d_2)}.$$

The estimate holds for every fixed $|\varepsilon| = 1$.

Proof. For each fixed ε with $|\varepsilon| = 1$, by the analytic characterization theorem, there exists a Carathéodory function

$$q_\varepsilon(v) = 1 + q_1 v + q_2 v^2 + \cdots, \quad \Re\{q_\varepsilon(v)\} > 0,$$

such that

$$\frac{\Upsilon'_\varepsilon(v) + \kappa v \Upsilon''_\varepsilon(v)}{\mathfrak{E}_{d_1, d_2}(v)} = \tau + (1-\tau)q_\varepsilon(v).$$

Comparing coefficients as in (3.5) and (3.6), we obtain

$$c_2 = \frac{A_2 + (1-\tau)q_1}{2(1+\kappa)}$$

and

$$c_3 = \frac{A_3 + (1-\tau)A_2 q_1 + (1-\tau)q_2}{3(1+2\kappa)}.$$

Therefore,

$$\begin{aligned} c_3 - \mu c_2^2 = & \frac{A_3}{3(1+2\kappa)} + \frac{(1-\tau)A_2 q_1}{3(1+2\kappa)} + \frac{(1-\tau)q_2}{3(1+2\kappa)} \\ & - \frac{\mu}{4(1+\kappa)^2} \left[A_2^2 + 2(1-\tau)A_2 q_1 + (1-\tau)^2 q_1^2 \right]. \end{aligned}$$

After rearranging, we get

$$\begin{aligned} c_3 - \mu c_2^2 = & \left[\frac{A_3}{3(1+2\kappa)} - \frac{\mu A_2^2}{4(1+\kappa)^2} \right] \\ & + (1-\tau)A_2 \left[\frac{1}{3(1+2\kappa)} - \frac{\mu}{2(1+\kappa)^2} \right] q_1 \\ & + \frac{1-\tau}{3(1+2\kappa)} \left[q_2 - \frac{3\mu(1-\tau)(1+2\kappa)}{4(1+\kappa)^2} q_1^2 \right]. \end{aligned}$$

Taking absolute values and using the triangle inequality, we obtain

$$\begin{aligned} |c_3 - \mu c_2^2| \leq & \left| \frac{A_3}{3(1+2\kappa)} - \frac{\mu A_2^2}{4(1+\kappa)^2} \right| \\ & + (1-\tau)|A_2| \left| \frac{1}{3(1+2\kappa)} - \frac{\mu}{2(1+\kappa)^2} \right| |q_1| \\ & + \frac{1-\tau}{3(1+2\kappa)} |q_2 - \nu q_1^2|, \end{aligned}$$

where

$$\nu = \frac{3\mu(1-\tau)(1+2\kappa)}{4(1+\kappa)^2}.$$

By Lemma 2.3, we obtain

$$\begin{aligned} |\Phi_{\mu,\varepsilon}(f)| = |c_3 - \mu c_2^2| \\ \leq \left| \frac{A_3}{3(1+2\kappa)} - \frac{\mu A_2^2}{4(1+\kappa)^2} \right| \\ + 2(1-\tau)|A_2| \left| \frac{1}{3(1+2\kappa)} - \frac{\mu}{2(1+\kappa)^2} \right| \\ + \frac{2(1-\tau)}{3(1+2\kappa)} \max \left\{ 1, \left| \frac{3\mu(1-\tau)(1+2\kappa)}{2(1+\kappa)^2} - 1 \right| \right\}. \end{aligned}$$

This completes the proof. $\square$

4. Conclusions

In this paper, we introduced a new subclass of harmonic functions associated with the normalized two-parameter Mittag–Leffler function through a second-order differential inequality involving the normalized Mittag–Leffler kernel. The proposed class established a natural connection between harmonic mapping theory and the geometric theory of Mittag–Leffler functions. First, an analytic characterization of the proposed harmonic class was established via the family of analytic functions $\Upsilon_\varepsilon = \eta + \varepsilon\xi$, which enabled the transfer of several geometric properties from analytic functions to harmonic mappings. Sufficient conditions for the starlikeness of the normalized Mittag–Leffler function were then derived and subsequently used to establish close-to-convexity results for both the analytic and harmonic κ -transforms. Furthermore, inclusion relations, coefficient estimates, growth

bounds, closure under convex combinations, and Fekete–Szegő inequalities were established for the proposed class. Several illustrative examples and graphical representations were also presented to demonstrate the geometric behavior of functions belonging to the class.

The results obtained in this work give a new directions for studying harmonic mappings related to the normalized Mittag–Leffler kernel and establish several geometric properties, such as, coefficient, growth, inclusion, close-to-convexity, and Fekete–Szegő results. In future studies may focus on extending the present work to generalized and multi-parameter Mittag–Leffler functions, as well as to q -operators, q -Mittag–Leffler functions, and fractional operators. In further directions includes the study of harmonic subclasses related to other special-function kernels and the investigation of additional geometric properties, such as differential subordination, Hankel determinants, and related coefficient problems.

Author contributions

All authors contributed equally to this work and have read and agreed to the published version of the manuscript.

Funding

The author extends their appreciation to the deputyship for Research & Innovation, Ministry of Education in Saudi Arabia for funding this research work through the project number (IFP-2022-2026-406).

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

References

1. J. Clunie, T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **9** (1984), 3–25. <https://doi.org/10.5186/aasfm.1984.0905>
2. P. Duren, *Harmonic mappings in the plane*, Cambridge: Cambridge University Press, 2004.
3. S. Ponnusamy, H. Yamamoto, H. Yanagihara, Variability regions for certain families of harmonic univalent mappings, *Complex Var. Elliptic Equ.*, **58** (2013), 23–34. <https://doi.org/10.1080/17476933.2010.551200>

4. N. Ghosh, A. Vasudevarao, Some basic properties of certain subclass of harmonic univalent functions, *Complex Var. Elliptic Equ.*, **63** (2018), 1687–1703. <https://doi.org/10.1080/17476933.2017.1403426>
5. D. Breaz, A. Durmus, S. Yalçın, L. I Cotirla, H. Bayram, Certain properties of harmonic functions defined by a second-order differential inequality, *Mathematics*, **11** (2023), 4039. <https://doi.org/10.3390/math11194039>
6. S. Nagpal, V. Ravichandran, Convolution properties of the harmonic Koebe function and its connection with 2-starlike mappings, *Complex Var. Elliptic Equ.*, **60**, (2015), 191–210. <https://doi.org/10.1080/17476933.2014.907284>
7. O. Mishra, A. Çetinkaya, J. Sokól, Harmonic close-to-convex mappings associated with Sălăgean q -differential operator, *Complex Var. Elliptic Equ.*, **70** (2025), 33–49. <https://doi.org/10.24193/subbmath.2025.1.03>
8. S. Çakmak, E. Yaşar, S. Yalcın, New subclass of the class of close-to-convex harmonic mappings defined by a third-order differential inequality, *Hacet. J. Math. Stat.*, **51** (2022), 172–186. <https://doi.org/10.15672/hujms.922981>
9. D. Breaz, Y. Polatoglu, N. Breaz, Generalized p -valent Janowski close-to-convex functions and their applications to the harmonic mappings, In: *Nonlinear analysis: stability, approximation, and inequalities*, New York: Springer, 2012, 79–89. https://doi.org/10.1007/978-1-4614-3498-6_6
10. S. Yalçın, H. Bayram, G. I. Oros, Some properties and graphical applications of a new subclass of harmonic functions defined by a differential inequality, *Mathematics*, **12** (2024), 2338. <https://doi.org/10.3390/math12152338>
11. O. Al-Refai, Some properties for a class of analytic functions defined by a higher-order differential inequality, *Turkish J. Math.*, **43** (2019), 2473–2493. <https://doi.org/10.3906/mat-1906-65>
12. S. Li, L. Ma, H. Tang, On Janowski type p -harmonic functions associated with generalized Sălăgean operator, *AIMS Math.*, **6** (2021), 569–583. <https://doi.org/10.3934/math.2021035>
13. K. R. Alhindi, Application of the q -derivative operator to a specialized class of harmonic functions exhibiting positive real part, *AIMS Math.*, **10** (2025), 1935–1944. <https://doi.org/10.3934/math.2025090>
14. S. Li, L. Ma, X. Niu, H. Tang, Some properties of Bazilevič harmonic functions, *AIMS Math.*, **10** (2025), 18824–18837. <https://doi.org/10.3934/math.2025841>
15. S. Çakmak, S. Yalçın, On a subclass of harmonic close-to-convex functions generated by the harmonic error function, *Math. Sci. Appl. E-Notes*, **14** (2026), 32–44. <https://doi.org/10.36753/mathenot.1838201>
16. L. Li, S. Ponnusamy, Disk of convexity of sections of univalent harmonic functions, *J. Math. Anal. Appl.*, **408** (2013), 589–596. <https://doi.org/10.1016/j.jmaa.2013.06.021>
17. L. Li, S. Ponnusamy, Injectivity of sections of univalent harmonic mappings, *Nonlinear Anal.*, **89** (2013), 276–283. <https://doi.org/10.1016/j.na.2013.05.016>
18. N. Ghosh, A. Vasudevarao, On a subclass of harmonic close-to-convex mappings, *Monatsh. Math.*, **188** (2019), 247–267. <https://doi.org/10.1007/s00605-017-1138-7>

19. Rajbala, J. K. Prajapat, Certain geometric properties of close-to-convex harmonic mappings, *Asian-Eur. J. Math.*, **14** (2021), 2150102. <https://doi.org/10.1142/S1793557121501023>
20. S. Çakmak, E. Yaşar, S. Yalçın, Some basic properties of a subclass of close-to-convex harmonic mappings, *TWMS J. Pure Appl. Math.*, **15** (2024), 163–173. <https://doi.org/10.30546/2219-1259.15.2.2024.01163>
21. E. Yaşar, S. Yalçın, Close-to-convexity of a class of harmonic mappings defined by a third-order differential inequality, *Turkish J. Math.*, **45** (2021), 678–694. <https://doi.org/10.3906/mat-2004-50>
22. H. M. Fraz, K. Ali, M. F. Nadeem, N. Ali, F. T. Tolasa, Predictive modeling of copper iodide properties using graph-theoretical descriptors, *Sci. Rep.*, **15** (2025), 39878. <https://doi.org/10.1038/s41598-025-23657-w>
23. G. M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, *C. R. Acad. Sci. Paris*, **137** (1903), 554–558. Available from: <https://archive.org/details/comptesrendusheb137acad/page/554/mode/2up>.
24. A. Wiman, Über den fundamentalsatz in der theorie der funktionen $E_\alpha(x)$, *Acta Math.*, **29** (1905), 191–201. <https://doi.org/10.1007/BF02403202>
25. I. Podlubny, *Fractional differential equations*, California: Academic Press, 1999.
26. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, In: *North-Holland mathematics studies*, Amsterdam: Elsevier, 2006.
27. D. Bansal, H. Orhan, Partial sums of Mittag–Leffler function, *J. Math. Inequal.*, **12** (2018), 423–431. <https://doi.org/10.7153/jmi-2018-12-31>
28. H. M. Srivastava, D. Bansal, Close-to-convexity of a certain family of q -Mittag–Leffler functions, *J. Nonlinear Var. Anal.*, **1** (2017), 61–69.
29. A. A. Attiya, Some applications of Mittag–Leffler function in the unit disk, *Filomat*, **30** (2016), 2075–2081. <https://doi.org/10.2298/FIL1607075A>
30. N. Taşar, F. M. Sakar, S. M. Aydoğan, G. I. Oros, Relations of harmonic starlike function subclasses with Mittag–Leffler function. *Axioms*, **13** (2024), 826. <https://doi.org/10.3390/axioms13120826>
31. M. U. Shah, B. Khan, M. Oliveira, S. Araci, Characterization of p -valent q -starlike functions through Hadamard product and Poisson distribution, *Appl. Math. Sci. Eng.*, **33** (2025), 2478040. <https://doi.org/10.1080/27690911.2025.2478040>
32. H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51** (1975), 109–116. <https://doi.org/10.1090/S0002-9939-1975-0369678-0>
33. C. Pommerenke, *Univalent functions*, Germany: Vandenhoeck and Ruprecht, 1975.
34. R. J. Libera, E. J. Złotkiewicz, Coefficient bounds for the inverse of a function with derivative in P , *Proc. Amer. Math. Soc.*, **87** (1983), 251–257. <https://doi.org/10.2307/2043698>



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)